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SHARPLY 2-TRANSITIVE GROUPS OF FINITE MORLEY RANK

TUNA ALTINEL, AYŞE BERKMAN AND FRANK O. WAGNER

ABSTRACT. A near-field of finite Morley rank and characteristic $\neq 2$ is an algebraically closed field. It follows that sharply 2-transitive permutation group of finite Morley rank of permutation characteristic 3 is the group of affine transformations of an algebraically closed field of characteristic 3.

INTRODUCTION

A *Frobenius group* is a group G together with a malnormal subgroup B , i.e. $B \cap B^g = \{1\}$ for all $g \in G \setminus B$. These groups arise naturally from transitive non-regular permutation groups such that only 1 fixes two distinct points, taking as B is the stabiliser of a single point. Borovik and Nesin have conjectured that a Frobenius group of finite Morley rank splits, i.e. that there is a normal subgroup N with $G = N \rtimes B$, and moreover that N is nilpotent.

A particular case of Frobenius groups are sharply two-transitive permutation groups. It is easy to see that they contain an involution: any permutation in G exchanging two points must have order 2. We shall call such a group *standard* if it is the group of affine transformations of some algebraically closed field K , i.e. of the form $K^+ \rtimes K^\times$.

Conjecture 1. A sharply 2-transitive permutation group of finite Morley rank is standard.

The conjecture can be decomposed into two parts:

- (1) A sharply 2-transitive permutation group of finite Morley rank splits.
- (2) A sharply 2-transitive split permutation group of finite Morley rank is standard.

If G is a sharply 2-transitive permutation group, then all involutions in G are conjugate. If B does not contain an involution, the *permutation characteristic* of G is 2. Otherwise, all product of two distinct involutions are conjugate, and the *permutation*

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characteristic of G is the order of ij , for any two distinct involutions i and j . For details, the reader may for instance consult [3, Chapters 11.3 and 11.4].

We shall show (1) in permutation characteristic 2 and 3, and (2) in permutation characteristic $\neq 2$. In particular, the conjecture holds in permutation characteristic 3.

1. PERMUTATION CHARACTERISTIC 2

Let G be connected sharply 2-transitive on X , of finite Morley rank, and B the stabiliser of $x \in X$. We want to show that G is split.

Suppose G has a non-trivial proper normal connected subgroup N . If $N \cap B = \{1\}$ (in particular if N is abelian) then $RM(N) \geq RM(n^B) = RM(B)$ for any $n \in N \setminus \{1\}$, and $2RM(B) \leq RM(NB) \leq RM(G) = 2RM(B)$. Thus $G = N \rtimes B$, and G is split. So we may assume G semisimple, and $N \cap B$ non-trivial. We may assume that N is minimal possible. Then N is simple, $G = NB$, and $(N, N \cap B)$ is a Frobenius group.

If G is of characteristic 2 and $i \notin N$ is an involution, then $G/N \cong B/(B \cap N)$ contains an involution, a contradiction. By [2] a 2-Sylow subgroup S of N is infinite, and $S^0 = B * T$. If $T \neq 1$, then $C_N(T)$ is generous in N and intersects $\bigcup_{n \in N} (B \cap N)^n$, a contradiction. Thus N is algebraic by [1], but an algebraic Frobenius group splits. The same proof works if G itself is simple of characteristic 2.

2. PERMUTATION CHARACTERISTIC $\neq 2$

Definition 2. A *near-field* is a skew field, except that the left distributive law $(y + z)x = yx + zx$ need not hold.

Definition 3. A *near-domain* is a near-field, except that additively it only is a *loop*, i.e.

- $x + 0 = 0 + x = x$ for all x ,
- $a + x = b$ has a unique solution for all a, b ,
- $x + a = b$ has a unique solution for all a, b

which satisfies in addition

- $a + b = 0$ implies $b + a = 0$, and
- for all a, b there is $d_{a,b}$ such that for all x one has $a + (b + x) = (a + b) + d_{a,b}x$.

Fact 4. [5] *A strictly 2-transitive permutation group is the group of affine transformations of a near-domain; it is split if and only if the near-domain is a near-field.*

Definition 5. The *kernel* $\ker(K)$ of a near-domain K is the set of elements with respect to which multiplication is left distributive:

$$\ker(K) = \{x \in K : \forall y, z \in K (y + z)x = yx + zx\}.$$

Remark 6. The prime field of a near-field is contained in the kernel.

