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Beginning teachers' perspectives on linear equations: A pilot quantitative comparison of Greek and Cypriot students

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In this paper we examine aspects of beginning primary teachers' understanding of linear equations. First-year teacher education students on a programme in Cyprus (12 Greek and 21 Cypriot) were shown a solution to the equation $x + 5 = 4x - 1$ comprising four rows of mathematically correct algebra but no commentary. They were asked to explain, in writing, the solution to a friend who had missed the lesson in which such equation solving processes had been taught. Analyses found almost all students, irrespective of nationality, writing about knowns and unknowns before offering a 'change the side and change the sign' rule. However, a major difference was that Cypriot students' accounts typically included an objective for the equation solving process, which the Greek students' did not.

Keywords: Linear equations, comparative research, Cyprus, Greece, teacher education.

Introduction

The topic of linear equations occupies an important position in students' learning. It "stands on the border between mathematics as concrete and inductive and mathematics as abstract and deductive", offering "one of the first authentic opportunities for them to connect their understanding of arithmetic to the symbolism of mathematics" (Andrews & Sayers, 2012, p.476). Yet, it is a difficult topic to teach well, because when learning arithmetic, learners typically come to see the equals sign as an instruction to operate (Kaput et al., 2007). This operational perspective (McNeil & Alibali, 2005) creates few problems with respect to arithmetical equations, with the unknown in one expression, because it supports a process of operation reversal (Herscovics & Linchevski, 1994; Kieran, 1992). However, non-arithmetical equations - the unknown in both expressions - requires a relational (Kieran, 1992) understanding of the equals sign as an assertion of equality between two expressions (Alibali et al., 2007; Filloy & Rojano, 1989) in order that they can operate on the unknown as an entity. In short, students find equation solving problematic because operational perspectives on expressions like $3x+1$ prevent their being construed as objects subject to, in relational terms, operations themselves (Kieran, 2004). Furthermore, to compound students' difficulties, teachers' practices frequently collude in the maintenance of an operational perspective on the equals sign (Haimes, 1996; Harel et al., 2008; Stephens, 2008), highlighting a need to evaluate the equations-related understandings that beginning teachers bring to their courses. This reflects the aim of this pilot study and an atypical approach, which is described below.

Perspectives on the teaching and learning of linear equations

Typically, research on equation solving has focused on approaches to the solution of non-arithmetical equations, not least because their solution poses few conceptual difficulties. In this respect, the most widely criticised is redistribution, a rote-learned, *change the side, change the sign* procedure (Nogueira de Lima & Tall, 2008), focused on transposing the equation so that the

unknown finishes on the left-hand side and a value on the right (Filloy & Rojano, 1989). The unknown's arbitrary leftwards movement perpetuates operational conceptions of the equals sign and fails to support students' understanding that such movement does not change the equation's equality (Capraro & Joffrion, 2006). Other approaches, like trial and improvement, support an understanding of the relational nature of the equals sign and the role of the unknown in context (Knuth et al., 2005). However, while it may be an appropriate initial strategy in a teaching sequence, it is inefficient and does not support the learning of general equation solving strategies (Filloy & Rojano, 1989). Other approaches (see Dickinson and Eade, 2004; Fong & Chong, 1995), present the equation as two rows of mathematical objects, one laid on top of the other, as in the representation of $2x + 10 = 4x + 2$ shown in Figure 1. Here, the authors claim, students can see easily how it reduces to $2x + 2 = 10$; an equation is amenable to an operation reversal procedure.

x	x	10		
x	x	x	x	2

Figure 1: A representation of $2x + 10 = 4x + 2$

However, while such approaches be procedurally helpful and support students' understanding of equations as manipulable objects, they may hinder students' understanding of the invariance of the solution. Moreover, despite Dickinson and Eade's (2004) optimistic arguments otherwise, they remain problematic with respect to negative coefficients (Marschall & Andrews, 2015).

