Variational strategies on the study of the existence and uniqueness theorem for ordinary differential equations

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In this paper, we study historical proofs of the existence and uniqueness theorem for the differential equation $y' = f(x, y)$. We analyze original works that played a part in the problematization of this concept, thus offering a rational reconstruction of its genesis which sheds light on its meaning. We will use results from the socioepistemologic theory to show that variational strategies are efficient in the analysis of the proof. We believe this epistemological analysis may help in the future for pedagogical designs.

Keywords: Socioepistemology, variation, existence, uniqueness, differential equation.

Introduction

We present the results of an ongoing research on the existence and uniqueness theorem for first order ordinary differential equations. The socioepistemological theory, theoretical basis of the research, faces, from a systemic view, a possible reconstruction of its meaning due to the variational strategies in its development, without limiting itself to a chronological reproduction of the contributions of the mathematical works, nor a reinterpretation of what we actually know of the theorem.

Variational strategies involved the construction of the mathematical notions related to the theorem mentioned above are analyzed. These strategies are part of a study program called "Variational Thinking and Language" that the socioepistemological program develops. From the study a different use of the actual knowledge is recognized, that is considered a potential element to start changing the relation with said knowledge from the objects to the actual practices (Fallas-Soto, 2015).

This notion of "from the objects to the actual practices" talks about constructing a new interpretation of the object (the theorem in this case) based on practices (Cantoral, 2013), with the notion of starting with the *problematization of knowledge*, from the Socioepistemology, finding the meanings of said knowledge at the moment of actual use. Then, the problematization from this view, consists in performing a double study whose elements, *historize* (historical reconstruction of knowledge) and *dialectize* (coordination of mathematical notions, examples, counterexamples and conceptions and misconceptions), are the base to study the evolution of the theorem throughout history and thus to analyze how its associated mathematical notions plays in order to construct the theorem from the point of what is known today.

Therefore, the research problem is linked to the *meaning* in mathematics, this because the way how the existence and uniqueness theorem is presented on textbooks does not appears to be deducted from the practices properly, nor also from the mathematization. Then, the problem, which are the principles that give meaning to the notions of existence and uniqueness as specific characteristics of the solution's nature? The hypothesis of this research work is to assume that the construction of the theorem can be a prediction model with the study of variation. It can be said that there's a common

thread that organizes the conditions for the existence and uniqueness, and is concected with the prediction idea that was the basis for the theorem along with its hypothesis.

The research focus for this research was to *mean the existence and uniqueness theorem* from a particular problematization of mathematical knowledge.

Methodology

The phases that describe the research are briefly described below: we study the textbooks used in the teaching of Differential Equations, particularly we analyze the demonstration. By showing that there is no problematization of this mathematical knowledge and only remains as a test of hypothesis, it is decided to study the genesis of this knowledge and its evolution. We offer a rational reconstruction of the main arguments used by mathematicians and in the part of conclusions we make a comparison of the origin of the theorem with the current didactic treatment.

A rational reconstruction for the existence and uniqueness for differential equations, in a first stage, a bibliographical research is presented with some of the mathematical works of the time that helped on the construction of this knowledge. These are:

- (Cauchy & Moigno, 1844), *"Leçons de Calcul Différentiel et de Calcul Intégral"*
- (Lipschitz, 1880), *"Lehrbuch der Analysis"*
- (Lipschitz, 1868) *"Disamina della possibilità d' integrare completamente un dato sistema di equazioni differenziali ordinarie"*
- (Peano, 1973 – new-edition) *"Sull' integrabilità delle equazioni differenziali di primo ordine".*
- (Picard, 1886) *"Cours d' Analyse"*

This proposal there will be reported the results obtained in the analysis of Cauchy & Moigno (1844) and Lipschitz (1880) papers, all of them for the study of the variational strategies. These five books were chosen because of Picard and Peano appear in the current textbooks contributing to the theorem. Then Picard (1886) in his work refers to Cauchy and Moigno in addition to Lipschitz.

On a second stage, a documental analysis, the elements used on the mathematical works for the study of the existence and uniqueness theorem are reconstructed. This gives contributions to generate implementation strategies that propitiate the construction of the theorem based on a pragmatic evolution of the practices. To obtain the conclusions of tis work, we perform a confrontation between the mathematical works (also used for teaching) that arose at the end of the XIX century and at the beginning of the XX century, with ideas from the textbooks of XX century and nowadays.

