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Turbulent and adaptive optics corrected point-spread functions as convolutive orders of the phase power spectral density

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ABSTRACT

Ground-based astronomy is severely limited by atmospheric turbulence, resulting in a large point-spread function (PSF) and poor imaging resolution. Even imaging with adaptive optics (AO) cannot correct the aberrated wavefront completely and residual turbulence still corrupts the observation. Thus the consequences of turbulence on the PSF are of primary interest when building any ground-based telescope. The power spectral density (PSD) of a spatially stationary turbulent phase carries all the information needed for describing the long-exposure PSF. We then develop an analytical description of the long-exposure PSF as a series expansion of the aberrated phase PSD. Our description of the PSF given the PSD of the phase is a simple theoretical way to describe the impact of turbulence on the PSF. We also show agreement with previous articles when restricting our model to its first expansion order. Finally, we derive applications of our formula to some particular cases, such as Kolmogorov or von Kármán models or the AO correction impact on the PSF.

Key words: instrumentation: adaptive optics – instrumentation: high angular resolution – methods: analytical.

1 INTRODUCTION

The point-spread function (PSF) is critical information for any optical system, since it is directly related to the imaging resolution. Diffraction-limited optics with a circular aperture results in the famous Airy pattern. However, in practice, the wavefront of the observed scene suffers from phase aberrations during its propagation through an inhomogeneous medium, reducing imaging performance. In the case of ground-based telescopes, the atmospheric turbulence distorts the wavefront and severely limits the resolution of the telescopes (Roddier 1981). Adaptive optics (AO) is a technique using deformable mirrors to reduce phase aberrations by flattening the wavefront back (Roddier 1999). Nevertheless, the AO correction is not perfect and some phase aberrations remain, especially the high spatial frequencies, which are not seen by the wavefront sensor and thus not corrected.

On the one hand, Roddier (1981) developed a relation to find the long-exposure PSF from the phase power spectral density (PSD). The formula is exact, but the PSD/PSF link is not direct, since it requires a double Fourier transform. On the other hand, Racine et al. (1999), Bloemhof et al. (2001) and Jolissaint & Veran (2002)

derived a first-order approximation of the Roddier method. The advantage is that it explains the direct link between the PSD and PSF, by cancelling the double Fourier transformation; however, the drawback is an inaccuracy due to the first-order approximation.

The two methods above illustrate the very close relation between the phase PSD and the PSF, each one with its advantages and drawbacks. In this article, we use the Roddier method to derive an exact expression for the PSF, with a direct analytical PSD/PSF link without requiring multiple Fourier transforms. Indeed, our method relies on a convolutional formalism, where the PSF is described as an infinite sum of convolutive orders of the phase PSD. Finally, when restricting our expression to its first order, we find the approximations of Racine et al., Bloemhof et al. and Jolissaint & Veran.

Straightforward applications involve practical cases where the fully turbulent, or AO partially corrected, phase is not known for each time step, but its PSD follows well-known statistics (e.g. Kolmogorov, von Kármán law, AO residual PSD). As a consequence, our model can easily be fed with available information on the turbulence. Our direct expression linking the PSD and PSF also allows for simple estimation of atmospheric parameters directly from the PSF.

Section 2 develops our analytical method, which involves writing the PSF in terms of orders of the residual aberrated phase PSD. This method is general and can be used for any spatially stationary

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phase. Section 3 shows some applications related to Kolmogorov or von Kármán statistics. We give an explicit dependence of the PSF shape on the Fried parameter r_0 (Fried 1966). We also derive a simple description for the halo for the long-exposure PSF using AO correction. Section 4 concludes our work. We discuss future applications of our model.

2 METHOD

2.1 Factorization of the optical transfer function

Let us use P to denote the entrance pupil transmission function (which may include static aberrations), h the long-exposure PSF and ψ the complex phasor (Goodman 1968), i.e. the complex scalar field when observing a point source. Scintillation is neglected, so the complex phasor has unit amplitude. The long-exposure optical transfer function (OTF) is given by the autocorrelation of the phasor on the pupil (Goodman):

$$\tilde{h}(\boldsymbol{\rho}/\lambda) = \frac{1}{S} \iint P(\mathbf{r})P(\mathbf{r} + \boldsymbol{\rho}) \langle \psi(\mathbf{r}, t)\psi(\mathbf{r} + \boldsymbol{\rho}, t) \rangle_t d\mathbf{r}, \quad (1)$$

