Convergence of the solutions of the discounted Hamilton-Jacobi equation: a counterexample

Bruno Ziliotto

Convergence of the solutions of the discounted Hamilton-Jacobi equation: a counterexample

Bruno Ziliotto*

Place du Maréchal de Lattre de Tassigny, 75016 Paris, France

Abstract

This paper provides a counterexample about the asymptotic behavior of the solutions of a discounted Hamilton-Jacobi equation, as the discount factor vanishes. The Hamiltonian of the equation is a 1-dimensional continuous and coercive Hamiltonian.

Résumé

Cet article fournit un contre-exemple à la convergence asymptotique des solutions d'une équation de Hamilton-Jacobi escomptée, lorsque le facteur d'escompte tend vers 0. Le Hamiltonien de cette équation est unidimensionnel, continu et coercitif.

Keywords: Hamilton-Jacobi equations, Viscosity solutions, Stochastic games, Zero-Sum games.

2010 MSC: 49L25, 91A15, 91A25, 91A05.

1. Introduction and main result

Let $n \geq 1$. Denote by $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. For $c \in \mathbb{R}$, consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = c \quad (E_0)$$

where the Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous and coercive in the momentum. In order to build solutions of the above equation, Lions, Papanicolaou and Varadhan [1] have introduced a technique called *ergodic approximation*. For $\lambda \in (0, 1]$, consider the discounted Hamilton-Jacobi equation

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = 0 \quad (E_\lambda) \tag{1}$$

By a standard argument, this equation has a unique viscosity solution $v_\lambda : \mathbb{T}^n \to \mathbb{R}$. Moreover, $(-\lambda v_\lambda)$ converges uniformly as λ vanishes to a constant $c(H)$ called the *critical value*. Set $u_\lambda := v_\lambda + c(H)/\lambda$. The

*CEREMADE, CNRS, Université Paris Dauphine, PSL University, Paris, France.
E-mail: ziliotto@math.cnrs.fr

family (u_λ) is equi-Lipschitz, and converges uniformly along subsequences towards a solution of (E_0), for $c = c(H)$. Note that (E_0) may have several solutions. Recently, under the assumption that H is convex in the momentum, Davini, Fathi, Iturriaga and Zavidovique [2] have proved that (u_λ) converges uniformly (towards a solution of (E_0)). In addition, they proved that the solution can be characterized using Mather measures and Peierls barriers.

Without the convexity assumption, the question of whether (u_λ) converges or not remained open. This paper solves negatively this question and provides a 1-dimensional continuous and coercive Hamiltonian for which (u_λ) does not converge[2].

Theorem 1. *There exists a continuous Hamiltonian* $H : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{R}$ *that is coercive in the momentum, such that* u_λ *does not converge as* λ *tends to 0.*

The example builds on a class of discrete-time repeated games called *stochastic games*. The main ingredient is to establish a connection between recent counterexamples to the existence of the limit value in stochastic games (see [4, 5]) and the Hamilton-Jacobi problem[3].

The remainder of the paper is structured as follows. Section 2 presents the stochastic game example. Section 3 shows that in order to prove Theorem 1, it is enough to study the asymptotic behavior of the stochastic game, when the discount factor vanishes. Section 4 determines the asymptotic behavior of the stochastic game.

[2] Note that for time-dependent Hamilton-Jacobi equations, several counterexamples about the asymptotic behavior of solutions have been pointed out in [3].

[3] Let us mention the work [6, 7, 8, 9] as other illustrations of the use of repeated games in PDE problems.

2. The stochastic game example

Given a finite set A, the set of probability measures over A is denoted by $\Delta(A)$. Given $a \in A$, the Dirac measure at a is denoted by δ_a.

