Joint Law of an Ornstein-Uhlenbeck Process and its Supremum

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Abstract

Let $(X_t)_{t\geqslant 0}$ be an Ornstein-Uhlenbeck process driven by a Brownian motion. We propose an expression for the joint density / distribution function $\mathbb{P}\left(X_t \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant t} X_u \leqslant m\right)$ where $m, x \in \mathbb{R}$, $x \leqslant m$. This law is expressed as an expansion involving parabolic cylinder functions. We obtain this law faster than with a Monte Carol's method. Numerical applications illustrate the interest of this result.

Keywords. Ornstein-Uhlenbeck process, joint law, supremum, endpoint, Fokker-Planck equation.

1 Introduction

The main purpose of this paper is to give an expression for the joint law of the endpoint and supremum of an Ornstein-Uhlenbeck process. The problem of finding such a joint law has attracted research since the late nineteenth century (see [Som94]). For a Brownian motion, the result is well known (see for example [KS91]). More recently, the authors of [EH14] give a method which allows to find this joint density / distribution function for the minimum and endpoints for a n-dimensional drifted Brownian motion. The explicit solution in the bi-dimensional case ($n = 2$) is given in [HKR98]. For $n > 2$, the method in [EH14] works in only some particular cases. [CPN18] and [Ngo16] study the case of a Lévy process, sum of a Brownian motion with drift and a compound Poisson process. [CP17] are interested in the joint law of a d-dimensional diffusion process ans the running supremum of the first component. They find a PDE satisfied by the joint density (but an Ornstein-Uhlenbeck process does not verify their assumptions). [DK06] study the general case of a Lévy process.
Let $(X_t)_{t\geqslant 0}$ be an Ornstein-Uhlenbeck process driven by a Brownian motion. Many results on the first passage time of an Ornstein-Uhlenbeck process have been obtained. In [APP05], the density function of the first hitting time to a fixed level by an Ornstein-Uhlenbeck process is given. Since hitting time and suprema are related, the cumulative distribution function of the supremum is obtained. Regarding the pair (process, its running maximum), [BS96] gives $\mathbb{P}(X_\tau \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant \tau} X_u \geqslant m)$ where τ is a $Exp(\lambda)$ r.v. independent of X. He also gives $\mathbb{P}(X_t \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant t} X_u \leqslant 0)$, $x \leqslant 0$. In this paper, we look for a more general case and we study the joint density / distribution function for an Ornstein-Uhlenbeck process driven by a Brownian motion, i.e. $\mathbb{P}(X_t \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant t} X_u \leqslant m)$ where $m, x \in \mathbb{R}$, $x \leqslant m$. This law is solution of a Fokker-Planck equation. To solve this equation, a spectrum method is used and makes Weber's parabolic cylinder functions appear (directly linked with the Hermite functions). Even if the joint law is an infinite expansion of special functions, it is easy to evaluate numerically and it is faster than the Monte Carlo's method. We then check that our result is consistent with those in [BS96] and [APP05].
The paper is organized as follows. In the next section, some results including the main theorem with the expression of the joint density / distribution are presented. Section 3 is devoted to the proof of this theorem. Finally, Section 4 is concerned with the law of the supremum and its joint density with the process.

2 Results

2.1 Context and notations

Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider an Ornstein-Uhlenbeck (OU in the following) process $(X_t)_{t\geqslant 0}$ solution of the following SDE :

$$\mathrm{d}X_t = -kX_t\,\mathrm{d}t + \mathrm{d}B_t, \quad X_0 \sim \mu$$

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where $k \in \mathbb{R}_+^*$ and μ is a probability measure. Suppose that X_0 and $(B_t)_{t\geqslant 0}$ are independent.
Let $\mathbb{P}_{x_0}$ be the law of the process $(X_t)_{t\geqslant 0}$ when $X_0 = x_0$.

2.2 Main theorem

Our aim is to compute the joint law of the process and its supremum. However, as it is not easy to compute the joint density, we are first interested in the joint density / distribution

$$\mathbb{P}\left(X_t \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant t} X_u \leqslant m\right) \text{ where } t \in \mathbb{R}_+^*, (m,x) \in \mathbb{R}^2, x \leqslant m.$$

The following theorem gives an expression for the joint density / distribution.

Theorem 2.1. *We assume that μ is a probability measure on $]-\infty, m]$, $m \in \mathbb{R}$ and set p_0 such that $\mu(\mathrm{d}x) = p_0(x)\,\mathrm{d}x$.*

(i) For all $x \leqslant m$ and $t \in \mathbb{R}_+$, the probability $\mathbb{P}\left(X_t \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant t} X_u \leqslant m\right)$ admits a density p :

$$\mathbb{P}\left(X_t \in \mathrm{d}x,\ \sup_{0\leqslant u\leqslant t} X_u \leqslant m\right) = p(m,x,t)\,\mathrm{d}x.$$

(ii) The density p solves the following Fokker-Planck equation

$$\forall m \in \mathbb{R}, \begin{cases} \partial_t p(m,x,t) &= k\partial_x(xp(m,x,t)) + \dfrac{1}{2}\partial^2_{x^2} p(m,x,t), & (x,t) \in]-\infty, m] \times \mathbb{R}_+^* \\ p(m,m,t) &= 0, & t \in \mathbb{R}_+^* \\ p(m,x,0) &= p_0(x), & x \in]-\infty, m]. \end{cases} \tag{1}$$

(iii) For $(m,x) \in \{(z,y) \in \mathbb{R}^2 \mid z \geqslant y\}, t \in \mathbb{R}_+^*$, the density p can be expressed as :

$$p(m,x,t) = -\sqrt{2k}\,\mathrm{e}^{-\frac{kx^2}{2}} \sum_{n=1}^{\infty} \mathrm{e}^{-k\nu_{n,m}t}\, \frac{D_{\nu_{n,m}}\left(-x\sqrt{2k}\right)\int_{-\infty}^{m} \mathrm{e}^{\frac{kz^2}{2}}\, p_0(z) D_{\nu_{n,m}}\left(-z\sqrt{2k}\right)\mathrm{d}z}{\nu_{n,m} D_{\nu_{n,m}-1}(-m\sqrt{2k})\partial_\nu D_{\nu_{n,m}}(-m\sqrt{2k})}, \tag{2}$$

where D_ν is the parabolic cylinder function and $(\nu_{j,m})_{j\geqslant 1}$ is the ordered sequence of positive zeros of $\nu \mapsto D_\nu(-m\sqrt{2k})$ (details are given in Appendix).

The proof of the theorem is postponed to Section 3. We start to study some particular cases and consequences.

Remark 2.2. *For $(m,x) \in \{(z,y) \in \mathbb{R}^2 \mid z \geqslant y\}, t \in \mathbb{R}_+^*$, when $X_0 = x_0 \leqslant m$,*

$$p(m,x,t) = -\sqrt{2k}\,\mathrm{e}^{\frac{k(x_0^2 - x^2)}{2}} \sum_{n=1}^{\infty} \mathrm{e}^{-k\nu_{n,m}t}\, \frac{D_{\nu_{n,m}}\left(-x_0\sqrt{2k}\right) D_{\nu_{n,m}}\left(-x\sqrt{2k}\right)}{\nu_{n,m} D_{\nu_{n,m}-1}(-m\sqrt{2k})\partial_\nu D_{\nu_{n,m}}(-m\sqrt{2k})}. \tag{3}$$

The expression of the joint density / distribution can be easily extended to an OU process depending on 3 parameters.