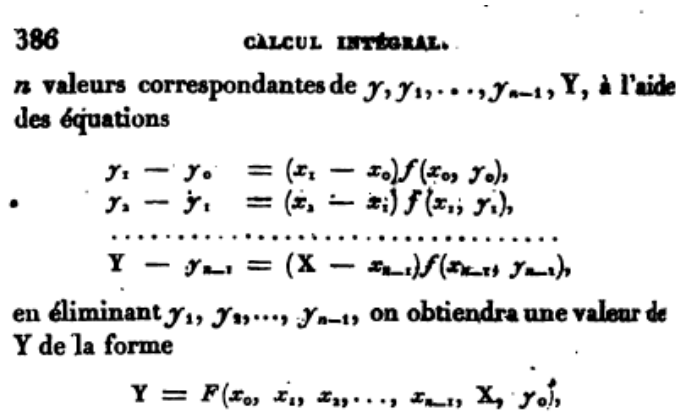
With respect to the variational strategies presents on this theorem, we are based on the PylVar (That its acronym in Spanish means Thinking and Variational Language) program that has been used throughout the years by some authors (Caballero, 2012), (Cabrera, 2009), (Cantoral, 2004; 2013a), (Cantoral & Farfan, 1998). This approach seeks to show the construction of mathematical knowledge from the study of motion, change and the variation of physical or natural phenomena. Our research work does not mention the modeling of phenomena related to the theorem of existence and uniqueness, however in the study of change is how the differential equations are born and we continue to deepen in current research. These works have permitted to explicit the variational strategies (practices on the study of change) that the Pylvar program describes, and that are discussed in Caballero (2012):

- Comparison: Associated to the action of establishing differences between states.
- Serialization: It is associated with the action of establishing relations between successive states. To study the changes to determine a certain pattern.
- Estimation: Starting from other knowlegde of changing states, proposing states for a short term.
- Prediction: The action of being able to determinate after analyzing some states to deduct posterior states. It means to anticipate to a certain rational state.

Results

We do not offer a mathematical proof but to show arguments that helped the mathematicians in formalizing this mathematical knowledge.

Cauchy & Moigno (1844) consider the differential equation $y' = f(x, y)$ with the hypothesis that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions with the initial condition (x_0, y_0). The authors prove the convergence of the sequence of points obtained by the fractions method.



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n valeurs correspondantes de $y, y_1, \ldots, y_{n-1}, Y$, à l'aide des équations

$$\begin{aligned} y_1 - y_0 &= (x_1 - x_0) f(x_0, y_0), \\ y_2 - y_1 &= (x_2 - x_1) f(x_1, y_1), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y - y_{n-1} &= (X - x_{n-1}) f(x_{n-1}, y_{n-1}), \end{aligned}$$

en éliminant $y_1, y_2, \ldots, y_{n-1}$, on obtiendra une valeur de Y de la forme

$$Y = F(x_0, x_1, x_2, \ldots, x_{n-1}, X, y_0),$$

Figure 1: Fractions method utilized by Cauchy & Moingo, page 386 of the lesson 26 of the work Leçons de Calcul Differentiel et de Calcul Intégral.

This method (this method of approximation was expounded by Euler in 1768 in his *Institutionum Calculi Integralis*) consists on determining successive states that depend on the preceding state parting on the study of the linearity and small variations that can be taken with the differential equation (as the rate of change that determines the slope of the line) and the initial condition (point by which the line passes). This procedure is shown in the following figures.

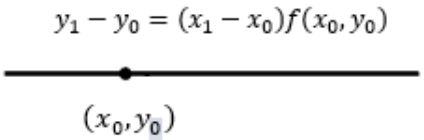

Figure 2: First iteration of the fractions method.

$$y_2 - y_0 = (x_1 - x_0)f(x_0, y_0) + (x_2 - x_1)f\big(x_1, y_0 + (x_1 - x_0)f(x_0, y_0)\big)$$

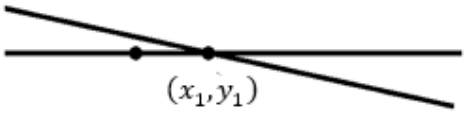

Figure 3: Second iteration of the fractions method

Therefore, the process is generalized and it is approximated that $y - y_0 = \pm\Theta A(x_n - x_0)$where $A$is an average of the $f(x_n, y_n)$and Θa value between 0 and 1. Then, the authors generalize the process and stating that $y_n$is equal to

$$y_n = y_0 + (x_n - x_0)f\big(x_0 + \theta(x_n - x_0), y_0 \pm \Theta A(x_n - x_0)\big)$$

