Introduction to the Renormalization Group

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Outline

DIRAC: You should put in a cutoff and get well defined equations. You should solve these equations and then take limits.

Wilson had his own particular way of looking at this: the Renormalization Group (his formulation). It does employ ultraviolet cutoffs (lattice or continuum) with well defined functional integral, analyzes the RG flow and takes limits. No infinities are ever met. The functional integral with cutoffs of course does solve the cutoff quantum field equations and in this sense Wilson is close to Dirac.

Wilson explained (see his Nobel Prize lecture) that when the fluctuations at the atomic scale average out we get a hydrodynamic picture (think of Landau's theory). But large scale fluctuations still remain as we approach a critical temperature and the hydrodynamic picture remains inadequate. In particular critical exponents, below the critical dimension, differ from the mean field vaues of the Landau-Ginzburg theory.

K. G. Wilson, Nobel Prize lecture 1982, Rev. Mod Phys, vol 55, 1983.

In a statistical system near a second order phase transition when the correlation length approaches infinity Wilson showed that the critical exponents are those of a continuum (no ultraviolet cutoff limit) scale invariant field theory. The two problems, that of approaching criticality and that of removing ultraviolet cutoffs to produce scale invariant field theories are related via a scaling limit which we will explain later.

The Renormalization Group in the sense of Wilson

In these lectures I will introduce the Renonormalization Group in a simple way and develop some of its applications. There will be two cutoffs: the volume cutoff and an ultraviolet cutoff. We will be mainly interested in removing the ultraviolet cutoff, known as the continuum limit. To make it as simple as possible I will work in the continuum.

The first lecture will be concerned with introducing essential ingredients and some approximate picture of whats going on. In the next two lectures I will give the essentials of RG analysis in the disrete framework and consider model results.

References: The lectures were based on the following references, all concerned with RG analysis directly in the continuum. They contain references to earlier relevant work.

Lecture 1: P.K.Mitter:The Exact Renormalization Group, Encyclopedia in Mathematical Physics, Elsevier 2006, http://arXiv:math-ph/0505008

Lectures 2, 3 : [BMS], Brydges, Mitter, Scoppola: Critical $\phi^4_{3,\epsilon}$, Comm. Math. Phys. **240**, 281-327 (2003).

A. Abdesselam:A complete Renormalization Group Trajectory between two Fixed Points, Comm. Math.Phys. **276**, 727-772 (2007).

Comment: Abdesselam detected an error in the paper of Brydges et al in the definition of some norms and produced a fix which happily left the estimates and theorems intact. A different fix is given here by choosing the measure space appropriately and the definition of norms of polymer activities (see later).

Euclidean field theory measures are naturally realized on the Schwartz spaces $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{D}'(\mathbb{R}^d)$. A Gaussian measure is specified by its covariance C which is a continuous, positive definite bilinear functional on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. Let $f \in \mathcal{S}(\mathbb{R}^d)$. The Fourier transform of a Gaussian measure of mean 0 and covariance C is given by

$$e^{\frac{-1}{2}C(f,f)} = \int_{\mathcal{S}'(\mathbb{R}^d)} d\mu_C(\phi)\, e^{i\phi(f)}$$

$$C(f,f) = (f, Cf)$$

We will now consider a simple example of a covariance, that of the massless scalar free field in dimension $d \geq 3$. In $d = 2$ there are logarithms which we want to avoid for the moment.

$$E(\phi(x)\phi(y)) = const.|x-y|^{-2[\phi]} = \int_{\mathbf{R}^d} dp\, e^{ip.(x-y)}\, \frac{1}{|p|^{d-2[\phi]}}$$

Here $[\phi] > 0$ is the (canonical) *dimension* of the field, which for the standard massless free field is $[\phi] = \frac{d-2}{2}$. The latter is positive for $d > 2$. However other choices are possible but in EQFT they are restricted by Osterwalder-Schrader positivity. This is assured if $[\phi] = \frac{d-\alpha}{2}$ with $0 < \alpha \leq 2$. If $\alpha < 2$ we get a generalized free field.

The covariance is singular for $x = y$ and this singularity is responsable for the ultraviolet divergences of quantum field theory. This singularity has to be initially cutoff and there are many ways to do this. A simple way to do this is as follows. Let $u(x)$ be a smooth, rotationally invariant, positive-definite function of compact support:

$$u(x) = 0 : \ |x| \geq 1$$

This can be achieved as follows. Let g be an $O(d)$ invariant C^∞ function of compact support:

$$g(x) = 0 : \ |x| \geq \frac{1}{2}$$

and choose

$$u(x) = (g \star g)(x)$$

Then

$$u(x) = 0 : \ |x| \geq 1$$

and moreover since $\hat{u}(p) = |\hat{g}(p)|^2$, u is positive definite.

Observe that

$$|x-y|^{-2\phi} = const. \int_0^\infty \frac{dl}{l}\, l^{-2[\phi]}\, u(\frac{x-y}{l})$$

as seen by scaling in l. Let $L > 1$. Define $\epsilon_N = L^{-N}$. Then $\epsilon_N \to 0$ as $N \to \infty$. We define the cutoff covariance by

$$C_{\epsilon_N}(x-y) = const. \int_{\epsilon_N}^\infty \frac{dl}{l}\, l^{-2[\phi]}\, u(\frac{x-y}{l})$$

$C_{\epsilon_N}(x-y)$ is positive definite and everywhere smooth. Being positive-definite it qualifies as the covariance of a Gaussian probability measure denoted $\mu_{C_{\epsilon_N}}$ on $\mathcal{S}'(\mathbb{R}^d)$ The covariance C_{ϵ_N} being smooth implies that the sample fields of the measure are $\mu_{C_{\epsilon_N}}$ almost everywhere sufficiently differentiable. This is easy to prove. A standard construction gives a version of the measure (obtained by taking the outer measure), supported on $\Omega \subset \mathcal{S}'(\mathbb{R}^d)$ of sufficiently differentiable functions. This measure is countably additive on the sigma field $\mathbb{B} = \mathbb{B}(\mathcal{S}') \cap \Omega$. Our measure space is thus $(\Omega, \mathbb{B}, \mu_{C_{\epsilon_N}})$

Let $L > 1$ be any real number. We define a scale transformation S_L on fields ϕ by

$$S_L\phi(x) = L^{-[\phi]}\, \phi(\frac{x}{L})$$

on covariances C by

$$S_L C(x-y) = L^{-2[\phi]}\, C(\frac{x-y}{L})$$

and on functions of fields $F(\phi)$ by

$$S_L F(\phi) = F(S_L\phi)$$

The scale transformations form a *multiplicative group* : $S_L^n = S_{L^n}$.

Define the *unit cutoff* covariance

$$C(x-y)=\int_1^\infty \frac{dl}{l}\, l^{-2[\phi]}\, u(\frac{x-y}{l})$$

and observe that

$$C_{\epsilon_N}(x-y)=S_{\epsilon_N}C(x-y)$$

Henceforth we will work with the unit cutoff covariance and when taking the continuum limit $\epsilon_N\to 0$ will rescale to unit cutoff.

