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# A geometric Morse-Novikov complex with infinite series coefficients

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ABSTRACT. Let  $M$  be a closed  $n$ -dimensional manifold,  $n > 2$ , whose first real cohomology group  $H^1(M; \mathbb{R})$  is non-zero. We present a general method for constructing a Morse 1-form  $\alpha$  on  $M$ , closed but non-exact, and a pseudo-gradient  $X$  such that the differential  $\partial^X$  of the Novikov complex of the pair  $(\alpha, X)$  has at least one incidence coefficient which is an infinite series. This is an application of our previous study of the homoclinic bifurcation of pseudo-gradients of multivalued Morse functions.

## 1. INTRODUCTION

Let us consider an  $n$ -dimensional closed smooth manifold  $M$  with a non-zero de Rham cohomology in degree one. Let  $u \in H_{dR}^1(M)$ ,  $u \neq 0$ , and let  $\alpha$  be a differential closed form in the class  $u$ . By Thom's transversality theorem with constraints [10, 11], generically in the class  $u$ , the local primitives of a closed 1-form  $\alpha$  are Morse functions (such an  $\alpha$  is named a Morse 1-form). The finite set  $Z(\alpha)$  made of its zeroes is graded by the Morse index:  $Z(\alpha) = \cup_{k=0}^n Z_k(\alpha)$ .

Taking the point of view that a Morse 1-form is nothing but a multivalued function (that is, a function well-defined up to an additive constant), S.P. Novikov developed a theory analogous to Morse theory [5, 6] (today referred to as *Morse-Novikov theory*) which leads to some homology denoted by  $H_*(M; u)$ . Here, we summarize it with the point of view adopted by J.-C. Sikorav [9].

The class  $u$  is seen as the morphism  $\pi_1(M) \rightarrow \mathbb{R}$  defined by  $u(\gamma) = \int_\gamma \alpha$  for every oriented loop  $\gamma$ . Let  $\tilde{u} : \mathbb{Z}\pi_1(M) \rightarrow \overline{\mathbb{R}}$  be the associate valuation: for  $\lambda \in \mathbb{Z}\pi_1(M)$ , that is a finite sum  $\sum_i n_i g_i$  with  $n_i \in \mathbb{Z}^*$  and  $g_i \in \pi_1(M)$ , one defines  $\tilde{u}(\lambda) = \max u(g_i)$ . The *universal* Novikov ring  $\Lambda_u$  is the completion of  $\mathbb{Z}\pi_1(M)$  for  $\tilde{u}(\lambda) \rightarrow -\infty$ . It is worth noticing that we get the same completion by replacing  $\pi_1(M)$  with the *fundamental groupoid* of  $M$ . This choice is made in what follows.

The Novikov complex  $N_*(\alpha, \partial^X)$  is the free  $\Lambda_u$ -module based on  $Z_*(\alpha)$ ; the differential  $\partial^X$  is detailed right below. For defining  $\partial^X$ , one chooses some (descending) pseudo-gradient  $X$  *adapted* to  $\alpha$ , meaning that  $X$  fulfils the following:

- 1)  $\langle \alpha(x), X(x) \rangle < 0$  for every  $x \in M$  apart from  $Z(\alpha)$ ;
- 2) for every  $p \in Z_k(\alpha)$ , the field  $-X$  is the Euclidean gradient of a local primitive  $h_p$  of  $\alpha$  in Morse coordinates  $(x_1, \dots, x_n)$  about  $p$  where  $h_p$  reads

$$h_p(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 \dots + x_n^2.$$

The zeroes of  $X$ , which coincide with the zeroes of  $\alpha$ , are *hyperbolic*. Therefore, for each  $p \in Z_k(\alpha)$ , we have a stable manifold  $W^s(p, X)$  of dimension  $n - k$  and an unstable manifold  $W^u(p, X)$  of dimension  $k$ . According to Kupka-Smale's theorem (see [8]), one can approximate  $X$  so that all stable and unstable manifolds are mutually transverse. For  $p \in Z_k(\alpha)$ , the value

of  $\partial^X$  in  $p$  reads:

$$\partial^X p = \sum_{q \in Z_{k-1}(\alpha)} \langle p, q \rangle^X q$$

where the *incidence coefficient*  $\langle p, q \rangle^X$  belongs to the Novikov ring  $\Lambda_u$ .

