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# Convergence of the solutions of the discounted Hamilton-Jacobi equation: a counterexample

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## Abstract

This paper provides a counterexample about the asymptotic behavior of the solutions of a discounted Hamilton-Jacobi equation, as the discount factor vanishes. The Hamiltonian of the equation is a 1-dimensional continuous and coercive Hamiltonian.

## Résumé

Cet article fournit un contre-exemple à la convergence asymptotique des solutions d'une équation de Hamilton-Jacobi escomptée, lorsque le facteur d'escompte tend vers 0. Le Hamiltonien de cette équation est unidimensionnel, continu et coercitif.

*Keywords:* Hamilton-Jacobi equations, Viscosity solutions, Stochastic games, Zero-Sum games.

*2010 MSC:* 49L25, 91A15, 91A25, 91A05.

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## 1. Introduction and main result

Let  $n \geq 1$ . Denote by  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  the  $n$ -dimensional torus. For  $c \in \mathbb{R}$ , consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = c \quad (E_0)$$

where the Hamiltonian  $H : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is jointly continuous and coercive in the momentum. In order to build solutions of the above equation, Lions, Papanicolaou and Varadhan [1] have introduced a technique called *ergodic approximation*. For  $\lambda \in (0, 1]$ , consider the discounted Hamilton-Jacobi equation

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = 0 \quad (E_\lambda) \tag{1}$$

By a standard argument, this equation has a unique viscosity solution  $v_\lambda : \mathbb{T}^n \rightarrow \mathbb{R}$ . Moreover,  $(-\lambda v_\lambda)$  converges uniformly as  $\lambda$  vanishes to a constant  $c(H)$  called the *critical value*. Set  $u_\lambda := v_\lambda + c(H)/\lambda$ . The

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family  $(u_\lambda)$  is equi-Lipschitz, and converges uniformly along subsequences towards a solution of  $(E_0)$ , for  
5  $c = c(H)$ . Note that  $(E_0)$  may have several solutions. Recently, under the assumption that  $H$  is convex  
in the momentum, Davini, Fathi, Iturriaga and Zavidovique [2] have proved that  $(u_\lambda)$  converges uniformly  
(towards a solution of  $(E_0)$ ). In addition, they proved that the solution can be characterized using Mather  
measures and Peierls barriers.

Without the convexity assumption, the question of whether  $(u_\lambda)$  converges or not remained open. This  
10 paper solves negatively this question and provides a 1-dimensional continuous and coercive Hamiltonian for  
which  $(u_\lambda)$  does not converge<sup>2</sup>.

**Theorem 1.** *There exists a continuous Hamiltonian  $H : \mathbb{T}^1 \times \mathbb{R} \rightarrow \mathbb{R}$  that is coercive in the momentum,  
such that  $u_\lambda$  does not converge as  $\lambda$  tends to 0.*

The example builds on a class of discrete-time repeated games called *stochastic games*. The main in-  
15 gredient is to establish a connection between recent counterexamples to the existence of the limit value in  
stochastic games (see [4, 5]) and the Hamilton-Jacobi problem<sup>3</sup>.

The remainder of the paper is structured as follows. Section 2 presents the stochastic game example. Section  
3 shows that in order to prove Theorem 1, it is enough to study the asymptotic behavior of the stochastic  
game, when the discount factor vanishes. Section 4 determines the asymptotic behavior of the stochastic  
20 game.

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<sup>2</sup>Note that for time-dependent Hamilton-Jacobi equations, several counterexamples about the asymptotic behavior of solu-  
tions have been pointed out in [3].

<sup>3</sup>Let us mention the work [6, 7, 8, 9] as other illustrations of the use of repeated games in PDE problems.

## 2. The stochastic game example

Given a finite set  $A$ , the set of probability measures over  $A$  is denoted by  $\Delta(A)$ . Given  $a \in A$ , the Dirac measure at  $a$  is denoted by  $\delta_a$ .

