The concept of infinity-different meanings through the centuries

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The concept of infinity and its use is one with different meanings through the centuries within various contexts reflecting mathematical historical development. This development is scarcely clear to pupils during school time and rarely stressed in teacher education although it offers a lot of potential to understand mathematics. The field of arithmetic is one example in which to study infinity within a range that student teachers are able to understand and that is useful for and in their future teaching. The paper focusses on the potential of this concept within the arithmetic field using an original article of Cantor and on examples, also from Hilbert, that stress different counting methods and various illustrations of infinity.

Keywords: Infinity, countable set, denumerable set, very small/large numbers.

Introduction

Like many other mathematical concepts ideas of infinity developed and diversified through the centuries. Well known is its appearance within the Elements of Euclid within the ninth chapter: There are more prime numbers than any given number of prime numbers, cf. Euclid (1997, IX, §20). This statement answers the question, if there is an end of prime numbers. The proof uses the fact that the product of arbitrary prime numbers added by 1 has a new prime number in its prime factorization. That is, to conceptualize the concept of infinity of prime numbers Euclid uses the idea that there is always another one apart from the given ones which has a somewhat operative aspect.

Also within geometric contexts there are ideas about infinity regarding the extensions of space, planes and lines as well as the number of points on lines and figures. There are considerations of the behavior of parallel lines regarding infinity, like for example the parallel axiom states.

Already these examples show that there exist different meanings of infinity depending on the various mathematical objects. Also, the meanings are verbalized without formulization. Doubtless does the use of very large numbers within everyday experiences come close to a sense of infinity. For example, the addition of two numbers, one of which is very much larger than the other, form a sum that does not differ very much from the large number in terms of the relative error concept.

Infinity and the use of it deserves a closer look because terms like “infinite”, “endless” and “unlimited” have a colloquial meaning that sometimes provides a reasonable starting point for the understanding of abstract patterns and sometimes not. In some instances infinity incorporates the imagination of a very large extension, endlessness is for many just a word for a very large but still finite set of objects or a very large extension like the ocean.

First-year students often show a vague notion of the term “infinity”, an observation that was done also by Woerner (2013). She even points out that a thorough understanding of infinity is neither a goal nor is it a step towards understanding mathematics. Dötschel (2011) even finds that the understanding of infinity does not vary much between teacher students and pupils of secondary level. In school they learned the lemniscate symbol “∞” and used it associated with the limit of sequences, series and – at the best – the differential quotient. At least in principle they know that it is not allowed to use this symbol like a number or a variable in all respects (like it was used partly in the 18th century), but one can observe a high degree of uncertainty. For instance, what does it mean when the differential quotient is interpreted in terms like “zero divided by zero”? And what is the outcome? Standard lectures like the ones for Analysis and Linear Algebra usually do not change the perspective and are continued with refined conceptions of limits and the definition of concepts like that of an infinitely dimensional space. Concerning the cardinality of sets there remains often the sketchy notation $|A| = \infty$ or the like.

Precisely because counting, the determination of a number, is a fundamental concept at all school levels, it is surprising that the issue of cardinality seems to be somewhat neglected. Nevertheless, it is not too difficult to provide a base of knowledge due to Georg Cantor. In many cases original mathematical treatises are definitely unsuitable within teacher education, but there are notable exceptions. One of these is the article of Cantor (1895), which is understandable in large parts. The reason for this is that students are able to acquire understanding with supporting examples on the side of the teacher which will be shown later in the paragraph.

Our paper describes the used examples during the seminar held in the summer term 2016 together with some details around the concept of infinity.

Various ways towards an understanding of infinity

The mathematical education of future elementary math teachers at the University of Erfurt includes among mathematical survey lectures a seminar on the basic principles of arithmetic and algebra. One of the goals of this seminar is to improve the student's understanding of (natural) numbers and their properties since this field will constitute one of the bases of their future teaching. During the seminar students are often encountering the question in how far they have an understanding of the set of natural numbers being infinite. The upcoming discussions circle around the question how one could find out “what is true”. And the discussion ends with the question why. The phenomenon of infinite many natural numbers very often brought about an astonished attitude on the part of the students who pondered about the reasoning of Aristotle: There will be always one number that can be added.