As far as we know, there are no published example<sup>1</sup> where some of the incidence coefficients are infinite series as the Novikov ring allows it. By contrast, it is known that if  $f : M \rightarrow \mathbb{R}$  is a Morse function, then for every  $c \gg 0$ , the differential of the Novikov complex  $N_*(\alpha + cdf, X)$  has all its incidence coefficients in  $\mathbb{Z}\pi_1(M)$ . Here,  $X$  denotes a pseudo-gradient adapted to  $\alpha + cdf$  which is  $C^1$ -close to a pseudo-gradient adapted to  $f$  [3, Lemma 3.7]. When  $n > 2$ , we present a general method for constructing some pairs  $(\alpha, X)$  such that the differential  $\partial^X$  of  $N_*(\alpha)$  have some incidence coefficients which are infinite series. Our method is based on our study [4] of the homoclinic bifurcations of the pseudo-gradient  $X$ .

**THEOREM.** *Let  $M$  be an  $n$ -dimensional closed manifold with  $n > 2$ . Let  $u \in H_{dR}^1(M)$  be a non-zero cohomology class and let  $\alpha$  be a Morse 1-form in the class  $u$  without centers, that is, with no zeroes of extremal index. For  $1 \leq k \leq n - 2$ , let  $\alpha'$  be any Morse 1-form obtained from  $\alpha$  by creating the birth of a pair of zeroes  $(q, p)$  of respective indices  $(k, k + 1)$ . Then, it is possible to build a pseudo-gradient  $X'$  adapted to  $\alpha'$  such that the incidence coefficient  $\langle p, q \rangle^{X'}$  is an infinite series in the Novikov ring  $\Lambda_u$ .*

**REMARK.** Observe that, when  $\alpha$  is a non-exact Morse 1-form, it is possible to cancel all zeroes of extremal indices [1, Remarque section 1.2]. Therefore, the assumption of having no centers is irrelevant.

## 2. PROOF

**2.1. MAKING A TUBE OF PERIODIC ORBITS.** Let  $X$  be a Kupka-Smale pseudo-gradient adapted to  $\alpha$ . After the no-center assumption, almost no orbit of  $X$  goes to (or comes from) one of its zeroes. A  $C^0$ -approximation of  $X$  near an accumulation point of such an orbit allows us to create not only one periodic orbit but a tube  $T \cong D^{n-1} \times S^1$  made of periodic orbits  $\{pt\} \times S^1$  (still name  $X$  the pseudo-gradient after this approximation). The (free) homotopy class of these periodic orbits is denoted by  $g$ . Note  $u(g) < 0$ .

**2.2. BIRTH OF A PAIR OF ZEROES IN CANCELLING POSITION.** Let  $B \subset T$ ,  $B \cong D^{n-1} \times D^1$ , be a polydisc bi-foliated by leaves of  $\alpha$  and orbits of  $X$ . Let  $h$  be a primitive of  $\alpha|_B$ . Let us insert a *birth model* into  $B$  to create a pair  $(p, q)$  of critical points of respective indices  $k + 1$  and  $k$  (see [2, chap. III] and also [12]). We impose this one-parameter deformation starting from  $h$  to be supported in  $B$ . Denote  $h'$  the function ending this deformation; set  $\alpha' = \alpha + dh' - dh$ . Let  $X_1$  be a pseudo-gradient adapted to  $\alpha'$  near  $p$  and  $q$  and coinciding with  $X$  outside  $B$ . There is exactly one orbit of  $X_1$  descending from  $p$  to  $q$  inside  $B$ . However, the pseudo-gradient  $X_1$  is not convenient for our theorem as some extra connecting orbit exists in  $T$  from  $p$  to  $q$ . Thus, we change  $X_1$  into  $X'$  as follows: make  $\partial T$  an attractor of  $X'|_T$  (while keeping  $X'|_{\partial T} = X_1|_{\partial T}$ ) and one of the periodic orbits in  $T \setminus B$  a repeller in order that any orbit of  $X'$  emanating from the

<sup>1</sup> A 3-dimensional example where  $u$  is a rational class has been given by A. Pajitnov in [7, Section 3].

bottom of  $B$ , namely  $\partial^- B := \{h' = h'(q) - \varepsilon\}$ ,  $0 < \varepsilon \ll 1$ , never returns to  $B$  in positive time. The outcome of this construction is that the Novikov complex  $N_*(\alpha', X')$  fulfils the following:  $\partial^{X'} q = 0$  and  $\partial^{X'} p = q$  for suitable orientations of the unstable manifolds. Moreover, as the boundary of  $T$  is invariant by  $X'$ , no zero of  $\alpha'$  other than  $p$  has a connecting orbit to  $q$ ; and no zero is connected to  $p$ .

