



HAL
open science

Schlesinger Foliation for Deformations of Foliations

Yohann Genzmer

► **To cite this version:**

Yohann Genzmer. Schlesinger Foliation for Deformations of Foliations. International Mathematics Research Notices, 2017, 10.1093/imrn/rnw316 . hal-01934801

HAL Id: hal-01934801

<https://hal.science/hal-01934801>

Submitted on 26 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SCHLESINGER FOLIATION FOR DEFORMATIONS OF FOLIATIONS.

YOHANN GENZMER

ABSTRACT. In this article, we show that for any deformation of analytic foliations, there exists a maximal analytic singular foliation on the space of parameters along the leaves of which the deformation is integrable.

This work was partially supported by ANR-13-JS01-0002-0.

1. INTRODUCTION AND STATEMENTS.

The Painlevé VI equation [1], that motivates this work, is a non-linear differential equation of order two that parametrizes the isomonodromic deformations of linear Fuchsian systems over the four-punctured complex sphere. This a particular case of the Schlesinger systems that parametrizes the integrable deformations of Garnier systems. In both case, a foliation on the space of parameters of the deformation whose leaves parametrize the integrable deformations is described.

Our purpose is to highlight this phenomenon in a general context.

Theorem. *Let $(X^m, \mathcal{F}^n) \xrightarrow{\pi} B^p$ be a proper deformation of analytic foliations with B^p as space of parameters. Then there exists a unique analytic singular foliation \mathcal{H} on B^p of maximal dimension among those that integrates the deformation. In particular, \mathcal{H} satisfies the following property: for any leaf L of \mathcal{H} , the restricted deformation*

$$(X^m, \mathcal{F}^n)|_{\pi^{-1}(L)} \xrightarrow{\pi} L$$

is integrable.

The definition of a *foliation integrating* a given deformation will appear below.

If we remove the maximality property, then the theorem becomes trivial. Indeed, the foliation of B^p by points satisfies its conclusion. Moreover, for a generic deformation, the foliation \mathcal{H} produced by the result above will be indeed the foliation by points. Nevertheless, in view of the example mentionned in the introduction, the foliation \mathcal{H} deserves to be called *the Schlesinger foliation* of the deformation. Finally, the main theorem holds in the real analytic class as well as in the complex one.

It might be possible that along some exceptional curves transverse to \mathcal{H} , the deformation is also integrable. Such a curve has to be more or less *isolated*: they cannot foliate the manifold B^p even locally. For instance, in the framework of the theory of Fuchsian systems, consider the six parameters family defined by

$$(1.1) \quad \frac{d}{dz} = \frac{A_1}{z - u_1} + \frac{A_2}{z - u_2} + \frac{A_3}{z - u_3}$$

where $A_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}$. Since the matrices A_i commute, the Schlesinger system [9, 10] reduces to

$$\frac{\partial A_i}{\partial u_j} = 0, \quad i, j = 1, 2, 3.$$

Thus, the Schlesinger foliation of this deformation is given by $da_1 = da_2 = da_3 = 0$. However, if $a_1 = a_2 = 0$ and $a_3 \neq 0$, then (1.1) degenerates toward a system which is conjugated to

$$\frac{d}{dz} = \frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{z - u_3}.$$

Therefore, at any point $(0, 0, a_3)$ with $a_3 \neq 0$, the family (1.1) is still integrable along the whole submanifold $a_1 = a_2 = 0$, which is not a leaf of the Schlesinger foliation. Hence, in general, the leaf of \mathcal{H} does not parametrize the maximal submanifold along which the deformation is integrable.

The main theorem can be compared to the following result of Kiso [4]: let L be a Lie algebra of holomorphic vector fields on a complex manifold $N \times M$ tangent to the projection on M . If L is simple and of finite dimension then there exists a maximal foliation on N along the leaves of which L is integrable. In our context, the Lie algebra underlying the foliation \mathcal{F}^n is in general not of finite dimension. Moreover, the fibration π is not trivial: the manifold supporting the foliation can also be deformed.