where S ensures normalization on the surface of the pupil, $\langle \cdot \rangle_t$ denotes the time average and \tilde{h} stands for the Fourier transform of the PSF h . We then write the phasor ψ as the complex exponential of the aberrated phase ϕ :

$$\psi(\mathbf{r}, t) = e^{i\phi(\mathbf{r}, t)}. \quad (2)$$

Roddiar (1981) has shown that, assuming Gaussian statistics of the phase, the long-exposure OTF can be written as

$$\tilde{h}(\boldsymbol{\rho}/\lambda) = \frac{1}{S} \iint P(\mathbf{r})P(\mathbf{r} + \boldsymbol{\rho}) e^{-\frac{1}{2}D_\phi(\mathbf{r}, \boldsymbol{\rho})} d\mathbf{r}, \quad (3)$$

where $D_\phi(\mathbf{r}, \boldsymbol{\rho})$ is the structure function of the residual phase, defined as

$$D_\phi(\mathbf{r}, \boldsymbol{\rho}) = \langle |\phi(\mathbf{r}, t) - \phi(\mathbf{r} + \boldsymbol{\rho}, t)|^2 \rangle_t. \quad (4)$$

Assuming a spatially stationary residual phase, the dependence on the position \mathbf{r} vanishes and only the separation $\boldsymbol{\rho}$ remains. This stationarity hypothesis is verified for a purely turbulent phase and is a reasonable approximation in the case of partial AO correction (Conan 1994). The approximation is better for larger aperture telescopes, since the non-stationarity of the residual phase essentially affects the edges of the pupil. The structure function then simplifies to

$$D_\phi(\mathbf{r}, \boldsymbol{\rho}) = D_\phi(\boldsymbol{\rho}) = \langle |\phi(\mathbf{r}, t) - \phi(\mathbf{r} + \boldsymbol{\rho}, t)|^2 \rangle_t. \quad (5)$$

The OTF can then be separated into two OTFs, the telescope one and the atmospheric one:

$$\tilde{h}(\boldsymbol{\rho}/\lambda) = \tilde{h}_T(\boldsymbol{\rho}/\lambda) \cdot \tilde{h}_A(\boldsymbol{\rho}/\lambda), \quad (6)$$

with the telescope OTF

$$\tilde{h}_T(\boldsymbol{\rho}/\lambda) = \frac{1}{S} \iint P(\mathbf{r})P(\mathbf{r} + \boldsymbol{\rho}) d\mathbf{r} \quad (7)$$

and the atmospheric OTF

$$\tilde{h}_A(\boldsymbol{\rho}/\lambda) = e^{-\frac{1}{2}D_\phi(\boldsymbol{\rho})}. \quad (8)$$

Even though this expression may give the PSF by a numerical Fourier transform, it does not exhibit an explicit dependence on the PSD of the phase. It is then necessary to modify the equation to find the direct link we are looking for.

2.2 Expression for the PSF

Assuming the phase variance is finite, let us define the autocorrelation of the phase as

$$B_\phi(\boldsymbol{\rho}) = \langle \langle \phi(\mathbf{r}, t)\phi(\mathbf{r} + \boldsymbol{\rho}, t) \rangle_t \rangle_r. \quad (9)$$

The equality

$$D_\phi(\boldsymbol{\rho}) = 2[B_\phi(\mathbf{0}) - B_\phi(\boldsymbol{\rho})] \quad (10)$$

follows directly. Using this identity and the series expansion of the exponential, one obtains the atmospheric OTF:

$$\tilde{h}_A(\boldsymbol{\rho}/\lambda) = e^{-B_\phi(\mathbf{0})} e^{B_\phi(\boldsymbol{\rho})} = e^{-B_\phi(\mathbf{0})} \sum_{n=0}^{+\infty} \frac{B_\phi(\boldsymbol{\rho})^n}{n!}. \quad (11)$$