Finally, in studies of teachers' unprompted approaches to equation solving, the balance scale has been the most widely reported, being the approach of choice in case studies from, for example, Canada (Haimes, 1996), Finland, Flanders and Hungary (Andrews & Sayers, 2012), New Zealand (Anthony & Burgess, 2014) and Poland (Marschall & Andrews, 2015). Here the solver manipulates, through addition or subtraction, weights on scale pans, while keeping the scales in balance. Its advocates argue that it helps students see the equation as a whole and not an instruction to operate (Warren & Cooper, 2005). Moreover, it supports an understanding of the need to do the same to both sides (Anthony & Burgess, 2014) and underpins the symbolic foundations of later algebraic formalisms (Andrews, 2003). Systematic attempts to evaluate the balance's efficacy have shown that it helps students to understand the principles of equations, solve non-arithmetical equations with understanding, particularly from the perspective of doing the same thing to both sides (Araya et al., 2010; Warren & Cooper, 2005) and facilitates students' acquisition of an appropriate vocabulary (Vlassis, 2002). Its critics argue that it cannot represent negatives in anything but a contrived way (Pirie & Martin, 1997), a criticism supported studies showing teachers simulating the tying of helium filled balloons to scales to counter the weight of objects in the scale pans (Anthony & Burgess, 2014).

In this paper we explore how students following an initial primary teacher education programme in Cyprus construe non-arithmetical linear equations. Due to Greek being the language of instruction, the programme includes both Cypriot and a high proportion of Greek students.

Methods

Shortly after the start of their course and before they had been exposed to university mathematics teaching, students were shown a solution to the equation $x + 5 = 4x - 1$ and asked to write a short

account of how they would explain it to someone who had missed the lesson in which it was introduced. The solution, with no additional narrative, was presented as follows

$$\begin{aligned}x + 5 &= 4x - 1 \\5 &= 3x - 1 \\6 &= 3x \\2 &= x\end{aligned}$$

A non-arithmetical equation was used for several reasons. Firstly, it could not be solved by means of a reversal of operations. Secondly, it should expose, in ways that an arithmetical equation could not, the underlying equations-related conceptions students bring to their courses. Thirdly, it would uncover the extent and depth of their equations-related procedural knowledge as, at each step, they would need not only to interpret and explain the solver's hidden thinking but decide what would need to be made explicit to the unknown learner. It was believed that this would be a more effective means of uncovering students' familiarity and understanding of the topic than a conventional test item and expose any pedagogical predisposition they bring to their course, as a result of their previous schooling.

Analysis

Students' transcripts were subjected to a constant comparison analysis (Fram, 2013), whereby a transcript was read and re-read to identify different equations-related understandings. This was followed by the next transcript being read and re-read in order to find further evidence of the original codes and any new ones not seen in the first. If new categories were found then the earlier transcript was re-read in case they had been previously missed. This process continued for all 33 transcripts, a number typically thought sufficient to achieve categorical saturation, and yielded seven categories of understanding, which can be seen in Table 1.

With respect to demonstrating the emergence of these categories, we turn to Ekaterini, a female Cypriot student, who wrote that

To solve this exercise we have to first set apart the known from the unknown numbers. The known numbers are the ones that don't include a letter, as for example, 5 and -1. Unknown numbers are the letters or the numbers that are accompanied by a letter, for example, $4x$ and x .

Ekaterini's comment about separating the known from the unknown implies an implicit objective for equation solving; in essence, solving an equation entails precisely that. In the same sentence can also be seen evidence of her awareness of the unknown and its role in equation solving. In the second and third sentences she goes further and defines an unknown. She then wrote (her parentheses):

To separate the two, the known numbers should be on one side of the equation, for example, on the left side, as we solved it in the class, while the unknown numbers should be on the other side. Later, I added $5 + 1$ (whose sign has changed because it moved to the other side of the equation) and subtracted $4x - x$ (again the sign changed because x has moved to the other side of the equation and when there is x or y alone this means $1x$ or $1y$). Finally, I reached $6=3x$ and so I divided 6 by 3 so that x is equal to 2.

Throughout this paragraph runs a rote procedure for equation solving invoking two simple instructions. The first is that knowns must be moved to one side of the equation and unknowns to the other. The second is that when an object moves from one side of an equation to the other its sign changes. Finally, having achieved this objective, an understanding of the role of inverse operations is invoked to divide the total of the knowns by the coefficient of the unknowns. Within this procedure, as with many of the students' suggestions, is evidence of flexibility in that it does not matter to which side of the equation which type of object travels, implying that it may be a matter of convenience.