### 2.1. Description of the game

25 Consider the following stochastic game  $\Gamma$ , described by:

- A state space  $K$  with two elements  $\omega_1$  and  $\omega_{-1}$ :  $K = \{\omega_1, \omega_{-1}\}$ ,
- An action set  $I = \{0, 1\}$  for Player 1,
- An action set  $J = \{2 - \sqrt{2} + 2^{-2n}, n \geq 1\} \cup \{2 - \sqrt{2}\}$  for Player 2,
- For each  $(k, i, j) \in K \times I \times J$ , a transition  $q(\cdot | k, i, j) \in \Delta(K)$  defined by:

$$\begin{aligned} q(\cdot | \omega_1, i, j) &= [ij + (1-i)(1-j)]\delta_{\omega_1} + [i(1-j) + (1-i)j]\delta_{\omega_{-1}}, \\ q(\cdot | \omega_{-1}, i, j) &= [i(1-j) + (1-i)j]\delta_{\omega_1} + [ij + (1-i)(1-j)]\delta_{\omega_{-1}}. \end{aligned}$$

- A payoff function  $g : K \times I \times J \rightarrow [0, 1]$ , defined by

$$g(\omega_1, i, j) = ij + 2(1-i)(1-j) \quad \text{and} \quad g(\omega_{-1}, i, j) = -ij - 2(1-i)(1-j).$$

30 Let  $k_1 \in K$ . The stochastic game  $\Gamma^{k_1}$  starting at  $k_1$  proceeds as follows:

- The initial state is  $k_1$ . At first stage, Player 2 chooses  $j_1 \in J$  and announces it to Player 1. Then, Player 1 chooses  $i_1 \in I$ , and announces it to Player 2. The payoff at stage 1 is  $g(k_1, i_1, j_1)$  for Player 1, and  $-g(k_1, i_1, j_1)$  for Player 2. A new state  $k_2$  is drawn from the probability  $q(\cdot | k_1, i_1, j_1)$  and announced to both players. Then, the game moves on to stage 2.

35

- At each stage  $m \geq 2$ , Player 2 chooses  $j_m \in J$  and announces it to Player 1. Then, Player 1 chooses  $i_m \in I$ , and announces it to Player 2. The payoff at stage  $m$  is  $g(k_m, i_m, j_m)$  for Player 1, and  $-g(k_m, i_m, j_m)$  for Player 2. A new state  $k_{m+1}$  is drawn from the probability  $q(\cdot | k_m, i_m, j_m)$  and announced to both players. Then, the game moves on to stage  $m + 1$ .

40 **Remark 2.** The action set of Player 2 can be interpreted as a set of randomized actions. Indeed, imagine that Player 2 has only two actions, 1 and 0. These actions are called *pure actions*. At stage  $m$ , if Player 2 chooses  $j_m \in J$ , this means that he plays 1 with probability  $j_m$ , and 0 with probability  $1 - j_m$ . Denote by  $\widetilde{j}_m \in \{0, 1\}$  his realized action. Player 1 knows  $j_m$  before playing, but does not know  $\widetilde{j}_m$ . If Player 1 chooses  $i_m \in I$  afterwards, then the realized payoff is  $g(k_m, i_m, \widetilde{j}_m)$ . Thus, the payoff  $g(k_m, i_m, j_m)$  represents the  
 45 expectation of  $g(k_m, i_m, \widetilde{j}_m)$ . Likewise, the transition  $q(\cdot | k_m, i_m, j_m)$  represents the law of  $q(k_m, i_m, \widetilde{j}_m)$ . The transition and payoff in  $\Gamma$  when players play pure actions can be represented by the following matrices:

Table 1: Transition and payoff functions in state  $\omega_1$  and  $\omega_{-1}$

|            |                      |                      |  |               |                     |                     |
|------------|----------------------|----------------------|--|---------------|---------------------|---------------------|
| $\omega_1$ | 1                    | 0                    |  | $\omega_{-1}$ | 1                   | 0                   |
| 1          | 1                    | $\overrightarrow{0}$ |  | 1             | -1                  | $\overleftarrow{0}$ |
| 0          | $\overrightarrow{0}$ | 2                    |  | 0             | $\overleftarrow{0}$ | -2                  |