Proof. Consider $n = 1 + \cdots + 1$, and $y, z \in K$. Then by right distributivity

$$(y + z)n = (y + z) + \cdots + (y + z) = (y + \cdots + y) + (z + \cdots + z) = yn + zn.$$

In characteristic 0 we obtain for $m = 1 \cdots + 1 \neq 0$:

$$\begin{aligned} (y + z)m^{-1}n &= (ym^{-1}m + zm^{-1}m)m^{-1}n = (ym^{-1} + zm^{-1})mm^{-1}n \\ &= (ym^{-1} + zm^{-1})n = ym^{-1}n + zm^{-1}n. \quad \square \end{aligned}$$

Note that this does not imply that the prime field is in the centre of K , even if the kernel is finite (and K connected), as conjugation is not an automorphism of K and need not stabilize the kernel.

Theorem 7. *A near-field K of finite Morley rank in characteristic $\neq 2$ is an algebraically closed field.*

Proof. If the kernel is infinite (in particular if $\text{char}(K) = 0$ or $Z(K)$ is infinite), this follows from [3] or [4].

In characteristic $p > 0$, note first that K is connected, as for any additive proper subgroup H of finite index the intersection $\bigcap_{x \in K^\times} xH$ is trivial, but equals a finite subintersection, and hence is of finite index, a contradiction.

Let A be an infinite abelian multiplicative subgroup; increasing A , we may assume that it is definable. Let M_0 be an A -minimal additive subgroup. Then M_0 is additively isomorphic to the additive group of an algebraically closed field K_0 , and A embeds multiplicatively into K_0^\times . In fact, for any $e_0 \in M_0 \setminus \{0\}$ and $a_1, \dots, a_n \in A$ such that $a_1 e_0 + \cdots + a_n e_0 = 0$, we have by right distributivity that $a_1^{e_0} + \cdots + a_n^{e_0} = 0$, so the addition induced on A by K_0 is the one inherited from K -addition on A^{e_0} . So we might replace A by A^{e_0} , M_0 by $e_0^{-1}M_0$ and e_0 by 1. Then $A \subseteq M_0$, field addition on K_0 is induced from K and field multiplication on K_0 is induced from K on $A \times K_0$, but does not necessarily agree with multiplication from K if the left factor is in $K_0 \setminus A$. In particular, A^0 is a good torus. But the next theorem implies that $Z(K^*)$ is infinite, which is the required contradiction. \square

Theorem 8. *Let G be a connected group of finite Morley rank whose connected definable abelian subgroups are good tori, of 2-Prüfer rank ≤ 1 . Then its 2-Sylow subgroup is connected and central.*

Proof. Recall that for a torus A we have $N_G(A)^0 = C_G(A)^0 = C_G(A)$, and A does not have an infinite family of definable subgroups. Moreover, $C_G(A)$ is generous in G .

We claim that a connected soluble multiplicative subgroup S is abelian: If A is the last non-trivial derived group, then A is central in S . So if N is the second last derived subgroup, N is nilpotent, and is the central product of a divisible abelian group with a connected nilpotent group of bounded exponent. If the latter were

infinite, it would contain an infinite elementary abelian subgroup, a contradiction. Thus N is abelian, and $N = A = S$. It follows that if A is connected maximal abelian, then $A = C_G(A)$, and A is almost self-normalizing and generous in G .

If A and A' are two non-conjugate maximal abelian connected subgroups of G with maximal intersection I , then in $C_G(I)^0$ both A and A' are generous, and there is a conjugate A'' of A' intersecting A in a generic point outside I . Contradiction, since $A'' \cap A > I$. It follows that all maximal connected abelian subgroups are conjugate. We shall call a maximal connected abelian subgroup of G a *full torus*; a connected subgroup H of G is *full* if it contains a full torus. Consider a non-trivial element $g \in G$. Then $C_G(g)$ is infinite, and contains a torus T , which we take maximal possible. So $C_G(T)$ is connected, and $g \in C_G(T)$. If $g \notin T$ then gT has an infinite centralizer C/T in $C_G(T)/T$, which contains an infinite connected abelian subgroup A/T . But then A is again abelian connected, whence a torus, and $g \in C_G(A/T) = C_G^0(A/T) \leq N_G(A)^0 = C_G(A)$. Thus $T < A \leq C_G(g)$, a contradiction. It follows that every element is contained in a torus T . But now T can be extended to a maximal connected abelian subgroup T' , which must be a full torus containing g .