Let  $\mathcal{S}_g$  be the stratum of pseudo-gradients adapted to  $\alpha'$  which are equal to  $X'$  outside  $T$  and have one homoclinic connecting orbit from  $p$  to  $p$  in  $T$  in the class  $g$  with the minimal defect of transversality. This is a codimension-one stratum which is canonically co-oriented.

**2.3. NOTATION.** Denote by  $\partial^+ B := \{h' = h'(p) + \varepsilon\}$  the top of  $B$ . It contains the so-called *belt sphere*  $\Sigma^+$  of  $p$ , that is,  $\Sigma^+ := W^s(p, X') \cap \partial^+ B$ . This sphere is  $(n - k - 2)$ -dimensional and co-oriented by the chosen orientation of  $W^u(p, X')$ . Moreover, it is the boundary of the closed  $(n - k - 1)$ -disc  $\Delta^+$  whose interior is made of points in  $\partial^+ B$  whose positive  $X'$ -flow goes to  $q$ . Similarly, set  $\partial^- B := \{h' = h'(q) - \varepsilon\}$ . Denote by  $\Delta^-$  the closure of  $W^u(p, X') \cap \partial^- B$  (a  $k$ -disc) and set  $\Sigma^- := W^u(q, X') \cap \partial^- B$ . The sphere  $\Sigma^-$  bounds  $\Delta^-$ .

Let  $\mathcal{L}^\pm$  denote the leaf of  $\alpha'$  in  $T$  which contains  $\partial^\pm B$ . The ball  $\partial^+ B$  shall be regarded as the union of a normal tube  $N(\Sigma^+)$  to  $\Sigma^+$  and a normal tube  $N(\Delta^+)$  to  $\Delta^+$  in  $\mathcal{L}^+$ . Moreover, there exists a larger tube  $N'(\Sigma^+)$  which contains all fibres of  $N(\Delta^+)$  which meet  $N(\Sigma^+)$ .

Let  $\rho : \mathcal{L}^- \rightarrow \mathcal{L}^+$  be the *first* holonomy diffeomorphism of the positive flow of  $X'$ . Denote by  $Desc : \mathcal{L}^+ \setminus \Delta^+ \rightarrow \mathcal{L}^- \setminus \Delta^-$  the *gradient descent* by  $X'$ . Set  $D := \rho(\Delta^-)$ ; it is a  $k$ -disc in  $\mathcal{L}^+$  disjoint from  $\partial^+ B$  whose boundary lies in  $W^u(q, X')$ .

Finally, let  $\mu_a$  be the fibre in  $a \in \Sigma^+$  of the tube  $N'(\Sigma^+)$ . By descent, the  $(k + 1)$ -disc  $\mu_a$  descends to  $\Gamma_a^-$ , a cylinder over  $\Delta^-$ , pinched along  $\partial\Delta^-$ . The cylinder  $\Gamma_a^-$  is foliated by *rays* which are traced by the normal fibres to  $\Delta^-$  in  $\mathcal{L}^-$ . The holonomy diffeomorphism  $\rho$  carries  $\Gamma_a^-$  and its foliation to a foliated pinched cylinder  $\Gamma_a^+$  whose leaves are still named rays.

**2.4. SOME ISOLATED SELF-SLIDE OF  $p$ ; HOMOCLINIC BIFURCATION.** In this setup, a *positive self-slide* of  $p$  consists in making an isotopy of  $D$  in  $\mathcal{L}^+$ , relative to  $\partial D$ , which crosses  $\Sigma^+$  exactly once and positively with respect to its co-orientation [4, Remark 3.2]. This isotopy lifts to a deformation  $(X'_s)_{s \in [0,1]}$  of  $X'_0 = X'$ . At the time of crossing (let us say  $s = \frac{1}{2}$ ), the homoclinic connecting orbit from  $p$  to  $p$  lies in the homotopy class  $g$ . The path  $(X'_s)_s$  is said to cross the codimension-one stratum  $\mathcal{S}_g$  positively in the space of pseudo-gradients adapted to  $\alpha'$ .

