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SHARPLY 2-TRANSITIVE GROUPS OF FINITE MORLEY RANK

TUNA ALTINEL, AYŞE BERKMAN, AND FRANK O. WAGNER

ABSTRACT. A near-field of finite Morley rank and characteristic $\neq 2$ is an algebraically closed field. It follows that a sharply 2-transitive permutation group of finite Morley rank of permutation characteristic 3 is the group of affine transformations of an algebraically closed field of characteristic 3.

INTRODUCTION

A *Frobenius group* is a group G together with a malnormal subgroup B , i.e. $B \cap B^g = \{1\}$ for all $g \in G \setminus B$; the group B is called the *Frobenius complement*. A Frobenius group G with Frobenius complement B *splits* if there is a normal subgroup N , called the *Frobenius kernel*, such that $G = N \rtimes B$. All finite Frobenius groups split [10]; moreover the Frobenius kernel is nilpotent [16]. This can be seen as a precursor to the Feit-Thompson theorem on the solubility of groups of odd order, which in turn is the starting point for the classification of the finite simple groups. As Tao has remarked, all known proofs of Frobenius' Theorem use group characters; removing character theory might lead to a new route to the classification of finite simple groups [15].

On the model-theoretic side, the classification of the finite simple groups has inspired Borovik's programme for the resolution of the *Algebraicity Conjecture* by Cherlin and Zilber, which asserts that a simple group of finite Morley rank should be an algebraic group over an algebraically closed field. This has been successful in case there is an infinite elementary abelian 2-subgroup [1]. However, since neither representation nor character theory are available in this context, there is no analogue of the Feit-Thompson Theorem, and there may well exist simple groups of finite Morley rank without involutions (the so-called *degenerate case*), which are impervious to Borovik's approach.

The interest of the study of Frobenius groups of finite Morley rank is thus two-fold: On one hand, following Tao, one hopes to reproduce Frobenius' Theorem in this context, and to climb up the ladder to eventually eliminate degenerate simple groups of finite Morley rank (i.e. prove a finite Morley rank version of the Feit-Thompson Theorem); on the other hand one seeks to develop methods in the finite Morley rank context which might help to find a character-free proof of Frobenius' Theorem for finite groups.

Frobenius groups arise naturally as transitive non-regular permutation groups such that only the identity fixes two distinct points, the Frobenius complement being the stabiliser of a single point. Borovik and Nesin have conjectured that a Frobenius group of finite Morley rank has a nilpotent Frobenius kernel (and in particular splits). A particular case of Frobenius groups are sharply two-transitive permutation groups. It is easy to see that they contain an involution: any permutation

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$g \in G$ exchanging two points x and y must have order 2; moreover g is the only permutation exchanging x and y by sharp 2-transitivity, and if g' is an involution exchanging x' and y' , then $g' = g^h$ for the unique $h \in G$ with $h(x') = x$ and $h(y') = y$. Thus all involutions are conjugate. Moreover, if the group is split, then the Frobenius kernel is abelian [13].

A sharply 2-transitive permutation group is called *standard* if it is the group of affine transformations of some algebraically closed field K , i.e. of the form $K^+ \rtimes K^\times$.

Conjecture 1. [6] An infinite sharply 2-transitive permutation group of finite Morley rank is standard. More precisely:

- (i) A sharply 2-transitive permutation group of finite Morley rank splits.
- (ii) A sharply 2-transitive split permutation group of finite Morley rank is standard.

If the Frobenius complement does not contain an involution, the *permutation characteristic* of G is 2. Otherwise, all products of two distinct involutions are conjugate, and the *permutation characteristic* of G is the order of ij , for any two distinct involutions i and j (or 0, if the order is infinite).

Kerby and Wefelscheid [12] have shown (i) in permutation characteristic 3; Cherlin, Grundhöfer, Nesin and Völklein [8] have shown (ii) in permutation characteristic 0. Note that an infinite sharply 2-transitive group of finite Morley rank is connected, and so is its Frobenius complement [6], i.e. they have no definable subgroups of finite index. We shall show (i) in permutation characteristic 2, and (ii) in permutation characteristic $\neq 2$. In particular, Conjecture 1 holds in permutation characteristic 3.

