# Convergence of the solutions of the discounted Hamilton-Jacobi equation: a counterexample 

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#### Abstract

This paper provides a counterexample about the asymptotic behavior of the solutions of a discounted Hamilton-Jacobi equation, as the discount factor vanishes. The Hamiltonian of the equation is a 1-dimensional continuous and coercive Hamiltonian.


## Résumé

Cet article fournit un contre-exemple à la convergence asymptotique des solutions d'une équation de HamiltonJacobi escomptée, lorsque le facteur d'escompte tend vers 0. Le Hamiltonien de cette équation est unidimensionnel, continu et coercitif.

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## 1. Introduction and main result

Let $n \geq 1$. Denote by $\mathbb{T}^{n}=\mathbb{R}^{n} / \mathbb{Z}^{n}$ the $n$-dimensional torus. For $c \in \mathbb{R}$, consider the Hamilton-Jacobi equation

$$
H(x, D u(x))=c \quad\left(E_{0}\right)
$$

where the Hamiltonian $H: \mathbb{T}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ is jointly continuous and coercive in the momentum. In order to build solutions of the above equation, Lions, Papanicolaou and Varadhan [1] have introduced a technique called ergodic approximation. For $\lambda \in(0,1]$, consider the discounted Hamilton-Jacobi equation

$$
\begin{equation*}
\lambda v_{\lambda}(x)+H\left(x, D v_{\lambda}(x)\right)=0 \quad\left(E_{\lambda}\right) \tag{1}
\end{equation*}
$$

By a standard argument, this equation has a unique viscosity solution $v_{\lambda}: \mathbb{T}^{n} \rightarrow \mathbb{R}$. Moreover, ( $-\lambda v_{\lambda}$ ) converges uniformly as $\lambda$ vanishes to a constant $c(H)$ called the critical value. Set $u_{\lambda}:=v_{\lambda}+c(H) / \lambda$. The

[^0]family $\left(u_{\lambda}\right)$ is equi-Lipschitz, and converges uniformly along subsequences towards a solution of $\left(E_{0}\right)$, for

5 $c=c(H)$. Note that ( $E_{0}$ ) may have several solutions. Recently, under the assumption that $H$ is convex in the momentum, Davini, Fathi, Iturriaga and Zavidovique [2] have proved that $\left(u_{\lambda}\right)$ converges uniformly (towards a solution of $\left(E_{0}\right)$ ). In addition, they proved that the solution can be characterized using Mather measures and Peierls barriers.
Without the convexity assumption, the question of whether $\left(u_{\lambda}\right)$ converges or not remained open. This paper solves negatively this question and provides a 1-dimensional continuous and coercive Hamiltonian for which $\left(u_{\lambda}\right)$ does not converge ${ }^{2}$.

Theorem 1. There exists a continuous Hamiltonian $H: \mathbb{T}^{1} \times \mathbb{R} \rightarrow \mathbb{R}$ that is coercive in the momentum, such that $u_{\lambda}$ does not converge as $\lambda$ tends to 0 .

The example builds on a class of discrete-time repeated games called stochastic games. The main ingredient is to establish a connection between recent counterexamples to the existence of the limit value in stochastic games (see [4, 5]) and the Hamilton-Jacobi problem ${ }^{3}$.
The remainder of the paper is structured as follows. Section 2 presents the stochastic game example. Section 3 shows that in order to prove Theorem 1, it is enough to study the asymptotic behavior of the stochastic game, when the discount factor vanishes. Section 4 determines the asymptotic behavior of the stochastic

[^1]
## 2. The stochastic game example

Given a finite set $A$, the set of probability measures over $A$ is denoted by $\Delta(A)$. Given $a \in A$, the Dirac measure at $a$ is denoted by $\delta_{a}$.