The left-hand side matrix stands for state  $\omega_1$ , and the right-hand side matrix stands for state  $\omega_{-1}$ . Consider the left-hand side matrix. Player 1 chooses a row (either 1 or 0), and Player 2 chooses a column (either 1 or 0). The payoff is given by the numbers: for instance,  $g(1, 1) = 1$  and  $g(1, 0) = 0$ . The arrow  
 50 means that when the corresponding actions are played, the state moves on to state  $\omega_{-1}$ ; otherwise, it stays in  $\omega_1$ . For instance,  $q(\cdot | \omega_1, 1, 1) = \delta_{\omega_{-1}}$  and  $q(\cdot | \omega_1, 1, 0) = \delta_{\omega_1}$ . The interpretation is the same for the right-hand side matrix. In the game  $\Gamma$ , Player 1 can play only pure actions (1 or 0), and Player 2 can play 1 with some probability  $j \in J$ .

This matrix representation is convenient to understand the strategic aspects of the game.

Let us now define formally *strategies*. In general, the decision of a player at stage  $m$  may depend on all the information he has: that is, the stage  $m$ , and all the states and actions before stage  $m$ . In this paper, it is sufficient to consider a restricted class of strategies, called *stationary strategies*. Formally, a stationary strategy for Player 1 is defined as a mapping  $y : K \times J \rightarrow I$ . The interpretation is that at stage  $m$ , if the current state is  $k$ , and Player 2 plays  $j$ , then Player 1 plays  $y(k, j)$ . Thus, Player 1 only bases his decision on the current state and the current action of Player 2. Denote by  $Y$  the set of stationary strategies for Player 1.

A stationary strategy for Player 2 is defined as a mapping  $z : K \rightarrow J$ . The interpretation is that at stage  $m$ , if the current state is  $k$ , then Player 2 plays  $z(k)$ . Thus, Player 2 only bases his decision on the current state. Denote by  $Z$  the set of stationary strategies for Player 2.

The sequence  $(k_1, i_1, j_1, k_2, i_2, j_2, \dots, k_m, i_m, j_m, \dots) \in H_\infty := (K \times I \times J)^{\mathbb{N}^*}$  generated along the game is called *history* of the game. Due to the fact that state transitions are random, this is a random variable. The law of this random variable depends on the initial state  $k_1$  and the pair of strategies  $(y, z)$ , and is denoted by  $\mathbb{P}_{y,z}^{k_1}$ . We will call  $g_m$  the  $m$ -stage random payoff  $g(k_m, i_m, j_m)$ . Let  $\lambda \in (0, 1]$ . The game  $\Gamma_\lambda^{k_1}$  is the game where the strategy set of Player 1 (resp. 2) is  $Y$  (resp.  $Z$ ), and the payoff is  $\gamma_\lambda^{k_1}$ , where

$$\gamma_\lambda^{k_1}(y, z) = \mathbb{E}_{y,z}^{k_1} \left( \sum_{m \geq 1} (1 - \lambda)^{m-1} g_m \right).$$

The goal of Player 1 is to maximize this quantity, while the goal of Player 2 is to minimize this quantity.

The game  $\Gamma_\lambda^{k_1}$  has a value, that is:

$$\min_{z \in Z} \max_{y \in Y} \gamma_\lambda^{k_1}(y, z) = \max_{y \in Y} \min_{z \in Z} \gamma_\lambda^{k_1}(y, z).$$

55 The value of  $\Gamma_\lambda^{k_1}$  is then defined as the above quantity, and is denoted by  $w_\lambda(k_1)$ . A strategy for Player 1 is *optimal* if it achieves the right-hand side maximum, and a strategy for Player 2 is *optimal* if it achieves the left-hand side minimum. The interpretation is that if players are rational they should play optimal strategies, and as a result Player 1 should get  $w_\lambda(k_1)$ , and Player 2 should get  $-w_\lambda(k_1)$ .

## 2.2. Asymptotic behavior of the discounted value

60 As we shall see in the next section, for each  $\lambda \in (0, 1]$ , one can associate a discounted Hamilton-Jacobi equation with  $c(H) = 0$ , such that its solution evaluated at  $x = 1$  is approximately  $w_\lambda(\omega_1)$ , for  $\lambda$  small

enough. Thus, the asymptotic behavior of this quantity needs to be studied.