Corollary 2.3. *Suppose the OU process $(X_t)_{t\geqslant 0}$ solution of the following SDE :*

$$\mathrm{d}X_t = (\phi - kX_t)\,\mathrm{d}t + \sqrt{\beta}\,\mathrm{d}B_t, \quad X_0 \sim \mu$$

where $\beta, k \in \mathbb{R}_+^*$ and $\phi \in \mathbb{R}$.
For $(m,x) \in \{(z,y) \in \mathbb{R}^2 \mid z \geqslant y\}, t \in \mathbb{R}_+^*$, the density p can be expressed as :

$$p(m,x,t) = -\sqrt{\frac{2k}{\beta}}\,\mathrm{e}^{-\frac{k\left(x-\frac{\phi}{k}\right)^2}{2\beta}} \sum_{n=1}^{\infty} \left(\int_{-\infty}^{m-\frac{\phi}{k}} \mathrm{e}^{\frac{kz^2}{2\beta}}\, p_0(z) D_{\nu_{n,m-\frac{\phi}{k}}}\left(-z\sqrt{\frac{2k}{\beta}}\right)\mathrm{d}z\right) \mathrm{e}^{-k\nu_{n,m-\frac{\phi}{k}}t}$$
$$\times \frac{D_{\nu_{n,m-\frac{\phi}{k}}}\left(-\left(x-\frac{\phi}{k}\right)\sqrt{2\frac{k}{\beta}}\right)}{\nu_{n,m-\frac{\phi}{k}} D_{\nu_{n,m-\frac{\phi}{k}}-1}\left(-\left(m-\frac{\phi}{k}\right)\sqrt{\frac{2k}{\beta}}\right)\partial_\nu D_{\nu_{n,m-\frac{\phi}{k}}}\left(-\left(m-\frac{\phi}{k}\right)\sqrt{\frac{2k}{\beta}}\right)}.$$

where D_ν is the parabolic cylinder function and $\left(\nu_{j,m-\frac{\phi}{k}}\right)_{j\geqslant 1}$ is the ordered sequence of positive zeros of $\nu \mapsto D_\nu\left(-\left(m-\frac{\phi}{k}\right)\sqrt{\frac{2k}{\beta}}\right)$.

Proof : Let set $U_t = X_{\frac{t}{\beta}} - \dfrac{\phi}{k}$ and $W_t = \sqrt{\beta} B_{\frac{t}{\beta}}$ (which is thus a standard Brownian motion). Then the dynamic of $(U_t)_{t \geqslant 0}$ is

$$\mathrm{d}U_t = -\frac{k}{\beta} U_t \, \mathrm{d}t + \mathrm{d}W_t, \quad U_0 = X_0 - \frac{\phi}{k} \in \mathbb{R}.$$

We conclude with Theorem 2.1. ■

2.3 Numerical results

Even though Formula (2) is an infinite sum of special functions, it is easy to evaluate numerically.
The function $x \mapsto p(m, x, t)$ is evaluated with expression (2) (or (3)) and is compared with a Monte Carlo's method using 5×10^5 simulations. Figures 1 and 2 plot $\mathbb{P}\left(X_t \in \mathrm{d}x, \sup_{0 \leqslant u \leqslant t} X_u \leqslant m\right)$ for different parameters and initial laws with $t = 1$.

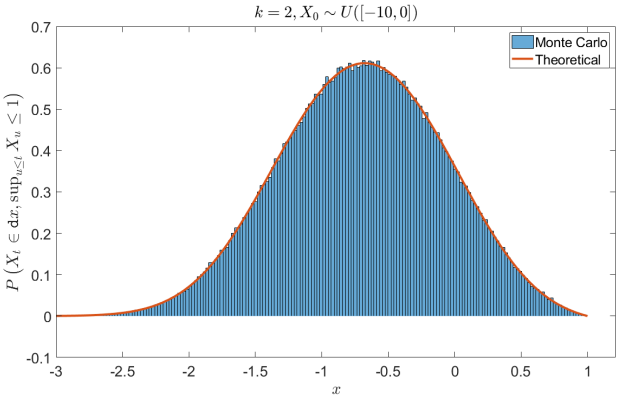

Figure 1: Comparison of expression (2) and the Monte Carlo's method when $m = 1$, $X_0 \sim \mathcal{U}([-10, 0])$ and $k = 2$.

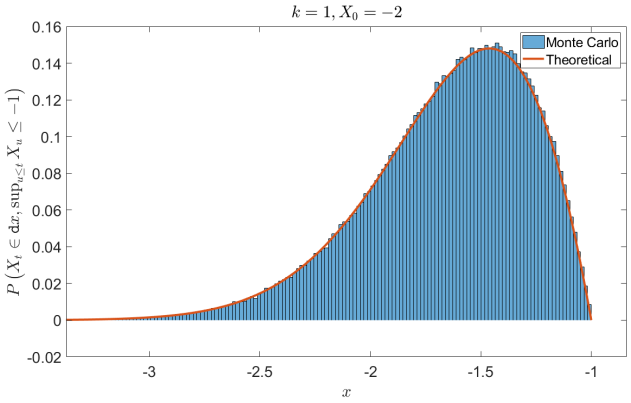

Figure 2: Comparison of expression (3) and the Monte Carlo's method when $m = -1$, $X_0 = -2$ and $k = 1$.

Last but not least, our formula is faster than the Monte Carlo's method. Indeed, with a simple core, obtaining the whole function $x \mapsto p(m, x, t)$ lasts around 23 sec whereas a Monte Carlo's method with 10^5 simulations lasts around 18 sec, 2.10^5 simulations 34 sec and 10^6 simulations 176 sec.

2.4 Particular case $m = 0$

When $m = 0$, the expression of the joint density / distribution can be simplified. Indeed, the zeros $(\nu_{n,0})_{n \geqslant 0}$ of $\nu \mapsto D_\nu(0)$ are the positive odd integers (see Appendix for more details). In this particular case, Formula (3) can be simplified.

Proposition 2.4. *We assume $X_0 = x_0 \leqslant 0$. For all $x \leqslant 0$ and $t \in \mathbb{R}_+^*$,*

$$\mathbb{P}_{x_0}\left(X_t \in \mathrm{d}x, \sup_{0 \leqslant s \leqslant t} X_s \leqslant 0\right) = \sqrt{\frac{k}{\pi(1 - \mathrm{e}^{-2kt})}} \left[\mathrm{e}^{-k\frac{(x - x_0 \, \mathrm{e}^{-kt})^2}{1 - \mathrm{e}^{-2kt}}} - \mathrm{e}^{-k\frac{(x + x_0 \, \mathrm{e}^{-kt})^2}{1 - \mathrm{e}^{-2kt}}}\right] \mathrm{d}x.$$

Proof : Without loss of generality, $k = 1$ is taken in the proof. We use the results given in Appendix.
With (29) and Proposition 6.2, we have :

$$p(0, x, t) = \frac{2 \, \mathrm{e}^{\frac{x_0^2 - x^2}{2}}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \mathrm{e}^{-(2n+1)t} \, \frac{D_{2n+1}\left(-x_0\sqrt{2}\right) D_{2n+1}\left(-x\sqrt{2}\right)}{(2n+1)!}.$$