### 2.1. Description of the game

Consider the following stochastic game $\Gamma$, described by:

- A state space $K$ with two elements $\omega_{1}$ and $\omega_{-1}: K=\left\{\omega_{1}, \omega_{-1}\right\}$,
- An action set $I=\{0,1\}$ for Player 1 ,
- An action set $J=\left\{2-\sqrt{2}+2^{-2 n}, n \geq 1\right\} \cup\{2-\sqrt{2}\}$ for Player 2,
- For each $(k, i, j) \in K \times I \times J$, a transition $q(. \mid k, i, j) \in \Delta(K)$ defined by:

$$
\begin{aligned}
q\left(. \mid \omega_{1}, i, j\right) & =[i j+(1-i)(1-j)] \delta_{\omega_{1}}+[i(1-j)+(1-i) j] \delta_{\omega_{-1}} \\
q\left(. \mid \omega_{-1}, i, j\right) & =[i(1-j)+(1-i) j] \delta_{\omega_{1}}+[i j+(1-i)(1-j)] \delta_{\omega_{-1}}
\end{aligned}
$$

- A payoff function $g: K \times I \times J \rightarrow[0,1]$, defined by

$$
g\left(\omega_{1}, i, j\right)=i j+2(1-i)(1-j) \quad \text { and } \quad g\left(\omega_{-1}, i, j\right)=-i j-2(1-i)(1-j)
$$

Let $k_{1} \in K$. The stochastic game $\Gamma^{k_{1}}$ starting at $k_{1}$ proceeds as follows:

- The initial state is $k_{1}$. At first stage, Player 2 chooses $j_{1} \in J$ and announces it to Player 1. Then, Player 1 chooses $i_{1} \in I$, and announces it to Player 2. The payoff at stage 1 is $g\left(k_{1}, i_{1}, j_{1}\right)$ for Player 1 , and $-g\left(k_{1}, i_{1}, j_{1}\right)$ for Player 2. A new state $k_{2}$ is drawn from the probability $q\left(. \mid k_{1}, i_{1}, j_{1}\right)$ and announced to both players. Then, the game moves on to stage 2.
- At each stage $m \geq 2$, Player 2 chooses $j_{m} \in J$ and announces it to Player 1. Then, Player 1 chooses $i_{m} \in I$, and announces it to Player 2. The payoff at stage $m$ is $g\left(k_{m}, i_{m}, j_{m}\right)$ for Player 1 , and $-g\left(k_{m}, i_{m}, j_{m}\right)$ for Player 2. A new state $k_{m+1}$ is drawn from the probability $q\left(. \mid k_{m}, i_{m}, j_{m}\right)$ and announced to both players. Then, the game moves on to stage $m+1$.
${ }_{40}$ Remark 2. The action set of Player 2 can be interpreted as a set of randomized actions. Indeed, imagine that Player 2 has only two actions, 1 and 0 . These actions are called pure actions. At stage $m$, if Player 2 chooses $j_{m} \in J$, this means that he plays 1 with probability $j_{m}$, and 0 with probability $1-j_{m}$. Denote by $\widetilde{j_{m}} \in\{0,1\}$ his realized action. Player 1 knows $j_{m}$ before playing, but does not know $\widetilde{j_{m}}$. If Player 1 chooses $i_{m} \in I$ afterwards, then the realized payoff is $g\left(k_{m}, i_{m}, \widetilde{j_{m}}\right)$. Thus, the payoff $g\left(k_{m}, i_{m}, j_{m}\right)$ represents the ${ }_{45}$ expectation of $g\left(k_{m}, i_{m}, \widetilde{j_{m}}\right)$. Likewise, the transition $q\left(. \mid k_{m}, i_{m}, j_{m}\right)$ represents the law of $q\left(k_{m}, i_{m}, \widetilde{j_{m}}\right)$. The transition and payoff in $\Gamma$ when players play pure actions can be represented by the following matrices:

Table 1: Transition and payoff functions in state $\omega_{1}$ and $\omega_{-1}$

| $\omega_{1}$ | 1 | 0 |
| :---: | :---: | :---: |
| 1 | 1 | $\overrightarrow{0}$ |
| 0 | $\overrightarrow{0}$ | 2 |


| $\omega_{-1}$ | 1 | 0 |
| :--- | :---: | :---: |
| 1 | -1 | $\overleftarrow{0}$ |
| 0 | $\overleftarrow{0}$ | -2 |

The left-hand side matrix stands for state $\omega_{1}$, and the right-hand side matrix stands for state $\omega_{-1}$. Consider the left-hand side matrix. Player 1 chooses a row (either 1 or 0), and Player 2 chooses a column (either 1 or 0 ). The payoff is given by the numbers: for instance, $g(1,1)=1$ and $g(1,0)=0$. The arrow means that when the corresponding actions are played, the state moves on to state $\omega_{-1}$; otherwise, it stays in $\omega_{1}$. For instance, $q\left(. \mid \omega_{1}, 1,1\right)=\delta_{\omega_{1}}$ and $q\left(. \mid \omega_{1}, 1,0\right)=\delta_{\omega_{-1}}$. The interpretation is the same for the right-hand side matrix. In the game $\Gamma$, Player 1 can play only pure actions ( 1 or 0 ), and Player 2 can play 1 with some probability $j \in J$.