Up to homotopy, the path  $(X'_s)_s$  is well-defined by the following data:

- a simple smooth path  $\gamma : [0, \frac{1}{2}] \rightarrow \mathcal{L}^+$ , starting non-tangentially to  $D$  from any point  $\gamma(0) \in \text{int}(D)$ , ending in  $a^+ := \gamma(\frac{1}{2})$  normally to  $\Sigma^+$  and which, for every  $s \in (0, \frac{1}{2})$ , avoids both of  $W^u(p, X') \cup \Delta^+$  and the repelling periodic orbit.
- a transverse framing  $\tau = (\tau_1, \dots, \tau_{n-2})$  to  $\gamma$  in  $\mathcal{L}^+$  such that  $\tau_1^k := \tau_1 \wedge \dots \wedge \tau_k$  is tangent to  $D$  in  $\gamma(0)$  and such that  $(\dot{\gamma}(\frac{1}{2}), \tau_1, \dots, \tau_{n-2})$  is an orthogonal framing positively normal to  $\Sigma^+$  in  $a^+$ .

Note that any point  $a^+ \in \Sigma^+$  is reachable from  $D$  by such a path. In what follows, we will specify  $\gamma$  more precisely near  $s \in \{0, \frac{1}{2}\}$ . For further use, we introduce  $a^- := \rho^{-1}(\gamma(0))$ , a point of  $\Delta^-$ . Let  $\tilde{\Sigma}^- := W^u(p, X') \cap \{h' = h'(p) - \varepsilon\}$  be the attaching  $k$ -sphere associated with  $p$  and denote by  $\tilde{a}^- \in \tilde{\Sigma}^-$  the point which descends to  $a^-$  through the flow of  $X'|_B$ .

Let  $Y$  be a vector field on  $\mathcal{L}^+$ , tangent to  $\Delta^+$ , whose support  $S$  is contained in a neighbourhood of  $\gamma([0, \frac{1}{2}]) \cup \partial^+ B$  and whose flow is noted  $Y^s$ ,  $s \in \mathbb{R}$ . The triple  $(\gamma, \tau, Y)$  may be chosen so that the following holds.

- (i) For every  $s \in [0, \frac{1}{2}]$ , we have  $Y^s(\gamma(0)) = \gamma(s)$  and  $\tau_1^k$  is tangent to  $Y^s(D)$  in  $\gamma(s)$ ; moreover, we impose the vector  $\dot{\gamma}(\frac{1}{2})$  to be tangent to  $\Delta^+$  pointing inwards.
- (ii) Denoting by  $\mathcal{T}$  the closure of  $\cup_{s=0}^{+\infty} Y^s(D)$ , named the *tongue*, the intersection  $\mathcal{T} \cap \partial^+ B$  is a corned disc  $\hat{\nu}_{a^+}$  contained in the fibre  $\mu_{a^+}$  of the tube  $N'(\Sigma^+)$ . Denote by  $\nu_{a^+}$  the corresponding fibre of  $N(\Sigma^+)$ .
- (iii) For every  $s \in [\frac{1}{2}, +\infty)$ , the intersection  $Y^s(D) \cap \partial^+ B$  is one fibre of  $N(\Delta^+)$  which meets  $N(\Sigma^+)$ . The distance of this fibre to  $\Sigma^+$  is noted  $r(s)$ ; set  $\eta = \max r(s)$ . A suitable choice of  $Y$  makes it as small as desired.
- (iv) The vector field  $Y$  is required to be tangent to the foliation of  $\Gamma_{a^+}^+$  by rays. This requirement forces the path  $\gamma$  to be tangent to a ray of  $\Gamma_{a^+}^+$  and to follow it entirely.
- (v) The support  $S$  satisfies the following condition:  $(S \setminus \partial^+ B)$  is contained in a small neighbourhood of  $\gamma([0, \frac{1}{2}])$ .