2.1. Description of the game

Consider the following stochastic game Γ, described by:

- A state space K with two elements ω_1 and ω_{-1}: $K = \{\omega_1, \omega_{-1}\}$,
- An action set $I = \{0, 1\}$ for Player 1,
- An action set $J = \left\{2 - \sqrt{2} + 2^{-2n}, n \geq 1\right\} \cup \left\{2 - \sqrt{2}\right\}$ for Player 2,
- For each $(k, i, j) \in K \times I \times J$, a transition $q(.\,|k, i, j) \in \Delta(K)$ defined by:

$$\begin{aligned} q(.\,|\omega_1, i, j) &= [ij + (1-i)(1-j)]\delta_{\omega_1} + [i(1-j) + (1-i)j]\delta_{\omega_{-1}}, \\ q(.\,|\omega_{-1}, i, j) &= [i(1-j) + (1-i)j]\delta_{\omega_1} + [ij + (1-i)(1-j)]\delta_{\omega_{-1}}. \end{aligned}$$

- A payoff function $g : K \times I \times J \to [0, 1]$, defined by

$$g(\omega_1, i, j) = ij + 2(1-i)(1-j) \quad \text{and} \quad g(\omega_{-1}, i, j) = -ij - 2(1-i)(1-j).$$

Let $k_1 \in K$. The stochastic game Γ^{k_1} starting at k_1 proceeds as follows:

- The initial state is k_1. At first stage, Player 2 chooses $j_1 \in J$ and announces it to Player 1. Then, Player 1 chooses $i_1 \in I$, and announces it to Player 2. The payoff at stage 1 is $g(k_1, i_1, j_1)$ for Player 1, and $-g(k_1, i_1, j_1)$ for Player 2. A new state k_2 is drawn from the probability $q(.\,|k_1, i_1, j_1)$ and announced to both players. Then, the game moves on to stage 2.

- At each stage $m \geq 2$, Player 2 chooses $j_m \in J$ and announces it to Player 1. Then, Player 1 chooses $i_m \in I$, and announces it to Player 2. The payoff at stage m is $g(k_m, i_m, j_m)$ for Player 1, and $-g(k_m, i_m, j_m)$ for Player 2. A new state k_{m+1} is drawn from the probability $q(. \,|k_m, i_m, j_m)$ and announced to both players. Then, the game moves on to stage $m+1$.

Remark 2. The action set of Player 2 can be interpreted as a set of randomized actions. Indeed, imagine that Player 2 has only two actions, 1 and 0. These actions are called *pure actions*. At stage m, if Player 2 chooses $j_m \in J$, this means that he plays 1 with probability j_m, and 0 with probability $1 - j_m$. Denote by $\widetilde{j_m} \in \{0, 1\}$ his realized action. Player 1 knows j_m before playing, but does not know $\widetilde{j_m}$. If Player 1 chooses $i_m \in I$ afterwards, then the realized payoff is $g(k_m, i_m, \widetilde{j_m})$. Thus, the payoff $g(k_m, i_m, j_m)$ represents the expectation of $g(k_m, i_m, \widetilde{j_m})$. Likewise, the transition $q(.\,|k_m, i_m, j_m)$ represents the law of $q(k_m, i_m, \widetilde{j_m})$. The transition and payoff in Γ when players play pure actions can be represented by the following matrices:

Table 1: Transition and payoff functions in state ω_1 and ω_{-1}

ω_1	1	0
1	1	$\overrightarrow{0}$
0	$\overrightarrow{0}$	2

ω_{-1}	1	0
1	-1	$\overleftarrow{0}$
0	$\overleftarrow{0}$	-2

The left-hand side matrix stands for state ω_1, and the right-hand side matrix stands for state ω_{-1}. Consider the left-hand side matrix. Player 1 chooses a row (either 1 or 0), and Player 2 chooses a column (either 1 or 0). The payoff is given by the numbers: for instance, $g(1, 1) = 1$ and $g(1, 0) = 0$. The arrow means that when the corresponding actions are played, the state moves on to state ω_{-1}; otherwise, it stays in ω_1. For instance, $q(.|\omega_1, 1, 1) = \delta_{\omega_1}$ and $q(.|\omega_1, 1, 0) = \delta_{\omega_{-1}}$. The interpretation is the same for the right-hand side matrix. In the game Γ, Player 1 can play only pure actions (1 or 0), and Player 2 can play 1 with some probability $j \in J$.