This matrix representation is convenient to understand the strategic aspects of the game.

Let us now define formally strategies. In general, the decision of a player at stage $m$ may depend on all the information he has: that is, the stage $m$, and all the states and actions before stage $m$. In this paper, it is sufficient to consider a restricted class of strategies, called stationary strategies. Formally, a stationary strategy for Player 1 is defined as a mapping $y: K \times J \rightarrow I$. The interpretation is that at stage $m$, if the current state is $k$, and Player 2 plays $j$, then Player 1 plays $y(k, j)$. Thus, Player 1 only bases his decision on the current state and the current action of Player 2. Denote by $Y$ the set of stationary strategies for Player 1.

A stationary strategy for Player 2 is defined as a mapping $z: K \rightarrow J$. The interpretation is that at stage $m$, if the current state is $k$, then Player 2 plays $z(k)$. Thus, Player 2 only bases his decision on the current state. Denote by $Z$ the set of stationary strategies for Player 2.

The sequence $\left(k_{1}, i_{1}, j_{1}, k_{2}, i_{2}, j_{2}, \ldots, k_{m}, i_{m}, j_{m}, \ldots\right) \in H_{\infty}:=(K \times I \times J)^{\mathbb{N}^{*}}$ generated along the game is called history of the game. Due to the fact that state transitions are random, this is a random variable. The law of this random variable depends on the initial state $k_{1}$ and the pair of strategies $(y, z)$, and is denoted by $\mathbb{P}_{y, z}^{k_{1}}$. We will call $g_{m}$ the $m$-stage random payoff $g\left(k_{m}, i_{m}, j_{m}\right)$. Let $\lambda \in(0,1]$. The game $\Gamma_{\lambda}^{k_{1}}$ is the game where the strategy set of Player 1 (resp. 2) is $Y$ (resp. $Z$ ), and the payoff is $\gamma_{\lambda}^{k_{1}}$, where

$$
\gamma_{\lambda}^{k_{1}}(y, z)=\mathbb{E}_{y, z}^{k_{1}}\left(\sum_{m \geq 1}(1-\lambda)^{m-1} g_{m}\right)
$$

The goal of Player 1 is to maximize this quantity, while the goal of Player 2 is to minimize this quantity. The game $\Gamma_{\lambda}^{k_{1}}$ has a value, that is:

$$
\min _{z \in Z} \max _{y \in Y} \gamma_{\lambda}^{k_{1}}(y, z)=\max _{y \in Y} \min _{z \in Z} \gamma_{\lambda}^{k_{1}}(y, z)
$$

${ }_{55}$ The value of $\Gamma_{\lambda}^{k_{1}}$ is then defined as the above quantity, and is denoted by $w_{\lambda}\left(k_{1}\right)$. A strategy for Player 1 is optimal if it achieves the right-hand side maximum, and a strategy for Player 2 is optimal if it achieves the left-hand side minimum. The interpretation is that if players are rational they should play optimal strategies, and as a result Player 1 should get $w_{\lambda}\left(k_{1}\right)$, and Player 2 should get $-w_{\lambda}\left(k_{1}\right)$.

### 2.2. Asymptotic behavior of the discounted value

As we shall see in the next section, for each $\lambda \in(0,1]$, one can associate a discounted Hamilton-Jacobi equation with $c(H)=0$, such that its solution evaluated at $x=1$ is approximately $w_{\lambda}\left(\omega_{1}\right)$, for $\lambda$ small
enough. Thus, the asymptotic behavior of this quantity needs to be studied.
Define $\lambda_{n}:=2^{-2 n}\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)^{-1}$ and $\mu_{n}:=2^{-2 n-1}\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)^{-1}$.
Proposition 3. The following hold:
(i) $w_{\lambda}\left(\omega_{-1}\right) \leq w_{\lambda}\left(\omega_{1}\right) \leq w_{\lambda}\left(\omega_{-1}\right)+2$
(ii) $\lim _{n \rightarrow+\infty} w_{\lambda_{n}}\left(\omega_{1}\right)=1 / \sqrt{2}$ and $\liminf _{n \rightarrow+\infty} w_{\mu_{n}}\left(\omega_{1}\right)>1 / \sqrt{2}$. Consequently, $\left(w_{\lambda}\left(\omega_{1}\right)\right)$ does not have a limit when $\lambda \rightarrow 0$.