The last requirement needs some preparation. By (i), the vector  $\dot{\gamma}(\frac{1}{2})$  is tangent to  $\Delta^+$  and points in the direction marked by the point  $\Delta^+ \cap \partial\nu_{a^+}$ . This vector determines a co-oriented equator  $E$  in the sphere  $\partial\nu_{a^+}$  (see [4, subsection 2.4]). Recall that the Morse model of  $p$  produces a canonical diffeomorphism from the meridian sphere  $\partial\nu_{a^+}$  to  $\tilde{\Sigma}^-$ . It maps  $E$  to an equator  $\tilde{E} \subset \tilde{\Sigma}^-$  which avoids the connecting orbit from  $p$  to  $q$  by the choice of  $\dot{\gamma}(\frac{1}{2})$ . Thus,  $\tilde{E}$  descends through the flow of  $X'|_B$  to a  $(k-1)$ -sphere  $E^-$  contained in the interior of  $\Delta^-$ . This  $E^-$  makes a decomposition  $\Delta^- = \delta^- \cup_{E^-} \delta^+$  where  $\delta^-$  is a  $k$ -disc and  $\delta^+$  is diffeomorphic to  $S^{k-1} \times [0, 1]$ . The signs of  $\delta^\pm$  reflect the co-orientation of  $E$  in  $\partial\nu_{a^+}$ , and hence, of  $E^-$  in  $\Delta^-$ . By moving  $\gamma(0)$  in  $D$ , we can fulfill the last requirement below.

- (vi) The point  $a^-$  lies in  $\delta^+$ . Moreover, we make the support  $S$  of  $Y$  disjoint from  $\rho(\delta^-)$ .

Finally, the isotopy  $(Y^s)_{s \in [0, 1]}$  lifts to the desired one-parameter deformation  $(X'_s)$  of  $X'$  realizing a self-slide of  $p$  in the homotopy class  $g$ .

LEMMA 1. *With the above requirements (i-vi), the self-slide at  $s = \frac{1}{2}$  is isolated. More precisely, for  $s \neq \frac{1}{2}$ , the pseudo-gradient  $X'_s$  has no homoclinic orbit.*

PROOF. The dynamics of  $\rho$  which moves  $\partial D$  away from  $S$  makes clear that  $W^u(q, X'_s)$  never meet  $\Delta^+$ . This implies that no  $s \in [0, 1]$  is a time of self-slide of  $q$ .

Concerning the self-slides of  $p$ , we argue as follows. By (ii), the corned disc  $\hat{\nu}_{a^+}$  descends to a corned cylinder  $C^- \subset \Gamma_{a^+}^-$ , also based on  $\Delta^-$  and pinched along  $\partial\Delta^-$ . Then, the holonomy diffeomorphism  $\rho$  carries  $C^-$  to a pinched corned cylinder  $C^+ \subset \Gamma_{a^+}^+$ . By (iv-v), the intersection  $C^+ \cap S$  is foliated by (truncated) rays of  $C^+$  starting from  $D \cap S$ . In particular,  $S \cap \mathcal{T}$  may be viewed as a *sub-tongue* of  $\mathcal{T}$ .

We are now ready to prove Lemma 1. We limit ourselves to prove that  $X'_s$  is never a self-slide of  $p$  in the class  $g^2$  when  $s \neq \frac{1}{2}$ , that is:

$$(*) \quad \forall s \neq \frac{1}{2}, \quad [(Y^s \rho) \circ Desc \circ_1 (Y^s \rho)(\Delta^-)] \cap \Sigma^+ = \emptyset.$$

Here, the operator  $\circ_1$  stands for  $Desc$  where it is defined, that is in  $\mathcal{L}^+ \setminus \Delta^+$ . The proof concerning the classes  $g^k$ ,  $k > 2$ , is similar with the same choices (i) - (vi).

When  $s < \frac{1}{2}$  and as long as the disc  $Y^s(D)$  does not touch the ball  $\partial^+ B$ , the dynamics of  $\rho \circ Desc$  implies (\*). When  $Y^s(D) \cap N(\Sigma^+)$  is non-empty, we define  $D_s^- := Desc \circ_1 Y^s(D)$ . When  $s < \frac{1}{2}$ , one checks that  $D_s^-$  is contained in the union of thin tube  $Desc(S \setminus \partial^+ B)$  and the part of the cylinder  $C^-$  over  $\delta^-$  (in the sense of the rays). Thus by (vi),  $\rho(D_s^-)$  does not approach the support  $S$  of  $Y$  and (\*) holds for every  $s \in [0, \frac{1}{2})$ .

