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The qualitative differences in generative characteristic of the aesthetic sensibility: Analysis of learners' problem solving

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There are some discussions about the relationship between creativity and the aesthetic sensibility. It has been a characteristic of mathematician and mathematical gifted that the aesthetic sensibility has a generative characteristic. On the other hand, some studies demonstrated that the generative characteristic also was available even by the non-mathematician or non-mathematical gifted. The purpose of this paper is to clarify the process to produce qualitative differences in the generative characteristic of the aesthetic sensibility through the analysis of learners' problem solving. For this purpose, two pairs of high school students and one pair of college students were observed during problem solving. As a result, it was clarified that it is critical whether learners have their own goal other than the given one to evoke the generative characteristic. Moreover, it was suggested that the difference of one's own goal is associated with qualitative difference in the generative characteristic.

Keywords: Aesthetic, generative, creativity, problem solving, qualitative study.

Introduction

There are two directions for the term “mathematical creativity” used widely: extraordinary creativity, known as big C, or everyday creativity, known as little c (Sriraman, Haavold & Lee, 2014, p.110). Everyday creativity also is important for mathematics educators. Silver (1997) said,

Although creativity is often viewed as being associated with the notions of “genius” or exceptional ability, it can be productive for mathematics educators to view creativity instead as an orientation or disposition toward mathematical activity that can be fostered broadly in general school population. (Silver, 1997, p.75)

In today's national curriculum in Japan, developing *all* students' creativity is one of the objectives in high school mathematics education. So, this paper uses the term “mathematical creativity” or “creativity” as the meaning used by Silver: everyday creativity.

Since Poincaré (1908/2003) pointed the importance of the aesthetic sensibility in mathematical discovery, the interest in the relationship between creativity and the aesthetic sensibility has risen in the field of mathematics education. Some studies have claimed that the aesthetic sensibility is one of characteristics of mathematician or mathematical gifted student (Dreyfus & Eisenberg, 1986; Hardy, 1956/1992; Krutetskii, 1976; Poincaré, 1908/2003; Silver & Metzger, 1989). As the bases for such a claim, these studies had drawn attention to the process that the aesthetic sensibility worked (Dreyfus & Eisenberg, 1986; Krutetskii, 1976; Silver & Metzger, 1989). In particularly, Silver & Metzger concluded that there were several characteristics of aesthetic sensibility during mathematical problem solving only by mathematicians. On the other hand, other studies disagree with above claims in that general learner, non-mathematician or non-mathematical gifted student, can have the aesthetic sensibility and their aesthetic sensibilities have the similar characteristics with mathematician or mathematical gifted student (Papert, 1978; Sinclair, 2006a). These studies demonstrated that the

generative characteristic of the aesthetic sensibility, which works as a guide in decision making during mathematical discovery, could work in general learners' mathematical problem solving.

If the difference of such a characteristic is not due to the mathematical talent, the questions have remained unanswered what critical factor causing difference is or how such difference is caused. The purpose of this paper is to clarify the process to produce qualitative differences in the generative characteristic of the aesthetic sensibility through the analysis of general learners' problem solving.

Theoretical background

The characteristics of the aesthetic sensibility in problem solving

There is not a clear and widely accepted definition of the term "aesthetic" in mathematics education. Poincaré (1908/2003) explained using both the form of mathematical objects and the sense of the perceiver as following:

It is the harmony of the different parts, there symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts. (Poincaré, 1908/2003, pp.30-31)

Hardy (1956/1992) and Dreyfus & Eisenberg (1986) defined it using subjective qualities like "economy", "simplicity" and "surprise". Without referring to its subjectivity, Hardy argued that mathematician would share them. In contrast to the above studies, Wells (1990) claimed that the aesthetic qualities are subjective and context-dependent.

On the other hand, some studies paid attention to the working process of the aesthetic sensibility, which is the ability to appreciate and respond to aesthetic qualities of mathematical objects, rather than strictly defining the term "aesthetic" (Papert, 1978; Silver & Metzger, 1989; Sinclair, 2006a, 2006b). Though claims about what the aesthetic is are various, the discussion about what characteristics the aesthetic sensibility have is a convergent as below. Papert (1978) was focused on the process of creation explained by Poincaré. Papert regarded the aesthetic widely and observed non-mathematicians' proving process. As a result, Papert concluded non-mathematician also could be guided by the aesthetic sensibility. Silver & Metzger (1989) classified the role of the aesthetic sensibility into two categories. First is "the guidance of decision making during problem solving" (p. 62). The viewpoint of this category can be regarded as the same as Papert's. Second is "the evaluation of the elegance of a completed solution" (p.62). In addition, Silver & Metzger observed mathematicians' problem solving and identified these roles in their problem solving. Similarly, Sinclair (2006b) classified characteristics of the aesthetic sensibility into three categories: the evaluative characteristic, the generative characteristic, the motivational characteristic. Moreover, Sinclair (2006a) observed learners' problem solving and problem posing and identified these characteristics of the aesthetic sensibility. Sinclair's generative characteristic and evaluative characteristic can be regarded as the same as Silver & Metzger's two roles.