If a 2-Sylow⁰-subgroup S is central, then $G/\text{dc}(S)$ has finite, whence trivial 2-Sylow subgroups, so S is connected. It hence suffices to show that S is central. If not, as $C_G(S)^0 = N_G(S)^0$, there is $a \in G$ with $S^a \neq S$. Choose two full tori A and A' containing distinct 2-Sylows⁰ $S \neq S'$ such that $A \cap A'$ is maximal possible. Replacing G by $C_G(A \cap A')^0/(A \cap A')$, we may assume that this intersection is trivial. In particular any two full tori with non-trivial intersection have the same 2-Sylow⁰. So for any $g \in G$, since any two full tori in $C_G(g)$ are conjugate and must contain g , there is a unique 2-Sylow⁰ in $C_G(g)$.

Consider involutions $i \in S$ and $i' \in S'$. Then i and i' invert the element $m = ii'$. Let S'' be the unique 2-Sylow⁰ in $C_G(m)$. Then i and i' normalize $C_G(m)$, whence also S'' . Now either $S'' \neq S$ or $S'' \neq S'$, and we may assume $S'' \neq S$. Then $i \notin C_G(S'')$, as otherwise $C_G(S'')/\text{dc}(S'')$ would contain an involution, whence a 2-torus, a contradiction.

Suppose $C_{C_G(S'')}(i)$ is infinite. Then it contains a torus T , and there is a full torus $A'' \geq S''T$. Moreover, $i \in C_G(T) = C_G(T)^0$, so there is a torus T' containing i and T . But then there is a full torus A^* containing T' . But $A'' \cap A^* \geq T$ so $S'' = S^*$, contradicting $i \notin C_G(S'')$. As i normalizes $C_G(S'')$ and centralizes only finitely many points, it inverts $C_G(S'')^0 = C_G(S'')$. It follows that $C_G(S'')$ is abelian, and equal to the full torus A'' ; moreover all elements of the coset iA'' are involutions. Note that $m \in C_G(S'') = A''$, and $i \in N_G(S'') = N_G(A'')$. For any non-trivial $g \in G$ the centralizer $C_G(g)$ contains a unique 2-Sylow⁰, and hence a unique full torus, which must contain g . It follows that $C_G(g)^0$ is this full torus. So the full tori of G are disjoint, and cover G .

Let N be a minimal normal definable subgroup of G . Then N cannot be abelian, so N is simple by connectedness of G . If N does not contain an involution, then any involution i of G gives rise to an involutive automorphism of N , whose set of fixed points is contained, up to finite index, in a torus of N , a contradiction to [Poizat]. It follows that N contains involutions, and we may replace G by N .

Any two involutions in $N_G(A'') \setminus A''$ invert S , so their product is in $C_G(S) = A''$ and they form a unique coset modulo A'' . We obtain a geometry on G whose points are the involutions, and whose lines are the cosets iA for a full torus A and involution $i \in N_G(A)$, i.e. precisely the involutions in $N_G(A(j))$, where $A = A(j)$ is the unique full torus containing j . Then j is the unique involution commuting with all the involutions of the line $\ell(j) = N_G(A(j))[2]$. Any two distinct lines $\ell(i)$ and $\ell(j)$ have empty intersection, or intersect in a coset of $A(i) \cap A(j) = \{1\}$ (note that the corresponding 2-Sylows must be distinct), i.e. in a single point. Any two distinct points lie on at least one common line, and in fact a unique common line. There is a polarity which associates an involution i to a line $\ell(i)$ and vice versa, and which preserves incidence. It follows that any two lines intersect in a unique point. Moreover, there are no isotropic points: $i \notin \ell(i)$ for all i . By Bachman's theorem, G is a group of linear transformations of a 3-dimensional vector space over an interpretable field K which preserve a symmetric bilinear form without isotropic vectors. But this contradicts stability. \square

Fact 9. [6] *A near-domain of characteristic 3 is a near-field.*

Corollary 10. *A near-domain of characteristic 3 of finite Morley rank is an algebraically closed field.*

Corollary 11. *A strictly 2-transitive group of finite Morley rank and (permutation) characteristic 3 is the group of affine transformations of an algebraically closed field of characteristic 3.*

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