This matrix representation is convenient to understand the strategic aspects of the game.

Let us now define formally *strategies*. In general, the decision of a player at stage m may depend on all the information he has: that is, the stage m, and all the states and actions before stage m. In this paper, it is sufficient to consider a restricted class of strategies, called *stationary strategies*. Formally, a stationary strategy for Player 1 is defined as a mapping $y : K \times J \to I$. The interpretation is that at stage m, if the current state is k, and Player 2 plays j, then Player 1 plays $y(k, j)$. Thus, Player 1 only bases his decision on the current state and the current action of Player 2. Denote by Y the set of stationary strategies for Player 1.

A stationary strategy for Player 2 is defined as a mapping $z : K \to J$. The interpretation is that at stage m, if the current state is k, then Player 2 plays $z(k)$. Thus, Player 2 only bases his decision on the current state. Denote by Z the set of stationary strategies for Player 2.

The sequence $(k_1, i_1, j_1, k_2, i_2, j_2, ..., k_m, i_m, j_m, ...) \in H_\infty := (K \times I \times J)^{\mathbb{N}^*}$ generated along the game is called *history* of the game. Due to the fact that state transitions are random, this is a random variable. The law of this random variable depends on the initial state k_1 and the pair of strategies (y, z), and is denoted by $\mathbb{P}^{k_1}_{y,z}$. We will call g_m the m-stage random payoff $g(k_m, i_m, j_m)$. Let $\lambda \in (0, 1]$. The game $\Gamma^{k_1}_\lambda$ is the game where the strategy set of Player 1 (resp. 2) is Y (resp. Z), and the payoff is $\gamma^{k_1}_\lambda$, where

$$\gamma^{k_1}_\lambda(y, z) = \mathbb{E}^{k_1}_{y,z}\left(\sum_{m \geq 1}(1-\lambda)^{m-1} g_m\right).$$

The goal of Player 1 is to maximize this quantity, while the goal of Player 2 is to minimize this quantity. The game $\Gamma^{k_1}_\lambda$ has a value, that is:

$$\min_{z \in Z} \max_{y \in Y} \gamma^{k_1}_\lambda(y, z) = \max_{y \in Y} \min_{z \in Z} \gamma^{k_1}_\lambda(y, z).$$

The value of $\Gamma^{k_1}_\lambda$ is then defined as the above quantity, and is denoted by $w_\lambda(k_1)$. A strategy for Player 1 is *optimal* if it achieves the right-hand side maximum, and a strategy for Player 2 is *optimal* if it achieves the left-hand side minimum. The interpretation is that if players are rational they should play optimal strategies, and as a result Player 1 should get $w_\lambda(k_1)$, and Player 2 should get $-w_\lambda(k_1)$.

2.2. Asymptotic behavior of the discounted value

As we shall see in the next section, for each $\lambda \in (0, 1]$, one can associate a discounted Hamilton-Jacobi equation with $c(H) = 0$, such that its solution evaluated at $x = 1$ is approximately $w_\lambda(\omega_1)$, for λ small enough. Thus, the asymptotic behavior of this quantity needs to be studied.

Define $\lambda_n := 2^{-2n}\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)^{-1}$ and $\mu_n := 2^{-2n-1}\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)^{-1}$.

Proposition 3. *The following hold:*

(i) $w_\lambda(\omega_{-1}) \leq w_\lambda(\omega_1) \leq w_\lambda(\omega_{-1}) + 2$

(ii) $\lim_{n\to+\infty} w_{\lambda_n}(\omega_1) = 1/\sqrt{2}$ *and* $\liminf_{n\to+\infty} w_{\mu_n}(\omega_1) > 1/\sqrt{2}$. *Consequently,* $(w_\lambda(\omega_1))$ *does not have a limit when* $\lambda \to 0$.