We use $\mathcal{F}\{f\}(\mathbf{u})$ to denote the Fourier transform operator applied to the function f , evaluated in \mathbf{u} . Using the linearity of the Fourier operator on the infinite sum, the PSF can be written as

$$h_A(-\mathbf{u}) = e^{-B_\phi(\mathbf{0})} \sum_{n=0}^{+\infty} \frac{\mathcal{F}\{B_\phi(\boldsymbol{\rho})^n\}(\mathbf{u})}{n!}, \quad (12)$$

where $\boldsymbol{\rho}/\lambda$ and \mathbf{u} are variables conjugated by the Fourier transform. Since the Fourier transform links multiplication and convolution, we define the convolutive orders of a function W as

$$\{\star^n W\}(\mathbf{u}) = \begin{cases} \delta(\mathbf{u}) & \text{if } n = 0, \\ W(\mathbf{u}) & \text{if } n = 1, \\ (W \star W \star \dots \star W)(\mathbf{u}) & \text{if } n \geq 2, \end{cases} \quad (13)$$

where $\delta(\mathbf{u})$ denotes the Dirac distribution. Then the PSF can be written as

$$h_A(-\mathbf{u}) = e^{-B_\phi(\mathbf{0})} \sum_{n=0}^{+\infty} \frac{\star^n \mathcal{F}\{B_\phi(\boldsymbol{\rho})\}(\mathbf{u})}{n!}. \quad (14)$$

By the Wiener–Khinchine theorem, the Fourier transform of the autocorrelation is equal to the PSD of the phase. We write the PSD of the phase as W_ϕ and, taking care of coordinate dilation for the Fourier transform,

$$h_A(-\mathbf{u}) = e^{-B_\phi(\mathbf{0})} \sum_{n=0}^{+\infty} \frac{\{\star^n W_\phi/\lambda^2\}(\mathbf{u}/\lambda)}{n!}. \quad (15)$$

This equation states that the atmospheric long-exposure PSF can be written as an infinite sum of convolutive orders of the phase PSD, under the assumption of spatially stationary phase. In other words, the long-exposure PSF can be retrieved when the turbulent phase for each time step is unknown but its PSD is well-known. This is the case for a long-exposure observation through a turbulent atmosphere, without or with AO. Then, using equation (6) to retrieve the total PSF, atmospheric and diffraction, we can write directly

$$h(\mathbf{u}) = h_T(\mathbf{u}) \star h_A(\mathbf{u}). \quad (16)$$

The convolutive formalism of equation (15) can be found in the literature, but for a short-exposure PSF only: see Sivaramakrishnan et al. (2002) and Perrin et al. (2003). These authors described the short-exposure PSF as a double infinite sum on the instantaneous phase. As we show in this article, based on Roddiar's method, the long-exposure PSF allows simplification to a unique infinite sum in the phase PSD.

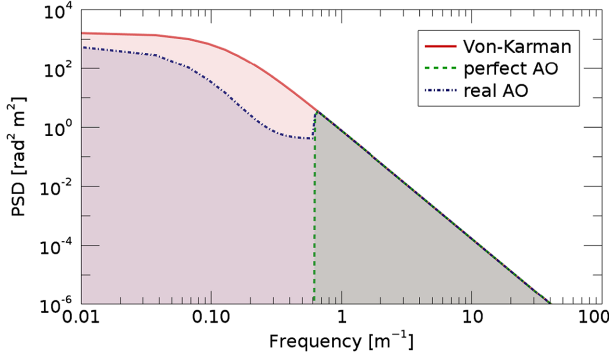


Figure 1. Three models of phase PSD considered in the following applications: von Kármán (plain), perfect AO as truncated Kolmogorov (dashed) and more realistic AO correction (dot-dashed).

3 APPLICATIONS

3.1 Factorization of the r_0 dependence

For a usual turbulence spectrum, such as Kolmogorov or von Kármán, there is a direct dependence of the PSF on the Fried parameter r_0 . Let us consider the von Kármán spectrum:

$$W_\phi(f) = 0.023r_0^{-5/3} \left(\frac{1}{L_0^2} + f^2 \right)^{-11/6}, \quad (17)$$

where L_0 is the external scale of the turbulence. The von Kármán spectrum for $L_0 = 8$ m is plotted in Fig. 1. Letting $L_0 \rightarrow \infty$, one would find the Kolmogorov spectrum.