In our seminar there was a focus on aspects of numbers, especially of the natural numbers. Apart from the Peano axioms, the aspect of ways of counting provides a reliable foundation, especially since the cardinalities of sets may be included. This approach promotes a formalization due to the concept of bijective mappings, which is of value in its own. The natural numbers can be regarded as cardinal numbers of finite sets. But what do students know about the cardinality of the set of the natural numbers? Being asked, students claim “infinite” and denote the lemniscate symbol which they know from limits of sequences or functions. In interviews they show an obvious uncertainty about the arithmetical behavior of the object “infinite”. In case of doubt they often suggest treating it like a “usual” number.

We feel responsible for giving a brief insight into the cultural heritage of the different approaches dealing with “infinity”. In particular we seek to show with what kind of caution Euclid, and Cantor too, got closer to proper descriptions, depending on the actual contexts. Especially we are interested in giving insight into the richness of mathematical thoughts and ideas: "It is possible to regard the history of the foundations mathematics as a progressive enlarging of the mathematical universe to include more and more infinities" (Rucker, 1982, p. 2). With regard to Cantor, we know that "... soon obtained a number of interesting results about actually infinite sets, most notably the result that the set of points on the real line constitutes a higher infinity than the set of all natural numbers. That is, Cantor was able to show that infinity is not an all or nothing concept: there are degrees of infinity." (Rucker, 1982, p. 9)

There are a lot of ways how to understand mathematical statements. Some point out their proof, some stress their genetic development, some point out their formal argument. Our actual understanding of infinity allows us to give statements like: the set of natural numbers or the set of natural numbers between 0 and 100. The first is an infinite set, the second is a finite one. Stressing the idea of a potential infinity which we could not grasp as a solid concept, we help ourselves by a stepwise approach knowing that we will never succeed. This very constructive standpoint or procedure permits a very simple activity and that is adding one, again and again: |, | |, | | |, … In this manner one can distinguish finite and infinite sets. For the first set the procedure ends with a certain number, for the second one there is no certain last number and it becomes clear that the procedure never ends. In both cases the counting is mathematically a 1-1-correspondence.

The different meanings of the term infinity show the richness of mathematics and its historical development. Needless to say that mathematical history does not develop in a regular and uniform way (Dieudonné, 1985, p. 16). Some epoch does not show any development in a field, in some there is a continuous change because of new developments. The fact that we use the word infinity the way we do with numbers goes back to Cantor (1895). It was the upcoming of new ideas, e.g. the idea of a set that changed the understanding of infinity.

How very much different this meaning is in contrast to the “old” Greek meaning shows when student teachers learn about it in their first mathematical lectures: infinity is hard to grasp and the use of it shows that school mathematics does not at all build a proper foundation. Because of its lack it is even more important to build a solid understanding during mathematical studies especially for student teachers as there are many potential links to basic notions of counting in their future teaching.

German mathematical education often refers to three basic experiences (“Grunderfahrungen“), by Winter (1996):

- perceiving phenomena of nature, society and culture;
- knowing (and appreciating) mathematical issues, represented by language, symbols, images and formulas;
- acquiring heuristic competencies.

We like to refer to Winter (1996) because he stresses a connection between everyday life experiences, heuristics and beginning formalization. In order to get aware of basic experiences and deepen the understanding there is a strategy necessary that gets students involved. Kattou et al. (2009) points out:

> In particular, academic programs offered to teachers should include mathematical knowledge regarding to infinity in combination with instructional approaches related to the concept. A proposed teaching approach could include the following steps: presentation with several typical tasks aimed at uncovering teachers' intuitions about the concept, discussion about infinity's applications in real life, introduction of the formal definition of infinity and the two aspects-potential and actual- and attempt to distinguish them in examples. (Kattou et al.2009).

Within this this context we pinpoint the following aspects:

1. The notion of infinity changed its meaning through the centuries. In the late 19th century the notion of aleph 0, aleph 1 and so forth came up.
2. The way infinity appears in mathematical textbooks follows the idea of Freudenthal's anti-didactical inversion. It is common to introduce infinity by using the lemniscate symbol, mostly just informing about it. The mathematical developments are neglected.
3. Some examples of infinite sets can be solved with simple steps used with finite sets. This presents an approach with a low barrier to student teachers and enhances their understanding of infinity.