We use (28) and then, with a Fubini theorem (everything is clearly integrable),

$$p(0, x, t) = -\frac{2 \, \mathrm{e}^{x_0^2}}{\pi\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^{2n+1} \, \mathrm{e}^{-(2n+1)t}}{(2n+1)!} \int_0^\infty \int_0^\infty (uv)^{2n+1} \, \mathrm{e}^{-(u^2+v^2)} \left(\mathrm{e}^{2\mathrm{i}vx_0} - \mathrm{e}^{-2\mathrm{i}vx_0}\right) \left(\mathrm{e}^{2\mathrm{i}ux} - \mathrm{e}^{-2\mathrm{i}ux}\right) \mathrm{d}u \, \mathrm{d}v.$$

We can invert integral and sum :

$$p(0,x,t) = -\frac{2\,\mathrm{e}^{x_0^2}}{\pi\sqrt{\pi}} \int_0^\infty \int_0^\infty \mathrm{e}^{-(u^2+v^2)} \left(\mathrm{e}^{2\mathrm{i}vx_0} - \mathrm{e}^{-2\mathrm{i}vx_0}\right) \left(\mathrm{e}^{2\mathrm{i}ux} - \mathrm{e}^{-2\mathrm{i}ux}\right) \mathrm{sh}(2uv\,\mathrm{e}^{-t})\,\mathrm{d}u\,\mathrm{d}v$$
$$= -\frac{\mathrm{e}^{x_0^2}}{\pi\sqrt{\pi}} \left[\int_{-\infty}^\infty \mathrm{e}^{-v^2+2\mathrm{i}vx_0} \int_{-\infty}^\infty \mathrm{e}^{-u^2+2\mathrm{i}ux+2uv\,\mathrm{e}^{-t}}\,\mathrm{d}u\,\mathrm{d}v - \int_{-\infty}^\infty \mathrm{e}^{-v^2+2\mathrm{i}vx_0} \int_{-\infty}^\infty \mathrm{e}^{-u^2+2\mathrm{i}ux-2uv\,\mathrm{e}^{-t}}\,\mathrm{d}u\,\mathrm{d}v\right].$$

Finally, formula (4.11.6) in [LS72] gives :

$$p(0,x,t) = -\frac{\mathrm{e}^{x_0^2}}{\pi} \left[\int_{-\infty}^\infty \mathrm{e}^{-v^2+2\mathrm{i}vx_0}\,\mathrm{e}^{(\mathrm{i}x+v\,\mathrm{e}^{-t})^2}\,\mathrm{d}v - \int_{-\infty}^\infty \mathrm{e}^{-v^2+2\mathrm{i}vx_0}\,\mathrm{e}^{(\mathrm{i}x-v\,\mathrm{e}^{-t})^2}\,\mathrm{d}v\right]$$
$$= -\frac{\mathrm{e}^{x_0^2-x^2}}{\sqrt{\pi(1-\mathrm{e}^{-2t})}} \left[\mathrm{e}^{-\frac{(x_0+x\,\mathrm{e}^{-t})^2}{1-\mathrm{e}^{-2t}}} - \mathrm{e}^{-\frac{(x_0-x\,\mathrm{e}^{-t})^2}{1-\mathrm{e}^{-2t}}}\right]$$
$$= \frac{1}{\sqrt{\pi(1-\mathrm{e}^{-2t})}} \left[\mathrm{e}^{-\frac{(x-x_0\,\mathrm{e}^{-t})^2}{1-\mathrm{e}^{-2t}}} - \mathrm{e}^{-\frac{(x+x_0\,\mathrm{e}^{-t})^2}{1-\mathrm{e}^{-2t}}}\right].$$

■

Remark 2.5. *This is formula (1.1.8) of Chapter 7 in [BS96]. Note that there is a typography in the book.*

3 Proof of the main theorem

This section is devoted to the proof of Theorem 2.1. The proof is divided into two parts. First, we prove the existence of the density and the fact that it is solution of the Fokker Planck Equation. The proof of the theorem relies on the second part namely resolution of this Fokker Planck equation. Without loss of generality, we take here $k = 1$.

Existence of the density and the Fokker-Planck equation

(i) The existence of the density is given by Lemma 2.1 in [DIRT13].

(ii) It is just Proposition 5.4.3.1 in [JYC09] with $h : t \mapsto m$.

Resolution of the Fokker-Planck equation

(iii) Now, we want to solve the Fokker-Planck equation. We use a spectrum method inspired by [BD08], [Gar04] and [RH89]. In that purpose, we introduce two operators.

Let us note $\mathcal{D}(\mathbf{L}_{\mathrm{FP}}) := \left\{ f \in H^1(]-\infty,m]) \mid \left[x \mapsto \frac{\partial}{\partial x}(xf(x)) + \frac{1}{2}f''(x)\right] \in L^2(]-\infty,m]), f(m) = 0 \right\}$.

The Fokker Planck operator $\mathbf{L}_{\mathrm{FP}}$ is defined by :

$$\mathbf{L}_{\mathrm{FP}} : \left| \begin{array}{ccc} \mathcal{D}(\mathbf{L}_{\mathrm{FP}}) & \longrightarrow & L^2(]-\infty,m]) \\ f & \longmapsto & \left[\mathbf{L}_{\mathrm{FP}} f : x \mapsto \frac{\partial}{\partial x}(xf(x)) + \frac{1}{2}f''(x)\right] \end{array} \right. .$$

The goal is to find a basis $(\tilde{\varphi}_n)_n$ of eigenvectors of $\mathbf{L}_{\mathrm{FP}}$ in order to write

$$p(m,x,t) = \sum_{n=1}^\infty c_n(t)\tilde{\varphi}_n(x).$$

The Fokker-Planck operator $\mathbf{L}_{\mathrm{FP}}$ is not hermitian. Then, to build this basis, we introduce the following operator $\mathbf{O}$, related to $\mathbf{L}_{\mathrm{FP}}$, defined by :

$$\mathbf{O} : \left| \begin{array}{ccc} \mathcal{D}(\mathbf{L}_{\mathrm{FP}}) & \longrightarrow & L^2(]-\infty,m]) \\ f & \longmapsto & \left[\mathbf{O} f : x \mapsto -\frac{1}{2}x^2 f(x) + \frac{1}{2}f''(x)\right] \end{array} \right. .$$

As $\mathbf{O}$ is hermitian, we can use classical results on hermitian operators decomposition.

Moreover, we have a correspondence for the spectrum. The function ϕ is an eigenfunction of $\mathbf{O}$ with eigenvalue μ if and only if $x \mapsto \mathrm{e}^{-\frac{x^2}{2}}\phi$ is an eigenfunction of $\mathbf{L}_{\mathrm{FP}}$ with eigenvalue $\mu + \frac{1}{2}$.