The proof of the above proposition is done in Section 4. As far as the proof of Theorem 1 is concerned, the key point is (ii). is Let us give here some piece of intuition for this result. Consider the game Γ' that is identical to Γ, except that Player 2's action set is $[0,1]$ instead of J. For each $\lambda \in (0,1]$, denote by w'_λ its discounted value. Because $J \subset [0,1]$, Player 2 is better off in the game Γ' compared to the game Γ: $w'_\lambda \leq w_\lambda$. Interpret now Γ and Γ' as games with randomized actions, as in Table 2. As λ vanishes, standard computations show that an (almost) optimal stationary strategy for Player 2 in $\Gamma'^{\omega_1}_\lambda$ is to play 1 with probability $p^*(\lambda) := 2-\sqrt{2}+\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)\lambda$ in both states ω_1 and ω_{-1}, and $(w_\lambda(\omega_1))$ converges to $\frac{1}{\sqrt{2}}$. Moreover, for all $n \geq 1$, $p^*(\lambda_n) \in J$. Thus, this strategy is available for Player 2 in Γ, and consequently $w_{\lambda_n}(\omega_1) = w'_{\lambda_n}(\omega_1) + O(\lambda_n)$, as n tends to infinity.
On the other hand, for all $n \geq 1$, $p^*(\mu_n) \notin J$, and the distance of $p^*(\mu_n)$ to J is larger than $\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)\mu_n/2$. Consequently, the distance of the optimal strategy in $\Gamma^{\omega_1}_{\mu_n}$ to the optimal strategy in $\Gamma'^{\omega_1}_{\mu_n}$ is of order μ_n. This produces a payoff difference of order μ_n at each stage, and thus of order 1 in the whole game. Thus, Player 2 is significantly disadvantaged in $\Gamma^{\omega_1}_{\mu_n}$ compared to $\Gamma'^{\omega_1}_{\mu_n}$, and the difference between $w_{\mu_n}(\omega_1)$ and $w'_{\mu_n}(\omega_1)$ is of order 1.

Remark 4. As we shall see in the following section, we have $\lim_{\lambda\to 0}\lambda w_\lambda(\omega_1) = \lim_{\lambda\to 0}\lambda w_\lambda(\omega_{-1}) = 0$.

The next section explains how to derive the counterexample and Theorem 1 from Proposition 3.

3. Link with the PDE problem and proof of Theorem 1

The following proposition expresses w_λ as the solution of a functional equation called *Shapley equation.*

Proposition 5. *Let $\lambda \in (0,1]$ and $u_\lambda := (1+\lambda)^{-1} w_{\lambda/(1+\lambda)}$. For each $r \in \{-1,1\}$, the two following equations hold:*

(i)

$$w_\lambda(\omega_r) \quad = \quad \min_{j\in J}\max_{i\in I}\left\{g(\omega_r,i,j) + (1-\lambda)\left[q(\omega_r|\omega_r,i,j)w_\lambda(\omega_r) + q(\omega_{-r}|\omega_r,i,j)w_\lambda(\omega_{-r})\right]\right\}$$

(ii)

$$\lambda u_\lambda(\omega_r) = \min_{j\in J}\max_{i\in I}\left\{g(\omega_r,i,j) + q(\omega_{-r}|\omega_r,i,j)\left[u_\lambda(\omega_{-r}) - u_\lambda(\omega_r)\right]\right\}$$

Proof. (a) The intuition is the following. Consider the game $\Gamma_\lambda^{\omega_r}$. At stage 1, the state is ω_r. The term g represents the current payoff, and the term $(1-\lambda)[...]$ represents the future optimal payoff, that is, the payoff that Player 1 should get from stage 2 to infinity. Thus, this equation means that the value of $\Gamma_\lambda^{\omega_r}$ coincides with the value of the one-stage game, where the payoff is a combination of the current payoff and the future optimal payoff. For a formal derivation of this type of equation, we refer to [10, VII.1., p. 392].