Our didactical approach is influenced by Vollrath (1987) who proposed a phase model showing the process of understanding mathematical concepts:

intuitive and content-related $\rightarrow$ formal / integrated $\rightarrow$ critical

We therefore stress the finding of variations of standard examples and of counting strategies on the side of the students. The integration of Cantor's text provides some formalism and fostered discussions about the historical circumstances which were not controversial, that is the conflict between Cantor and Kronecker e.g.

The following paragraph presents examples that proved useful within elementary school teacher education.

Methods of counting

In many cases it turns out difficult to provide an original text to students with the expectation of an adequate comprehension. But just mathematical topics that lead to very fundamental issues may prove appropriate in order to their connection with intuition and imagination. Cantor (1895) develops a concept of elementary set theory, which includes transfinite cardinalities and their arithmetic properties. During teaching it became evident, that important parts of this text are quite understandable and can be an opportunity to discuss an historical treatise and express own reflections.

Before we start investigating into various ways of counting infinite sets we observe that there is no uniform definition of the concept infinity. The word occurs as an adjective to characterize sets especially. We concentrate therefore on the arithmetic field.

The following sections present a couple of examples that may foster the understanding of infinite sets.

Counting as one-to-one correspondence

The concept of a set, as introduced from Cantor (1895), surely fits into these frameworks. To him we owe the so-called "naïve" definition of a set.

By an "aggregate" we are to understand any collection into a whole *M* of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of *M*.

A counting or numerating of a finite set M with exactly *n* elements means, that every number 1, 2, 3, …, *n* is assigned to exactly one element of the set. This is linked to the concepts of maps and functions and more over bijectivity.

The set ℚ is countably infinite

This follows out of a scheme in which every positive rationale number shows one time and is arranged like this:

1/1	1/2	1/3	1/4	1/5	1/6	…
2/1	2/2	2/3	2/4	2/5	2/6	…
3/1	3/2	3/3	3/4	3/5	3/6	…
4/1	4/2	4/3	4/4	4/5	4/6	…
5/1	5/2	5/3	5/4	5/5	5/6	…
6/1	6/2	6/3	6/4	6/5	6/6	…
…	…	…	…	…	…	…

Table 1: Cantor's first diagonal method

The way the scheme is counted goes back to Cantor and is called the "diagonal method".

Following the presentation of the scheme students were invited to vary it and write down their proposals. Are there other suitable paths? What do they have in common? Furthermore, how could repetitions of numbers be avoided? In the scheme above every positive rational number is repeated infinitely. Does this cause problems? What options do we have to be represented by a reduced table of fractions? Is this already an indicative of the countability of even "larger" sets? After all most students could design various methods for counting even all rational numbers, for instance by designing spiral paths or the like.

The set ℝ is uncountable

Suppose that there is an enumeration

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...$$

of the interval [0, 1[, which is a subset of ℝ, and the numbers are represented in the decimal system, i.e.

$$\alpha_1 = 0,\underline{\alpha_{11}}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{15}.....$$

$$\alpha_2 = 0,\alpha_{21}\underline{\alpha_{22}}\alpha_{23}\alpha_{24}\alpha_{25}.....$$

$$\alpha_3 = 0,\alpha_{31}\alpha_{32}\underline{\alpha_{33}}\alpha_{34}\alpha_{35}.....$$

$$\alpha_4 = 0,\alpha_{41}\alpha_{42}\alpha_{43}\underline{\alpha_{44}}\alpha_{45}.....$$

$$.....$$

with digits $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, ... \in \{0, 1, ..., 9\}$ for every positive natural numbers i.