Hilbertian basis of $L^2(]-\infty,m])$

We prove that any function of $L^2(]-\infty,m])$ can be decomposed on an hilbertian basis of eigenvectors of $\mathbf{O}$. Since it is not easy to prove this directly with the operator $\mathbf{O}$, it is classic (see proof of Theorem VIII.20 in [Bré83] for instance) to introduce the operator $\mathbf{T}$ defined as :

$$\mathbf{T} : \left| \begin{array}{ccc} L^2(]-\infty,m]) & \longrightarrow & L^2(]-\infty,m]) \\ g & \longmapsto & f_g, \end{array} \right.$$

where f_g is solution of

$$\left\{ \begin{array}{rcl} \frac{1}{2}f_g'' - \frac{1}{2}x^2 f_g & = & g \\ f_g(m) & = & 0 \\ f_g & \in & H^1(]-\infty,m]). \end{array} \right. \tag{4}$$

To prove that $\mathbf{T}$ is well defined, we need to introduce the following space V and we give some properties on this space. Let us note

$$V := \{v \in L^2(]-\infty,m]) \mid v' \in L^2(]-\infty,m]), v(m) = 0, x \mapsto xv \in L^2(]-\infty,m])\}$$

endowed with the scalar product $\langle u,v\rangle_V = \langle u',v'\rangle_{L^2(]-\infty,m])} + \langle xu,xv\rangle_{L^2(]-\infty,m])}$. The space V is a weighted Sobolev space. With its scalar product, the space V is then an Hilbert space on $\mathbb{R}$.

The following lemma generalizes in our context a classical inequality for Sobolev spaces.

Lemma 3.1. *If $f \in V$, there exists a constant $C \in \mathbb{R}$ such that*

$$\|f\|_{L^2(]-\infty,m])} \leqslant C \|f\|_V . \tag{5}$$

Proof : If $m \geqslant -1$, we have

$$\begin{aligned} \|f\|^2_{L^2(]-\infty,m])} &\leqslant \int_{-\infty}^{-1} f^2 x^2 \times \underbrace{\frac{1}{x^2}}_{\leqslant 1} \mathrm{d}x + \underbrace{\tilde{C}\|f'\|^2_{L^2([-1,m])}}_{\text{Poincaré's inequality because } f(m)=0} \\ &\leqslant C\|f\|_V^2 . \end{aligned}$$

If $m < -1$, it is easy to adapt the previous proof . ■

We prove that the operator $\mathbf{T}$ is well defined thanks to the Lax-Milgram theorem. For a fixed function $g \in L^2(]-\infty,m])$, the weak formulation of (4) is, for $\varphi \in H^1_0(]-\infty,m])$,

$$\left\{ \begin{array}{rcl} a(f,\varphi) & = & l(\varphi) \\ f(m) & = & 0 \\ f & \in & H^1(]-\infty,m]) \end{array} \right. \tag{6}$$

with $a(f,\varphi) = -\frac{1}{2}\left(\int_{-\infty}^m f'\varphi' + \int_{-\infty}^m x^2 f\varphi\right)$ et $l(\varphi) = \int_{-\infty}^m g\varphi$.

We check the hypothesis of the Lax-Milgram's theorem. The application a is clearly bilinear and coercive. It is continuous thanks to the Cauchy-Schwarz inequality. Finally, (5) gives that l is a linear continuous form on V.

The Lax-Milgram's theorem allows to conclude that there exists an unique solution $f \in V$ to $a(f,\varphi) = l(\varphi)$ and then the operator $\mathbf{T}$ is well defined.

Proposition 3.2. *The space $L^2(]-\infty,m])$ admits an hilbertian basis of eigenvectors of $\mathbf{O}$ for the scalar product associated to the L^2-norm.*

Proof : First, we prove that $L^2(]-\infty,m])$ admits an hilbertian basis of eigenvectors of $\mathbf{T}$. We prove that the operator $\mathbf{T}$ is self-adjoint and compact so that we have an hilbertian basis.

The operator T is self-adjoint. It is just an integration by parts.

The operator T is compact. To prove that the operator $\mathbf{T}$ is linear, we use the uniqueness of the solution implied by Lax-Milgram's theorem. Then, $\mathbf{T}(g) = f_g \in V$ and with (5)

$$\|\mathbf{T}(g)\|_{L^2(]-\infty,m])} = \|f_g\|_{L^2(]-\infty,m])} \leqslant C \|f_g\|_V .$$

However, multiplying (4) by f_g and integrating, we have

$$a(f_g, f_g) = l(f_g) \Rightarrow \frac{1}{2}\|f_g\|_V^2 \leqslant C\|f_g\|_V \Rightarrow \|f_g\|_V \leqslant 2C.$$

Then, $\mathbf{T}$ is continuous. The compacity of $\mathbf{T}$ follows from the compact injection of V in $L^2(]-\infty, m])$ (this can be proved using Corollary IV.26 in [Bré83]).

Conclusion $\mathbf{T}$ is a self-adjoint compact operator. Then, by Theorem VI.11. in [Bré83], as $L^2(]-\infty, m])$ is a separable space, $L^2(]-\infty, m])$ admits an hilbertian basis of eigenvectors of $\mathbf{T}$ for the scalar product of $L^2(]-\infty, m])$.

However, the eigenvectors of $\mathbf{T}$ are the eigenvectors of $\mathbf{O}$ with inverse eigenvalues and we can conclude the proof. ■

Eigenvectors of O

Let ϕ be an eigenvector of $\mathbf{O}$ with eigenvalue μ. We have

$$\frac{1}{2}\phi''(x) - \frac{1}{2}x^2\phi(x) = \mu\phi(x).$$

We then apply the variable change $z = x\sqrt{2}$ and $h : z \mapsto \phi\left(\frac{z}{\sqrt{2}}\right) = \phi(x)$. The equation for h is

$$h''(z) + \left(-\frac{1}{4}z^2 - \mu\right)h(z) = 0.$$

As indicated in Appendix (see Section 6.1), the solutions for this equation can be written as $h = \alpha h_1 + \beta h_2$ where $h_1(z) = D_{-\mu-\frac{1}{2}}(z)$ and $h_2(z) = D_{-\mu-\frac{1}{2}}(-z)$ and D is the parabolic cylinder function defined in (26). Unlike h_1, h_2 is integrable on $-\infty$.

Then, we take $\phi : x \mapsto D_{-\mu-\frac{1}{2}}\left(-x\sqrt{2}\right)$.

As $\phi \in \mathcal{D}(\mathbf{L}_{\mathrm{FP}})$, we must have $\phi(m) = 0$, i.e.

$$D_{-\mu-\frac{1}{2}}\left(-m\sqrt{2}\right) = 0. \tag{7}$$

Eigenvalues of O

Thanks to the previous paragraph, we have $\mathrm{Sp}(\mathbf{O}) = \left\{-\nu - \frac{1}{2} \mid D_\nu(-m\sqrt{2}) = 0\right\}$. We prove that the spectrum of $\mathbf{O}$ is included in $\left]-\infty, -\frac{1}{2}\right]$.

We recall that

$$\langle \mathbf{O}f, f\rangle_{L^2(]-\infty,m])} = -\frac{1}{2}\left(\int_{-\infty}^{m} x^2 f^2(x)\,\mathrm{d}x + \int_{-\infty}^{m} f'^2(x)\,\mathrm{d}x\right).$$

Let us study $\mathrm{Sp}(\mathbf{O})$ for the different values of m.

• **if** $m \leqslant -1$ **:** We easily obtain $\langle \mathbf{O}f, f\rangle_{L^2(]-\infty,m])} < -\frac{1}{2}\langle f, f\rangle$ and so $\mathrm{Sp}(\mathbf{O}) \subset \left]-\infty, -\frac{1}{2}\right[$.