Now define a *fluctuation covariance* Γ_L

$$\Gamma_L(x-y) = \int_1^L \frac{dl}{l}\, l^{-2[\phi]}\, u(\frac{x-y}{l})$$

$\Gamma_L(x-y)$ is smooth, positive-definite and of range L:

$$\Gamma_L(x-y) = 0 : \ |x-y| \geq L$$

It generates a key scaling decomposition

$$C(x-y) = \Gamma_L(x-y)\ + S_L C(x-y)$$

Iterating this we get

$$C(x-y)=\sum_{n=0}^{\infty}\Gamma_n(x-y)$$

where

$$\Gamma_n(x-y)=S_{L^n}\Gamma_L(x-y)=L^{-2n[\phi]}\,\Gamma_L(\frac{x-y}{L^n})$$

The $\Gamma_n(x-y)$ are positive definite C^∞ functions of range L^{n+1}.

$$\Gamma_n(x-y)=0:\ |x-y|\geq L^{n+1}$$

This gives a *Finite Range Decomposition* into a sum over increasing length scales. Being positive-definite the Γ_n qualify as covariances of Gaussian probability measures , and therefore $\mu_C = \bigotimes_{n=0}^{\infty} \mu_{\Gamma_n}$. Correspondingly introduce a family of independent Gaussian random fields ζ_n, called *fluctuation fields*, distributed according to μ_{Γ_n} . Then

$$\phi = \sum_{n=0}^{\infty} \zeta_n$$

The fluctuation fields ζ_n are slowly varying over length scales L^n. An estimate using a Tchebycheff inequality shows that for any $\gamma > 0$

$$|x-y| \le L^n \Rightarrow E\Big(|\zeta_n(x) - \zeta_n(y)| \ge \gamma\Big) \le const.\gamma^{-2}$$

which reveals the slowly varying nature of ζ_n on scale L^n. This is an example of a *Finite Range Multiscale Decomposition* of a Gaussian random field.

The above implies that the μ_C integral of a function can be written as a multiple integral over the fields ζ_n . We calculate it by integrating out the fluctuation fields ζ_n step by step going from shorter to longer length scales. This can be accomplished by the iteration of a single transformation T_L, a *renormalization group transformation*, as follows.

Let $F(\phi)$ be a function of fields. Then we define a RG transformation $F \to T_L F$ by

$$(T_L F)(\phi) = S_L \mu_{\Gamma_L} * F(\phi) = \int d\mu_{\Gamma_L}(\zeta)\, F(\zeta + S_L \phi)$$

Thus the renormalization group transformation consists of a convolution with the fluctuation measure followed by a rescaling

Semigroup property : The discrete RG transformations form a *semigroup*

$$T_L T_{L^n} = T_{L^{n+1}}$$

for all $n \geq 0$. To prove this we must first see how scaling commutes with convolution with a measure. We have the property :

$$\mu_{\Gamma_L} * S_L F = S_L \mu_{S_L \Gamma_L} * F$$

To see this observe first that if ζ is a Gaussian random field distributed with covariance Γ_L then the Gaussian field $S_L\zeta$ is distributed according to $S_L\Gamma_L$. This can be checked by computing the covariance of $S_L\zeta$. Now the lefthand side of previous equation is just the integral of $F(S_L\zeta + S_L\phi)$ with respect to $d\mu_{\Gamma_L}(\zeta)$. By the previous observation this is the integral of $F(\zeta + S_L\phi)$ with respect to $d\mu_{S_L\Gamma_L}(\zeta)$, and the latter is the right hand side.

Now we check the semigroup property:

$$\begin{aligned} T_L T_{L^n} F = S_L \mu_{\Gamma_L} * S_{L^n} \mu_{\Gamma_{L^n}} * F = S_L S_{L^n} \mu_{S_{L^n}\Gamma_L} * \mu_{\Gamma_{L^n}} * F \\ = S_{L^{n+1}} \mu_{\Gamma_{L^n} + S_{L^n}\Gamma_L} * F = S_{L^{n+1}} \mu_{\Gamma_{L^{n+1}}} * F \\ = T_{L^{n+1}} F \end{aligned} \tag{1}$$

We have used the fact that $\Gamma_{L^n} + S_{L^n}\Gamma_L = \Gamma_{L^{n+1}}$. This is because $S_{L^n}\Gamma_L$ has the representation with integration interval changed to $[L^n, L^{n+1}]$.

T_L has an unique invariant measure, namely μ_C : For any bounded function F

$$\int d\mu_C T_L F = \int d\mu_C F$$

To understand this recall the earlier observation that if ϕ is distributed according to the covariance C, then $S_L\phi$ is distributed according to $S_L C$. Now $\Gamma_L + S_L C = C$. Therefore

$$\begin{aligned}\int d\mu_C T_L F &= \int d\mu_C S_L \mu_{\Gamma_L} * F \\ &= \int d\mu_{S_L C} \mu_{\Gamma_L} * F \\ &= \int d\mu_C F \qquad (2)\end{aligned}$$

T_L is a *contraction semigroup* on $L^p(d\mu_C)$ for $1 \leq p < \infty$.
Suppose $p = 2$. Then

$$\begin{aligned}
||T_L F||^p_{L^p(d\mu_C)} &= \int d\mu_C(\phi)|T_L F(\phi)|^p \\
&= \int d\mu_C(\phi)\Big|\int d\mu_\Gamma(\zeta)F(\zeta + S_L\phi)\Big|^p \\
&\leq \int d\mu_C(\phi)\int d\mu_\Gamma(\zeta)|F(\zeta + S_L\phi)|^p \\
&= \int d\mu_C(\phi)T_L|F(\phi)|^p = \int d\mu_C(\phi)|F(\phi)|^p \\
&= ||F||^p_{L^p(d\mu_C)} \qquad (3)
\end{aligned}$$

Wick Monomials

Define

$$\Delta_C F(\phi) = \int d\mu_C(\zeta)\, D^2 F(\phi; \zeta, \zeta)$$

This is a second functional derivative

$$\Delta_C F = \int dxdy\ C(x-y)\partial_{\phi(x)}\partial_{\phi(y)}$$

Let $p_{n,m}(\phi(x))$ be a local monomial of m fields with n derivatives. Then we define its Wick ordering by

$$: p_{n,m}(\phi(x)) :_C = e^{-\frac{1}{2}\Delta_C}\, p_{n,m}(\phi(x))$$

Properties of Wick products can be derived from this formula.

Wick monomials are Eigen functions of T_L :

Let $:p_{n,m}:(\phi(x))$ be a C Wick ordered local monomial of m fields with n derivatives. Define $P_{n,m}(X) = \int_X dx \ :p_{n,m}:_C(x)$. The $P_{n,m}(X)$ play the role of eigenfunctions of the RG transformation T_L *upto a scaling of volume*:

$$T_L P_{n,m}(X) = L^{d-m[\phi]-n} P_{n,m}(L^{-1}X)$$

We define

$$dim[P_{n,m}] = d - m[\phi] - n$$

This is easy to see if we realise (F below is a monomial in fields and its derivatives)

$$(T_L F)(\phi) = S_L \mu_{\Gamma_L} * F(\phi) = S_L(e^{\frac{1}{2}\Delta_{\Gamma_L}} F)$$

so that

$$(T_L : F :_C)(\phi) = S_L(e^{\frac{1}{2}\Delta_{\Gamma_L}} e^{-\frac{1}{2}\Delta_C} F) = S_L(e^{-\frac{1}{2}\Delta_{(C-\Gamma_L)}} F)$$

$$= S_L(e^{-\frac{1}{2}\Delta_{S_L C}} F) = S_L : F :_{S_L C}$$

Hence

$$(T_L : F :_C) = L^{-dim[F]} : F(\frac{\cdot}{L}) :_C$$

They are classified as expanding (*relevant*), contracting (*irrelevant*) or central (*marginal*) depending on whether the exponent of L on the right hand side is positive, negative or zero.