Let us now extract the r_0 dependence from the PSD. We define Y_ϕ as the PSD of the phase with unit r_0 , and scaled by the $1/\lambda^2$ factor. Then, by the definition of Y_ϕ ,

$$\frac{W_\phi(f)}{\lambda^2} = r_0^{-5/3} Y_\phi(f). \quad (18)$$

The convolutive orders of Y_ϕ can be computed numerically only once for a given L_0 . The PSF for different values of r_0 is then a weighted sum of these components. Plugging Y_ϕ into equation (15) gives the PSF

$$h_A(-\mathbf{u}) = e^{-B_\phi(0)} \sum_{n=0}^{+\infty} \frac{r_0^{-5n/3}}{n!} \{ \star^n Y_\phi \} (\mathbf{u}/\lambda). \quad (19)$$

The impact of the strength of the turbulence on the PSF is then more explicit: for high r_0 (weak turbulence), the high-order convolutive terms have little importance. Consequently, the shape of the PSF is dominated by low convolutive orders. For low r_0 (strong turbulence), the higher convolutive orders gain more importance in describing the PSF shape.

For a practical application, the infinite sum is truncated up to a number N_{\max} of orders. One would like to get an idea of this number of terms to consider in order to have an accurate PSF model. The number N_{\max} depends on r_0 , but also on the evolution of the Y_ϕ convolutive orders with n . Let us call E the integral of Y_ϕ , or, equivalently, the phase variance for unit r_0 . We find a semi-empirical approximation (see explanations in Appendix) for values of \mathbf{u} in the halo:

$$\{ \star^n Y_\phi \} (\mathbf{u}/\lambda) \approx E^{n-1} Y_\phi(\mathbf{u}/\lambda). \quad (20)$$

Dropping the term $n = 0$ (particular behaviour of the Dirac at $\mathbf{u} = \mathbf{0}$), we obtain a rough estimate of the PSF behaviour as regards

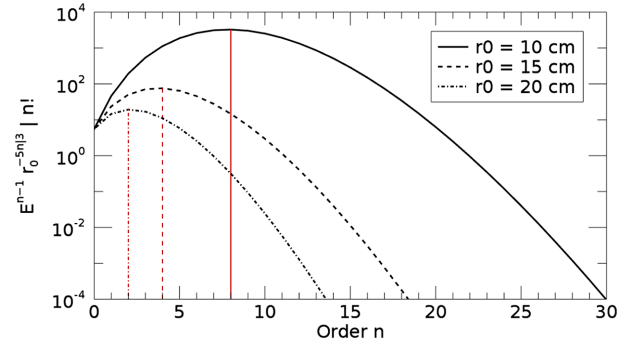


Figure 2. $E^{n-1}r_0^{-5n/3}/n!$ versus n , given for $E = 0.18$. The different curves are plotted for different values of the Fried parameter r_0 . All curves pass a maximum before decreasing. The vertical lines help in visualizing the N_{crit} criterion for which the maximum of the curves is reached.

orders:

$$\forall \mathbf{u} \in \{\text{halo}\}, \quad h_A(-\mathbf{u}) \approx Y_\phi(\mathbf{u}/\lambda) e^{-B_\phi(0)} \sum_{n=1}^{+\infty} \frac{E^{n-1} r_0^{-5n/3}}{n!}. \quad (21)$$

The terms of the series are decreasing in magnitude when the following constraint on n is satisfied:

$$\frac{E^n r_0^{-5(n+1)/3}}{(n+1)!} < \frac{E^{n-1} r_0^{-5n/3}}{n!} \iff n > E r_0^{-5/3} - 1. \quad (22)$$

Let us define N_{crit} as the smallest integer satisfying the inequality. One cannot truncate the sum before N_{crit} or one would miss terms of higher magnitude. This basic constraint gives a lower bound on the number of terms to be considered in the series for an accurate description of the PSF: mathematically, we should ensure $N_{\max} > N_{\text{crit}}$. The criterion can be visualized in Fig. 2 for different values of r_0 and a chosen $E = 0.18$ (corresponding to the perfect AO correction up to the cut-off frequency $f_{\text{AO}} = 0.625$ m $^{-1}$; see Section 3.3). Once again, we see that more terms are needed for high phase variance $\sigma^2 = B_\phi(0) = E r_0^{-5/3}$. In the case of AO correction, for a given r_0 , E depends mainly on the AO cut-off frequency.