2. DEFORMATION OF FOLIATIONS AND KODAIRA-SPENCER MAP.

2.1. Deformations of foliations. Let X^m be an analytic manifold and Θ_{X^m} its tangent sheaf. Let E be a coherent subsheaf of Θ_{X^m} . The stalk of E at any point p is finitely generated as $\mathcal{C}^\omega(X^m)_p$ -module - the analytic functions on B^p - and we denote by $\text{rank}_p(E)$ the minimal number of elements of a generating family. The *singular locus* of E is defined by

$$\text{Sing}(E) = \{p \in X^m \mid \dim_{\mathbb{R}} \{v(p) \mid v \in E_p\} < \text{rank}_p(E)\}.$$

Since E is coherent, the singular locus is an analytic subset of X^m [7]. The sheaf is said *regular* if $\text{Sing}(E) = \emptyset$. The integer

$$d = \max_{p \in X^m} \dim_{\mathbb{R}} \{v(p) \mid v \in E_p\} \leq m.$$

is called the dimension of E and $m - d$ its codimension.

Definition 1. An analytic foliation on X^m is a coherent subsheaf \mathcal{F} of Θ_{X^m} which is *integrable*, i.e.,

$$[\mathcal{F}_p, \mathcal{F}_p] \subset \mathcal{F}_p \quad \text{for any } p \in X^m \setminus \text{Sing}(\mathcal{F}).$$

Definition 2. A deformation of foliation, denoted by $(X^m, \mathcal{F}^n) \xrightarrow{\pi} B^p$, is the data of a proper map $\pi : X^m \rightarrow B^p$ that is an analytic deformation of manifold [5] and of a regular foliation \mathcal{F}^n tangent to the fibers of π , i.e., $\mathcal{F}^n \subset \ker d\pi$. The integer m is the dimension of X^m , p the dimension of B^p and n the dimension of \mathcal{F}^n .

The following lemma is a direct consequence of the Frobenius theorem [2] and states that a deformation of foliation is locally trivial in the total space.

Lemma 3. Let $(X^m, \mathcal{F}^n) \xrightarrow{\pi} B^p$ be a deformation of foliation and $x \in X^n$. There exists an isomorphism ϕ of deformations of foliation defined on an open neighborhood $U \ni x$ such that the following diagram commutes

$$\begin{array}{ccc} (U, \mathcal{F}^n|_U) & \xrightarrow{\phi} & (\mathbb{R}^{m-p} \times B^p, L^n) \\ & \searrow \pi & \swarrow \text{pr}_2 \\ & \pi(U) & \end{array}$$

In the second term of the above commutative diagram, the foliation L^n of $\mathbb{R}^{m-p} \times B^p$ is given by the fibers of the projection

$$\Pi : \begin{cases} \mathbb{R}^{m-p} \times B^p & \rightarrow \mathbb{R}^{m-p-n} \times B^p \\ (x_1, \dots, x_{m-p}, \tau) & \rightarrow (x_1, \dots, x_{m-p-n}, \tau) \end{cases} .$$

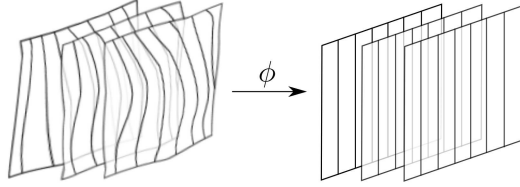


FIGURE 2.1. Local trivialization of a deformation of foliation.

Definition 4. Let $(X^m, \mathcal{F}^n) \xrightarrow{\pi} B^p$ be a deformation of foliation. A foliation \mathcal{H} of B^p is said to integrate the deformation if, for any point $\tau \in B^p \setminus \text{Sing}(\mathcal{H})$, there exists a neighborhood $U \ni \tau$ and a regular foliation \mathcal{G} in $\pi^{-1}(U)$ such that $\dim \mathcal{G} = n + \dim \mathcal{H}$ and

$$\mathcal{F}^n|_{\pi^{-1}(U)} \subset \mathcal{G} \subset \pi^* \mathcal{H}|_U .$$

In particular, for any leaf L of \mathcal{H} , the restricted deformation

$$\pi : \left(\pi^{-1}(L), \mathcal{F}^n|_{\pi^{-1}(L)} \right) \rightarrow L$$

is completely integrable. In Figure (2.1), the deformation is locally completely integrable: the foliation \mathcal{H} has one leaf that is \mathcal{H} itself and the foliation \mathcal{G} of the definition above is given in the coordinates of Lemma 3 by the fibers of the projection

$$p : \begin{cases} \mathbb{R}^{m-p} \times B^p & \rightarrow \mathbb{R}^{m-p-n} \\ (x_1, \dots, x_{m-p}, \tau) & \rightarrow (x_1, \dots, x_{m-p-n}) \end{cases} .$$

Our goal is to show the existence of a unique maximal integrating foliation.