Define  $\lambda_n := 2^{-2n} \left( \frac{3}{4} - \frac{1}{\sqrt{2}} \right)^{-1}$  and  $\mu_n := 2^{-2n-1} \left( \frac{3}{4} - \frac{1}{\sqrt{2}} \right)^{-1}$ .

65 **Proposition 3.** *The following hold:*

(i)  $w_\lambda(\omega_{-1}) \leq w_\lambda(\omega_1) \leq w_\lambda(\omega_{-1}) + 2$

(ii)  $\lim_{n \rightarrow +\infty} w_{\lambda_n}(\omega_1) = 1/\sqrt{2}$  and  $\liminf_{n \rightarrow +\infty} w_{\mu_n}(\omega_1) > 1/\sqrt{2}$ . *Consequently,  $(w_\lambda(\omega_1))$  does not have a limit when  $\lambda \rightarrow 0$ .*

The proof of the above proposition is done in Section 4. As far as the proof of Theorem 1 is concerned,  
70 the key point is (ii). Let us give here some piece of intuition for this result. Consider the game  $\Gamma'$  that is identical to  $\Gamma$ , except that Player 2's action set is  $[0, 1]$  instead of  $J$ . For each  $\lambda \in (0, 1]$ , denote by  $w'_\lambda$  its discounted value. Because  $J \subset [0, 1]$ , Player 2 is better off in the game  $\Gamma'$  compared to the game  $\Gamma$ :  $w'_\lambda \leq w_\lambda$ . Interpret now  $\Gamma$  and  $\Gamma'$  as games with randomized actions, as in Table 2. As  $\lambda$  vanishes, standard computations show that an (almost) optimal stationary strategy for Player 2 in  $\Gamma'^{\omega_1}$  is to play 1  
75 with probability  $p^*(\lambda) := 2 - \sqrt{2} + \left( \frac{3}{4} - \frac{1}{\sqrt{2}} \right) \lambda$  in both states  $\omega_1$  and  $\omega_{-1}$ , and  $(w_\lambda(\omega_1))$  converges to  $\frac{1}{\sqrt{2}}$ . Moreover, for all  $n \geq 1$ ,  $p^*(\lambda_n) \in J$ . Thus, this strategy is available for Player 2 in  $\Gamma$ , and consequently  $w_{\lambda_n}(\omega_1) = w'_{\lambda_n}(\omega_1) + O(\lambda_n)$ , as  $n$  tends to infinity.

On the other hand, for all  $n \geq 1$ ,  $p^*(\mu_n) \notin J$ , and the distance of  $p^*(\mu_n)$  to  $J$  is larger than  $\left( \frac{3}{4} - \frac{1}{\sqrt{2}} \right) \mu_n / 2$ . Consequently, the distance of the optimal strategy in  $\Gamma_{\mu_n}^{\omega_1}$  to the optimal strategy in  $\Gamma_{\mu_n}'^{\omega_1}$  is of order  $\mu_n$ . This  
80 produces a payoff difference of order  $\mu_n$  at each stage, and thus of order 1 in the whole game. Thus, Player 2 is significantly disadvantaged in  $\Gamma_{\mu_n}^{\omega_1}$  compared to  $\Gamma_{\mu_n}'^{\omega_1}$ , and the difference between  $w_{\mu_n}(\omega_1)$  and  $w'_{\mu_n}(\omega_1)$  is of order 1.

**Remark 4.** As we shall see in the following section, we have  $\lim_{\lambda \rightarrow 0} \lambda w_\lambda(\omega_1) = \lim_{\lambda \rightarrow 0} \lambda w_\lambda(\omega_{-1}) = 0$ .

85 The next section explains how to derive the counterexample and Theorem 1 from Proposition 3.

### 3. Link with the PDE problem and proof of Theorem 1

The following proposition expresses  $w_\lambda$  as the solution of a functional equation called *Shapley equation*.