3.2 Validation for von Kármán turbulence

In order to test the validity of the formula, we consider a von Kármán spectrum for the turbulent phase, given equation (17). On one hand, the PSF is generated using the atmospheric OTF as an exponential of the phase structure function equation (8). On the other hand, we use our expression giving the PSF directly with the PSD of the phase equation (15). Fig. 3 shows the PSF for $N_{\max} = 5$ and $N_{\max} = 30$ convolutive orders. For a wavelength of $\lambda = 1600$ nm, convergence requires approximately seven orders. This is in accordance with our minimal criterion equation (22), which gives $N_{\text{crit}} = 4$ at this wavelength. For a shorter wavelength ($\lambda = 800$ nm), the PSD is higher and the convergence is much slower and requires approximately 30 orders. This criterion gives $N_{\text{crit}} = 22$; it confirms the fact that the criterion is only a lower bound, while in practice many more orders can be required to reach convergence.

This test shows the validity of our formula with respect to the Roddier method for the OTF. This example illustrates the difficulty in approximating the PSF, especially for shorter wavelengths, by the first order of the PSD for a full von Kármán spectrum. The method is more suited for the low-energy PSD obtained with AO correction, as developed in the following subsection.

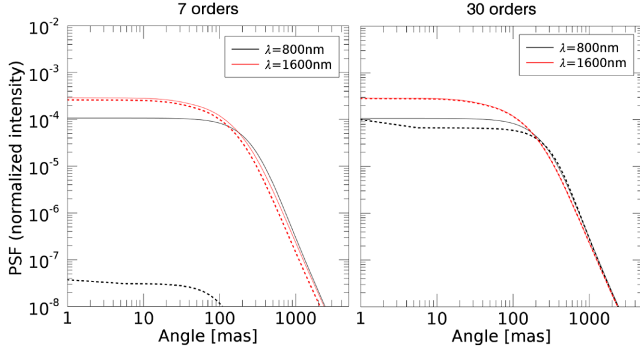


Figure 3. PSF computed from a von Kármán spectrum, $L_0 = 8$ m and $r_0 = 12$ cm at $\lambda = 400$ nm. Coloured lines show the PSF for two wavelengths: $\lambda = 800$ nm and $\lambda = 1600$ nm. Plain: PSF computed by the Roddier OTF method. Dashed: PSF computed by the convolutive orders method. Left panel: seven orders. Right panel: 30 orders.

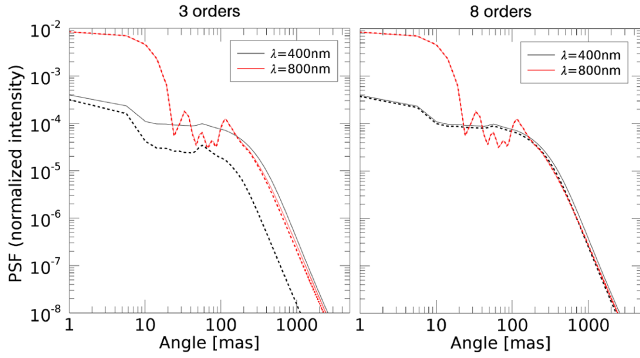


Figure 4. PSF computed for the given AO spectrum. Plain curve: PSF computed by the OTF method. Dashed: PSF computed by the convolutive orders method. Left panel: three orders. Right panel: 8 orders.

3.3 Truncated Kolmogorov and adaptive optics

Let us consider a simplified adaptive optics system with a linear number of actuators N_{AO} in the pupil plane of aperture diameter D . All spatial frequencies of the turbulent phase below the AO cut-off frequency $f_{AO} = N_{AO}/(2D)$ are assumed to be perfectly corrected by the AO system. All frequencies above f_{AO} are not seen by the AO system and follow Kolmogorov statistics. The PSD of the residual phase can be written as

$$W_\phi(f) = \begin{cases} 0 & \text{if } f < f_{AO}, \\ 0.023r_0^{-5/3} f^{-11/3} & \text{if } f \geq f_{AO}. \end{cases} \quad (23)$$

This shape of the AO-corrected PSD is plotted in green in Fig. 1, for a chosen AO cut-off frequency $f_{AO} = 0.625 \text{ m}^{-1}$, corresponding to $N_{AO} = 10$ actuators for a $D = 8$ m aperture. In practice, the centre of the PSF is dominated by highly energetic low-frequency modes, such as residual tip-tilt. Our simple model of a perfect AO high-pass filter does not take these residues into account, inducing poor fidelity in the centre of the computed PSF. However we can expect a correct description of the halo. Indeed, Fig. 4 shows good agreement between the two PSF computation methods – using a numerical Fourier transform equation (8) or using the convolutive orders equation (15). Let us note that the r_0 factorization in Section 3.1 is still applicable for numerical computation efficiency with our method for convolutive orders of the PSD. The atmospheric PSF is then convolved by h_T to take into account the telescope contribution, equation (16). At first order, we retrieve the shape developed in