Ultraviolet cutoff removal

Let $\Lambda = [-R/2, R/2]^d$ be a torus in $\mathbb{R}^d$ of side length R. We define the *partition function*:

$$Z_{\epsilon_N}(\Lambda) = \int d\mu_{C_{\epsilon_N}}(\phi)\, e^{-V_0(\Lambda,\, \phi,\, \tilde{\xi}_N,\, \tilde{g}_N,\, \tilde{\mu}_N)}$$

where

$$V_0(X, \phi) = \int_X dx\, (\xi\, |\nabla\phi(x)|^2 + g_0\, \phi^4(x) + \mu_0\, \phi^2(x))$$

with g_0, μ_0 replaced by $\tilde{g}_N, \tilde{\mu}_N$.

By dimensional analysis we can write

$$\tilde{\xi}_N = \epsilon_N^{(2[\phi]-d+2)}\xi, \quad \tilde{g}_N = \epsilon_N^{(4[\phi]-d)} g, \quad \tilde{\mu}_N = \epsilon_N^{(2[\phi]-d)}\mu$$

where g, ξ, μ are dimensionless parameters.

Recall that $[\phi] = \frac{d-\alpha}{2}$ and $0 < \alpha \leq 2$. Then

$$4[\phi] - d = d - 2\alpha, \quad 2[\phi] - d = -\alpha, \quad 2[[\phi] - d + 2 = d - \alpha$$

$d_c = 2\alpha$ is the critical dimension. Define $\epsilon = d_c - d = 2\alpha - d$. Then we have

Let $\epsilon_N = L^{-N}$. Then we have

$$\tilde{g}_N = L^{N\epsilon} g, \quad \tilde{\mu_N} = L^{N\alpha}\mu, \quad \tilde{\xi_N} = L^{-N(2-\alpha)}\xi$$

For superrenormalizable theories (like massive $\phi^4_3,\ \alpha = 2$) the dimensional coupling constant $\tilde{g}_N$ is held fixed whereas the dimensionless coupling $g \to 0$. There is no coupling constant renormalization, only a mass renormalization. However we are interested in these lectures in *critical theories*. The dimensionless couplings will hit RG fixed points whereas the dimensional couplings $\tilde{g}_N,\ \tilde{\mu}_N \to \infty$.

Now ϕ distributed according to C_{ϵ_N} equals in distribution $S_{\epsilon_N}\phi$ distributed according to C, the unit cutoff covariance. Therefore choosing $\epsilon_N = L^{-N}$ we get

$$\begin{aligned} Z_{\epsilon_N}(\Lambda) &= \int d\mu_C(\phi)\; e^{-V_0(\Lambda,\, S_{\epsilon_N}\phi,\, \tilde{\xi}_N,\, \tilde{g}_N,\, \tilde{\mu}_N)} \\ &= \int d\mu_C(\phi) e^{-V_0(\Lambda_N,\, \phi,\, \xi\, g,\, \mu)} \\ &= Z(\Lambda_N) \end{aligned} \tag{4}$$

where $\Lambda_N = [-L^N\frac{R}{2}, L^N\frac{R}{2}]^d$.

Thus the field theory problem of ultraviolet cutoff removal i.e. taking the limit $\epsilon_N \to 0$ has been reduced to the study of a statistical mechanical model in a very large volume. The latter will have to be analyzed via RG iterations.

We proceed to a scale by scale analysis: Because μ_C is an invariant measure of T_L we have for the partition function $Z(\Lambda_N)$ in the volume Λ_N

$$Z(\Lambda_N) = \int d\mu_C(\phi) z_0(\Lambda_N, \phi) = \int d\mu_C(\phi)\, T_L z_0(\Lambda_N, \phi)$$

The integrand on the right hand side is a new function of fields which because of the final scaling live in the smaller volume Λ_{N-1}. This leads to the definition :

$$z_1(\Lambda_{N-1}, \phi) = T_L z_0(\Lambda_N, \phi)$$

Iterating the above transformation we get for all $0 \leq n \leq N$

$$z_{n+1}(\Lambda_{N-n-1}, \phi) = T_L z_n(\Lambda_{N-n}, \phi)$$

After N iterations we get

$$Z(\Lambda_N) = \int d\mu_C(\phi) z_N(\Lambda_0, \phi)$$

where Λ_0 is the unit cube. To take the $N \to \infty$ we have to control the infinite sequence of iterations

Because V_0 was local z_0 has a factorization property for unions of sets with disjoint interiors. This is no longer the case for z_1 and the subsequent z_n. The idea is to extract out a local part and *also* consider the remainder. The local part leads to a flow of coupling constants and the (unexponentiated) remainder is an irrelevant term. This operation and its mathematical control is an essential feature of RG analysis. This corresponds to Wilson's intuition.

To take the $N \to \infty$ we have to control the infinite sequence of iterations. To do this we have to set up the *coordinates* which represent which represent local parts and irrelevant terms what I said in the previous frame. It will be enough to control the flow of these coordinates to compute and control correlation functions. I will take this up in the last lecture.

The continuous RG and the flow of coupling constants

The fastest way of computing approximate RG flows of the local part is by methods of the *Continuous RG*. The discrete semigroup of the previous section has a natural continuous counterpart. Just take L to be a continuous parameter, $L = e^t : t \geq 0$ and write by abuse of notation T_t, S_t, Γ_t instead of T_{e^t} etc. The continuous transformations T_t

$$T_t F = S_t \, \mu_{\Gamma_t} * F$$

give a semigroup

$$T_t T_s = T_{t+s}$$

T_t is a contraction on $L^2(d\mu_C)$ with μ_C as invariant measure. The generator $\mathcal{L}$ is defined by

$$\mathcal{L}F = \lim_{t \to 0^+} \frac{T_t - 1}{t} F$$

whenever this limit exists. This restricts F to a suitable subspace $\mathcal{D}(\mathcal{L}) \subset L^2(d\mu_C)$. $\mathcal{D}(\mathcal{L})$ contains for example polynomials in fields as well as twice differentiable bounded cylindrical functions. The generator $\mathcal{L}$ can be easily computed.

Define $(D^n F)(\phi; \zeta_1, \ldots, \zeta_n)$ as the n-th tangent map at ϕ along directions $\zeta_1, \ldots, gz_n$. The functional Laplacian $\Delta_{\dot\Gamma}$ is defined by

$$\Delta_{\dot\Gamma} F(\phi) = \int d\mu_{\dot\Gamma}(\zeta)\, (D^2 F)(\phi; \zeta, \zeta)$$

where $\dot\Gamma = u$.

Define a vector field $\mathcal{X}$

$$\mathcal{X}F = \frac{d}{dt}\Big|_{t=0} F(e^{-t[\phi]}\phi(e^{-t}\cdot)) \tag{5}$$

Then an easy computaion gives

$$\mathcal{L} = \frac{1}{2}\Delta_{\dot{\Gamma}} + \mathcal{X} \tag{6}$$

T_t is a semigroup with $\mathcal{L}$ as generator. Therefore $T_t = e^{t\mathcal{L}}$. Let $F_t(\phi) = T_t F(\phi)$. Then F_t satisfies the linear PDE

$$\frac{\partial F_t}{\partial t} = \mathcal{L} F_t \tag{7}$$

with initial condition $F_0 = F$. This evolution equation assumes a more familiar form if we write $F_t = e^{-V_t}$, V_t being known as the *effective potential*.

We get

$$\frac{\partial V_t}{\partial t} = \mathcal{L} V_t - \frac{1}{2}(V_t)_\phi \cdot (V_t)_\phi \tag{8}$$

where

$$(V_t(\phi))_\phi \cdot (V_t(\phi))_\phi = \int d\mu_{\dot{\Gamma}}(\zeta)((DV_t)(\phi;\zeta))^2 \tag{9}$$

and $V_0 = V$. This infinite dimensional non-linear PDE is a version of Wilson's *flow equation*. Note that the linear semigroup T_t acting on functions induces a semigroup $\mathcal{R}_t$ acting non-linearly on effective potentials giving a trajectory $V_t = \mathcal{R}_t V_0$.