Now one can define a number $\beta = 0,\beta_1\beta_2\beta_3 ... \in [0, 1[$ such that

$$\beta_i = \begin{cases} 1, & if \quad \alpha_{ii} \neq 1 \\ 7, & if \quad \alpha_{ii} = 1 \end{cases}$$

for all positive integers i. Obviously, the representation of β possesses at least one decimal digit that differs from α_i, namely, the i-th digit. Therefore β cannot occur in the enumeration above, which is inevitably incomplete. Now, if the given interval is already uncountable, then all the more the real numbers are. This scheme originates from Cantor, too, and is called the second diagonal method.

To promote an adequate understanding, students did vary this scheme in a written form, also regarding other b-adic representations. At this point, the fundamental significance of place value systems in general is to be clarified. During teaching lessons students were encouraged to replace the digits by other symbols such as letters or notes from sheet music etc., and it has become clear, that the relating interpretations ("the entity of 'texts' is uncountable") can foster an adequate understanding in the sense that students are able to make a transfer.

Hilbert's Hotel

The cardinality of the set of the natural numbers is denoted by $\aleph_0$. In set theory several properties of this first transfinite cardinality are elaborated, as there are

$$1 + \aleph_0 = \aleph_0,$$

$$n + \aleph_0 = \aleph_0$$

for any $n \in \mathbb{N}$, as well as

$$2\aleph_0 = \aleph_0$$

and

$$n\aleph_0 = \aleph_0,$$

again for any natural number n. To illustrate this, the thought experiment of Hilbert's hotel is helpful: Suppose that there is a hotel with an unlimited number of single rooms, which are numbered according to the natural numbers. The hotel is fully occupied and one other person is knocking on the door. Will the hotel be able to accommodate this person, too? In the classical version each present guest moves up to the room that is numbered one greater as yet. In this way the first room (numbered by 0 or 1) becomes available and no one has to leave the hotel. The situation is very similar if two or a finite number of new guests ask to come in: The present guests move up in the rooms that are numbered n greater than now.

A bit more challenging is the arrival of a “Hilbertian bus” with an infinite number of passengers, named or numbered due to the natural numbers. In this case a constant moving up will not be successful. But the past guests could double their initial room number, and every passenger gets an oddly numbered room. This is not the only option available, students should contribute alternatives. More general, if there are two or a finite number n of “Hilbertian busses”, one can multiply every original room number with $n + 1$ and assign the passengers of the first bus those rooms, which numbers are congruent $n + 1$ modulo 1. The occupants of the second bus move into the rooms that are numbered by natural numbers congruent $n + 1$ modulo 2 and so on. Students are expected to formulate a proper mapping rule and to come up with their own ideas relating alternatives.

Where is the border line? Even a “Hilbertian bus-fleet” of infinite number of “Hilbertian busses” numbered according the natural numbers, is still not able to overstrain the hotel. For example, one can assign a double index to every passenger due to his bus number and his seat number within this bus. Now Cantor does the work by applying his first diagonal method to this matrix structure. Also here students could consider a formula or a formal description of an algorithm.

Students varied the above solutions in several and diverse ways. For example, prime numbers were used and alternating methods of simultaneous counting. Of course, the most important task is the clarification of the impossibility of lodging an uncountable amount of recent arrivals.

The above considerations go along with the equation

$$\aleph_0 + \aleph_0 + \aleph_0 + \ldots = \aleph_0,$$

where the number of the summands on the left side is countable.

The given examples above, which were part of the studies of our futures teachers, have certain potential to support understanding.

Conclusions with respect to understanding the concept infinity

We referred to Cantor especially when we stressed the 1-1-coresspondance (or mapping) and some insights of arithmetic rules including infinity. Since all examples are rather basic but initially unknown to most of the students they gained competencies with counting and the notion of bijectivity. It is important to realize that the arithmetic rules known from the basic arithmetic operations may vary, depending on the context. Another example, but in a different relationship is the “double distributivity” in case of unions and intersections of sets. The phenomenon “infinity” holds in itself ambiguities which contradict common sense at first glance. It is of great educational value to become acquainted with some of them, namely in two respects: in terms of general education, which should be a concern of mathematics education and for the purpose of educating “good” teachers. The well-educated primary teacher is then in the position to react properly when children ask smart questions or questions that show insight but do not use proper wording. Pupils occasionally may achieve even philosophical significance – so long as the teacher recognizes its meaning.

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