The above results demonstrate that whether the aesthetic sensibility does work in problem solving is due not to only mathematical talent but also to something else. The existence of this "something else" is consistent with the claim of Papert. However, there has been not enough study done concerning this point. In particular, few studies have attempted to observe learners' generative characteristic of the aesthetic sensibility.

Based on the above background, this paper defines the generative characteristic of the aesthetic sensibility (GCA) as a guide in decision making during mathematical discovery, and clarifies the process to produce qualitative differences in the GCA.

Four categories of the generative characteristic of the aesthetic sensibility.

Poincaré (1908/2003) regarded the aesthetic sensibility as a thing working in the unconscious level. In contrast, some studies limited the discussion to the conscious level (e.g. Papert, 1978; Silver & Metzger, 1989; Sinclair, 2006a). This paper also limits a discussion to the conscious level.

Papert regarded reasoning without conviction or logic but with pleasure as “the problem of guidance” (p.109) by the aesthetic sensibility. Sinclair (2006b) associated such non-conviction reasoning to intuition as “capitalising on intuition” (p.94). Moreover, Sinclair (2006b) identified additional three categories of the GCA based on the mentions by mathematicians. First category is “playing with or ‘getting a feel for’ a situation” (p.94). This means exploration “in that the one playing is seeking to identify organizing themes and structures and to arrange the objects being played with in a meaningful, expressive way” (p.95). That is, it can be interpreted as pursuing these goals without depending on the goals of the given problem. Second category is “establishing intimacy” (p.94). This means, for example, to give a name to the considered subject. Third category is “enjoying the craft” (p.95). This is interpreted as consideration using mastered tools. Although the question remains whether it is reasonable that intuition is regarded as one of the aesthetic generating, this classification by Sinclair (2006b) can be used as viewpoints for extraction of the GCA from one’s behavior in problem solving.

A study on the generative characteristic of the aesthetic sensibility in general learners’ problem solving

Participants

Two pairs of high school students and a pair of college students, who had several mathematical knowledge and mathematical experience, were selected as participants, and observed during solving a problem. One pair of high school students belonged to 10th grade (Pair H1), another pair belonged to 11th grade (Pair H2). One of college students belonged to third year and another belonged to fourth year of mathematics teacher-training course (Pair C). Although they were all better than the average learners in Japan, they were not so good as mathematician or mathematical gifted student.

Pair H1 and Pair H2 belonged to the same high school in Japan. Pair H1 had learned double radical signs. However, they had learned about a particular type like $\sqrt{3 + 2\sqrt{2}}$ which could be transformed into other form without a double radical sign. In addition, they had not learned the relationship between the roots and the coefficient of the quadratic equation. The other hand, Pair H2 had learned same type double radical signs with Pair H1. This pair had learned the relationship between the roots and the coefficient of the quadratic equation.

Pair C did not belong to the same high school with Pair H1 and Pair H2. They had learned double radical signs and the relationship between the roots and the coefficient of the quadratic equation. Although a student belonged to third year had never “studied” mathematics in college, another student belonged to fourth year had “studied” mathematics in college for a half year. So, fourth grade

student was expected to have experienced mathematical discovery and to show the experience in the process of problem solving.

Procedure of the study

Because the GCA is “involved in the actual process of inquiry, in the discovery and the invention of solutions or ideas” (Sinclair, 2006b, p.93), participants were observed their problem solving behavior in following process.

Each pair calculated in order to clear some concrete double radical signs like $\sqrt{3 + 2\sqrt{2}}$ as warm-up. Then, observer showed another type which cannot be cleared double radical signs like $\sqrt{5 + 2\sqrt{2}}$. After that, each pair solved a problem about an abstract double radical sign (it is shown below). After they finished to solve, they were interviewed about how to solve it.

In order to analyze verbal report during solving problem as data, participants were asked to solve a problem while consulting in pairs. By this setting, it was expected to provide simultaneous and nature verbal report. All participants’ verbal report was recorded on a IC recorder. (Only Pair H2 was recorded on a video camera, too.)

A problem

In this study, a following problem was chosen.

Find the conditions for clearance of a double radical sign from $\sqrt{p + 2\sqrt{q}}$. However, p and q belongs in positive rational number. Moreover, q is not the square of rational number.