• **if** $-1 < m \leqslant 0$ **:** We have

$$\begin{aligned}\int_{-\infty}^{m} f^2 &= \int_{-\infty}^{-1} f^2 + \int_{-1}^{m} f^2 \\ &\leqslant \int_{-\infty}^{-1} f^2 x^2 \,\mathrm{d}x + \int_{-1}^{m} \left(\int_x^m f'(t)\,\mathrm{d}t\right)^2 \mathrm{d}x \\ &\overset{\text{C.S.}}{\leqslant} \int_{-\infty}^{-1} f^2 x^2 \,\mathrm{d}x + \|f'\|^2_{L^2(]-\infty,m])} \int_{-1}^{m} (m-x)^2 \,\mathrm{d}x \\ &= \int_{-\infty}^{-1} f^2 x^2 \,\mathrm{d}x + \|f'\|^2_{L^2(]-\infty,m])} \frac{(m+1)^3}{3},\end{aligned}$$

$$\langle f, f\rangle < -2\langle \mathbf{O}f, f\rangle_{L^2(]-\infty,m])}.$$

Then, if $-1 < m \leqslant 0$, $\langle \mathbf{O}f, f\rangle_{L^2(]-\infty,m])} < -\frac{1}{2}\langle f, f\rangle$ and so $\mathrm{Sp}(\mathbf{O}) \subset \left]-\infty, -\frac{1}{2}\right[$.

• **if** $m > 0$ **:** Let μ be an eigenvalue of $\mathbf{O}$. Using a proof by contradiction, let us assume that $\mu \geqslant -\frac{1}{2}$. Then, by (7) and formula (19) in Appendix, as $H_{-\mu-\frac{1}{2}}(-m) = 0$, we would have

$$\sum_{i=0}^{+\infty} \frac{\Gamma\left(\frac{i+\mu+\frac{1}{2}}{2}\right)}{i!}(2m)^i = 0.$$

This is absurd because all the terms are non-negative. Then, we have necessarily $\mu < -\frac{1}{2}$ when $m > 0$.

Finally, for all $m \in \mathbb{R}$, $\mathrm{Sp}(\mathbf{O}) \subset \left]-\infty, -\frac{1}{2}\right[$ and hence $\left\{\nu \mid D_\nu(-m\sqrt{2}) = 0\right\} \subset \mathbb{R}_+^*$.

Expression of p

We have found the eigenvectors of $\mathbf{O}$ namely the functions $x \mapsto D_{\nu_{n,m}}\left(-x\sqrt{2}\right)$ with $(\nu_{n,m})_{n\in\mathbb{N}^*}$ the ordered positive roots of $\nu \mapsto D_\nu\left(-m\sqrt{2}\right)$.

With a normalisation, we finally have an hilbertian basis $(\tilde{e}_n)_{n\in\mathbb{N}^*}$ of $L^2\left(]-\infty, m]\right)$ with the scalar product of $L^2\left(]-\infty, m]\right)$. It is defined by

$$\tilde{e}_n : x \mapsto \frac{D_{\nu_{n,m}}\left(-x\sqrt{2}\right)}{\sqrt{\int_{-\infty}^{m} D^2_{\nu_{n,m}}\left(-y\sqrt{2}\right)\mathrm{d}y}}.$$

Now, it remains to go back to $\mathbf{L}_{\mathrm{FP}}$ to find the form of p.

Let us note $\tilde{\varphi}_n(x) := \mathrm{e}^{-\frac{x^2}{2}}\,\tilde{e}_n(x)$. We recall that $\tilde{\varphi}_n$ is an eigenfunction of $\mathbf{L}_{\mathrm{FP}}$ with eigenvalue $-\nu_{n,m}$.

It is easy to prove that $(\tilde{\varphi}_n)_{n\in\mathbb{N}^*}$ is an hilbertian basis of $L^2(]-\infty, m], \mathrm{e}^{x^2/2})$ with the scalar product

$$\langle f, g\rangle_0 = \int_{-\infty}^{m} \mathrm{e}^{x^2} f(x)g(x)\,\mathrm{d}x.$$

We look for a solution p in $L^2(]-\infty, m], \mathrm{e}^{x^2/2})$. We can then decompose $x \mapsto p(m, x, t)$ on the basis of $(\tilde{\varphi}_n)_{n\in\mathbb{N}^*}$:

$$p(m, x, t) = \sum_{n=1}^{\infty} c_n(t)\tilde{\varphi}_n(x).$$

However,

$$p(m, x, 0) = \sum_{n=1}^{\infty} c_n(0)\tilde{\varphi}_n(x).$$

Then,

$$\begin{aligned} c_n(0) &= \langle p(m,\cdot,0), \tilde{\varphi}_n \rangle_0 \\ &= \int_{-\infty}^{m} \mathrm{e}^{z^2}\, p(m,z,0)\tilde{\varphi}_n(z)\,\mathrm{d}z \\ &= \frac{1}{\sqrt{\int_{-\infty}^{m} D^2_{\nu_{n,m}}\left(-y\sqrt{2}\right)\mathrm{d}y}} \int_{-\infty}^{m} \mathrm{e}^{\frac{z^2}{2}}\, p(m,z,0) D_{\nu_{n,m}}\left(-z\sqrt{2}\right)\mathrm{d}z. \end{aligned}$$

It remains to determinate $c_n(t)$.

On one hand, $\dfrac{\partial p}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n'(t)\tilde{\varphi}_n(x)$.

On the other hand, $\dfrac{\partial p}{\partial t}(x,t) = \mathbf{L}_{\mathrm{FP}} p(m,x,t) = \sum_{n=1}^{\infty} c_n(t)\mathbf{L}_{\mathrm{FP}}\tilde{\varphi}_n(x) = -\sum_{n=1}^{\infty} c_n(t)\nu_{n,m}\tilde{\varphi}_n(x)$.

By uniqueness of the decomposition, we have

$$c_n'(t) = -\nu_{n,m} c_n(t).$$

So

$$c_n(t) = c_n(0)\,\mathrm{e}^{-\nu_{n,m}t}.$$

Finally,

$$\begin{aligned} p(m,x,t) &= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{\int_{-\infty}^{m} D^2_{\nu_{n,m}}\left(-y\sqrt{2}\right)\mathrm{d}y}} \int_{-\infty}^{m} \mathrm{e}^{\frac{z^2}{2}}\, p(m,z,0) D_{\nu_{n,m}}\left(-z\sqrt{2}\right)\mathrm{d}z \right) \\ &\qquad \times \mathrm{e}^{-\nu_{n,m}t}\,\mathrm{e}^{-\frac{x^2}{2}}\, \frac{D_{\nu_{n,m}}\left(-x\sqrt{2}\right)}{\sqrt{\int_{-\infty}^{m} D^2_{\nu_{n,m}}\left(-y\sqrt{2}\right)\mathrm{d}y}} \\ &= \mathrm{e}^{-\frac{x^2}{2}} \sum_{n=1}^{\infty} \left(\int_{-\infty}^{m} \mathrm{e}^{\frac{z^2}{2}}\, p(m,z,0) D_{\nu_{n,m}}\left(-z\sqrt{2}\right)\mathrm{d}z \right) \mathrm{e}^{-\nu_{n,m}t}\, \frac{D_{\nu_{n,m}}\left(-x\sqrt{2}\right)}{\int_{-\infty}^{m} D^2_{\nu_{n,m}}\left(-y\sqrt{2}\right)\mathrm{d}y}. \end{aligned}$$

We can conclude with Proposition 6.3.