Articulating an awareness of the unknown	31
Defining what is meant by an unknown	15
Offering an implicit objective for equation solving	18
Offering an explicit objective for equation solving	1
Offering a rote procedure for equation solving	29
Articulating an understanding of the role of inverse operations	26
Offering an unspecified process for equation solving	4

Table 1: The seven categories of understanding elicited from the data

With respect to the remaining two categories, only one student offered an explicit objective for equation solving. In this case, Irene, a Cypriot female, wrote that the “question in this equation is to find ‘x’ and what value it has”. Finally, four students wrote of an unspecified procedure. For example, Moira, a Greek female wrote, “I would tell the student that we separate the known from the unknown numbers and then make the calculations”. From her comment we inferred three perspectives on equation solving; an implicit objective tied to separating the known from the unknown, an awareness of the unknown and an unspecified procedure. The seven categories of response, along with their respective frequencies, can be seen in Table 1.

Results

The figures of Tables 2 and 3 show which of the seven categories identified by the constant comparison analysis were found in the accounts of the Greek and Cypriot students respectively. From these can be inferred both similarities and differences. With respect to the former, several similarities were identified. Firstly, with a single exception in each country, students' explanations showed an explicit awareness of the unknown. Secondly, a very high proportion of students from both countries - only four students did not - offered a rote procedure for solving the equation and of these, all focused on the mantra, change the side change the sign. Thirdly, only five students did not write in ways indicative of their understanding inverse operations, although this was typically seen with respect to explaining how the solution is reached from the point where $6=3x$, as in Chloe's comment that “after reaching $6=3x$, we divided 6 by 3 so that x will be by itself”. Fourthly, around half of all students in each group defined what they meant by an unknown

Greek student	A	B	C	D	E	F	G	H	I	J	K	L	12
Aware of unknowns	1	1	1	1	1	1	1	1	1		1	1	11
Defines unknowns		1			1	1			1		1	1	6
Implicit objectives						1					1		2
Rote procedure	1	1	1		1	1			1	1	1	1	9
Inverse operations		1	1		1	1	1			1		1	7
Unspecified process				1			1	1					3
Explicit objectives													0
	2	4	3	2	4	5	3	2	3	2	4	4	3.17

Table 2: Distribution of Greek students' equations-related insights

With respect to differences, only two Greek students offered any sense of objective, albeit implicit, for the equation solving process compared with 17 of the Cypriot, of which only one, student F, offered an explicit goal.

Cypriot student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	21	
Aware of unknowns	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	1	1	1	1	1	20
Defines unknowns	1		1	1				1							1	1	1		1		1	9	
Implicit objectives	1	1	1	1				1	1		1	1	1	1	1		1	1	1	1	1	16	
Rote procedure	1	1	1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	20	
Inverse operations	1	1		1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	1	19	
Unspecified process				1																		1	
Explicit objectives						1																1	
	5	4	4	5	3	4	3	5	3	3	4	3	4	4	5	4	5	4	5	4	5	4.1	

Table 3: Distribution of Cypriot students' equations-related insights

The figures of Tables 2 and 3 show also that Greek students' data yielded fewer codes per student, 3.17, than their Cypriot counterparts mean of 4.10. Indeed, at the upper end of the spectrum the accounts of seven Cypriot students, A, D, H, O, Q, S and U, yielded five categories of response, compared with that of just one, F, Greek student. At the lower end, the accounts of four Greek students, A, D, H and J, yielded only two categories each, compared with zero Cypriot students. These differences were statistically significant in two ways. Firstly, a t-test showed differences in the mean number of codes were unlikely to be due to chance ($t = 2.95$, $p = 0.006$). Secondly, a chi-square test performed on the data in Table 4 confirmed that variation in the number of codes yielded by each of the Greek students and Cypriot students respectively were unlikely to be due to chance ($\chi^2 = 9.15$, $df = 3$, $p = 0.027$). This difference in the codes, we argue is likely to be a consequence of differences in how the two systems introduced their students to linear equations; a possibility warranted by, for example, evidence that Cypriot students tended to specify objectives in their accounts in ways that their Greek colleagues typically did not. Finally, of the eight students whose accounts yielded five codes, seven yielded the same five. That is, they indicated an awareness of and defined the unknown; they offered implicit objectives and a rote procedure alongside an awareness of inverse operations related to division. In short, the most complete responses were typically the same.