2.2. The sheaf of basic vector fields.

Definition 5. A vector field v is said to be basic for \mathcal{F} if and only if

$$[v, \mathcal{F}] \subset \mathcal{F} .$$

Definition 6. A vector field v is said to be projectable if and only if there exists a vector field w on B^p such that the following diagram commutes

$$\begin{array}{ccc} X^m & \xrightarrow{\pi} & B^p \\ \downarrow v & & \downarrow w \\ TX^m & \xrightarrow{d\pi} & TB^p \end{array} .$$

It is said to be vertical if $w = 0$, i.e., $d\pi(v) = 0$.

We denote by \mathcal{B}^π the sheaf of basic and projectable vector fields and \mathcal{B}^0 the subsheaf of basic and vertical vector fields. The sheaf \mathcal{B}^0 is a sheaf of modules over the ring of local first integrals of \mathcal{F}^n . In the coordinates given by Lemma 3, a section of the quotient sheaf $\mathcal{B}^0/\mathcal{F}^n$ is written

$$\sum_{i=1}^{m-p-n} a(x_1, \dots, x_{m-p-n}, \tau) \frac{\partial}{\partial x_i} .$$

Hence, $\mathcal{B}^0/\mathcal{F}^n$ is a free. It will be more convenient to work with a $\mathcal{C}^\omega(X^m)$ -module, which is why, we will consider the product $\mathcal{B}^0/\mathcal{F}^n \otimes \mathcal{C}^\omega(X^m)$. It is also a locally free sheaf over $\mathcal{C}^\omega(X^m)$.

2.3. The basic Kodaira-Spencer map. Let w be a vector field on B^p defined on a small open set W of B^p . According to Lemma 3, there exists a covering $\{U_i\}_{i \in I}$ of a neighborhood in X^n of $\pi^{-1}(W)$ such that the deformation is trivialized on any U_i by some conjugacy ϕ_i . For any U_i , the vector field $v_i = d\phi_i^{-1}(0, w)$ is a section of \mathcal{B}^π on U_i that projects on w . Considering the image of the family $\{v_i \otimes 1 - v_j \otimes 1\}_{i,j}$ in $H^1(\pi^{-1}(W), \mathcal{B}^0/\mathcal{F}^n \otimes \mathcal{C}^\omega(X^m))$, we obtain a $\mathcal{C}^\omega(B^p)$ -morphism of sheaves

$$\partial\mathcal{F}^n : \Theta_{B^p} \rightarrow R^1\pi_* \left(\mathcal{B}^0/\mathcal{F}^n \otimes \mathcal{C}^\omega(X^m) \right)$$

which is called the *basic Kodaira-Spencer map* of the deformation. Notice that this is not the standard Kodaira-Spencer map as defined in [6] since, it does not measure the infinitesimal directions of triviality, but the infinitesimal directions of integrability.

Lemma 7. $\ker \partial\mathcal{F}^n$ is a foliation of B^p .

Proof. The sheaf $\mathcal{B}^0/\mathcal{F}^n \otimes \mathcal{C}^\omega(X^m)$ is a coherent sheaf of $\mathcal{C}^\omega(X^m)$ -modules. Since π is proper, $R^1\pi_* \mathcal{B}^0/\mathcal{F}^n \otimes \mathcal{C}^\omega(X^m)$ is a coherent sheaf of $\mathcal{C}^\omega(X^m)_{B^p}$ -modules, and so is $\ker \partial\mathcal{F}^n$ [3]. Suppose w_1 and w_2 belongs to the kernel of $\partial\mathcal{F}^n$. By construction, there exist two families $\{v_i^\epsilon\}_i$, $\epsilon = 1, 2$ of sections of \mathcal{B}^π such that for any i and ϵ , the vector field v_i^ϵ projects on w_ϵ and such that for any i, j and ϵ ,

$$v_i^\epsilon - v_j^\epsilon = t_{ij}^\epsilon \in \mathcal{F}^n .$$

The difference of their Lie brackets is written

$$[v_i^1, v_i^2] - [v_j^1, v_j^2] = [v_j^1, t_{ij}^2] - [v_i^2, t_{ij}^1] + [t_{ij}^1, t_{ij}^2]$$