For  $s > \frac{1}{2}$ ,  $D_s^-$  is now an annulus whose boundary is made of two  $(k-1)$ -spheres, one of which is  $Desc(\partial D)$  and the other is  $\partial \Delta^-$ . By [4, Lemma 3.8], when  $s$  goes to  $\frac{1}{2}$  from above, it approaches any compact domain given in the interior of the annulus  $\delta^+$  in the  $C^1$ -topology. As a consequence of (iv), when  $s - \frac{1}{2}$  is small enough the intersection of  $\rho(D_s^-)$  with  $S$  is the graph  $G_{u_s}$  in the cylinder  $C^+$  of some positive function  $u_s : \rho(\delta^+) \cap S \rightarrow \mathbb{R}_{>0}$ . Since the flow of  $Y$  preserves the order on each orbit,  $[(Y^s \rho)(D_s^-)] \cap S$  (which lies in the tongue) is in front position with respect to  $[Y^{1/2}(D)] \cap S$  on each flow line of  $Y$ . This claim holds true when  $s > \frac{1}{2}$  is very close to  $\frac{1}{2}$  and it remains true up to 1 by the flow rule. Therefore,  $[(Y^s \rho)(D_s^-)] \cap S$  is still beyond  $[Y^{1/2}(D)] \cap S$  in the tongue, and hence, it cannot meet  $\Sigma^+$ .  $\square$

**2.5. THE CHARACTER FUNCTION.** This function is a natural continuous function  $\chi : \mathcal{S}_g \rightarrow \mathbb{R}$  which takes positive *and* negative values on each connected component of  $\mathcal{S}_g$  [4, Theorem 1.1]. For a pseudo-gradient  $Y \in \mathcal{S}_g$ , the value  $\chi(Y)$  only depends on the 1-jet of the holonomy  $\mathfrak{h}$  of  $Y$  along the unique homoclinic orbit  $\ell_Y$  from a zero of  $Y$  to itself in the homotopy class  $g$ . The construction and properties of  $\chi(Y)$  are given in [4, Section 2]. Denote by  $\mathcal{S}_g^+$  (resp.  $\mathcal{S}_g^-$ ) the non-empty open set in  $\mathcal{S}_g$  where  $\chi > 0$  (resp.  $\chi < 0$ ). When crossing  $\mathcal{S}_g$  transversely, the change of the differential in the Morse-Novikov complex is different according to the crossing: it can be positive or negative; through  $\mathcal{S}_g^+$  or  $\mathcal{S}_g^-$ . A positive crossing through  $\mathcal{S}_g^+$  implies a change by multiplication with the infinite series  $1 + g + g^2 + \dots$ .

In our setting, the considered pseudo-gradient is  $X'_{1/2}$  and the involved homoclinic orbit  $\ell$  is the orbit of  $X'_{1/2}$  which leaves the box  $B$  at  $a^-$  and enters  $B$  at  $a^+$ . We wish  $\chi(X'_{1/2}) > 0$ . By the construction, the character reads  $\chi(X'_{1/2}) = \lambda \omega_+ + \omega_-$  with  $\lambda > 0$  and  $\omega_{\pm} \in [-1, +1]$ . Each term<sup>2</sup> in this formula depends on  $\mathfrak{h}$  only. The term  $\omega_-$  is positive by the choice of  $\dot{\gamma}(\frac{1}{2})$  (which fixes the decomposition of  $\Delta^- = \delta^- \cup_{E^-} \delta^+$ ) and by taking  $a^- \in \delta^+$ . The term  $\omega_+$  is not estimated but the factor  $\lambda$  may be made as small as desired by the choice of the holonomy of  $Y$  along  $\gamma([0, \frac{1}{2}])$  once  $\gamma(0) = \rho(a^-)$  and  $\gamma(\frac{1}{2}) = a^+$  are fixed. And this choice is free. As a consequence, by [4, Theorem 3.6], we have:

$$(*) \quad \begin{cases} \partial^{X'_1} q = 0, \\ \partial^{X'_1} p = (1 + g + g^2 + g^3 + \dots) (\partial^{X'_0} p) = (1 + g + g^2 + g^3 + \dots) q. \end{cases}$$

Some comment about (\*) is necessary here. In the theorem which is referred to, (\*) is stated only for the germ of  $(X'_s)_s$  at  $s = \frac{1}{2}$  with truncation of the incidence coefficients. The support of a representative of the germ depends on the order of the truncation, the higher the order of truncation the smaller the domain of the representative. Here, we are in a very particular

<sup>2</sup>The terms  $\omega^+$  and  $\omega^-$  are named the *latitudes* in [4] and the factor  $\lambda$  is called the *holonomy factor*.

situation: by Lemma 1 the self-slide is isolated. As a consequence, the truncation in question is not needed. This finishes the proof of our theorem.

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