However, our results do not advance the search for a finite Morley rank version of the Feit-Thompson Theorem, nor address the question of degenerate simple groups of finite Morley rank, since they are based on the study of involutions whose existence is guaranteed by sharp 2-transitivity. They might, however, contribute to the study of the *odd characteristic* case of the Algebraicity Conjecture (where the group contains a Prüfer 2-subgroup), which is still open.

For background material on groups of finite Morley rank the reader should consult [6], in particular Chapters 11.3 and 11.4 on Frobenius and sharply 2-transitive permutation groups.

1. PERMUTATION CHARACTERISTIC 2

In [6] Borovik and Nesin study mostly Frobenius groups whose Frobenius complement contains an involution. The following result elucidates what happens in permutation characteristic 2. It relies on the even case of the Algebraicity Conjecture.

Recall that a definable subgroup $H \leq G$ is *generous* if $\bigcup_{g \in G} H^g$ is generic in G . If G is connected, any two generic subsets have generic intersection. Hence any two generous subgroups can be conjugated to intersect non-trivially. Examples of generous subgroups include a malnormal subgroup B (as $RM(\bigcup_{g \in G} B^g) = RM(G/B) + RM(B) = RM(G)$) and the centralizer of a *decent torus*, i.e. a definable divisible abelian subgroup with dense torsion [7].

Theorem 2. *Let G be a connected Frobenius group with Frobenius complement B . If B does not contain an involution, then G has a normal definable connected subgroup N containing all involutions, such that $N \cap B = \{1\}$.*

Proof. Note that B is definable by [6, Proposition 11.19]. By [5] a 2-Sylow subgroup S of G is infinite. If S has a non-trivial abelian divisible subgroup T , then its *definable hull* $\text{dc}(T)$, the smallest definable group containing T , is a decent torus, its centralizer $C_G(T) = C_G(\text{dc}(T))$ is

generous, and after conjugation intersects B non-trivially. But then $T \leq B$ by malnormality, a contradiction. It follows that S has bounded exponent.

Let N be a definable connected normal subgroup of G . If $N \cap B$ is trivial, then either $G = N \rtimes B$ splits or BN/N is malnormal in G/N by [6, Lemma 11.37]; note that BN/N does not contain an involution. If moreover $N \leq M \trianglelefteq G$ with $BN \cap M = N$, then $B \cap M = B \cap BN \cap M = B \cap N = \{1\}$. We now choose N of maximal Morley rank possible. If $N < M \trianglelefteq G$ with $B \cap M = \{1\}$, then $RM(M/N) \geq RM((mN)^B/N) = RM(B) > 0$. Thus N is maximal definable normal with $B \cap N = \{1\}$; if it contains all involutions of G we are done.

Otherwise we can divide out by N and suppose that G has no definable normal subgroup intersecting B trivially. In particular it has no definable abelian normal subgroup. Hence G has a definable infinite minimal simple normal subgroup N (which may be G itself). Then $N \cap B$ is malnormal, whence generous in N , as is $(N \cap B)^g$ for any $g \in G$. It follows that there is $n \in N$ with $(N \cap B) \cap (N \cap B)^{gn} \neq \{1\}$. Then $gn \in B$ by malnormality, and $G = BN$. Moreover, N contains an involution, and is algebraic by [1]. But an algebraic Frobenius group splits [6, Lemma 11.39]. If M is a Frobenius kernel of N , then $M \cap B = M \cap N \cap B = \{1\}$ and $G = BN = B(B \cap N)K = BK$. Thus $G = M \rtimes B$ splits. \square

Remark 3. A simple B -invariant section of N does not contain an involution. If it did, it would be algebraic by [1], but a simple algebraic group cannot be a Frobenius complement [6, Corollary 11.40].