Equations like the above are very difficult to control rigorously.However they may solved in formal perturbation theory when the initial V_0 is small via the presence of small parameters. In particular they give rise easily to perturbative flow equations for coupling constants. These approximate perturbative flows are very useful for getting a prelimnary view of the flow. Discrete versions of these flows figure as an input in non-perturbative analysis.

Flow in second order perturbation theory: We will simplify by working in infinite volume (no infrared divergences can arise because $\dot{\Gamma}(x-y) = u(x-y)$ is of fast decrease). Now suppose that we are in standard ϕ^4 theory with $[\phi] = \frac{d-2}{2}$ and $d > 2$. We want to show that

$$V_t = \int dx \left(\xi_t : |\nabla\phi(x)|^2 : + g_t : \phi(x)^4 : + \mu_t : \phi(x)^2 : \right) \tag{10}$$

satisfies the flow equation in second order modulo irrelevant terms provided the parameters flow correctly. We will ignore field independent terms. The Wick ordering is with respect to the covariance C of the invariant measure.

Notice that we have ignored a ϕ^6 term which is actually relevant in $d = 3$ for the above choice of $[\phi]$. This is because we will only discuss the $d = 3$ case for the model discussed at the end of this computation and for this case the ϕ^6 term is irrelevant. We will assume that ξ_t, μ_t are of $O(g^2)$.

Plug in the above in the flow equation. $\lambda_t^{n,m} : P_{n,m} :$ represent one of the terms above with m fields and n derivatives. Because $\mathcal{L}$ is the generator of the semigroup T_t we have

$$(\frac{\partial}{\partial t} - \mathcal{L})\lambda_t^{n,m} : P_{n,m} := (\frac{d\lambda_t^{n,m}}{dt} - (d - m[\phi] - n)\lambda_t^{n,m}) : P_{n,m} : \quad (11)$$

Next turn to the non-linear term in the flow equation and insert the ϕ^4 term (the others are already of $O(g^2)$). This produces a double integral of $\dot{\Gamma}(x-y) : \phi(x)^3 :: \phi(y)^3 :$ which after complete Wick ordering gives

$$-\frac{g_t^2}{2} 16 \int dxdy\, \dot{\Gamma}(x-y)\big(: \phi(x)^3 \phi(y)^3 : +$$

$$+9C(x-y) : \phi(x)^2 \phi(y)^2 : +36C(x-y)^2 : \phi(x)\phi(y) : +$$
$$+6C(x-y)^3\big)$$

Consider the non-local ϕ^4 term. We can localize it by writing

$$: \phi(x)^2\phi(y)^2 := \frac{1}{2} : \left(\phi(x)^4 + \phi(y)^4 - (\phi(x)^2 - \phi(y)^2)^2\right) : \quad (12)$$

The local part gives a ϕ^4 contribution and the the last term above gives rise to an irrelevant contribution because it produces additional derivatives. The coefficients are well defined because $C, \dot{\Gamma}$ are smooth and $\dot{\Gamma}(x-y)$ is of finite range. Now the non-local ϕ^2 term is similarly localized. It gives a relevant local ϕ^2 contribution as well as a marginal $|\nabla\phi|^2$ contribution.

Same principle applies to the non-local ϕ^6 contribution and generates further irrelevant terms. By matching see that that the flow equation is satisfied in second order up to irrelevant terms (these would have to be compensated by adding additional terms in V_t) provided

$$\frac{dg_t}{dt} = (4-d)g_t - ag_t^2 + O(g_t^3)$$
$$\frac{d\mu_t}{dt} = 2\mu_t - bg_t^2 + O(g_t^3)$$
$$\frac{d\xi_t}{dt} = cg_t^2 + O(g_t^3) \tag{13}$$

where a, b, c are positive constants.

We see from the above formulae that up to second order in g^2 as $t \to \infty$, $g_t \to 0$ for $d \geq 4$. In fact for $d \geq 5$ the decay rate is $O(e^{-t})$ and for $d = 4$ the rate is $O(t^{-1})$. However to see if V_t converges we have to also discuss the μ_t, ξ_t flows. It is clear that in general the μ_t flow will diverge. This is fixed by choosing the initial μ_0 to be the *bare critical mass*.

This is obtained by integrating upto time t and then expressing μ_0 as a function of the entire g trajectory up to time t. And then assume that μ_t is uniformly bounded and take $t \to \infty$. This gives the critical mass as

$$\mu_0 = b \int_0^\infty ds\, e^{-2s} g_s^2 = \mu_c(g_0) \tag{14}$$

This integral converges for all cases discussed above. With this choice of μ_0 we get

$$\mu_t = b \int_0^\infty ds\, e^{-2s} g_{s+t}^2 \tag{15}$$

and this exists for all t and converges to zero (for $d \geq 4$ as $t \to \infty$.

Now consider the perturbative ξ flow. It is easy to see from the above that for $d \geq 4$, ξ_t converges as $t \to \infty$. We have not discussed the $d = 3$ case because the perturbative g fixed point is of $O(1)$. But we can tackle this problem in another way.

Take $[\phi] = \frac{d-\alpha}{2}$ with $0 < \alpha < 2$ We also restrict $d < 4$. The *critical dimension* is $d_c = 2\alpha$. These considerations emerge from a generalization of Dyson's long range model to higher dimensions (Aizenman and Fernandez). Define

$$\epsilon = 2\alpha - d > 0$$

Let $d = 3$ and $\alpha = \frac{3+\epsilon}{2}$. Let $\epsilon > 0$ be sufficiently small. The flow equations are now:

$$\frac{dg_t}{dt} = \epsilon g_t - a g_t^2 + O(g_t^3)$$
$$\frac{d\mu_t}{dt} = \alpha \mu_t - b g_t^2 + O(g_t^3)$$
$$\frac{d\xi_t}{dt} = -\frac{1-\epsilon}{2} + c g_t^2 + O(g_t^3) \qquad (16)$$

where a, b, c are positive constants.

We have an attractive fixed point $g_* = O(\epsilon)$ of the g flow and the convergence is exponentially fast. The critical bare mass μ_0 can be determined as before and is given by

$$\mu_0 = b \int_0^\infty ds\, e^{-\alpha s} g_s^2 = \mu_c(g_0) \tag{17}$$

This integral converges for all cases discussed above. With this choice of μ_0 we get

$$\mu_t = b \int_0^\infty ds\, e^{-\alpha s} g_{s+t}^2 \tag{18}$$

and this exists for all t and converges to $\mu_\star = O(g_\star^2)$ as $t \to \infty$.

The ξ_t flow converges to a fixed point $\xi_\star = O(g_\star^2)$. The existence of this fixed point and the critical mass was proved by Brydges, Mitter and Scoppola [CMP(2003)] by a rigorous analysis of the discrete RG flow. Later Abdesselam [CMP 2007] proved the existence of a RG trajectory connecting the unstable Gaussian fixed point to the stable nontrivial fixed point.

In the next two lectures I will consider rigorous discrete RG analysis. This model will crop up as an example.