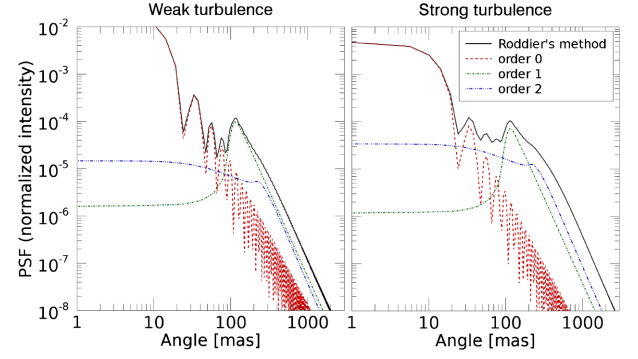


Figure 5. Impact of W_ϕ convolutive orders on the PSF. Each order is also convolved with the pupil PSF, called h_T in the text. Left: $r_0 = 20$ cm. Right: $r_0 = 10$ cm. The three orders $n \in \{0, 1, 2\}$ are respectively plotted as coloured dashed lines. The PSF based on the Roddier method, from the OTF in $e^{-D_\phi/2}$, is plotted in plain black.

Racine et al. (1999), with a coherent core made of the Airy pattern and a turbulent halo depending on the residual PSD. The difference is that our method includes all convolutive orders of the PSD in the description of the PSF. Higher orders can be of high magnitude when the PSD is of high energy E or for strong turbulence (Fig. 5). In particular, the convolution folds back energy from the PSF halo into the corrected area, reducing contrast in this zone. Magnitudes of higher orders can be reduced (and the contrast in the corrected area then increased) by decreasing the integral E of Y_ϕ , which is strongly dependent on the cut-off frequency f_{AO} and the wavelength λ (Fig. 4). A few orders, $N_{max} \simeq 3-8$ in our example, are required to describe the AO-corrected PSF, since there is little energy in the PSD, to be compared with the $N_{max} > 30$ orders required to describe the pure von Kármán turbulent phase. Our criterion gives respectively $N_{crit} = 6$ for $\lambda = 400$ nm and $N_{crit} = 1$ for $\lambda = 800$ nm. This is in agreement with the PSFs plotted in Fig. 4, since convergence is reached at $N_{max} = 8$ for $\lambda = 400$ nm and $N_{max} = 3$ for $\lambda = 800$ nm.

3.4 Modelling errors in the AO system

We previously considered a perfect AO system up to the cut-off frequency, with a phase PSD identically null for $f < f_{AO}$. However, in practice a residue of phase PSD remains at low frequencies, due to errors of the AO system such as wavefront sensor (WFS) noise, corrective-loop delay or deformable mirror fitting error. These error terms can be measured or modelled (Fusco et al. 2006) for a finer description of the AO partial correction. Plugging these errors into our PSD, we can obtain a better approximation of the PSF, especially at the central peak. Methods based directly on Roddier's OTF formula have already been implemented (Rigaut, Véran & Lai 1998; Jolissaint 2010; Correia et al. 2017). Our convolutive formalism then offers a direct understanding of the link between the phase PSD and the PSF in the results provided by Roddier's OTF-based algorithms. For extreme AO correction, we find that the PSF behaves as the phase PSD (first-order approximation), whereas the following PSD convolutive orders are required for lower AO correction quality.

We choose here to model the residual phase PSD due to AO errors by a Moffat function (Moffat 1969). The shape of the residual PSD can be written as

$$W_\phi(f) = \begin{cases} A(1 + f^2/\alpha^2)^{-\beta} + B & \text{if } f < f_{AO}, \\ 0.023r_0^{-5/3} f^{-11/3} & \text{if } f > f_{AO}, \end{cases} \quad (24)$$