The proof of the above proposition is done in Section 4. As far as the proof of Theorem 1 is concerned, the key point is (ii). is Let us give here some piece of intuition for this result. Consider the game $\Gamma^{\prime}$ that is identical to $\Gamma$, except that Player 2's action set is $[0,1]$ instead of $J$. For each $\lambda \in(0,1]$, denote by $w_{\lambda}^{\prime}$ its discounted value. Because $J \subset[0,1]$, Player 2 is better off in the game $\Gamma^{\prime}$ compared to the game $\Gamma$ : $w_{\lambda}^{\prime} \leq w_{\lambda}$. Interpret now $\Gamma$ and $\Gamma^{\prime}$ as games with randomized actions, as in Table 2. As $\lambda$ vanishes, standard computations show that an (almost) optimal stationary strategy for Player 2 in $\Gamma_{\lambda}^{\prime \omega_{1}}$ is to play 1 with probability $p^{*}(\lambda):=2-\sqrt{2}+\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right) \lambda$ in both states $\omega_{1}$ and $\omega_{-1}$, and $\left(w_{\lambda}\left(\omega_{1}\right)\right)$ converges to $\frac{1}{\sqrt{2}}$. Moreover, for all $n \geq 1, p^{*}\left(\lambda_{n}\right) \in J$. Thus, this strategy is available for Player 2 in $\Gamma$, and consequently $w_{\lambda_{n}}\left(\omega_{1}\right)=w_{\lambda_{n}}^{\prime}\left(\omega_{1}\right)+O\left(\lambda_{n}\right)$, as $n$ tends to infinity.
On the other hand, for all $n \geq 1, p^{*}\left(\mu_{n}\right) \notin J$, and the distance of $p^{*}\left(\mu_{n}\right)$ to $J$ is larger than $\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right) \mu_{n} / 2$. Consequently, the distance of the optimal strategy in $\Gamma_{\mu_{n}}^{\omega_{1}}$ to the optimal strategy in $\Gamma_{\mu_{n}}^{\omega_{1}}$ is of order $\mu_{n}$. This produces a payoff difference of order $\mu_{n}$ at each stage, and thus of order 1 in the whole game. Thus, Player 2 is significantly disadvantaged in $\Gamma_{\mu_{n}}^{\omega_{1}}$ compared to $\Gamma_{\mu_{n}}^{\prime \omega_{1}}$, and the difference between $w_{\mu_{n}}\left(\omega_{1}\right)$ and $w_{\mu_{n}}^{\prime}\left(\omega_{1}\right)$ is of order 1 .

Remark 4. As we shall see in the following section, we have $\lim _{\lambda \rightarrow 0} \lambda w_{\lambda}\left(\omega_{1}\right)=\lim _{\lambda \rightarrow 0} \lambda w_{\lambda}\left(\omega_{-1}\right)=0$.

The next section explains how to derive the counterexample and Theorem 1 from Proposition 3.

## 3. Link with the PDE problem and proof of Theorem 1

The following proposition expresses $w_{\lambda}$ as the solution of a functional equation called Shapley equation.

Proposition 5. Let $\lambda \in(0,1]$ and $u_{\lambda}:=(1+\lambda)^{-1} w_{\lambda /(1+\lambda)}$. For each $r \in\{-1,1\}$, the two following equations hold:
(i)

$$
w_{\lambda}\left(\omega_{r}\right)=\min _{j \in J} \max _{i \in I}\left\{g\left(\omega_{r}, i, j\right)+(1-\lambda)\left[q\left(\omega_{r} \mid \omega_{r}, i, j\right) w_{\lambda}\left(\omega_{r}\right)+q\left(\omega_{-r} \mid \omega_{r}, i, j\right) w_{\lambda}\left(\omega_{-r}\right)\right]\right\}
$$

(ii)

$$
\lambda u_{\lambda}\left(\omega_{r}\right)=\min _{j \in J} \max _{i \in I}\left\{g\left(\omega_{r}, i, j\right)+q\left(\omega_{-r} \mid \omega_{r}, i, j\right)\left[u_{\lambda}\left(\omega_{-r}\right)-u_{\lambda}\left(\omega_{r}\right)\right]\right\}
$$