4 Law of the supremum and its joint density with the process

Natural extensions of this work is to check or extend the known results. First, we express the cumulative distribution function of the supremum. Then, we find PDE for the joint density of the process and its supremum.

4.1 Cumulative distribution function of the supremum

From Equation (3), it is easy to obtain the law of the supremum. The result is consistent with the law obtained using the link between the supremum of a process X and the hitting time and the expression in [APP05].

Proposition 4.1 (Supremum law)**.** *For $m \in \mathbb{R}$ and $t \in \mathbb{R}_+^*$, the law of the supremum when $X_0 = x_0$ is given by*

$$\mathbb{P}_{x_0}\left(\sup_{0 \leqslant s \leqslant t} X_s < m \right) = -\,\mathrm{e}^{\frac{k(x_0^2 - m^2)}{2}} \sum_{n=1}^{\infty} \mathrm{e}^{-k\nu_{n,m}t}\, \frac{D_{\nu_{n,m}}\left(-x_0\sqrt{2k}\right)}{\nu_{n,m}\partial_\nu D_{\nu_{n,m}}(-m\sqrt{2k})}, \tag{8}$$

where $(\nu_{n,m})_{n\in\mathbb{N}^*}$ is defined as in Theorem 2.1.

Proof : It is just integration and the application of Proposition 6.2. ■

Remark 4.2. *Using the hitting time density named $p^{(k)}_{x_0\to m}$ in [APP05] (Eq. 31) and the link between hitting time and supremum, we can also find the expression* (8)*. Indeed,*

$$\mathbb{P}_{x_0}\left(\sup_{s \leqslant t} X_s < m \right) = \mathbb{P}_{x_0}(T_m > t) = \int_t^{+\infty} p^{(k)}_{x_0\to m}(s)\,\mathrm{d}s.$$

4.2 Results on the joint density

With the expression of the joint density / distribution, we can find a system of PDEs satisfied by the joint density $\partial_m p$ of the process and its supremum. We take here $k = 1$.

Proposition 4.3. *In the deterministic case where $X_0 = x_0$, the density p satisfies the following PDEs :*

$$\forall m \in \mathbb{R}, \forall t \in \mathbb{R}_+^*, \begin{cases} -\partial_t \partial_m p(m,x,t) - \partial_x(x\partial_m p(m,x,t)) + \frac{1}{2}\partial_{xx}^2 \partial_m p(m,x,t) = 0, \qquad x < m, & (9) \\ -m\partial_m p(m,m,t) - \frac{1}{2}\partial_m^2 p(m,m,t) - \partial_x \partial_m p(m,m,t) = 0, & (10) \\ -m\partial_m p(m,m,t) + \partial_t \partial_m \mathbb{P}_{x_0}\left(\sup_{0\leqslant s\leqslant t} X_s \leqslant m\right) - \frac{1}{2}\partial_x \partial_m p(m,m,t) = 0. & (11) \end{cases}$$

Proof : For this proof, it is more convenient to use the Hermite function H instead of the parabolic cylinder function D.
With the regularity of H (see Appendix) and the regularity of $m \mapsto \nu_{n,m}$ (see Proposition 6.4), we can prove that the function p is regular with respect to x, m and t. Moreover, the order of the further partial derivatives does not matter.
Hence, from (1), we have the equation (9).
For the other equations, let us write the partial derivatives of p.
We denote

$$h_n(m,t) := \frac{\mathrm{e}^{-\nu_{n,m}t}\, H_{\nu_{n,m}}(-x_0)}{\nu_{n,m} H_{\nu_{n,m}-1}(-m)\partial_\nu H_{\nu_{n,m}}(-m)}.$$

Then, we have :

$$p(m,x,t) = -\,\mathrm{e}^{m^2-x^2} \sum_{n=1}^{\infty} h_n(m,t) H_{\nu_{n,m}}(-x),$$

and

$$\mathbb{P}_{x_0}\left(\sup_{0\leqslant s\leqslant t} X_s \leqslant m\right) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{h_n(m,t)}{\nu_{n,m}} \partial_x H_{\nu_{n,m}}(-m).$$

We denote in the following $\nu'_{n,m} = \partial_m \nu_{n,m}$ (see Proposition 6.4). We have

$$\partial_m p(m,x,t) = -\,\mathrm{e}^{m^2-x^2}\left[\sum_{n=1}^{\infty} h_n(m,t)\left(2mH_{\nu_{n,m}}(-x) + \partial_m\left(H_{\nu_{n,m}}(-x)\right)\right) + \sum_{n=1}^{\infty} \partial_m h_n(m,t) H_{\nu_{n,m}}(-x)\right]. \tag{12}$$

Then, with Proposition 6.6, we have

$$\partial_m p(m,m,t) = -\sum_{n=1}^{\infty} h_n(m,t)\partial_x H_{\nu_{n,m}}(-m) \tag{13}$$

and

$$\partial_m \partial_x p(m,m,t) = \sum_{n=1}^{\infty} h_n(m,t)\left(4m\partial_x H_{\nu_{n,m}}(-m) + \nu'_{n,m}\left(\partial_\nu \partial_x H\right)_{\nu_{n,m}}(-m)\right) + \sum_{n=1}^{\infty} \partial_m h_n(m,t)\partial_x H_{\nu_{n,m}}(-m). \tag{14}$$

With (22), we have

$$\partial_m \partial_t\left(\mathbb{P}\left(\sup X_s \leqslant m\right)\right) = \frac{1}{2}\sum_{n=1}^{\infty} h_n(m,t)\left[2m\partial_x H_{\nu_{n,m}}(-m) + \nu'_{n,m}\left(\partial_\nu \partial_x H\right)_{\nu_{n,m}}(-m)\right] \tag{15}$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} \partial_m h_n(m,t)\partial_x H_{\nu_{n,m}}(-m). \tag{16}$$

Equation (12) and Proposition 6.7 imply that

$$\partial_{m^2}^2 p(m,m,t) = \sum_{n=1}^{\infty} h_n(m,t)\left[-6m\partial_x H_{\nu_{n,m}}(-m) - 2\nu'_{n,m}\left(\partial_\nu \partial_x H\right)_{\nu_{n,m}}(-m)\right] \tag{17}$$

$$- 2\sum_{n=1}^{\infty} \partial_m h_n(m,t)\partial_x H_{\nu_{n,m}}(-m). \tag{18}$$

Combining (13), (14) and (17), we obtain (10).
Combining (13), (15) and (14), we obtain (11). ■

Remark 4.4. *It is Theorem 3.1 in [CP17]. This theorem is given for processes written as $X_t = A(X_t) + B_t$, where A is a bounded function of class $\mathcal{C}^1$. Even if we are not under this hypothesis (for an OU process, $A(x) = -kx$), the theorem is verified.*

5 Acknowledgements

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6 Appendix

6.1 Definitions and well-known properties on Hermite functions and Parabolic cylinder functions

We recall here the definitions and some known results about parabolic cylinder functions and Hermite functions. These results can be found in chapters 4 and 10 of [LS72] and in Section 7 of [APP05].