(b) Evaluating the previous equation at $\lambda/(1+\lambda)$ yields

$$w_{\frac{\lambda}{1+\lambda}}(\omega_r) \quad = \quad \min_{j\in J}\max_{i\in I}\left\{g(\omega_r,i,j) + \frac{1}{1+\lambda}\left[q(\omega_r|\omega_r,i,j)w_{\frac{\lambda}{1+\lambda}}(\omega_r) + q(\omega_{-r}|\omega_r,i,j)w_{\frac{\lambda}{1+\lambda}}(\omega_{-r})\right]\right\}$$

Using the fact that $q(\omega_r|\omega_r,i,j) = 1 - q(\omega_{-r}|\omega_r,i,j)$ yields the result.

□

For $r \in \{1, -1\}$ and $p \in \mathbb{R}$, define $H_r : \mathbb{R} \to \mathbb{R}$ by

$$H_r(p) := \begin{cases} -\min_{j \in J} \max_{i \in I} \{g(\omega_r, i, j) - rp \cdot ([i(1-j) + (1-i)j]\}, & \text{if } |p| \leq 2, \\ H_r\left(2\frac{p}{|p|}\right) + |p| - 2 & \text{if } |p| > 2. \end{cases}$$

For $x \in [-1, 1]$ and $p \in \mathbb{R}$, let

$$H(x, p) := |x| H_1(|p|) + (1 - |x|) H_{-1}(|p|). \tag{2}$$

Note that the definition of H_1 and H_{-1} for $|p| > 2$ ensures that $\lim_{|p| \to +\infty} H_1(p) = \lim_{|p| \to +\infty} H_{-1}(p) = +\infty$, thus $\lim_{|p| \to +\infty} H(p) = +\infty$. Note also that for all $x \in [-1, 1]$, $H_1(x, .)$ is increasing on $[-2, 2]$ and $H_{-1}(x, .)$ is decreasing on $[-2, 2]$.

Thanks to Proposition 5 (ii) and Proposition 3 (i), we have $\lambda u_\lambda(\omega_1) + H_1(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = 0$ and $\lambda u_\lambda(\omega_{-1}) + H_{-1}(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = 0$.

For $x \in [-1, 1]$, let $u_\lambda(x) = |x| u_\lambda(\omega_1) + (1 - |x|) u_\lambda(\omega_{-1})$. Let $x \in (-1, 1) \setminus \{0\}$. Proposition 3 (i) implies that $w_\lambda(\omega_{-1}) \leq w_\lambda(\omega_1)$, thus $u_\lambda(\omega_{-1}) \leq u_\lambda(\omega_1)$ and $|Du_\lambda(x)| = u_\lambda(\omega_1) - u_\lambda(\omega_{-1})$. Consequently, Proposition 5 (ii) yields

$$\lambda u_\lambda(x) + H(x, Du_\lambda(x)) = 0. \tag{3}$$

Note that the above equation is identical to equation (3). The reason why we use the notation u_λ and not v_λ is that, as we shall see, $c(H) = 0$, thus u_λ coincides with v_λ.

Extend u_λ and $H(., p)$ ($p \in \mathbb{R}$) as 2-periodic functions defined on $\mathbb{R}$. The Hamiltonian H is continuous and coercive in the momentum, and the above equation holds in a classical sense for all $x \in \mathbb{R} \setminus \mathbb{Z}$.

For $x \in \mathbb{R}$, denote by $D^+ u_\lambda(x)$ (resp., $D^- u_\lambda(x)$) the super-differential (resp., the sub-differential) of u_λ at x. Let us show that u_λ is a viscosity solution of (3) on $\mathbb{R}$. By 2-periodicity, it is enough to show that this is a viscosity solution for $x = 0$ and $x = 1$.

Let us start by $x = 0$. We have $D^+ u_\lambda(0) = \emptyset$ and $D^- u_\lambda(0) = [u_\lambda(\omega_{-1}) - u_\lambda(\omega_1), u_\lambda(\omega_1) - u_\lambda(\omega_{-1})]$.

Let $p \in D^- u_\lambda(0)$. Then $H_{-1}(p) \geq H_{-1}(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = -\lambda u_\lambda(\omega_{-1})$, thus $\lambda u_\lambda(0) + H(0, p) \geq 0$. Consequently, u_λ is a viscosity solution at $x = 0$.