Proof. (a) The intuition is the following. Consider the game $\Gamma_{\lambda}^{\omega_{r}}$. At stage 1 , the state is $\omega_{r}$. The term $g$ represents the current payoff, and the term $(1-\lambda)[\ldots]$ represents the future optimal payoff, that is, the payoff that Player 1 should get from stage 2 to infinity. Thus, this equation means that the value of $\Gamma_{\lambda}^{\omega_{r}}$ coincides with the value of the one-stage game, where the payoff is a combination of the current payoff and the future optimal payoff. For a formal derivation of this type of equation, we refer to [10, VII.1., p. 392].
(b) Evaluating the previous equation at $\lambda /(1+\lambda)$ yields

$$
w_{\frac{\lambda}{1+\lambda}}\left(\omega_{r}\right)=\min _{j \in J} \max _{i \in I}\left\{g\left(\omega_{r}, i, j\right)+\frac{1}{1+\lambda}\left[q\left(\omega_{r} \mid \omega_{r}, i, j\right) w_{\frac{\lambda}{1+\lambda}}\left(\omega_{r}\right)+q\left(\omega_{-r} \mid \omega_{r}, i, j\right) w_{\frac{\lambda}{1+\lambda}}\left(\omega_{-r}\right)\right]\right\}
$$

Using the fact that $q\left(\omega_{r} \mid \omega_{r}, i, j\right)=1-q\left(\omega_{-r} \mid \omega_{r}, i, j\right)$ yields the result.

For $r \in\{1,-1\}$ and $p \in \mathbb{R}$, define $H_{r}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
H_{r}(p):= \begin{cases}-\min _{j \in J} \max _{i \in I}\left\{g\left(\omega_{r}, i, j\right)-r p \cdot([i(1-j)+(1-i) j]\},\right. & \text { if }|p| \leq 2 \\ H_{r}\left(2 \frac{p}{|p|}\right)+|p|-2 & \text { if }|p|>2\end{cases}
$$

For $x \in[-1,1]$ and $p \in \mathbb{R}$, let

$$
\begin{equation*}
H(x, p):=|x| H_{1}(|p|)+(1-|x|) H_{-1}(|p|) \tag{2}
\end{equation*}
$$

Note that the definition of $H_{1}$ and $H_{-1}$ for $|p|>2$ ensures that $\lim _{|p| \rightarrow+\infty} H_{1}(p)=\lim _{|p| \rightarrow+\infty} H_{-1}(p)=+\infty$, thus $\lim _{|p| \rightarrow+\infty} H(p)=+\infty$. Note also that for all $x \in[-1,1], H_{1}(x,$.$) is increasing on [-2,2]$ and $H_{-1}(x,$. is decreasing on $[-2,2]$.
Thanks to Proposition 5 (ii) and Proposition 3 (i), we have $\lambda u_{\lambda}\left(\omega_{1}\right)+H_{1}\left(u_{\lambda}\left(\omega_{1}\right)-u_{\lambda}\left(\omega_{-1}\right)\right)=0$ and $\lambda u_{\lambda}\left(\omega_{-1}\right)+H_{-1}\left(u_{\lambda}\left(\omega_{1}\right)-u_{\lambda}\left(\omega_{-1}\right)\right)=0$.

For $x \in[-1,1]$, let $u_{\lambda}(x)=|x| u_{\lambda}\left(\omega_{1}\right)+(1-|x|) u_{\lambda}\left(\omega_{-1}\right)$. Let $x \in(-1,1) \backslash\{0\}$. Proposition 3 (i) implies that $w_{\lambda}\left(\omega_{-1}\right) \leq w_{\lambda}\left(\omega_{1}\right)$, thus $u_{\lambda}\left(\omega_{-1}\right) \leq u_{\lambda}\left(\omega_{1}\right)$ and $\left|D u_{\lambda}(x)\right|=u_{\lambda}\left(\omega_{1}\right)-u_{\lambda}\left(\omega_{-1}\right)$. Consequently, Proposition 5 (ii) yields

$$
\begin{equation*}
\lambda u_{\lambda}(x)+H\left(x, D u_{\lambda}(x)\right)=0 . \tag{3}
\end{equation*}
$$

Note that the above equation is identical to equation (3). The reason why we use the notation $u_{\lambda}$ and not $v_{\lambda}$ is that, as we shall see, $c(H)=0$, thus $u_{\lambda}$ coincides with $v_{\lambda}$.