Corollary 4. *An infinite sharply 2-transitive permutation group of finite Morley rank and permutation characteristic 2 splits.*

Proof. Let G be the group, B its Frobenius complement, and $N \trianglelefteq G$ the definable normal subgroup given by Theorem 2. If $i \in N$ is an involution, then $RM(N) \geq RM(i^B) = RM(B)$. Thus $2RM(B) \leq RM(BN) \leq RM(G) = 2RM(B)$. It follows that $G = N \rtimes B$ splits. \square

2. CENTRALITY OF THE SYLOW 2-SUBGROUP

Theorem 5. *Let G be a connected group of finite Morley rank whose connected definable abelian subgroups are decent tori. Then its 2-Sylow subgroup is connected and central.*

Proof. If A is a decent torus, then $C_G(A)$ is generous in G and A does not have an infinite family of definable subgroups [7]. Moreover, $N_G(A)^0 = C_G(A)^0 = C_G(A)$ by [3].

We claim that a connected soluble subgroup S is abelian: If A is the last non-trivial derived group, then A is central in S . So if N is the second last derived subgroup, N is connected nilpotent, and is the central product of a divisible abelian group with a connected nilpotent group of bounded exponent. If the latter were infinite, it would contain an infinite elementary abelian subgroup, a contradiction. So N is abelian, and $N = A = S$. Thus if A is connected maximal abelian, then $A = C_G(A)$, and A is almost self-normalizing and generous in G .

If A and A' are two non-conjugate maximal abelian connected subgroups of G with maximal intersection I , then in $C_G(I)^0$ both A and A' are generous, and there is a $C_G(I)^0$ -conjugate of A' intersecting A in a generic point outside I , contradicting maximality of I . It follows that all maximal connected abelian subgroups are conjugate. We shall call a maximal connected abelian subgroup of G a *full torus*; a connected subgroup H of G is *full* if it contains a full torus. Consider a non-trivial element $g \in G$. Then $C_G(g)$ is infinite [2] and contains a decent torus T , which we take maximal possible. So $C_G(T)$ is connected, and $g \in C_G(T)$. If $g \notin T$ then gT has an infinite

centralizer C/T in $C_G(T)/T$, which contains an infinite connected abelian subgroup A/T . But then A is again abelian connected, whence a torus, and

$$g \in C_{C_G(T)}(A/T) = C_{C_G(T)}^0(A/T) \leq N_{C_G(T)}(A)^0 = C_G(A).$$

Thus $T < A \leq C_G(g)$, a contradiction. It follows that every element is contained in a torus T . But now T can be extended to a maximal connected abelian subgroup T' , which must be a full torus containing g . This shows that if H is a full connected definable subgroup of G and $g \in H$, there is a full torus $T \leq H$ containing g .

Let S be the connected component of a Sylow 2-subgroup (in short, a 2-Sylow⁰). Suppose for a contradiction that S is not central. As $C_G(S)^0 = N_G(S)^0$, there is $a \in G$ with $S^a \neq S$. Choose two full tori A and A' containing distinct 2-Sylows⁰ $S \neq S'$ such that $A \cap A'$ is maximal possible. Replacing G by $C_G(A \cap A')^0 / (A \cap A')$, we may assume that this intersection is trivial. In particular any two full tori with non-trivial intersection have the same 2-Sylow⁰. So for any $g \in G$, since any two full tori in $C_G(g)^0$ are conjugate and must contain g , there is a unique 2-Sylow⁰ in $C_G(g)^0$, which is also the unique 2-Sylow⁰ in $C_G(g)$.

Consider involutions $i \in S$ and $i' \in S'$. Then i and i' invert the element $m = ii'$. Let S'' be the unique 2-Sylow⁰ in $C_G(m)$. Then i and i' normalize $C_G(m)$, whence also S'' . Now either $S'' \neq S$ or $S'' \neq S'$, and we may assume $S'' \neq S$. Then $i \notin C_G(S'')$, as otherwise $C_G(S'')/\text{dc}(S'')$ would contain an involution, whence a 2-torus, a contradiction.