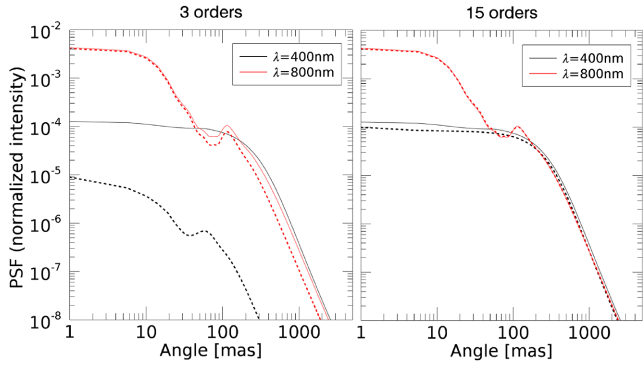


Figure 6. PSF computed for the given AO spectrum. Plain curve: PSF computed by the OTF method. Dashed: PSF computed by the convolutive orders method. Left panel: three orders. Right panel: 15 orders.

where A , B , α and β are the Moffat parameters associated with the error budget for the corrected phase. The shape of the PSD for arbitrary values of Moffat parameters is plotted in blue in Fig. 1. The resulting PSD is in between the von Kármán uncorrected turbulence and perfect AO correction.

Fig. 6 shows the convolutive orders associated with the discussed PSD, for $N_{\max} = 3$ and $N_{\max} = 15$. The number of terms required is higher than in the case of a perfect AO system, due to higher energy under the PSD, but lower than in the case of uncorrected turbulence.

4 CONCLUSIONS

For long-exposure observations, highly time-variable phase aberrations average out and only the statistical behaviour of the turbulence remains. We consequently derived a general method to compute the long-exposure PSF, given only the PSD of the turbulent phase. When restricting our model to its first-order approximation, we find agreement with previous works on the subject (Racine et al. 1999; Bloemhof et al. 2001). We have shown that the higher convolutive orders have more impact on the PSF for stronger turbulence and must be taken into account for low-Strehl systems.

A hypothesis of the spatial stationarity of phase over the pupil is required, which is in agreement with Kolmogorov and von Kármán statistics. Our method is also applicable to the partially corrected phase with adaptive optics. Finally, we extracted the r_0 dependence for a better understanding of the PSF shape variation with turbulence strength. It is then possible to obtain a direct estimation of r_0 given the PSF halo or, reciprocally, an estimation of the PSF halo given r_0 . We plan to use this model for an accurate description of the halo for an AO-corrected PSF. The link we make between the PSD and the PSF can be used as a direct measurement of the Fried parameter r_0 from a PSF observed for a star. Finally, the strong correlation between the AO-corrected PSD and the shape of the PSF can provide a tool for AO system diagnostics (Jolissaint & Veran 2002).

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APPENDIX A: ON THE CONVOLUTIVE APPROXIMATION

Let us consider a PSD shape verifying the two following hypotheses:

- (i) the majority of the PSD energy is in the centre ($|\mathbf{u}| \ll 1$) rather than in the halo ($|\mathbf{u}| \gg 1$);
- (ii) the halo is slowly varying.

These hypotheses are verified for the $-11/3$ power law of the Kolmogorov spectrum, in particular.

Let us now consider \mathbf{u} , a coordinate in the halo. The PSD auto-convolution can then be written as

$$Y_\phi \star Y_\phi(\mathbf{u}) = \iint_{\mathbb{R}^2} Y_\phi(\mathbf{t})Y_\phi(\mathbf{u} - \mathbf{t}) d\mathbf{t}. \quad (\text{A1})$$

The \mathbb{R}^2 domain is then separated into three subdomains:

- (i) $|\mathbf{t}| \ll 1$, where $Y_\phi(\mathbf{t})$ is large due to the first hypothesis and $Y_\phi(\mathbf{u} - \mathbf{t})$ is slowly varying due to the second one (when choosing $\mathbf{u} \gg 1$);
- (ii) $|\mathbf{u} - \mathbf{t}| \ll 1$, where the opposite behaviour occurs;
- (iii) the remaining domain of \mathbb{R}^2 , called \mathcal{A} , where both $Y_\phi(\mathbf{t})$ and $Y_\phi(\mathbf{u} - \mathbf{t})$ are small due to the first hypothesis.