Extend $u_{\lambda}$ and $H(., p)(p \in \mathbb{R})$ as 2-periodic functions defined on $\mathbb{R}$. The Hamiltonian $H$ is continuous and coercive in the momentum, and the above equation holds in a classical sense for all $x \in \mathbb{R} \backslash \mathbb{Z}$.

For $x \in \mathbb{R}$, denote by $D^{+} u_{\lambda}(x)$ (resp., $D^{-} u_{\lambda}(x)$ ) the super-differential (resp., the sub-differential) of $u_{\lambda}$ at $x$. Let us show that $u_{\lambda}$ is a viscosity solution of (3) on $\mathbb{R}$. By 2-periodicity, it is enough to show that this is a viscosity solution for $x=0$ and $x=1$.
Let us start by $x=0$. We have $D^{+} u_{\lambda}(0)=\emptyset$ and $D^{-} u_{\lambda}(0)=\left[u_{\lambda}\left(\omega_{-1}\right)-u_{\lambda}\left(\omega_{1}\right), u_{\lambda}\left(\omega_{1}\right)-u_{\lambda}\left(\omega_{-1}\right)\right]$.
Let $p \in D^{-} u_{\lambda}(0)$. Then $H_{-1}(p) \geq H_{-1}\left(u_{\lambda}\left(\omega_{1}\right)-u_{\lambda}\left(\omega_{-1}\right)\right)=-\lambda u_{\lambda}\left(\omega_{-1}\right)$, thus $\lambda u_{\lambda}(0)+H(0, p) \geq 0$. Consequently, $u_{\lambda}$ is a viscosity solution at $x=0$. $-\lambda u_{\lambda}$ converges to $c(H)$. Proposition 3 (ii) implies that $\left(-\lambda_{n} u_{\lambda_{n}}(1)\right)$ converges to 0 , thus $c(H)=0$. Still by Proposition 3 (ii), ( $\left.u_{\lambda}(1)\right)$ does not have a limit when $\lambda$ tends to 0 : Theorem 1 is proved.

## 4. Proof of Proposition 3

### 4.1. Proof of (i)

Consider Proposition 5 (i) for $r=1$. Take $j=1 / 2 \in J$. It yields

$$
\begin{align*}
w_{\lambda}\left(\omega_{1}\right) & \leq \max _{i \in I}\left\{1+(1-\lambda)\left(\frac{1}{2} w_{\lambda}\left(\omega_{1}\right)+\frac{1}{2} w_{\lambda}\left(\omega_{-1}\right)\right)\right\} \\
& =1+\frac{1}{2}(1-\lambda)\left(w_{\lambda}\left(\omega_{1}\right)+w_{\lambda}\left(\omega_{-1}\right)\right) \tag{4}
\end{align*}
$$

Take $i=1 / 2$. This yields

$$
\begin{equation*}
w_{\lambda}\left(\omega_{1}\right) \geq \frac{1}{2}+\frac{1}{2}(1-\lambda)\left(w_{\lambda}\left(\omega_{1}\right)+w_{\lambda}\left(\omega_{-1}\right)\right) \tag{5}
\end{equation*}
$$

For $r=-1$, taking $j=1 / 2$ and then $i=1 / 2$ produce the following inequalities:

$$
\begin{equation*}
w_{\lambda}\left(\omega_{-1}\right) \leq-\frac{1}{2}+\frac{1}{2}(1-\lambda)\left(w_{\lambda}\left(\omega_{1}\right)+w_{\lambda}\left(\omega_{-1}\right)\right) \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
w_{\lambda}\left(\omega_{-1}\right) \geq-1+\frac{1}{2}(1-\lambda)\left(w_{\lambda}\left(\omega_{1}\right)+w_{\lambda}\left(\omega_{-1}\right)\right) \tag{7}
\end{equation*}
$$

Combining (5) and (6) yield $w_{\lambda}\left(\omega_{1}\right) \geq w_{\lambda}\left(\omega_{-1}\right)+1 \geq w_{\lambda}\left(\omega_{-1}\right)$. Combining (4) and (7) yield $w_{\lambda}\left(\omega_{-1}\right) \geq$ $w_{\lambda}\left(\omega_{1}\right)-2$, and (i) is proved.