We shall now go back to the discrete RG. C is the unit cutoff covariance and Λ_N is the torus with side length L^N. Recall that after iterating n times we obtained:

$$\int d\mu_C(\phi)\mathcal{Z}_0(\Lambda_N, \phi) = \int d\mu_C(\phi)\mathcal{Z}_n(\Lambda_{N-n}, \phi)$$

where

$$\mathcal{Z}_n(\Lambda_{N-n}, \phi) = \int d\mu_{\Gamma_L}(\zeta)\mathcal{Z}_{n-1}(\Lambda_{N-n+1}, \zeta + S_L\phi)$$

At the end of N steps the volume reduces to the unit cube (unit block) and we take the $N \to \infty$.

Discrete RG analysis

We want to analyze the generic step. The first problem that we want to take care of is nonlocality. The starting density has a locality property which we lose after one iteration. Thus if $X,\ Y$ are two subsets with disjoint interiors then

$$\mathcal{Z}_0(X \cup Y)) = \mathcal{Z}_0(X)\mathcal{Z}_0(Y))$$

but after one iteration this is no longer true

$$\mathcal{Z}_1(X \cup Y)) \neq \mathcal{Z}_1(X)\mathcal{Z}_1(Y))$$

This is resolved by introducing a new representation for the densities-the so called polymer gas representation- and the finite range of the fluctuation field correlations.

Pave $\mathbb{R}^d$ with *unit blocks* (unit cubes). Then $\Lambda \subset \mathbb{R}^d$ has the induced paving. A *polymer* X is a connected subset of blocks. A *polymer activity* K is a map $(X, \phi) \to K(X, \phi) \in \mathbb{R}$. The field ϕ has been restricted to X.

At any given step n of the sequence of RG transformations the densities will be given coordinates g_n, μ_n, K_n. Here g_n, μ_n are the evolved parameters of the local potential V_n. and K_n is a so called irrelevant (contracting) term characterized as a polymer activity. The density $\mathcal{Z}_n(\Lambda_{N-n}, \phi)$ can be expressed in terms of these coordinates in a polymer gas representation.

Polymer Gas Representation

$$\mathcal{Z}_n(\Lambda_{N-n},\phi) = \sum_{N\geq 0}\frac{1}{N!}e^{-V_n(X_c,\phi)}\sum_{X_1,..X_N}\prod_{j=1}^{N}K_n(X_j,\phi)$$

where $X_c = \Lambda_{N-n}/\cup_{j=1}^{n} X_j$, and the sum is over mutually disjoint connected polymers X_j in Λ_{N-n}.

Note that the local potential V_n depends on (evolved) coupling g_n and mass parameter μ_n.

This representation is *stable* under RG.

The nth RG transformation induces a map:

$$f_{N-n}: \ (K_{n-1}, V_{n-1}) \to (K_n, V_n)$$

The subscript $N-n$ is there because the initial partition function is defined in a finite volume Λ_N and consequently the nth RG transformation is a map of polymer activities supported on subsets of Λ_{N-n+1} to polymer activities supported on subsets of Λ_{N-n} Now let X be a connected polymer. For every polymer X it can be shown that $\lim_{N\to\infty} f_{N-n}(K, V)(X, \phi)$ exists pointwise in X. We can study the action of the RG on the coordinates in this pointwise infinite volume limit.

The renormalization group step T_L involves fluctuation integration and then rescaling. The density depends on $\zeta + \phi$ and we integrate with measure $d\mu_{\Gamma_L}(\zeta)$. Replace ϕ by $\phi + \zeta$ in $V_n,\ K_n$. We now prepare the integrand a bit before actually doing the integral.

$V_n(X_c, \zeta + \phi)$ is local. We consider also an *arbitrary* local potential $\tilde{V}_n(X_c, \phi)$ *which depends only on* ϕ. We can write

$$\exp -V_n(X_c, \zeta + \phi) = \prod_{\Delta \subset X_c} \exp -V_n(\Delta) =$$

$$\prod_{\Delta \subset X_c} [P_n(\Delta, \zeta, \phi) + \exp -\tilde{V}_n(\Delta, \phi)]$$

where

$$P_n(\Delta, \zeta, \phi) = \exp -V_n(\Delta, \zeta + \phi) - \exp -\tilde{V}_n(\Delta, \phi)$$

Expand out and glue together the $P_n(\Delta)$ with the polymer activities K_n. This will create new polymer activities. Finally remember that the fluctuation covariance Γ_L has finite range L. So we should glue together these new $1-$ polymers into disjoint connected $L-$ polymers built out of connected $L-$ blocks (cubes of side length L). We call it taking the L-closure.

The reblocking operation is a nonlinear map. But the essence of it is captured by its linearization, the nonlinear parts being very small if the polymer activities are small.

Linearized reblocking

Let X be a connected 1-polymer. Its L-closure will be denoted $\bar{X}^L$. It is the smallest connected L- polymer which contains X. Then we define the linearised by:

$$(\mathcal{B}_1 K)(Y) = \sum_{X:\bar{X}^L = Y} e^{\tilde{V}(Y\setminus X)} K(X)$$

.

The net result is a new representation:

$$\mathcal{Z}_n(\Lambda_{N-n}, \zeta + \phi) = \sum_{N \geq 0} \frac{1}{N!} e^{-\tilde{V}_n(Y_c, \phi)} \sum_{Y_1, .. Y_N} \prod_{j=1}^{N} \mathcal{B}K_n(Y_j, \zeta, \phi)$$

The sum is now over mutually disjoint connected L- polymers in Λ_{N-n} . $\mathcal{B}K_n$ is a non-linear functional of K_n, $\tilde{V}_n$ which depends on ϕ, ζ. $\tilde{V}_n$ is a yet to be chosen local potential *which depends only on* ϕ . Its linearization was defined earlier.

The *fluctuation* map $S_L \mu_{\Gamma_n} *$ integrates out the ζ and then rescales. The integral sails through $\exp -\tilde{V}_n(Y_c, \phi)$ which is independent of ζ. Then it *factorizes* over the product of polymer activities because of the finite range property of Γ_n since the connected L- polymers are separated by a distance $\geq L$. Thus the polymer representation is preserved after fluctuation integration. Then we rescale to get back to $1-$ polymers.

The fluctuation integration plus rescaling has given a map

$$V_n \to \tilde{V}_{n,L} = S_L \tilde{V}_n \quad \tilde{V}_{n,L}(\Delta,\ \phi) = \tilde{V}_n(L\Delta,\ S_L\phi)$$

$$K_n \to \mathcal{F}K_n \quad \mathcal{F}K_n(X,\phi) = \int d\mu_{\Gamma_L}(\zeta)\mathcal{B}K(LX,\zeta,S_L\phi)$$

We shall now do one more *crucial* step to produce our final *renormalization map* .

Extraction

The representation that we have given is not unique because $\tilde{V}_n$ is upto us to choose. A change in $\tilde{V}_n$ changes $\mathcal{F}K_n$. For example choose $\tilde{V}_n = V_n$. Then subtract out the (localized) expanding parts of $\mathcal{F}K_n$, and absorb them in $V_{n,L}$ thus producing a flow of parameters. The new subtracted polymer activities have good contraction properties measured in appropriate norms. This procedure is known as *Extraction*

This subtraction operation on $\mathcal{F}K_n(X,\phi)$ needs only to be be done for *small* sets $X: \ |X| \leq 2^d$, because *large* sets provide contracting contributions measured in appropriate norms. The new subtracted polymer activities have good contraction properties (*irrelevant terms*).