Hermite functions For $\nu \in \mathbb{R}$, the Hermite function H_ν is defined, for all $x \in \mathbb{R}$ as

$$H_\nu(x) = \frac{1}{2\Gamma(-\nu)} \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!} \Gamma\left(\frac{i-\nu}{2}\right) (2x)^i. \tag{19}$$

The Hermite function satisfies the following relations:

$$\partial_x H_\nu(x) = 2\nu H_{\nu-1}(x), \tag{20}$$

$$H_{\nu+1}(x) - 2xH_\nu(x) + 2\nu H_{\nu-1}(x) = 0. \tag{21}$$

Combining (20) and (21), we have :

$$\partial^2_{x^2} H_\nu(x) = -2\nu H_\nu(x) + 2x\partial_x H_\nu(x). \tag{22}$$

When $\mathrm{Re}(\nu) > -1$,

$$H_\nu(x) = \frac{2\,\mathrm{e}^{x^2}}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2} (2t)^\nu \cos\left(2xt - \frac{\nu\pi}{2}\right) \mathrm{d}t. \tag{23}$$

In particular, we have :

$$H_\nu(0) = \frac{2^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}. \tag{24}$$

Then, for $n \in \mathbb{N}$,

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \qquad H_{2n+1}(0) = 0. \tag{25}$$

Remark 6.1 (Zeros of the function $\nu \mapsto H_\nu(0)$)**.** *The zeros $(\nu_{n,0})_n$ of the function $\nu \mapsto H_\nu(0)$ are known. They are the positive odd integers. Indeed,* (24) *indicates that $H_\nu(0) \neq 0$ when $\nu \notin \mathbb{N}$. Therefore, the zeros are among $\mathbb{N}$. Equation* (25) *allows to conclude.*

Parabolic cylinder function The equation

$$y''(x) + \left(-\frac{x^2}{4} + \frac{1}{2} + \nu\right) y(x) = 0 \tag{26}$$

admits two real solutions. The parabolic cylinder function D_ν is defined as the solution of (26) with the property that $y(x) \underset{x\to+\infty}{\sim} x^\nu\, \mathrm{e}^{-x^2/4}$. The other independent solution of (26) is $x \mapsto D_\nu(-x)$.

We have the following relation

$$\partial_x D_\nu(x) = \nu D_{\nu-1}(x) - \frac{x}{2} D_\nu(x). \tag{27}$$

When $\mathrm{Re}(\nu) > -1$,

$$D_\nu(x\sqrt{2}) = \frac{2^{\frac{\nu}{2}+1}\, \mathrm{e}^{\frac{x^2}{2}}}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2}\, t^\nu \cos\left(2xt - \frac{\nu\pi}{2}\right) \mathrm{d}t. \tag{28}$$

We have, for $n \in \mathbb{N}$,

$$\int_{-\infty}^0 D_n^2\left(-x\sqrt{2}\right) \mathrm{d}x = n!\frac{\sqrt{\pi}}{2}. \tag{29}$$

The parabolic cylinder function and the Hermite function are linked by the following relation :

$$D_\nu(x\sqrt{2}) = 2^{-\frac{\nu}{2}}\, \mathrm{e}^{-\frac{x^2}{2}}\, H_\nu\left(x\right).$$

6.2 Useful results

In this part, we note ν_α a zero of $\nu \mapsto D_\nu(-\alpha\sqrt{2})$. This is equivalent to a zero of $\nu \mapsto H_\nu(-\alpha)$.
Thanks to (23), we have, for $x \in \mathbb{R}$ and when $\mathrm{Re}(\nu) > -1$,

$$\partial_x H_\nu(x) = 2xH_\nu(x) - \frac{2\,\mathrm{e}^{x^2}}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2} (2t)^{\nu+1} \sin\left(2xt - \frac{\nu\pi}{2}\right) \mathrm{d}t, \tag{30}$$

$$\partial_\nu H_\nu(x) = \frac{2\,\mathrm{e}^{x^2}}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2} (2t)^{\nu} \left[\ln(2t) \cos\left(2xt - \frac{\nu\pi}{2}\right) + \frac{\pi}{2} \sin\left(2xt - \frac{\nu\pi}{2}\right)\right] \mathrm{d}t, \tag{31}$$

$$\partial_x\partial_\nu H_\nu(x) = 2x\partial_\nu H_\nu(x) - \frac{2\,\mathrm{e}^{x^2}}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-t^2} (2t)^{\nu+1} \left[\ln(2t) \sin\left(2xt - \frac{\nu\pi}{2}\right) - \frac{\pi}{2} \cos\left(2xt - \frac{\nu\pi}{2}\right)\right] \mathrm{d}t. \tag{32}$$

Thanks to (26), we can prove by direct induction that $x \mapsto D_\nu(x)$ is a $\mathcal{C}^\infty(\mathbb{R})$ function. Then, so is $x \mapsto H_\nu(x)$. With the previous expressions, we can prove that all the derivatives with respect to ν exist and are continuous if $\nu \in \mathbb{R}_+$. Then, $\nu \mapsto H_\nu(x)$ is a $\mathcal{C}^\infty(\mathbb{R}_+)$ function. The cross partial derivatives with respect to ν and x also exist and are continuous. Then, we have the symmetry of second derivatives (Schwarz's theorem).
We give here several results useful in the proofs.

Proposition 6.2. *For all $\nu, \alpha \in \mathbb{R}$, we have :*

$$\int_{-\infty}^{\alpha} \mathrm{e}^{-\frac{x^2}{2}} D_\nu(-x\sqrt{2})\,\mathrm{d}x = \frac{\mathrm{e}^{-\frac{\alpha^2}{2}}}{\sqrt{2}} D_{\nu-1}(-\alpha\sqrt{2}).$$

Proof : We prove the relation on H. Let us take $\nu, \alpha \in \mathbb{R}$. We have with (21) et (20) :

$$\begin{aligned}\int_{-\infty}^{\alpha} \mathrm{e}^{-x^2} H_\nu(-x)\,\mathrm{d}x &= -\int_{-\infty}^{\alpha} \mathrm{e}^{-x^2} \left(2xH_{\nu-1}(-x) + 2(\nu-1)H_{\nu-2}(-x)\right)\mathrm{d}x \\ &= \int_{-\infty}^{\alpha} \partial_x \left(\mathrm{e}^{-x^2} H_{\nu-1}(-x)\right)\mathrm{d}x.\end{aligned}$$

Finally,

$$\int_{-\infty}^{\alpha} \mathrm{e}^{-x^2} H_\nu(-x)\,\mathrm{d}x = \mathrm{e}^{-\alpha^2} H_{\nu-1}(-\alpha).$$

■

Proposition 6.3. *For all $\alpha \in \mathbb{R}$,*

$$\int_{-\infty}^{\alpha} D^2_{\nu_\alpha}(-x\sqrt{2})\,\mathrm{d}x = -\frac{\nu_\alpha}{\sqrt{2}} D_{\nu_\alpha-1}(-\alpha\sqrt{2})\partial_\nu D_{\nu_\alpha}(-\alpha\sqrt{2}).$$

Proof : We apply the method given in [Sle68].
Let set

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (\nu, x) & \longmapsto & D_\nu(-x\sqrt{2}) \end{array} \right. .$$

Then, thanks to (26), the function f is solution of the following equation, for all $\nu, x \in \mathbb{R}$:

$$\partial^2_{x^2} f(\nu, x) + \left(-x^2 + 1 + 2\nu\right) f(\nu, x) = 0.$$

We derive this equation with respect to ν and using the regularity of H :

$$\partial^2_{x^2}\partial_\nu f(\nu, x) + 2f(\nu, x) + \left(-x^2 + 1 + 2\nu\right) \partial_\nu f(\nu, x) = 0.$$