### 4.2. Proof of (ii)

For $\left(i, i^{\prime}\right) \in\{0,1\}^{2}$, consider the strategy $y$ of Player 1 that plays $i$ in $\omega_{1}$ and $i^{\prime}$ in $\omega_{-1}$ (regardless of
125 Player 2's actions), and the strategy $z$ of Player 2 that plays $a$ in state $\omega_{1}$, and $b$ in state $\omega_{-1}$. Denote $\gamma_{\lambda}^{i, i^{\prime}}(a, b):=\gamma_{\lambda}^{\omega_{1}}(y, z)$ (resp., $\widetilde{\gamma}_{\lambda}^{i, i^{\prime}}(a, b):=\gamma_{\lambda}^{\omega_{-1}}(y, z)$ ), the payoff in $\Gamma_{\lambda}^{\omega_{1}}$ (resp., $\Gamma_{\lambda}^{\omega_{-1}}$ ), when $(y, z)$ is played.

Proposition 6. The following hold:
1.

$$
\begin{gathered}
\gamma_{\lambda}^{0,0}(a, b)=\frac{-2(a-b-\lambda+b \lambda)}{\lambda(a+b+\lambda-a \lambda-b \lambda)} \\
\gamma_{\lambda}^{1,1}(a, b)=-\frac{a-b+\lambda b}{\lambda(a+b+\lambda-a \lambda-b \lambda-2)} \\
\gamma_{\lambda}^{1,0}(a, b)=\frac{2 a+2 b+2 \lambda-a b-a \lambda-2 b \lambda+a b \lambda-2}{\lambda(b-a+\lambda a-b \lambda+1)} \\
\gamma_{\lambda}^{0,1}(a, b)=-\frac{2 a+2 b-a b-2 b \lambda+a b \lambda-2}{\lambda(a-b-a \lambda+b \lambda+1)}
\end{gathered}
$$

2.     - $\gamma_{\lambda}^{0,0}$ is decreasing with respect to $a$ and increasing with respect to $b$.

- $\gamma_{\lambda}^{1,1}$ is increasing with respect to a and decreasing with respect to $b$.
- $\gamma_{\lambda}^{1,0}$ is increasing with respect to $a$ and $b$.
- $\gamma_{\lambda}^{0,1}$ is decreasing with respect to $a$ and $b$.

Proof. 1. The payoffs $\gamma_{\lambda}^{0,0}(a, b)$ and $\widetilde{\gamma}_{\lambda}^{0,0}(a, b)$ satisfy the following recursive equation:

$$
\begin{aligned}
\gamma_{\lambda}^{0,0}(a, b) & =a(1-\lambda) \widetilde{\gamma}_{\lambda}^{0,0}(a, b)+(1-a)\left(2+(1-\lambda) \gamma_{\lambda}^{0,0}(a, b)\right) \\
\widetilde{\gamma}_{\lambda}^{0,0}(a, b) & =a(1-\lambda) \gamma_{\lambda}^{0,0}(a, b)+(1-a)\left(-2+(1-\lambda) \widetilde{\gamma}_{\lambda}^{0,0}(a, b)\right)
\end{aligned}
$$

Combining these two relations give the first equality. The three other equalities can be derived in a similar fashion.
2. These monotonicity properties are simply obtained by deriving $\gamma_{\lambda}^{i, i^{\prime}}$ with respect to $a$ and $b$.

For $\lambda \in(0,1]$, set $p^{*}(\lambda):=2-\sqrt{2}+\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right) \lambda$. Define a strategy $y$ of Player 1 in the following way:

- in state $\omega_{1}$, play 0 if $j \leq p^{*}(\lambda)$, play 1 otherwise,
- in state $\omega_{-1}$, play 1 if $j \leq p^{*}(\lambda)$, play 0 otherwise.