Relevant parts

Let $P(\phi(x))$ be a *local* polynomial, which means that it is a polynomial in $\phi(x)$ and derivatives of $\phi(x)$ at x. We consider a change in V of the form

$$V_F(Y) = \sum_P \int_Y dx\ \alpha_P(x) P(\phi(x))$$

where the sum ranges over finitely many *local* polynomials and, for each such P, $\alpha_P(x)$ has the form

$$\alpha_P(x) = \sum_{X \supset x} \alpha_P(X, x)$$

such that $\alpha_P(X, x) = 0$ if x is not in the interior of X and $\alpha_P(X, x) = 0$ if $X \not\subset \Lambda$. The corresponding change in K is given in terms of

The corresponding change in K is given in terms of

$$F(X) = \sum_P \int dx \, \alpha_P(X,x) P(\phi(x))$$

$$F(X,\Delta) = \sum_P \int_\Delta dx \, \alpha_P(X,x) P(\phi(x))$$

Given V_{F_0} and V_{F_1} as above, with V_{F_0} field independent, there exists $\mathcal{E}(K, F_0, F_1)$ such that in the polymer representation we have a map

$$V \to V'_{F_1} = V - V_F$$

$$K \to \mathcal{E}(K, F_0, F_1)$$

and the linearized extraction is

$$\mathcal{E}_1(K, F_0, F_1) = K - (F_0 + F_1)e^{-V}$$

This procedure produces our final RG map f_{n+1}

$$f_V(V_n, K_n) = V_{n+1}, \quad f_K(V_n, K_n) = K_{n+1}$$

Using second order perturbation theory,

$$K_n = e^{-V_n} Q_n + R_n$$

Q_n is a second order contribution. It is form invariant and depends on g_n, μ_n and a non-local kernel w_n which converges fast to a fixed point kernel w_*. R_n is a remainder (formally of third order).

Normalization conditions.

We will say a polymer activity J is normalized, if for all small sets,

$$J(X,0) = 0$$

$$D^2 J(X,0;1,1) = 0$$

$$D^2 J(X,0;1,x_\mu) = 0$$

$$D^4 J(X,0;1,1,1,1) = 0$$

Fixing relevant parts

Fixing the relevant parts F_R. of the polymer activity $\mathcal{F}R$. Define

$$R^\sharp(X,\phi) = \int d\mu_{\Gamma_L}(\zeta) R(LX,\zeta,S_L\phi)$$

$$J(X,\phi) = R^\sharp(X,\phi) - \tilde{F}_R(X,\phi)e^{-\tilde{V}(X,\phi)}$$

$$\tilde{F}_R(X,\phi) = \sum_P \int_X d^3x\, \tilde{\alpha}_P(X) P(\phi(x),\partial\phi(x))$$

and fix the coefficients by imposing that J is *normalized*.

Q is an explicit polymer activity which we will call the *second order polymer activity'*. It is motivated by second order perturbation theory in powers of g and is defined as follows: Q is supported on connected polymers X, $|X| \leq 2$. We write

$$Q(X, \phi) = Q(X, \phi; C, \mathbf{w}, g) = g^2 \sum_{j=1}^{3} n_j Q^{(j,j)}(\tilde{X}, \phi; C, w^{(4-j)})$$

where $(n_1, n_2, n_3) = (48, 36, 8)$ and $\mathbf{w} = (w^{(1)}, w^{(2)}, w^{(3)})$ is a triple of integral kernels to be obtained inductively.

Here

$$X = \Delta \to \tilde{X} = \Delta \times \Delta$$

$$X = \Delta_1 \times \Delta_2 \to \tilde{X} = (\Delta_1 \times \Delta_2) \cup (\Delta_2 \times \Delta_1)$$

$$Q^{(m,m)}(\tilde{X}) = \frac{1}{2}\int_{\tilde{X}} dxdy : (\phi^m(x) - \phi^m(y))^2 :_C w^{(4-m)}(x-y) : m = 1, 2$$

$$Q^{(3,3)}(\tilde{X}) = \frac{1}{2}\int_{\tilde{X}} dx\, dy : \phi^3(x)\phi^3(y) :_C \; w^{(1)}(x-y)$$

The kernels w^j obey a recursion relation and converge rapidly to a fixed point kernel in some Banach space norm. To speed things up we just set $w = w_*$. Then $u_n = (g_n, \mu_n, R_n)$ represents a point on the RG trajectory. The RG map produces a discrete flow:

$$u_{n+1} = f(u_n)$$

The flow map in components is:

$$g_{n+1} = f_g(u_n) = L^{\epsilon} g_n(1 - L^{\epsilon} a g_n) + \xi_n(u_n)$$

$$\mu_{n+1} = f_{\mu}(u_n) = L^{\alpha}\mu_n - L^{2\epsilon} b g_n^2 + \rho_n(u_n)$$

$$R_{n+1} = f_R(u_n) =: U_{n+1}(u_n)$$

The coefficient a is positive. We have an approximate flow $\bar{g}_n$ obtained by ignoring the remainder ξ_n. This approximate flow generated by second order perturbation theory has an attractive fixed point $\bar{g} = O(\epsilon)$, for ϵ sufficiently small.

Let $\tilde{g}_n = g_n - \bar{g}$. Then $v_n = (\tilde{g}_n, \mu_n, R_n)$ are the new coordinates. Then

$$\tilde{g}_{n+1} = f_g(v_n) = (2 - L^{\epsilon})\tilde{g}_n + \tilde{\xi}_n(v_n)$$
$$\mu_{n+1} = f_\mu(v_n) = L^{\alpha}\mu_n + \tilde{\rho}_n(v_n)$$
$$R_{n+1} = f_R(v_n) =: U(v_n)$$

are the new flow equations.

$\gamma(\epsilon) = 2 - L^{\epsilon} = 1 - O(\log L)\epsilon < 1$, for sufficiently small ϵ (with L fixed. $\gamma(\epsilon)$ is a contraction factor. Also we will see that the R evolution has a contraction factor. The μ evolution is dangerous because of the L^{α} factor.

NORMS *Important fact*

Important fact : μ_C supported on subspace Ω of smooth functions. On this subspace we can give various norms (Sobolev norms, C^k norms etc thus producing (uncompleted) normed spaces. Suppose $\mathcal{L}(X, u)$ is a $\mathbb{C}$ valued bounded linear functional on $C^k(X) = C^k(\Omega|X)$ considered as a normed space. Then its dual defined with the norm $||\mathcal{L}(X)|| = \sup_{||u||_{C^k(X)} \leq 1} |\mathcal{L}(X, u)|$ is complete and thus defines a Banach space.

The proof of completeness only appeals to the completeness of the field $\mathbb{C}$.

Norms:

Large field regulator:

$$G_\kappa(X,\phi) = e^{\kappa\|\phi\|^2_{X,1,\sigma}}$$

where

$$\|\phi\|^2_{X,1,\sigma} = \sum_{1\le|\alpha|\le\sigma} \|\partial^\alpha\phi\|^2_X$$

Here $\|\phi\|_X$ is the L^2 norm and α is a multi-index. We take $\sigma > d/2+2$ so that this norm can be used in Sobolev inequalities to control ϕ and its first two derivatives pointwise. $d=3$ for the model.

Stability

$$S_L\, \mu_{\Gamma_L} \star G_\kappa(X,\phi) \leq 2^{|X|} G_\kappa(L^{-1}X,\phi)$$

Derivatives

We will measure the size of derivatives of polymer activities, by the norm:

$$\|(D^n K)(X,\phi)\| = \sup_{\|f_j\|_{C^2(X)} \leq 1\ \forall j} |(D^n K)(X,\phi; f^{\times n})|$$

This defines a bounded multinear functional on the normed space $C^2(X)$. This is the space $\Omega|X$ endowed with the C^2 norm. The space of $\mathbb{C}$ valued bounded multilinear functionals is a Banach space.