		Number of codes per student				
		2	3	4	5	
Greek		4	3	4	1	12
Cypriot		0	5	9	7	21
		4	8	13	8	33

Table 5: Distribution of total number of codes per student per country

Discussion

The aim of this pilot study, the quantitative analyses for which are presented in this paper, was to explore the equations-related understanding primary teacher education students bring to their courses and, in so doing, evaluate the effectiveness of a simple to implement tool for later comparative use. In this instance comparison was made possible by the fact that courses in Cyprus are taught in Greek, making them accessible to Greek students. The results are methodologically encouraging but, acknowledging the fact that all respondents were prospective teachers, mathematically worrying, albeit with some qualifying strengths.

Methodological encouragement stems from the evidence that students responded positively to the invitation and produced written accounts sufficient to expose their perspectives on or conceptions of linear equations. It was also encouraging that the tool was able to discriminate between the two cultural groups in its highlighting similarities and differences in students' accounts that, we infer, reflect systemic differences in the ways in which linear equations had been experienced by these two sets of students as learners of school mathematics. In short, the tool proved fit-for-purpose.

The mathematical disappointment derives in part from the lack of any evidence of students holding a relational (Kieran, 1992) conception of the equals sign, in that nothing said by any student indicated an understanding of the equals sign as an assertion of equality between two expressions (Alibali et al., 2007; Filloy & Rojano, 1989). Mathematical disappointment also derives from the very high proportion of students in both countries who seemed to construe equation solving as a rote process of 'change the side and change the sign'. That is, the majority of students appeared to have a procedural rather than a conceptual perspective on equations in which symbols are moved around "with a kind of additional 'magic' to get the correct solution" (Nogueira de Lima & Tall, 2008, p.4). Indeed, even those students whose accounts yielded the most categories of response presented procedural perspectives with implicit objectives and rote procedures. However, in contrast with international research showing the balance as teachers' preferred representation (Andrews & Sayers, 2012; Anthony & Burgess, 2014; Haimes, 1996; Marschall & Andrews, 2015), no reference to the balance was made by any student. Indeed, with the exception of their implicit awareness of inverse operations, which we discuss below, nothing written by any student indicated a narrative based on performing the same action to both sides of the equation.

Despite the negatives, there were some interesting positives. The majority of students, particularly the Cypriot, indicated an awareness of the role of inverse operations. In every case this occurred at the point in the solution where the step connecting $6=3x$ to $2 = x$ was discussed. Here, students indicated, albeit implicitly, an awareness that division was the inverse operation to invoke, insights that seem to confound the mechanical procedure of 'change the side, change the sign'. Also, despite the highly procedural nature of their accounts, only two students, one from each country, did not

demonstrate an awareness of the unknown. That is, students not only used an appropriate vocabulary but were generally aware of the function of the unknown in the equation solving process. Indeed, around half of all students from both groups offered a definition, typically implicitly, of the unknown, as in Carissa's account in which she wrote, "I separated the known from the unknown numbers. Unknown $x + 5$ (known) = unknown $4x - 1$ (known)".

Finally, Cypriot students' accounts yielded more response categories than their Greek colleagues, typically due to their tendency to offer objectives, again implicit, which their Greek colleagues did not. We speculate that such a difference may reflect cultural teaching norms; while the evidence of these students' accounts indicates the outcome of procedural teaching this particular finding suggests that Cypriot teachers may warrant their procedural approaches to a topic in ways that their Greek colleagues may not. In sum, the responses from both sets of students indicated deep-set procedural perspectives on linear equations. It was clear that they had understood the task in that all their explanations were valid but, with both the Cypriot and the Greek curricula advocating that students learn a relational mathematics, the implications of this study for teacher education are profound.

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