The rationale behind this strategy can be found in Section 2.2.
For all $n \geq 1$, define

$$
\lambda_{n}:=\frac{2^{-2 n}}{\frac{3}{4}-\sqrt{2}} \quad \text { and } \quad \mu_{n}:=\frac{2^{-2 n-1}}{\frac{3}{4}-\sqrt{2}}
$$

## Proposition 7. The following hold:

1. 

$$
\lim _{n \rightarrow+\infty} \min _{z \in Z} \gamma_{\lambda_{n}}(y, z)=\frac{1}{\sqrt{2}}
$$

2. 

$$
\lim _{n \rightarrow+\infty} \min _{z \in Z} \gamma_{\mu_{n}}(y, z)=\frac{5}{2 \sqrt{2}}-1>\frac{1}{\sqrt{2}}
$$

Proof. 1. For all $\left(i, i^{\prime}\right) \in\{0,1\}$,

$$
\lim _{n \rightarrow+\infty} \gamma_{\lambda_{n}}^{i, i^{\prime}}\left(p^{*}\left(\lambda_{n}\right), p^{*}\left(\lambda_{n}\right)\right)=\frac{1}{\sqrt{2}}
$$

and the result follows.
2. Let $z$ be a strategy of Player 2, and $a=z\left(\omega_{1}\right)$ and $b=z\left(\omega_{-1}\right)$.

Note that the interval $\left(p^{*}\left(\mu_{n} / 2\right), p^{*}\left(2 \mu_{n}\right)\right)$ does not intersect $J$.
The following cases are distinguished:
Case 1. $a \leq p^{*}\left(\mu_{n}\right)$ and $b \leq p^{*}\left(\mu_{n}\right)$, thus $a \leq p^{*}\left(\mu_{n} / 2\right)$ and $b \leq p^{*}\left(\mu_{n} / 2\right)$
We have $\gamma_{\mu_{n}}^{\omega_{1}}(y, z)=\gamma_{\mu_{n}}^{0,1}(a, b) \geq \gamma_{\mu_{n}}^{0,1}\left(p^{*}\left(\mu_{n} / 2\right), p^{*}\left(\mu_{n} / 2\right)\right) \underset{n \rightarrow+\infty}{\rightarrow} \frac{5}{4} \sqrt{2}-1$
Case 2. $a \leq p^{*}\left(\mu_{n}\right)$ and $b \geq p^{*}\left(\mu_{n}\right)$, thus $a \leq p^{*}\left(\mu_{n} / 2\right)$ and $b \geq p^{*}\left(2 \mu_{n}\right)$
We have $\gamma_{\mu_{n}}^{\omega_{1}}(y, z)=\gamma_{\mu_{n}}^{0,0}(a, b) \geq \gamma_{\mu_{n}}^{0,0}\left(p^{*}\left(\mu_{n} / 2\right), p^{*}\left(2 \mu_{n}\right)\right) \underset{n \rightarrow+\infty}{\rightarrow}-\frac{1+2 \sqrt{2}}{8(-2+\sqrt{2})}$

Case 3. $a \geq p^{*}\left(\mu_{n}\right)$ and $b \leq p^{*}\left(\mu_{n}\right)$, thus $a \geq p^{*}\left(2 \mu_{n}\right)$ and $b \leq p^{*}\left(\mu_{n} / 2\right)$

We have $\gamma_{\mu_{n}}^{\omega_{1}}(y, z)=\gamma_{\mu_{n}}^{1,1}(a, b) \geq \gamma_{\mu_{n}}^{1,1}\left(p^{*}\left(2 \mu_{n}\right), p^{*}\left(\mu_{n} / 2\right)\right) \underset{n \rightarrow+\infty}{\rightarrow}(-1 / 16) \frac{-25+14 \sqrt{2}}{\sqrt{2}-1}$
Case 4. $a \geq p^{*}\left(\mu_{n}\right)$ and $b \geq p^{*}\left(\mu_{n}\right)$, thus $a \geq p^{*}\left(2 \mu_{n}\right)$ and $b \geq p^{*}\left(2 \mu_{n}\right)$
We have $\gamma_{\mu_{n}}^{\omega_{1}}(y, z)=\gamma_{\mu_{n}}^{1,0}(a, b) \geq \gamma_{\mu_{n}}^{1,0}\left(p^{*}\left(2 \mu_{n}\right), p^{*}\left(2 \mu_{n}\right)\right) \underset{n \rightarrow+\infty}{\rightarrow}-2+2 \sqrt{2}$
Among these cases, the smallest limit is $\frac{5}{4}(\sqrt{2}-1)$, and the result follows.

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[^1]:    ${ }^{2}$ Note that for time-dependent Hamilton-Jacobi equations, several counterexamples about the asymptotic behavior of solutions have been pointed out in [3].
    ${ }^{3}$ Let us mention the work $[6,7,8,9]$ as other illustrations of the use of repeated games in PDE problems.