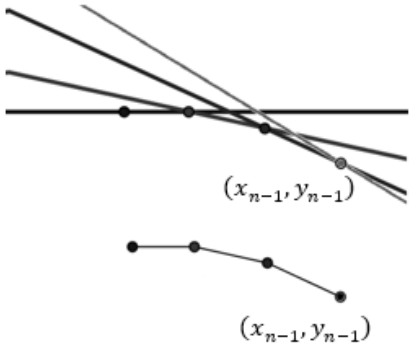

Figure 4: N-th iteration obtaining a numerical approximation of the solution.

From which $y_n$corresponds, practically, to the same value that $y_0$if the difference $x_n - x_0$is small. In other words, if $x_n \to x_0$then $y_n \to y_0$.

Besides, convergence is studied in the following form as part of a stability of the system. If a small increment $\varsigma_0$is added to y_0, then $y_n$will have an increment ς_n. In order for it to converge, this last increment has to be as small as ς_0. This seeks the stability of the function $y$to guarantee the existence. Again, the small variation plays a fundamental role to compare states and thus determine a local prediction on each iteration to determine a final global prediction (estimation) of the system.

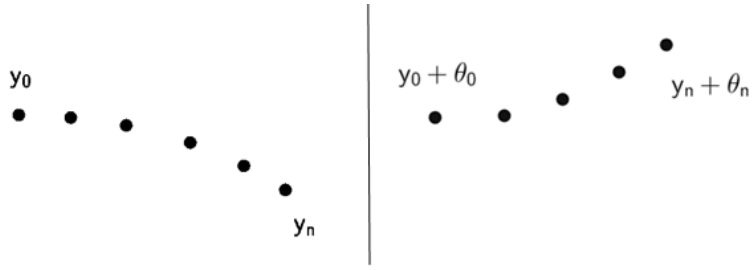

Figure 5: Case where the increment of $y_n$changes considerably with respect to the increment of y_0.

This result is studied even further by Lipchitz (1880) when working with systems of equations and the uniqueness of the solution. By looking at figure 05, what is really happening is that from the first

iteration two tangent lines are obtained that correspond, respectively, to each of the solutions. Because of this, on the point (x_1, y_1), and another one close to it, say (x_1, η_1), with $\eta_1 = y_1 + \theta_1$such that $\eta_1 - y_1$is close to zero, two tangent lines are determined, given by

$$y - y_1 = (x - x_1)f(x_1, y_1)$$

and

$$y - \eta_1 = (x - x_1)f(x_1, \eta_1)$$

Which are depicted in the following representation

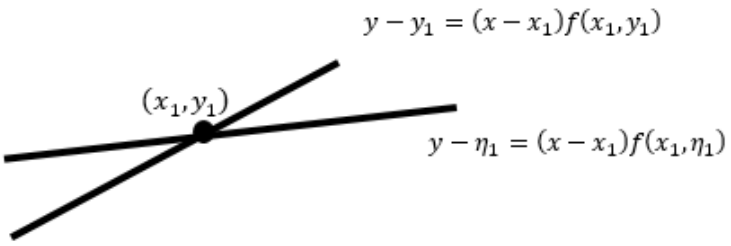

Figure 6: The existence of two solutions for the equation.

Then, if we study the difference between these two lines, we see that is the same that subtracting their two respective slopes, this because $\eta_1$is a value that is close to y_1. Notice that the distance between the two lines is determined by

$$|f(x_1, \eta_1) - f(x_1, y_1)|$$

That is why the Lipschitz condition plays a very important role, due to the fact that this difference would be bounded by

$$|f(x_1, \eta_1) - f(x_1, y_1)| < M|\eta_1 - y_1|$$

where the constant $M$is the bound of $\frac{\partial f}{\partial y}$ that not necessarily is a continuous function. Therefore, if this condition holds, we would have that $|f(x_1, \eta_1) - f(x_1, y_1)| = 0$, being a unique solution to the equation. If the Lipschitz condition does not hold, we cannot guarantee uniqueness, it can or cannot exists.