Consider now the case $x = 1$. We have $D^+u_\lambda(1) = [u_\lambda(\omega_{-1}) - u_\lambda(\omega_1), u_\lambda(\omega_1) - u_\lambda(\omega_{-1})]$ and $D^-u_\lambda(1) = \emptyset$. Let $p \in D^+u_\lambda(1)$. Then $H_1(p) \leq H_1(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = -\lambda u_\lambda(\omega_1)$, thus $\lambda u_\lambda(1) + H(1,p) \geq 0$. Consequently, u_λ is a viscosity solution at $x = 1$.

Let us now conclude the proof of Theorem 1. Because H is 2-periodic, equation (3) can be considered as written on $\mathbb{T}^1$.

As noticed before, equation (3) is identical to equation (1). Therefore, as stated in the introduction, $-\lambda u_\lambda$ converges to $c(H)$. Proposition 3 (ii) implies that $(-\lambda_n u_{\lambda_n}(1))$ converges to 0, thus $c(H) = 0$. Still by Proposition 3 (ii), $(u_\lambda(1))$ does not have a limit when λ tends to 0: Theorem 1 is proved.

4. Proof of Proposition 3

4.1. Proof of (i)

Consider Proposition 5 (i) for $r = 1$. Take $j = 1/2 \in J$. It yields

$$\begin{aligned} w_\lambda(\omega_1) &\leq \max_{i \in I} \left\{ 1 + (1-\lambda)\left(\frac{1}{2} w_\lambda(\omega_1) + \frac{1}{2} w_\lambda(\omega_{-1})\right)\right\} \\ &= 1 + \frac{1}{2}(1-\lambda)\left(w_\lambda(\omega_1) + w_\lambda(\omega_{-1})\right). \end{aligned} \tag{4}$$

Take $i = 1/2$. This yields

$$w_\lambda(\omega_1) \geq \frac{1}{2} + \frac{1}{2}(1-\lambda)\left(w_\lambda(\omega_1) + w_\lambda(\omega_{-1})\right). \tag{5}$$

For $r = -1$, taking $j = 1/2$ and then $i = 1/2$ produce the following inequalities:

$$w_\lambda(\omega_{-1}) \leq -\frac{1}{2} + \frac{1}{2}(1-\lambda)\left(w_\lambda(\omega_1) + w_\lambda(\omega_{-1})\right), \tag{6}$$

and

$$w_\lambda(\omega_{-1}) \geq -1 + \frac{1}{2}(1-\lambda)\left(w_\lambda(\omega_1) + w_\lambda(\omega_{-1})\right). \tag{7}$$

Combining (5) and (6) yield $w_\lambda(\omega_1) \geq w_\lambda(\omega_{-1}) + 1 \geq w_\lambda(\omega_{-1})$. Combining (4) and (7) yield $w_\lambda(\omega_{-1}) \geq w_\lambda(\omega_1) - 2$, and (i) is proved.

4.2. Proof of (ii)

For $(i, i') \in \{0,1\}^2$, consider the strategy y of Player 1 that plays i in ω_1 and i' in ω_{-1} (regardless of Player 2's actions), and the strategy z of Player 2 that plays a in state ω_1, and b in state ω_{-1}. Denote $\gamma_\lambda^{i,i'}(a,b) := \gamma_\lambda^{\omega_1}(y,z)$ (resp., $\widetilde{\gamma}_\lambda^{i,i'}(a,b) := \gamma_\lambda^{\omega_{-1}}(y,z)$), the payoff in $\Gamma_\lambda^{\omega_1}$ (resp., $\Gamma_\lambda^{\omega_{-1}}$), when (y,z) is played.