Taylor series norm

Let $h > 0$ be a real parameter. We define the following set of norms:

$$\|K(X,\phi)\|_h = \sum_{j=0}^{n_0} \frac{h^j}{j!} \|(D^j K)(X,\phi)\|$$

Convergence in the $||\cdot||_h$ *norm is the pointwise convergence of Taylor series coefficients*.

Large field norm

$$\|K(X,\phi)\|_h \leq c\, G_\kappa(X,\phi)$$

The smallest constant c defines the norm

$$\|K(X)\|_{h,G_\kappa} = \sup_{\phi\in\Omega} \|K(X,\phi)\|_h G_\kappa^{-1}(X,\phi)$$

In our model we have $\bar{g}$, the approximate fixed point of the flow $O(\epsilon)$ and we choose $h = \bar{g}^{-\frac{1}{4}}$.

Kernel norms:

$$\|K(X)\|_{h_\star} = \sum_{j=0}^{n_0} \frac{h_\star^j}{j!} \|(D^j K)(X,0)\|$$

In our model we choose $h_\star = L^{\frac{\alpha}{2}}$. Moreover $n_0 = 9$. The kernel norms are useful for controlling the remainder contributions to the flow coefficiens in the local potential.

Our final norm is

$$||K|| = \sup_{\Delta} \sum_{X \supset \Delta} \}\mathcal{A}(X)||K(X)||_{h,G_\kappa}$$

where

$$\}\mathcal{A}(X) = 2^{p|X|} L^{(d+2)|X|}$$

where $p \geq 0$. This norm gives us a Banach space of Polymer activities. It says that that bigger the polymer the smaller is its contribution.

We define small connected sets by $X : |X| \leq 2^d$. Large sets are those connected sets which are not small. $\bar{X}^L$ is the L-closure of X, the smallest connected union of L-blocks containing X. Fix $p \geq 0$ and let L be sufficiently large depending on p. We have the important property: For all connected 1-polymers X

$$\}\mathcal{A}(\bar{X}^L) \leq c_p \}\mathcal{A}(LX)$$

For X a *large* connected 1-polymer we have

$$\}\mathcal{A}(\bar{X}^L) \leq c_p L^{-d-1} \}\mathcal{A}(LX)$$

It has the *important property* : large sets contribute contracting (by a factor $L^{-(d+1)}$) contributions to the fluctuation map. Hence the relevant (expanding) parts have to be only extracted from small sets : $|X| \leq 2^d$.

We measure the polymer activity R_n in a norm $|||\cdot|||$, where

$$|||R_n|| = max\{|R_n|, \epsilon^2||R_n||\}$$

Define a Banach space E consisting of elements $v = (\tilde{g},\ \mu,\ R)$ with the (box) norm

$$||v|| = \max\{(\epsilon)^{-3/2}|\tilde{g}|, \epsilon^{-(2-\delta)}|\mu|, \epsilon^{-(11/4-\eta)}|||R|||\}$$

where $\delta, \eta > 0$ are very small numbers.

Let $v_n = (\tilde{g}_n,\ \mu_n,\ R_n)$ and let $B(r) \subset E$ be a closed ball of radius r centered at the origin. Then our next theorem says

Theorem 2.1 (stability): Let $v_n \in B(1)$. Then

$$|\tilde{\xi}(v_n)| \le C_L \epsilon^{11/4-\eta}, \quad |\tilde{\rho}(v_n)| \le C_L \epsilon^{11/4-\eta}$$

These are estimates for the error terms in the g_n, μ_n flow. Moreover $R_{n+1} = U_{n+1}(v_n)$ has the bound

$$|||U_{n+1}(v_n)||| \le L^{-1/4} \epsilon^{11/4-\eta}$$

On the right hand side we have a contraction factor.

Theorem 2.2 : (Lipshitz continuity):

Let $v,\ v' \in B(1/4)$. Then we have Lipshitz continuity:

$$|\tilde{\xi}(v) - \tilde{\xi}(v')| \leq \epsilon^{11/4-\eta}||v - v'||$$

$$|\tilde{\rho}(v) - \tilde{\rho}(v')| \leq \epsilon^{5/2-\eta}||v - v'||$$

$$|||U(v) - U(v')||| \leq L^{-1/4}\epsilon^{11/4-\eta}||v - v'||$$

Write the flow equations in integral form: after n steps of the renormalization map we get

$$\tilde{g}_k = \gamma(\epsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1} \gamma(\epsilon)^{k-1-j} \tilde{\xi}(v_j), \ \ 1 \leq k \leq n$$

and the reversed flow for μ

$$\mu_k = L^{-\alpha(n-k)} \mu_n - \sum_{j=k}^{n-1} L^{-\alpha(j+1-k)} \tilde{\rho}(v_j), \ \ 0 \leq k \leq n-1$$

We want to solve for a bounded flow. So fix $|\mu_n| \leq M$ and let $n \to \infty$ in the reversed μ flow equation. We must show that such a flow exists.

Therefore we have to solve

$$\tilde{g}_k = \gamma(\epsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1} \gamma(\epsilon)^{k-1-j} \tilde{\xi}(v_j), \ \ 1 \le k \le n$$

$$\mu_k = -\sum_{j=k}^{n-1} L^{-\alpha(j+1-k)} \tilde{\rho}(v_j), \ \ 0 \le k \le n-1$$

$$R_k = U(v_{k-1})$$

In particular

$$\mu_0 = -\sum_{j=0}^{n-1} L^{-\alpha(j+1)}\tilde{\rho}(v_j)$$

Note that this is a discrete analogue of the approximate continuous backward flow determining μ_0 we considered earlier.

Existence of bounded RG flow

We consider a Banach space **E** of sequences $\mathbf{v} = (v_0, v_1, v_2, ...)$, with $v_n \in E$, supplied with the norm

$$||\mathbf{v}|| = \sup_{n \geq 0} ||v_n||$$

$\mathbf{B}(r) \subset \mathbf{E}$ is a closed ball of radius r. Let $v_0 = (\tilde{g}_0, \mu_0, 0)$.

Theorem 3

Existence of global bounded RG trajectory:

There exists an initial mass μ_0 such that for $v_0 \in B(1/32)$, $v_k = f(v_{k-1}) \in B(1/4)$ for all $k \geq 1$.

Write the RG flow in the integral form in the space of sequences **E**:

$$v_k = F_k(\mathbf{v}) : \; F_k = (F_k^g, F_k^\mu, F_k^R)$$

where the right hand side side is defined by the right hand side of the integral flow equations. If we define the sequence

$$\mathbf{F}(\mathbf{v}) = (F_0(\mathbf{v}), F_1(\mathbf{v}), ...)$$

then the integral flow equation can be written as a fixed point equation:

$$\mathbf{v} = \mathbf{F}(\mathbf{v})$$

This fixed point equation for the entire trajectory

$$\mathbf{v} = \mathbf{F}(\mathbf{v})$$

has a unique bounded solution under the conditions of Theorem 3 by virtue of Liphsitz continuity. That the flow is bounded follows from the

Lemma 3.1 :

$$\mathbf{v} \in \mathbf{B}(\frac{1}{32}) \Rightarrow \mathbf{F}(\mathbf{v}) \in \mathbf{B}(\frac{1}{16}))$$

Use Theorem 2.1 to prove this. Unique solution now follows from Lemma 3.2 below which asserts Lipshitz continuity in a closed ball in the space of sequences:

Lemma 3.2 : For $\mathbf{v}, \mathbf{v}' \in \mathbf{B}(\frac{1}{4})$,

$$||\mathbf{F}(\mathbf{v}) - \mathbf{F}(\mathbf{v}')|| \leq \frac{1}{2}||\mathbf{v} - \mathbf{v}'||$$

Use Theorem 2.2 to prove this.