Suppose $C_{C_G(S'')}(i)$ is infinite. Then it contains a torus T , and there is a full torus $A'' \geq S''T$. Moreover, all full tori in $C_G(i)^0$ contain i , and there is one, say A^* , which also contains T . But $A'' \cap A^* \geq T$ so $S'' = S^*$, contradicting $i \notin C_G(S'')$. As i normalizes $C_G(S'')$ and centralizes only finitely many points, it inverts $C_G(S'')^0 = C_G(S'')$. It follows that $C_G(S'')$ is abelian, and $C_G(S'') = C_G(A'') = A''$; moreover all involutions in $N_G(A'') \setminus A''$ invert A'' and are in the same coset iA'' modulo A'' ; conversely all elements of the coset iA'' are involutions inverting A'' . Note that $m \in C_G(S'') = A''$, and $i \in N_G(S'') = N_G(A'')$; if $j, k \in A''$ are two involutions, they commute with i and invert A . So $ik \in C_G(A) \cap A'' = A \cap A'' = \{1\}$, and a full torus contains a unique involution. For any non-trivial $g \in G$ the centralizer $C_G(g)$ contains a unique 2-Sylow⁰, and hence a unique full torus, which must contain g . It follows that $C_G(g)^0$ is this full torus. So the full tori of G are disjoint, and cover G .

Let N be a minimal normal definable subgroup of G . Then N cannot be abelian, so N is simple by connectedness of G . If N does not contain an involution, then any involution i of G gives rise to an involutive automorphism of N . If $F = C_N(i)$ is the subgroup of fixed points and I the set of points inverted by i , then N decomposes uniquely as $F \cdot I$ and all conjugates of F intersect I trivially by [14]. Hence F is connected, and a torus of N . But the conjugates of F cover N , so $I = \{1\}$ and $N = F$, a contradiction. It follows that N contains involutions, and we may replace G by N .

We obtain a geometry on G whose points are the involutions, and whose lines are the cosets iA for a full torus A and involution $i \in N_G(A)$, i.e. sets $\ell(j) = N_G(A(j))[2] \setminus \{j\}$, where $A = A(j)$ is the unique full torus containing j . Then j is the unique involution commuting with all the involutions of $\ell(j)$. Any two distinct lines $\ell(i)$ and $\ell(j)$ have empty intersection, or intersect in a coset of $A(i) \cap A(j) = \{1\}$ (note that the corresponding 2-Sylows must be distinct), i.e. in a single point. Any two distinct points i, j lie on at least one common line $\ell(k)$, where $A(k) = C_G^0(ij)$. There is a polarity which associates an involution i to a line $\ell(i)$ and vice versa, and which preserves incidence. It follows that any two lines intersect in a unique point, and any two points lie on a unique line. Moreover, there are no isotropic points: $i \notin \ell(i)$ for all i . By Bachman's theorem [4], G is a group of

linear transformations of a 3-dimensional vector space over an interpretable field K which preserve a symmetric bilinear form without isotropic vectors. But this contradicts stability.

It follows that S is central. But then $G/\text{dc}(S)$ has finite Sylow 2-subgroups, which must be trivial by [5]. Thus S is the connected central Sylow 2-subgroup of G . \square

Deloro and Wiscons have recently obtained Theorem 5 as a corollary of a more general theorem on the 2-structure of a connected group of finite Morley rank [9, Corollary B2].

3. PERMUTATION CHARACTERISTIC $\neq 2$

Definition 6. A *near-field* is a skew field, except that the left distributive law $(y+z)x = yx + zx$ need not hold.

Fact 7. [11] *A split sharply 2-transitive permutation group is the group of affine transformations of a near-field; the near-field characteristic is equal to the permutation characteristic.*

Definition 8. The *kernel* $\ker(K)$ of a near-field K is the set of elements with respect to which multiplication is left distributive:

$$\ker(K) = \{x \in K : \forall y, z \in K (y+z)x = yx + zx\}.$$

Remark 9. The prime field of a near-field is contained in the kernel.