**Proposition 5.** Let  $\lambda \in (0, 1]$  and  $u_\lambda := (1 + \lambda)^{-1} w_{\lambda/(1+\lambda)}$ . For each  $r \in \{-1, 1\}$ , the two following equations hold:

(i)

$$w_\lambda(\omega_r) = \min_{j \in J} \max_{i \in I} \{g(\omega_r, i, j) + (1 - \lambda) [q(\omega_r | \omega_r, i, j) w_\lambda(\omega_r) + q(\omega_{-r} | \omega_r, i, j) w_\lambda(\omega_{-r})]\}$$

(ii)

$$\lambda u_\lambda(\omega_r) = \min_{j \in J} \max_{i \in I} \{g(\omega_r, i, j) + q(\omega_{-r} | \omega_r, i, j) [u_\lambda(\omega_{-r}) - u_\lambda(\omega_r)]\}$$

90 *Proof.* (a) The intuition is the following. Consider the game  $\Gamma_\lambda^{\omega_r}$ . At stage 1, the state is  $\omega_r$ . The term  $g$  represents the current payoff, and the term  $(1 - \lambda)[\dots]$  represents the future optimal payoff, that is, the payoff that Player 1 should get from stage 2 to infinity. Thus, this equation means that the value of  $\Gamma_\lambda^{\omega_r}$  coincides with the value of the one-stage game, where the payoff is a combination of the current payoff and the future optimal payoff. For a formal derivation of this type of equation,  
95 we refer to [10, VII.1., p. 392].

(b) Evaluating the previous equation at  $\lambda/(1 + \lambda)$  yields

$$w_{\frac{\lambda}{1+\lambda}}(\omega_r) = \min_{j \in J} \max_{i \in I} \left\{ g(\omega_r, i, j) + \frac{1}{1 + \lambda} [q(\omega_r | \omega_r, i, j) w_{\frac{\lambda}{1+\lambda}}(\omega_r) + q(\omega_{-r} | \omega_r, i, j) w_{\frac{\lambda}{1+\lambda}}(\omega_{-r})] \right\}$$

Using the fact that  $q(\omega_r | \omega_r, i, j) = 1 - q(\omega_{-r} | \omega_r, i, j)$  yields the result.

□



For  $r \in \{1, -1\}$  and  $p \in \mathbb{R}$ , define  $H_r : \mathbb{R} \rightarrow \mathbb{R}$  by

$$H_r(p) := \begin{cases} -\min_{j \in J} \max_{i \in I} \{g(\omega_r, i, j) - rp \cdot ([i(1-j) + (1-i)j])\}, & \text{if } |p| \leq 2, \\ H_r\left(2\frac{p}{|p|}\right) + |p| - 2 & \text{if } |p| > 2. \end{cases}$$

For  $x \in [-1, 1]$  and  $p \in \mathbb{R}$ , let

$$H(x, p) := |x|H_1(|p|) + (1 - |x|)H_{-1}(|p|). \quad (2)$$

Note that the definition of  $H_1$  and  $H_{-1}$  for  $|p| > 2$  ensures that  $\lim_{|p| \rightarrow +\infty} H_1(p) = \lim_{|p| \rightarrow +\infty} H_{-1}(p) = +\infty$ , thus  $\lim_{|p| \rightarrow +\infty} H(p) = +\infty$ . Note also that for all  $x \in [-1, 1]$ ,  $H_1(x, \cdot)$  is increasing on  $[-2, 2]$  and  $H_{-1}(x, \cdot)$  is decreasing on  $[-2, 2]$ .

Thanks to Proposition 5 (ii) and Proposition 3 (i), we have  $\lambda u_\lambda(\omega_1) + H_1(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = 0$  and  $\lambda u_\lambda(\omega_{-1}) + H_{-1}(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = 0$ .

For  $x \in [-1, 1]$ , let  $u_\lambda(x) = |x|u_\lambda(\omega_1) + (1 - |x|)u_\lambda(\omega_{-1})$ . Let  $x \in (-1, 1) \setminus \{0\}$ . Proposition 3 (i) implies that  $w_\lambda(\omega_{-1}) \leq w_\lambda(\omega_1)$ , thus  $u_\lambda(\omega_{-1}) \leq u_\lambda(\omega_1)$  and  $|Du_\lambda(x)| = u_\lambda(\omega_1) - u_\lambda(\omega_{-1})$ . Consequently, Proposition 5 (ii) yields

$$\lambda u_\lambda(x) + H(x, Du_\lambda(x)) = 0. \quad (3)$$

Note that the above equation is identical to equation (3). The reason why we use the notation  $u_\lambda$  and not  $v_\lambda$  is that, as we shall see,  $c(H) = 0$ , thus  $u_\lambda$  coincides with  $v_\lambda$ .