Proposition 6. *The following hold:*

1.

$$\gamma_\lambda^{0,0}(a,b) = \frac{-2(a-b-\lambda+b\lambda)}{\lambda(a+b+\lambda-a\lambda-b\lambda)}$$

$$\gamma_\lambda^{1,1}(a,b) = -\frac{a-b+\lambda b}{\lambda(a+b+\lambda-a\lambda-b\lambda-2)}$$

$$\gamma_\lambda^{1,0}(a,b) = \frac{2a+2b+2\lambda-ab-a\lambda-2b\lambda+ab\lambda-2}{\lambda(b-a+\lambda a-b\lambda+1)}$$

$$\gamma_\lambda^{0,1}(a,b) = -\frac{2a+2b-ab-2b\lambda+ab\lambda-2}{\lambda(a-b-a\lambda+b\lambda+1)}$$

2.
- $\gamma_\lambda^{0,0}$ *is decreasing with respect to* a *and increasing with respect to* b.
- $\gamma_\lambda^{1,1}$ *is increasing with respect to* a *and decreasing with respect to* b.
- $\gamma_\lambda^{1,0}$ *is increasing with respect to* a *and* b.
- $\gamma_\lambda^{0,1}$ *is decreasing with respect to* a *and* b.

Proof. 1. The payoffs $\gamma_\lambda^{0,0}(a,b)$ and $\widetilde{\gamma}_\lambda^{0,0}(a,b)$ satisfy the following recursive equation:

$$\begin{aligned} \gamma_\lambda^{0,0}(a,b) &= a(1-\lambda)\widetilde{\gamma}_\lambda^{0,0}(a,b) + (1-a)(2+(1-\lambda)\gamma_\lambda^{0,0}(a,b)) \\ \widetilde{\gamma}_\lambda^{0,0}(a,b) &= a(1-\lambda)\gamma_\lambda^{0,0}(a,b) + (1-a)(-2+(1-\lambda)\widetilde{\gamma}_\lambda^{0,0}(a,b)) \end{aligned}$$

Combining these two relations give the first equality. The three other equalities can be derived in a similar fashion.

2. These monotonicity properties are simply obtained by deriving $\gamma_\lambda^{i,i'}$ with respect to a and b.

□

For $\lambda \in (0,1]$, set $p^*(\lambda) := 2 - \sqrt{2} + \left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)\lambda$. Define a strategy y of Player 1 in the following way:

- in state ω_1, play 0 if $j \leq p^*(\lambda)$, play 1 otherwise,
- in state ω_{-1}, play 1 if $j \leq p^*(\lambda)$, play 0 otherwise.

The rationale behind this strategy can be found in Section 2.2.

For all $n \geq 1$, define

$$\lambda_n := \frac{2^{-2n}}{\frac{3}{4} - \sqrt{2}} \quad \text{and} \quad \mu_n := \frac{2^{-2n-1}}{\frac{3}{4} - \sqrt{2}}.$$

Proposition 7. *The following hold:*

1.

$$\lim_{n \to +\infty} \min_{z \in Z} \gamma_{\lambda_n}(y, z) = \frac{1}{\sqrt{2}}$$

2.

$$\lim_{n \to +\infty} \min_{z \in Z} \gamma_{\mu_n}(y, z) = \frac{5}{2\sqrt{2}} - 1 > \frac{1}{\sqrt{2}}$$

Proof. 1. For all $(i, i') \in \{0, 1\}$,

$$\lim_{n \to +\infty} \gamma_{\lambda_n}^{i,i'}(p^*(\lambda_n), p^*(\lambda_n)) = \frac{1}{\sqrt{2}},$$

and the result follows.

2. Let z be a strategy of Player 2, and $a = z(\omega_1)$ and $b = z(\omega_{-1})$.

Note that the interval $(p^*(\mu_n/2), p^*(2\mu_n))$ does not intersect J.