Stable manifold and non-trivial fixed point.
Let f^k be the $k-$ *fold* composition of the map f. The *stable (critical) manifold* of f is defined by

$$W^s(f) = v \in E(1/32) : \ f^k(v) \in E(1/4) \, \forall k \geq 0$$

Write $v = (v_1, v_2)$ with $v_1 = (\tilde{g}, R, 0)$ and $v_2 = \mu$. Initially $v_{1,0} = (\tilde{g}_0, 0, 0)$ and $v_{2,0} = \mu_0$. Theorem 3 says that for $v \in E(1/32)$, there exists v_2 such that $f^k(v) \in E(1/4) : \ \forall k \geq 0$.

Theorem 4 (Stable manifold theorem)

$W^s(f)$ is the graph $\{v_1, h(v_1)\}$ of a function $v_2 = h(v_1)$ with h Lipshitz continuous. Moreover iterations of f restricted to $W^s(f)$ contracts distances and ttherefore has a unique fixed point which attracts all points of $W^s(f)$.

Corollary: The theorem has established that the critical mass $\mu_0 = h(\tilde{g}_0) = \mu_c(g_0)$ is a Lipshitz continuous function. Moreover $v_n \to v_*$ in the ball $E(1/4)$. If $\tilde{g}_* = g * -\bar{g}$ is one of the coordinates of v_* then $g_* \neq 0$ since $v_* \in E(1/4)$ and therefore

$$|g_* - \bar{g}| \leq \frac{1}{4}\epsilon^{3/2}$$

and we know that $\bar{g} = O(\epsilon)$. So our fixed point is nontrivial.

Correlation functions

We have seen the convergence of the coordinates $v_n = (g_n, \mu_n, R_n$ of the RG trajectory to the fixed point $v_* = (g_*, \mu_*, R_*)$ in the Banach space E provided the initial mass μ_0 is chosen to lie on the critical curve $\mu_0 = \mu_c(g_0)$ with initial $R_0 = 0$. This is sufficient to prove the existence of the ultraviolet (scaling limit) for correlation functions.

Recall: $C_{\epsilon_0} = C$ is the unit cutoff covariance. Let $j(s)(x)$ be a test function in $\mathbb{R}^3$. It is a C^∞ function of compact support, and s is a collection of parameters. We define the generating function of the ϵ cutoff theory by

$$
\begin{aligned}
Z_{\epsilon_N}(\Lambda, j(s)) &= \int d\mu_C(\phi)\, e^{-V_0(\Lambda,\, S_{\epsilon_N}\phi,\, \tilde{\xi}_N,\, \tilde{g}_N,\, \tilde{\mu}_N)+\phi(j)} \\
&= \int d\mu_C(\phi) e^{-V_0(\Lambda_N,\, \phi,\, \xi\, g,\, \mu)+\phi(j)} \\
&= Z(\Lambda_N, j_N) \qquad (19)
\end{aligned}
$$

where $\Lambda_N = [-L^N \frac{R}{2}, L^N \frac{R}{2}]^d$ and

$$
j_N(s)(x) = \epsilon_N^{d-[\phi]} j(s)(\epsilon_N x) = L^{-N(d-[\phi])} j(s)(L^{-N} x)
$$

Translating in the field ϕ gives

$$Z(\epsilon_N, \Lambda_0, j(s)) = e^{-1/2(j_N, C * j_N)} \int d\mu_C(\phi)\, \mathcal{Z}_0(\Lambda_N,\ \phi + C * j_N)$$

Applying the RG transformation once gives:

$$Z(\epsilon_N, \Lambda_0, j(s)) = e^{-1/2(j_N(s), C * j_N(s))} \int d\mu_C(\phi) \mathcal{Z}_1(\Lambda_{N-1}, \phi + S_{L^{-1}} C * j_N(s)$$

where

$$S_{L^{-1}}(C * j_N(s))(x) = L^{[\phi]}(C * j_N(s))(Lx)$$

Iterating N times gives

$$Z(\epsilon_n, \Lambda_0, j(s)) = e^{-1/2(j_N(s), C*j_N)(s))} \int d\mu_C(\phi)\mathcal{Z}_N(\Delta, \phi + S_{L^{-N}} C*j_N(s))$$

where Δ is a unit block (unit cube). Easy to check

$$(j_N(s), C * j_N)(s)) = (j(s), C_{\epsilon_N} j(s))$$

$$S_{L^{-N}} C * j_N(s)(x) = (C_{\epsilon_N} * j)(x)$$

We will look at the 2-point function: Let $j(s) = s_1 j_1 + s_2 j_2$. Taking partial derivatives with respect to s_1, s_2 at $s_1 = s_2 = 0$ and dividing out by the normalizing factor (vacuum energy) gives

$$< \phi(j_1)\phi(j_2) >_{\epsilon_N,\Lambda_0} = (j_1, C_{\epsilon_N} * j_2) -$$

$$\frac{1}{Z_N(0)} \int d\mu_C(\phi) \, (D^2 \mathcal{Z}_N)(\Delta, \phi; C_{\epsilon_N} * j_1, C_{\epsilon_N} * j_2)$$

$$Z_N(0) = \int d\mu_C(\phi) \mathcal{Z}_N(\Delta, \phi)$$

Since we are on a unit block we have simple expressions

$$\mathcal{Z}_N(\Delta,\phi) = e^{\Omega_N}\,\bar{\mathcal{Z}}_N(\Delta,\phi)$$

$$\bar{\mathcal{Z}}_N(\Delta,\phi) = e^{-V_N(\Delta,\phi)} + K_N(\Delta,\phi)$$

and Ω_N is the total extracted vacuum energy which divides out in the normalized Schwinger functions.

As $N \to \infty$, we have $C_{\epsilon_N} \to C_f$ the free continuum covariance with no cutoff. Moreover $V_N \to V_*$ where $V_*(\Delta, \phi) = V(\Delta, \phi, g_*, \mu_*)$ and $K_N \to K_*$ in an open ball in the Banach space E. The norms are such that the two functional derivatives of K_* smeared with test functions are easily estimated. In fact, j is a C^∞ function of compact support in $\mathbb{R}^3$, and C_f is in L^1_{loc}. Let $\mathrm{supp}\, j \subset U$ where U is a compact set. Then it is easy to see

$$||(C_f \star j)||_{C^2(\Delta)} \leq ||C_f||_{L^1(U)}||j||_{C^2(\mathbb{R}^3)}$$

We can estimate the two functional derivatives on $K_\star$ by

$$const.\epsilon^{1/2}\, ||K_\star||_{h,G,\mathcal{A}}||j_1||_{C^2}||j_2||_{C^2}$$

and

$$||K_\star||_{h,G,\mathcal{A}} \leq ||Q_\star e^{-V_\star}||_{h,G,\mathcal{A}} + ||R_\star||_{h,G,\mathcal{A}}$$
$$\leq const.(\epsilon^{1/2} + \epsilon^{7/4})$$

by estimates obtained in [BMS].

Hence the term with two functional derivatives on $K_\star$ is estimated as

$$const.\, \epsilon \,||j_1||_{C^2}||j_2||_{C^2}$$

The derivatives on V_* can be similarly estimated so that the correction term to the free covariance is of $O(\epsilon)$. A more careful estimate by arranging the supports of j_i appropriately actually shows that the correction term decays faster than the free covariance. The upshot is that the ultraviolet cutoff limit $N \to \infty$ or $\epsilon_N \to 0$ exists for the connected truncated Schwinger functions and is dominated by the free covariance.