Extend  $u_\lambda$  and  $H(\cdot, p)$  ( $p \in \mathbb{R}$ ) as 2-periodic functions defined on  $\mathbb{R}$ . The Hamiltonian  $H$  is continuous and coercive in the momentum, and the above equation holds in a classical sense for all  $x \in \mathbb{R} \setminus \mathbb{Z}$ .

For  $x \in \mathbb{R}$ , denote by  $D^+u_\lambda(x)$  (resp.,  $D^-u_\lambda(x)$ ) the super-differential (resp., the sub-differential) of  $u_\lambda$  at  $x$ . Let us show that  $u_\lambda$  is a viscosity solution of (3) on  $\mathbb{R}$ . By 2-periodicity, it is enough to show that this is a viscosity solution for  $x = 0$  and  $x = 1$ .

Let us start by  $x = 0$ . We have  $D^+u_\lambda(0) = \emptyset$  and  $D^-u_\lambda(0) = [u_\lambda(\omega_{-1}) - u_\lambda(\omega_1), u_\lambda(\omega_1) - u_\lambda(\omega_{-1})]$ .

Let  $p \in D^-u_\lambda(0)$ . Then  $H_{-1}(p) \geq H_{-1}(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = -\lambda u_\lambda(\omega_{-1})$ , thus  $\lambda u_\lambda(0) + H(0, p) \geq 0$ . Consequently,  $u_\lambda$  is a viscosity solution at  $x = 0$ .

110 Consider now the case  $x = 1$ . We have  $D^+u_\lambda(1) = [u_\lambda(\omega_{-1}) - u_\lambda(\omega_1), u_\lambda(\omega_1) - u_\lambda(\omega_{-1})]$  and  $D^-u_\lambda(1) = \emptyset$ .  
 Let  $p \in D^+u_\lambda(1)$ . Then  $H_1(p) \leq H_1(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = -\lambda u_\lambda(\omega_1)$ , thus  $\lambda u_\lambda(1) + H(1, p) \geq 0$ . Consequently,  $u_\lambda$  is a viscosity solution at  $x = 1$ .

Let us now conclude the proof of Theorem 1. Because  $H$  is 2-periodic, equation (3) can be considered as written on  $\mathbb{T}^1$ .

115 As noticed before, equation (3) is identical to equation (1). Therefore, as stated in the introduction,  $-\lambda u_\lambda$  converges to  $c(H)$ . Proposition 3 (ii) implies that  $(-\lambda_n u_{\lambda_n}(1))$  converges to 0, thus  $c(H) = 0$ . Still by Proposition 3 (ii),  $(u_\lambda(1))$  does not have a limit when  $\lambda$  tends to 0: Theorem 1 is proved.

#### 4. Proof of Proposition 3

##### 4.1. Proof of (i)

120 Consider Proposition 5 (i) for  $r = 1$ . Take  $j = 1/2 \in J$ . It yields

$$\begin{aligned} w_\lambda(\omega_1) &\leq \max_{i \in I} \left\{ 1 + (1 - \lambda) \left( \frac{1}{2} w_\lambda(\omega_1) + \frac{1}{2} w_\lambda(\omega_{-1}) \right) \right\} \\ &= 1 + \frac{1}{2} (1 - \lambda) (w_\lambda(\omega_1) + w_\lambda(\omega_{-1})). \end{aligned} \quad (4)$$

Take  $i = 1/2$ . This yields

$$w_\lambda(\omega_1) \geq \frac{1}{2} + \frac{1}{2} (1 - \lambda) (w_\lambda(\omega_1) + w_\lambda(\omega_{-1})). \quad (5)$$