The following cases are distinguished:

Case 1. $a \leq p^*(\mu_n)$ and $b \leq p^*(\mu_n)$, thus $a \leq p^*(\mu_n/2)$ and $b \leq p^*(\mu_n/2)$

We have $\gamma_{\mu_n}^{\omega_1}(y, z) = \gamma_{\mu_n}^{0,1}(a, b) \geq \gamma_{\mu_n}^{0,1}(p^*(\mu_n/2), p^*(\mu_n/2)) \underset{n \to +\infty}{\to} \frac{5}{4}\sqrt{2} - 1$

Case 2. $a \leq p^*(\mu_n)$ and $b \geq p^*(\mu_n)$, thus $a \leq p^*(\mu_n/2)$ and $b \geq p^*(2\mu_n)$

We have $\gamma_{\mu_n}^{\omega_1}(y, z) = \gamma_{\mu_n}^{0,0}(a, b) \geq \gamma_{\mu_n}^{0,0}(p^*(\mu_n/2), p^*(2\mu_n)) \underset{n \to +\infty}{\to} -\frac{1 + 2\sqrt{2}}{8(-2 + \sqrt{2})}$

Case 3. $a \geq p^*(\mu_n)$ and $b \leq p^*(\mu_n)$, thus $a \geq p^*(2\mu_n)$ and $b \leq p^*(\mu_n/2)$

We have $\gamma_{\mu_n}^{\omega_1}(y,z) = \gamma_{\mu_n}^{1,1}(a,b) \geq \gamma_{\mu_n}^{1,1}(p^*(2\mu_n), p^*(\mu_n/2)) \underset{n\to+\infty}{\to} (-1/16)\dfrac{-25+14\sqrt{2}}{\sqrt{2}-1}$

Case 4. $a \geq p^*(\mu_n)$ and $b \geq p^*(\mu_n)$, thus $a \geq p^*(2\mu_n)$ and $b \geq p^*(2\mu_n)$

We have $\gamma_{\mu_n}^{\omega_1}(y,z) = \gamma_{\mu_n}^{1,0}(a,b) \geq \gamma_{\mu_n}^{1,0}(p^*(2\mu_n), p^*(2\mu_n)) \underset{n\to+\infty}{\to} -2+2\sqrt{2}$
Among these cases, the smallest limit is $\frac{5}{4}(\sqrt{2}-1)$, and the result follows.

□

Acknowledgments

The author is very grateful to Pierre Cardaliaguet, Andrea Davini, Abraham Neyman, Sylvain Sorin and Maxime Zavidovique for helpful discussions.
This project benefited from the support of the FMJH Program PGMO VarPDEMFG and from the support of EDF, Thales, Orange and Criteo. It was also partly funded by the grant "PEPS Jeunes Chercheurs" from CNRS.

References

[1] P.-L. Lions, G. Papanicolaou, S. Varadhan, Homogenization of hamilton-jacobi equations, Unpublished preprint.

[2] A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique, Convergence of the solutions of the discounted hamilton–jacobi equation, Inventiones mathematicae 206 (1) (2016) 29–55.

[3] G. Barles, P. Souganidis, Some counterexamples on the asymptotic behavior of the solutions of hamilton–jacobi equations, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics 330 (11) (2000) 963–968.

[4] G. Vigeral, A zero-sum stochastic game with compact action sets and no asymptotic value, Dynamic Games and Applications 3 (2) (2013) 172–186.

[5] B. Ziliotto, Zero-sum repeated games: counterexamples to the existence of the asymptotic value and the conjecture maxmin= lim v (n), The Annals of Probability 44 (2) (2016) 1107–1133.

[6] R. Kohn, S. Serfaty, A deterministic-control-based approach motion by curvature, Communications on pure and applied mathematics 59 (3) (2006) 344–407.

[7] R. Kohn, S. Serfaty, A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations, Communications on Pure and Applied Mathematics 63 (10) (2010) 1298–1350.

[8] C. Imbert, S. Serfaty, Repeated games for non-linear parabolic integro-differential equations and integral curvature flows, Discrete Contin. Dyn. Syst 29 (4) (2011) 1517–1552.

[9] B. Ziliotto, Stochastic homogenization of nonconvex hamilton-jacobi equations: a counterexample, Communications on Pure and Applied Mathematics 70 (9) (2017) 1798–1809.

[10] J. Mertens, S. Sorin, S. Zamir, Repeated games, CORE DP 9420-22, 1994.