Fig. A1 shows an example of these domains for a given \mathbf{u} and a PSD shape as described in Section 3.4. The convolutive integral of Y_ϕ is separated into the three domains described above. The integral over the first domain is

$$\iint_{|\mathbf{t}| \ll 1} Y_\phi(\mathbf{t})Y_\phi(\mathbf{u} - \mathbf{t}) d\mathbf{t} \simeq Y_\phi(\mathbf{u}) \iint_{|\mathbf{t}| \ll 1} Y_\phi(\mathbf{t}) d\mathbf{t}. \quad (\text{A2})$$

The approximation we just made relies on the fact that $|\mathbf{u}| \gg 1$ is in the halo and $|\mathbf{t}| \ll 1$, so $\mathbf{u} - \mathbf{t}$ is still in the halo. Slow variations in the halo (second hypothesis) allow us to approximate $Y_\phi(\mathbf{u} - \mathbf{t}) \simeq Y_\phi(\mathbf{u})$. We make the same reasoning in the second domain, with a

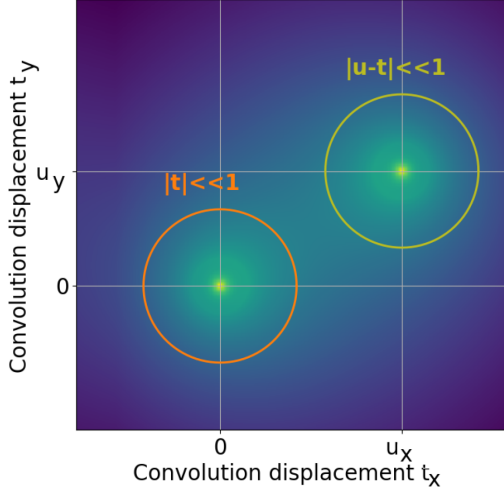


Figure A1. Visualization of the three different domains for a given convolutive shift \mathbf{u} . The colour map shows the product $Y_\phi(\mathbf{t})Y_\phi(\mathbf{u} - \mathbf{t})$. The intensity plot is on a logarithmic scale.

coordinate change $\mathbf{x} = \mathbf{u} - \mathbf{t}$, so we find

$$\iint_{|\mathbf{u}-\mathbf{t}| \ll 1} Y_\phi(\mathbf{t})Y_\phi(\mathbf{u} - \mathbf{t}) d\mathbf{t} \simeq Y_\phi(\mathbf{u}) \iint_{|\mathbf{x}| \ll 1} Y_\phi(\mathbf{x}) d\mathbf{x}. \quad (\text{A3})$$

The first integrals in the two cases are then identical. Regarding the third domain, it corresponds to the convolution of two small terms (first hypothesis) and consequently is assumed to be much less energetic than the other domains. We simply write it as

$$\iint_{\mathcal{A}} Y_\phi(\mathbf{t})Y_\phi(\mathbf{u} - \mathbf{t}) d\mathbf{t} = \eta(\mathbf{u})E, \quad (\text{A4})$$

where $\eta(\mathbf{u})$ is a small-valued function and E denotes the integral of Y_ϕ on \mathbb{R}^2 . Summing the partial convolutions over the three domains, one obtains the full convolution

$$Y_\phi \star Y_\phi(\mathbf{u}) \simeq 2Y_\phi(\mathbf{u}) \left[\iint_{|\mathbf{t}| \ll 1} Y_\phi(\mathbf{t}) d\mathbf{t} \right] + \eta(\mathbf{u})E. \quad (\text{A5})$$

Finally, using the first hypothesis, the integral above contains the majority of the energy. We can then write

$$\iint_{|\mathbf{t}| \ll 1} Y_\phi(\mathbf{t}) d\mathbf{t} = \kappa E, \quad (\text{A6})$$

where $\kappa \in]0, 1]$ represents the ratio of energy in the area $|\mathbf{t}| \ll 1$. It depends on the shape of Y_ϕ ; in practice, we find $\kappa \simeq 0.85$ for our AO truncated Kolmogorov. Using κ , the convolution can be rewritten as

$$Y_\phi \star Y_\phi(\mathbf{u}) \simeq 2\kappa Y_\phi(\mathbf{u})E + \eta(\mathbf{u})E. \quad (\text{A7})$$

If we neglect the $\eta(\mathbf{u})E$ term, one can easily iterate by recurrence over the convolutive orders to find

$$\{\star^n Y_\phi\}(\mathbf{u}) \simeq (2\kappa E)^{n-1} Y_\phi(\mathbf{u}). \quad (\text{A8})$$

Errors in this equation come from the different approximations we made and the modification of shape for $\{\star^n Y_\phi\}$, especially in the centre ($\mathbf{u} \ll 1$). One should recall that this expression is only true in order of magnitude, since the approximations propagate through the convolutive orders.

This paper has been typeset from a \TeX/L\AA\TeX file prepared by the author.