For  $r = -1$ , taking  $j = 1/2$  and then  $i = 1/2$  produce the following inequalities:

$$w_\lambda(\omega_{-1}) \leq -\frac{1}{2} + \frac{1}{2} (1 - \lambda) (w_\lambda(\omega_1) + w_\lambda(\omega_{-1})), \quad (6)$$

and

$$w_\lambda(\omega_{-1}) \geq -1 + \frac{1}{2} (1 - \lambda) (w_\lambda(\omega_1) + w_\lambda(\omega_{-1})). \quad (7)$$

Combining (5) and (6) yield  $w_\lambda(\omega_1) \geq w_\lambda(\omega_{-1}) + 1 \geq w_\lambda(\omega_{-1})$ . Combining (4) and (7) yield  $w_\lambda(\omega_{-1}) \geq w_\lambda(\omega_1) - 2$ , and (i) is proved.

4.2. Proof of (ii)

For  $(i, i') \in \{0, 1\}^2$ , consider the strategy  $y$  of Player 1 that plays  $i$  in  $\omega_1$  and  $i'$  in  $\omega_{-1}$  (regardless of  
 125 Player 2's actions), and the strategy  $z$  of Player 2 that plays  $a$  in state  $\omega_1$ , and  $b$  in state  $\omega_{-1}$ . Denote  
 $\gamma_\lambda^{i, i'}(a, b) := \gamma_\lambda^{\omega_1}(y, z)$  (resp.,  $\tilde{\gamma}_\lambda^{i, i'}(a, b) := \gamma_\lambda^{\omega_{-1}}(y, z)$ ), the payoff in  $\Gamma_\lambda^{\omega_1}$  (resp.,  $\Gamma_\lambda^{\omega_{-1}}$ ), when  $(y, z)$  is played.

**Proposition 6.** *The following hold:*

1.

$$\begin{aligned}\gamma_\lambda^{0,0}(a, b) &= \frac{-2(a - b - \lambda + b\lambda)}{\lambda(a + b + \lambda - a\lambda - b\lambda)} \\ \gamma_\lambda^{1,1}(a, b) &= -\frac{a - b + \lambda b}{\lambda(a + b + \lambda - a\lambda - b\lambda - 2)} \\ \gamma_\lambda^{1,0}(a, b) &= \frac{2a + 2b + 2\lambda - ab - a\lambda - 2b\lambda + ab\lambda - 2}{\lambda(b - a + \lambda a - b\lambda + 1)} \\ \gamma_\lambda^{0,1}(a, b) &= -\frac{2a + 2b - ab - 2b\lambda + ab\lambda - 2}{\lambda(a - b - a\lambda + b\lambda + 1)}\end{aligned}$$

2. •  $\gamma_\lambda^{0,0}$  is decreasing with respect to  $a$  and increasing with respect to  $b$ .  
 •  $\gamma_\lambda^{1,1}$  is increasing with respect to  $a$  and decreasing with respect to  $b$ .  
 130 •  $\gamma_\lambda^{1,0}$  is increasing with respect to  $a$  and  $b$ .  
 •  $\gamma_\lambda^{0,1}$  is decreasing with respect to  $a$  and  $b$ .

*Proof.* 1. The payoffs  $\gamma_\lambda^{0,0}(a, b)$  and  $\tilde{\gamma}_\lambda^{0,0}(a, b)$  satisfy the following recursive equation:

$$\begin{aligned}\gamma_\lambda^{0,0}(a, b) &= a(1 - \lambda)\tilde{\gamma}_\lambda^{0,0}(a, b) + (1 - a)(2 + (1 - \lambda)\gamma_\lambda^{0,0}(a, b)) \\ \tilde{\gamma}_\lambda^{0,0}(a, b) &= a(1 - \lambda)\gamma_\lambda^{0,0}(a, b) + (1 - a)(-2 + (1 - \lambda)\tilde{\gamma}_\lambda^{0,0}(a, b))\end{aligned}$$

Combining these two relations give the first equality. The three other equalities can be derived in a similar fashion.

- 135 2. These monotonicity properties are simply obtained by deriving  $\gamma_\lambda^{i, i'}$  with respect to  $a$  and  $b$ .