We multiply by f

$$f(\nu, x)\partial^2_{x^2}\partial_\nu f(\nu, x) + 2f^2(\nu, x) + \left(-x^2 + 1 + 2\nu\right) f(\nu, x)\partial_\nu f(\nu, x) = 0.$$

We integrate with respect to x :

$$\begin{aligned}\int_{-\infty}^{\alpha} f^2(\nu,x)\,\mathrm{d}x &= -\frac{1}{2}\left[\int_{-\infty}^{\alpha} f(\nu,x)\partial^2_{x^2}\partial_\nu f(\nu,x)\,\mathrm{d}x + \int_{-\infty}^{\alpha}\left(-x^2+1+2\nu\right) f(\nu,x)\partial_\nu f(\nu,x)\,\mathrm{d}x\right]\\ &= -\frac{1}{2}\left[[f(\nu,x)\partial_x\partial_\nu f(\nu,x)]_{-\infty}^{\alpha} - \int_{-\infty}^{\alpha}\partial_x f(\nu,x)\partial_x\partial_\nu f(\nu,x)\,\mathrm{d}x - \int_{-\infty}^{\alpha}\partial^2_{x^2} f(\nu,x)\partial_\nu f(\nu,x)\,\mathrm{d}x\right]\\ &= -\frac{1}{2}\left[f(\nu,\alpha)\partial_x\partial_\nu f(\nu,\alpha) - \int_{-\infty}^{\alpha}\partial_x\left\{\partial_x f(\nu,x)\partial_\nu f(\nu,x)\right\}\mathrm{d}x\right]\\ &= -\frac{1}{2}\left[f(\nu,\alpha)\partial_x\partial_\nu f(\nu,\alpha) - [\partial_x f(\nu,x)\partial_\nu f(\nu,x)]_{-\infty}^{\alpha}\right]\\ &= -\frac{1}{2}\left[f(\nu,\alpha)\partial_x\partial_\nu f(\nu,\alpha) - \partial_x f(\nu,\alpha)\partial_\nu f(\nu,\alpha)\right].\end{aligned}$$

Then, writing using the function D,

$$\int_{-\infty}^{\alpha} D^2_{\nu_\alpha}(-x\sqrt{2})\,\mathrm{d}x = \frac{-\sqrt{2}}{2}\partial_x D_{\nu_\alpha}(-\alpha\sqrt{2})\partial_\nu D_{\nu_\alpha}(-\alpha\sqrt{2}).$$

The conclusion follows with (27). ■

Proposition 6.4. *For all $\alpha\in\mathbb{R}$, there exists $g:\alpha\mapsto\nu_\alpha$, continuously differentiable on a neighbourhood of α. We note $\nu'_\alpha = g'(\alpha)$. Then*

$$\nu'_\alpha = \frac{\partial_x H_{\nu_\alpha}(-\alpha)}{\partial_\nu H_{\nu_\alpha}(-\alpha)}.$$

Proof : Let set

$$f:\left|\begin{array}{ccc}\mathbb{R}^2 & \longrightarrow & \mathbb{R}\\ (\nu,x) & \longmapsto & D_\nu(-x\sqrt{2})\end{array}\right. .$$

The function f is a continuously differentiable function. We have $f(\nu_\alpha,\alpha)=0$. We also have $\partial_\nu f(\nu_\alpha,\alpha)\neq 0$ because of Proposition 6.3 and the fact that $\displaystyle\int_{-\infty}^{\alpha} D^2_\nu(-x\sqrt{2})\,\mathrm{d}x \neq 0$.
Then, thanks to the implicit function theorem, there exists an open set U of $\mathbb{R}$ containing α such that there exists an unique continuously differentiable function $g:U\mapsto\mathbb{R}$ such that $g(\alpha)=\nu_\alpha$ and for all $x\in U$, $f(g(x),x)=0$. Moreover, the partial derivative of g in U is given by

$$g'(x) = -\frac{\partial_x f(g(x),x)}{\partial_\nu f(g(x),x)}.$$

■

Proposition 6.5. *We have*

$$\begin{aligned}\partial_m(H_{\nu_m}(x)) = \mathrm{e}^{x^2-m^2}\,\partial_x H_{\nu_m}(-m) - \frac{4\nu'_m\,\mathrm{e}^{x^2}}{\sqrt{\pi}}\int_0^{\infty}\mathrm{e}^{-t^2}(2t)^{\nu_m}\sin((m+x)t)\times\\ \left[\ln(2t)\sin\left((x-m)t-\frac{\nu_m\pi}{2}\right) - \frac{\pi}{2}\cos\left((x-m)t-\frac{\nu_m\pi}{2}\right)\right]\mathrm{d}t.\end{aligned}$$

Proof : Using Proposition 6.4 twice and (30) and (31), we have :

$$\begin{aligned}\partial_m(H_{\nu_m}(x)) &= \nu'_m\partial_\nu H_{\nu_m}(x)\\ &= \nu'_m\partial_\nu H_{\nu_m}(x) + \mathrm{e}^{x^2-m^2}\,\partial_x H_{\nu_m}(-m) - \mathrm{e}^{x^2-m^2}\,\partial_x H_{\nu_m}(-m)\\ &= \nu'_m\partial_\nu H_{\nu_m}(x) + \mathrm{e}^{x^2-m^2}\,\partial_x H_{\nu_m}(-m) - \mathrm{e}^{x^2-m^2}\,\nu'_m(\partial_\nu H)_{\nu_m}(-m)\\ &= \mathrm{e}^{x^2-m^2}\,\partial_x H_{\nu_m}(-m) - \frac{4\nu'_m\,\mathrm{e}^{x^2}}{\sqrt{\pi}}\int_0^{\infty}\mathrm{e}^{-t^2}(2t)^{\nu_m}\sin((m+x)t)\times\\ &\qquad\left[\ln(2t)\sin\left((x-m)t-\frac{\nu_m\pi}{2}\right) - \frac{\pi}{2}\cos\left((x-m)t-\frac{\nu_m\pi}{2}\right)\right]\mathrm{d}t.\end{aligned}$$

■

In particular, with $x=-m$, we have the following proposition :

Proposition 6.6. *For all $m \in \mathbb{R}$,*

$$\partial_m(H_{\nu_m}(x))|_{x=-m} = \partial_x H_{\nu_m}(-m).$$

We also have the following proposition :

Proposition 6.7. *We have*

$$\partial^2_{m^2}(H_{\nu_m}(x))|_{x=-m} = 2\nu'_m \partial_\nu \partial_x H_{\nu_m}(-m) + 2m\partial_x H_{\nu_m}(-m).$$

Proof : It is the same kind of calculus as in the proof of Proposition 6.5. We use the formula (32) and the fact that

$$\begin{aligned}\partial^2_{m^2}(H_{\nu_m}(x))|_{x=-m} = &-2m\partial_x H_{\nu_m} + \partial_m\left(\partial_x H_{\nu_m}(-m)\right)\\ &- \frac{2\nu'_m\,\mathrm{e}^{m^2}}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-t^2}(2t)^{\nu_m+1}\left[\ln(2t)\sin\left(-2mt-\frac{\nu_m\pi}{2}\right) - \frac{\pi}{2}\cos\left(-2mt-\frac{\nu_m\pi}{2}\right)\right]\mathrm{d}t.\end{aligned}$$

■

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