There are multiple conclusions in this problem as following. The participants were not informed what conditions were appropriate as the conclusion. Therefore, they were also asked value judgments for determining their finding as a conclusion. In contrast to that the multiple solution problems need participants to solve problem by more than one way — general learners usually do not so, such open-ended problems need participants’ value judgments more naturally.

(Conclusion 1): If a double radical sign can be transformed into following expression:

$$\sqrt{p + 2\sqrt{q}} = \sqrt{\alpha} + \sqrt{\beta},$$

then right side of above equation can be transformed into following expression:

$$\sqrt{\alpha} + \sqrt{\beta} = \sqrt{(\sqrt{\alpha} + \sqrt{\beta})^2} = \sqrt{\alpha + \beta + 2\sqrt{\alpha\beta}}.$$

From the above transformations, a necessary condition for clearance of a double radical sign from $\sqrt{p + 2\sqrt{q}}$ is the existence of α and β belonging to positive rational number such that $p = \alpha + \beta$ and $q = \alpha\beta$. Conversely, if α and β belonging to positive rational number such that $p = \alpha + \beta$ and $q = \alpha\beta$ exist, then the double radical sign of $\sqrt{p + 2\sqrt{q}}$ can be clear as following:

$$\begin{aligned}\sqrt{p + 2\sqrt{q}} &= \sqrt{\alpha + \beta + 2\sqrt{\alpha\beta}} \\ &= \sqrt{(\sqrt{\alpha} + \sqrt{\beta})^2} = \sqrt{\alpha} + \sqrt{\beta}.\end{aligned}$$

(Conclusion 2): Existence of α and β belonging to positive rational number such that $p = \alpha + \beta$ and $q = \alpha\beta$ is equivalence with that α and β are the roots of the quadratic equation $x^2 - px + q = 0$ in x . From this, the latter condition is also a conclusion of above problem.

In the following, D expresses the discriminant of the quadratic equation. If $D = p^2 - 4q \geq 0$, then

$$x = \frac{p \pm \sqrt{p^2 - 4q}}{2} > 0.$$

From this, if D can be expressed as the square of rational number like k^2 ($k \in \mathbb{Q}^+$), then $x^2 - px + q = 0$ has positive rational roots in x . Conversely, if D can be expressed as k^2 ($k \in \mathbb{Q}, 0 < k < p$), then the two roots of quadratic equation $x^2 - px + q = 0$ are

$$x = \frac{p \pm k}{2}.$$

These sum and product are

$$\frac{p+k}{2} + \frac{p-k}{2} = p, \quad \frac{p+k}{2} \times \frac{p-k}{2} = \frac{p^2 - k^2}{4} = \frac{4q}{4} = q.$$

From these,

$$\begin{aligned}\sqrt{p + 2\sqrt{q}} &= \sqrt{\frac{p+k}{2} + \frac{p-k}{2} + 2\sqrt{\frac{p+k}{2} \times \frac{p-k}{2}}} \\ &= \sqrt{\left(\sqrt{\frac{p+k}{2}} + \sqrt{\frac{p-k}{2}}\right)^2} = \sqrt{\frac{p+k}{2}} + \sqrt{\frac{p-k}{2}}.\end{aligned}$$

From above, if D can be expressed as k^2 ($k \in \mathbb{Q}, 0 < k < p$), then the double radical sign of $\sqrt{p + 2\sqrt{q}}$ can be clear.

(Conclusion 3): Looking back at the conclusion 2 can provide next developmental conclusion. That is, the conditions for clearance of a double radical sign from $\sqrt{p - 2\sqrt{q}}$ is that D can be expressed as k^2 ($k \in \mathbb{Q}, 0 < k < p$), too.

(Conclusion 4): Moreover, seeing $4q$ as $(2\sqrt{q})^2$ can provide another perspective. That is, if $2\sqrt{q}$ in the $\sqrt{p \pm 2\sqrt{q}}$ is replaced to \sqrt{q} , then a conclusion of finding the conditions for clearance of a double radical sign from $\sqrt{p \pm \sqrt{q}}$ is that $p^2 - q$ can be expressed as k^2 ($k \in \mathbb{Q}, 0 < k < p$).

Results

Overview of three pairs' problem solving is shown in Table 1.

	Pair H1 (H1-1&H1-2)	Pair H2 (H2-1&H2-2)	Pair C (C1&C2)
Conclusion 1	04:00	H2-1 02:20/ H2-2 06:02	08:16
Conclusion 2	16:20	H2-1 05:37/ H2-2 06:59	
Conclusion 3, 4			
Finish	16:20	51:16	38:00

Table 1: Overview of three pairs' problem solving

In Table 1, each of the values show the time participants spent to get to the conclusion shown to far left, and diagonals show that participants did not arrive at the conclusion. Each participant of Pair H2 had got to the conclusion on their own before they started to consult each other. Therefore, each time of Pair H2 was shown in Table 1. Line of the "Finish" shows the time which each pair had spent solving problem. For instance, Pair H2 and Pair C kept pursuing more exact condition than their conclusion in cooperation each other. Solving process of each pair are as follows. However, the symbols used below are the same as those used by participants.