□

For  $\lambda \in (0, 1]$ , set  $p^*(\lambda) := 2 - \sqrt{2} + \left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)\lambda$ . Define a strategy  $y$  of Player 1 in the following way:

- in state  $\omega_1$ , play 0 if  $j \leq p^*(\lambda)$ , play 1 otherwise,
- in state  $\omega_{-1}$ , play 1 if  $j \leq p^*(\lambda)$ , play 0 otherwise.

The rationale behind this strategy can be found in Section 2.2.

For all  $n \geq 1$ , define

$$\lambda_n := \frac{2^{-2n}}{\frac{3}{4} - \sqrt{2}} \quad \text{and} \quad \mu_n := \frac{2^{-2n-1}}{\frac{3}{4} - \sqrt{2}}.$$

140 **Proposition 7.** *The following hold:*

1.

$$\lim_{n \rightarrow +\infty} \min_{z \in Z} \gamma_{\lambda_n}(y, z) = \frac{1}{\sqrt{2}}$$

2.

$$\lim_{n \rightarrow +\infty} \min_{z \in Z} \gamma_{\mu_n}(y, z) = \frac{5}{2\sqrt{2}} - 1 > \frac{1}{\sqrt{2}}$$

*Proof.* 1. For all  $(i, i') \in \{0, 1\}$ ,

$$\lim_{n \rightarrow +\infty} \gamma_{\lambda_n}^{i, i'}(p^*(\lambda_n), p^*(\lambda_n)) = \frac{1}{\sqrt{2}},$$

and the result follows.

2. Let  $z$  be a strategy of Player 2, and  $a = z(\omega_1)$  and  $b = z(\omega_{-1})$ .

Note that the interval  $(p^*(\mu_n/2), p^*(2\mu_n))$  does not intersect  $J$ .

The following cases are distinguished:

145 **Case 1.**  $a \leq p^*(\mu_n)$  and  $b \leq p^*(\mu_n)$ , thus  $a \leq p^*(\mu_n/2)$  and  $b \leq p^*(\mu_n/2)$

We have  $\gamma_{\mu_n}^{\omega_1}(y, z) = \gamma_{\mu_n}^{0,1}(a, b) \geq \gamma_{\mu_n}^{0,1}(p^*(\mu_n/2), p^*(\mu_n/2)) \xrightarrow{n \rightarrow +\infty} \frac{5}{4}\sqrt{2} - 1$

**Case 2.**  $a \leq p^*(\mu_n)$  and  $b \geq p^*(\mu_n)$ , thus  $a \leq p^*(\mu_n/2)$  and  $b \geq p^*(2\mu_n)$

We have  $\gamma_{\mu_n}^{\omega_1}(y, z) = \gamma_{\mu_n}^{0,0}(a, b) \geq \gamma_{\mu_n}^{0,0}(p^*(\mu_n/2), p^*(2\mu_n)) \xrightarrow{n \rightarrow +\infty} -\frac{1 + 2\sqrt{2}}{8(-2 + \sqrt{2})}$

**Case 3.**  $a \geq p^*(\mu_n)$  and  $b \leq p^*(\mu_n)$ , thus  $a \geq p^*(2\mu_n)$  and  $b \leq p^*(\mu_n/2)$

150 We have  $\gamma_{\mu_n}^{\omega_1}(y, z) = \gamma_{\mu_n}^{1,1}(a, b) \geq \gamma_{\mu_n}^{1,1}(p^*(2\mu_n), p^*(\mu_n/2)) \xrightarrow{n \rightarrow +\infty} (-1/16) \frac{-25 + 14\sqrt{2}}{\sqrt{2} - 1}$

**Case 4.**  $a \geq p^*(\mu_n)$  and  $b \geq p^*(\mu_n)$ , thus  $a \geq p^*(2\mu_n)$  and  $b \geq p^*(2\mu_n)$

We have  $\gamma_{\mu_n}^{\omega_1}(y, z) = \gamma_{\mu_n}^{1,0}(a, b) \geq \gamma_{\mu_n}^{1,0}(p^*(2\mu_n), p^*(2\mu_n)) \xrightarrow{n \rightarrow +\infty} -2 + 2\sqrt{2}$

Among these cases, the smallest limit is  $\frac{5}{4}(\sqrt{2} - 1)$ , and the result follows.

□

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