Conclusions

This study, from the socioepistemological point of view, broadens the knowledge on the Existence and Uniqueness Theorem for Ordinary Differential Equations, but most of all it shows the kind of practices (variational strategies) that play a role in the justification of both existence and uniqueness. The fractions method, absent in most textbooks on Differential Equations, is present in textbooks on numerical methods. This is the second time this phenomenon arises. A similar situation arose in the prediction based on Taylor series. This findings, are worth mentioning, are derived from an appropriate problematization of the mathematical knowledge. Two things that gave genesis to this problem were the looking for a formalization on the proof of the theorem and also the determination of the minimum quantity of hypothesis that guarantee the existence and uniqueness of the solution.

Additionally, the initial questions referred to the inverse tangent problem and the different examples present in the mathematical discourse for school were answered, but these time with the support different resources: variational, numerical, analytical and visual. All these was obtained thanks to a

documental analysis that was based on original mathematical works. On the other hand, it was possible to discuss other constructs, such as convergence, the Lipschitz condition and the continuity of the functions $f(x, y)$ and $\frac{\partial f}{\partial y}$, with respect to y.

Besides, the variational strategies on the construction of this theorem were:

- Comparison: The states that correspond to the numerical solution of the differential equation are compared. Besides, the solution is compared before and after of the small variation to determine its uniqueness.
- Serialization: When finding a relation between a state and another, starting with the initial conditions, in a lot of cases are possible to obtain an analytical solution to the equation, while in other cases it is only possible to predict the value that the solution will take in the next state (numerical solution), both cases with the support of the study of the patterns between one state and the other.
- Estimation: When knowing the initial values and unknowing the next value that determines the numerical solution of the equation, is when the linear approximation enters to determine the next value or state.
- Prediction: This theorem corresponds to a predictive model, it is utilized to predict the existence of the solution, and with the small variation we are certain of the convergence and uniqueness of the solution.

When performing a study such as this one, it is possible to study the rupture between most contemporary textbooks against the works reported on the mathematical studies of the past. Besides, it can be observed that some rationalities arise that will help to design teaching and learning activities by means of the use of teaching variables or control variables to modify, keeping in mind the present construction of these meanings. Reconstructing these meanings helped us to understand other problems related with differential equations, such as the stability of a system of differential equations, and to construct other visual interpretations.

References

Caballero, M. (2012). *Un estudio de las dificultades en el desarrollo del pensamiento y lenguaje variacional en profesores de bachillerato* (Tesis de Maestría). México: Cinvestav.

Cabrera, L. (2009). *El pensamiento y lenguaje variacional y el desarrollo de competencias. Un estudio en el marco de la Reforma Integral de Bachillerato* (Tesis de Maestría). México: Cinvestav.

Cantoral, R., Farfán, R. (1998). Pensamiento y lenguaje variacional en la introducción al análisis. *Epsilon, 42*(14), 353−369.

Cantoral, R. (2004). Desarrollo del pensamiento y lenguaje variacional, una mirada socioepistemológica. *Acta Latinoamericana de Matemática Educativa 17,* 1–9.

Cantoral, R. (2013a). *Desarrollo del pensamiento y lenguaje variacional*. Distrito Federal, México: Secretaria de Educación Pública.

Cantoral, R. (2013). *Teoría socioepistemológica de la matemática educativa. Estudios sobre la construcción social del conocimiento*. Barcelona: Gedisa.

Cauchy, A., & Moigno. (1844). *Leçons de calcul différentiel et de calcul intégral. Tome II : Calcul intégral, 1ère partie*. Paris: Bachelier.

Fallas-Soto, R. (2015). *Existencia y unicidad: Estudio socioepistemológico de la solución de las ecuaciones diferenciales ordinarias de primer orden* (Tesis de maestría no publicada). Cinvestav, México.

Lipschitz, R. (1868). Disamina della pknossibilità d' integrare completamente un dato sistema di equazioni differenziali ordinarie. *Annali di Matematica Pura ed Applicata, 2*(2), 288–302.

Lipschitz, R. (1880). *Lehrbuch der analysis*. Bonn, Deutschland: Verlag Von Max Cohen & Sohn.

Peano, G. (1973). Sull' integrabilità delle equazioni differenziali di primo ordine. *Atti della Reale Accademia delle Scienze di Torino 21* (1885-1886), 677–685. (Reprinted in *Peano 1957−1959, 1*, 74–81).

Picard, E. (1886). *Cours d' analyse*. Faculté des Sciences de Paris.