(Pair H1)

After confirming the question, each immediately got to the relationship: $p = a + b, q = ab$. H1-1 continued more investigations from the reason that " p and q cannot be found when these are the large numbers. I want another one easily puts out with only p and q .", and continued for a further inquiry. In addition, H1-2 said "it is not good to use the new a and b ", and continued to explore. H1-1 associated the factorization from the above equation, and flashed that it will go well if he can factorize as following: $x^2 + px + q = (x + a)(x + b)$. H1-1 started to think about that H1-2 had been questioned: the case $x^2 + px + q = 0$ has the rational solutions. H1-1 paid attention to the discriminant of the quadratic equation, but dismissed this idea. H1-1 dazzled that if the root in the quadratic formula could be clear, it will go well. H1-2 agreed this. They established conclusion that $D = p^2 - 4q$ is square number, and finished the solving.

(Pair H2)

After confirming the question, H2-1 and H2-2 went ahead the discussion using a specific example. However, they did not use the peculiarities of example, generalized immediately once outlook was obtained. Up to this point, they worked on the problem at each, and got to a conclusion 2 through the conclusion 1. However, they had not been convinced that the condition they got was the one they sought for. When the observer was urged to check the progress of each other, they decided to consider in cooperation about the conditions with the p and q , which H2-1 had considered. H2-1 was looking for a simple conclusion than conclusion 2. Ultimately, they made out that D can be expressed as the square of rational number is necessary and sufficient condition for clearance of a double radical sign without deriving another "simple" conclusion.

(Pair C)

After confirming the question, C1 remembered following condition as a formula: $p = a + b, q = ab$. Then, they considered they could regard this formula as conclusion. They made sure that $a \geq 0, b \geq 0$, and could regard the formula as conclusion. By the "intervention" of the observer: asking them about example, they began to consider whether concrete double radical signs can be clear based on above "conclusion". C1 noticed that x, y were rational number when a double radical sign could be cleared, and he said, "I guess I should add another conditions". C2 considered that the example which could not be clear a double radical sign was really not able to clear through a specific calculation. As a result, they confirmed that it could not clear. In response to the results, they concluded as following:

Condition (i): $p = a + b, q = ab$.

Condition (ii): $a \geq 0, b \geq 0$.

Condition (iii): a, b are rational number.

The analysis

The analysis was carried out in the following procedure. First, making transcripts of participant's problem solving process. Next, judging whether some of the four GCA proposed by Sinclair (2006b) can be seen through the observation of problem solving behavior and the interpretation of the intention of the behavior. The interpretation of the intention is based on the transcripts of the problem solving process and the explanations of their process that the participant did after solution. Finally, comparing the GCA of each pair. As a result, the GCA were seen as following Table 2.

the GCA	Pair H1	Pair H2	Pair C
Removing new symbols (for Conclusion 2)	○	○	/
Pursuit of exactness (for Conclusion 2)	/	○	○
Pursue of simplicity (for other Conclusions)	/	○	/

Table 2: The GCA of three pairs in problem solving process

Discussion: The qualitative differences in the generative characteristic of the aesthetic sensibility

From a comparison of the three pairs, mainly three of qualitative differences in the GCA in problem solving were observed. First is the point whether they attempt to remove new symbols a and b . Both Pair H1 and Pair H2 discussed about this point, but Pair C did not. Second is the point whether they confirm the exactness of the conclusions. Pair H1 did not check it, and the remaining two pairs did. However, there was a difference in the method used by pair H2 and pair C. Pair H2 was showed exactness by considering that the found condition was necessary and sufficient condition. On the other hand, pair C confirmed the exactness by considering a concrete example. Third is the point whether they attempt to further improve the conclusion that D could be expressed as the square of rational number. This had done only pair H2. However, this pair could not obtain conclusion.

From above results, it can be presumed that the qualitative difference of the GCA in problem solving depends on participants' goal in problem solving. For instance, pair H1 wanted not only to find condition, but also find condition without new characters a and b . Pair H2 also had this goal, but pair C did not. As a result, only pair C did not pursue another conclusion. Therefore, in order to evoke the GCA it is critical whether learners have one's own goal other than the given one. In addition, even if

learners have same goal such as exactness, the difference in means of the “exactness” can cause qualitative differences of the GCA.

From above discussion, if we can make learners to have one’s own goal, it is possible that we can evoke learners’ GCA. However, it is not clear in this paper that what kind of goal is desirable for mathematical creation, and how can we make learners to have one’s own goal. Therefore, a further study of these points should be conducted.

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