Proof. Consider $n = 1 + \dots + 1$, and $y, z \in K$. Then by right distributivity

$$(y+z)n = (y+z) + \dots + (y+z) = (y + \dots + y) + (z + \dots + z) = yn + zn.$$

In characteristic 0 we obtain for $m = 1 \dots + 1 \neq 0$:

$$\begin{aligned} (y+z)m^{-1}n &= (ym^{-1}m + zm^{-1}m)m^{-1}n = (ym^{-1} + zm^{-1})mm^{-1}n \\ &= (ym^{-1} + zm^{-1})n = ym^{-1}n + zm^{-1}n. \quad \square \end{aligned}$$

Note that this does not imply that the prime field is in the centre of K , even if the kernel is finite (and K connected), as conjugation is not an automorphism of K and need not stabilize the kernel.

Theorem 10. *An infinite near-field K of finite Morley rank in characteristic $\neq 2$ is an algebraically closed field.*

Proof. If the kernel is infinite (in particular if $\text{char}(K) = 0$ or $Z(K^\times)$ is infinite), this follows from [6] or [8].

In characteristic $p > 0$, note first that K is additively connected, as for any additive proper subgroup H of finite index the intersection $\bigcap_{x \in K^\times} xH$ is trivial, but equals a finite subintersection, and hence is of finite index, a contradiction. There is thus a unique type of maximal Morley rank, so K^\times is multiplicatively connected as well.

Let A be a definable connected infinite abelian multiplicative subgroup, and M_0 an A -minimal additive subgroup. Then M_0 is additively isomorphic to the additive group of an algebraically closed field K_0 , and A embeds multiplicatively into K_0^\times . In fact, for any $e_0 \in M_0 \setminus \{0\}$ and $a_1, \dots, a_n \in A$ such that $a_1 e_0 + \dots + a_n e_0 = 0$, we have by right distributivity that $a_1^{e_0} + \dots + a_n^{e_0} = 0$, so the addition induced on A by K_0 is the one inherited from K -addition on A^{e_0} . So we might replace A by A^{e_0} , M_0 by $e_0^{-1} M_0$ and e_0 by 1. Then $A \subseteq M_0 = K_0^+$, and field multiplication on K_0 is induced from K on $A \times K_0$, but does not necessarily agree with multiplication from K if the left factor is in $K_0 \setminus A$. In particular, A is a good torus by [17]. By Theorem 5 the centre $Z(K^\times)$ contains the Sylow 2-subgroup, which is infinite in permutation characteristic different from 2. We finish by the first paragraph. \square

Corollary 11. *A sharply 2-transitive group of finite Morley rank and permutation characteristic 3 is the group of affine transformations of an algebraically closed field of characteristic 3.*

Proof. By [12] a sharply 2-transitive permutation group of permutation characteristic 3 splits. Now use Fact 7 and Theorem 10. \square

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TUNA ALTINEL, UNIVERSITÉ DE LYON; UNIVERSITÉ LYON 1; CNRS; INSTITUT CAMILLE JORDAN UMR 5208, 21 AVENUE CLAUDE BERNARD, 69622 VILLEURBANNE-CEDEX, FRANCE
E-mail address: altinel@math.univ-lyon1.fr

AYŞE BERKMAN, MIMAR SINAN GÜZEL SANATLAR ÜNİVERSİTESİ; MSGSÜ MATEMATİK BÖLÜMÜ, ŞİLAHŞÖR CAD. 71, BOMONTI, ŞİŞLİ, İSTANBUL 34380, TURKEY
E-mail address: ayse.berkman@msgsu.edu.tr

FRANK O. WAGNER, UNIVERSITÉ DE LYON; UNIVERSITÉ LYON 1; CNR UMR 5208, INSTITUT CAMILLE JORDAN, 21 AVENUE CLAUDE BERNARD, 69622 VILLEURBANNE-CEDEX, FRANCE
E-mail address: wagner@math.univ-lyon1.fr