Since $[\mathcal{B}^\pi, \mathcal{F}^n] \subset \mathcal{F}^n$ and since the map $d\pi$ commutes with the Lie bracket, one has

$$\partial\mathcal{F}^n([w_1, w_2]) = \overline{\{[v_i^1, v_i^2] \otimes 1 - [v_j^1, v_j^2] \otimes 1\}_{i,j}} = 0 .$$

So $\ker \partial \mathcal{F}^n$ is integrable and thus is a foliation. \square

3. INTEGRATING FOLIATIONS OF DEFORMATIONS.

Consider the trivial projection $\mathbb{R}^{m-p} \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ where the source is foliated by L^n given by the fibers of

$$\begin{cases} \mathbb{R}^{m-p} \times \mathbb{R}^p & \rightarrow & \mathbb{R}^{m-p-n} \times \mathbb{R}^p \\ (x_1, \dots, x_{m-p}, \tau) & \rightarrow & (x_1, \dots, x_{m-p-n}, \tau). \end{cases}$$

Let $\{T_i\}_{i=1, \dots, l}$ be an involutive family of germs of vector fields inducing a germ of regular foliation of dimension l in (\mathbb{R}^p, τ) . The family

$$\{\partial_{x_{m-p-n+1}}, \dots, \partial_{x_{m-p}}, T_1, \dots, T_l\}$$

where T_i is seen as vector field in $\mathbb{R}^{m-p} \times \mathbb{R}^p$ is involutive and defines an integrating dimension $n+l$ foliation \mathcal{G} . Notice that \mathcal{G} is not unique but does depend only on the *basic part* of T_i : for any family of vector fields $\{X_i\}_{i=1..l}$ tangent to $\{\partial_{x_{m-p-n+1}}, \dots, \partial_{x_{m-p}}\}$, the distribution

$$\{\partial_{x_{m-p-n+1}}, \dots, \partial_{x_{m-p}}, T_1 + X_1, \dots, T_l + X_l\}$$

induces the same foliation \mathcal{G} . This remark is the key of the proof of the propositions below.

Proposition 8. *Let $(X^m, \mathcal{F}^n) \xrightarrow{\pi} B^p$ be a deformation of foliation and w be a vector field in B^p defined near τ with $w(\tau) \neq 0$. The two following properties are equivalent:*

- (1) *the foliation induced by w integrates the deformation.*
- (2) $\partial \mathcal{F}^n(w) = 0$.

Proof. Suppose that there exists a regular foliation \mathcal{G} of dimension $n+1$ in X^m such that $\mathcal{F}^n|_{\pi^{-1}(U)} \subset \mathcal{G} \subset \pi^* \mathcal{H}$ where \mathcal{H} is the foliation of B^p induced by w in a neighborhood $U \ni \tau$. Applying the Frobenius result to \mathcal{G} yields a covering $\{U_i\}_{i \in I}$ of $\pi^{-1}(U)$ and a family of conjugacies $\{\phi_i\}_{i \in I}$ such that the following diagram commutes

$$\begin{array}{ccc} (U_i, \mathcal{G}|_{U_i}) & \xrightarrow{\phi_i} & (\mathbb{R}^{m-p} \times B^p, \mathbb{R}^{m-p} \times \mathcal{H}) \\ & \searrow \pi & \swarrow \text{pr}_2 \\ & \pi(U_i) & \end{array}$$

By construction, the vector field $v_i = d\phi_i^{-1}(0, w)$ is basic for \mathcal{F}^n and projects on w . Thus $\partial \mathcal{F}^n(w) = \overline{\{v_i \otimes 1 - v_j \otimes 1\}_{i,j}}$. Moreover, $v_i - v_j$ is vertical and tangent to \mathcal{G} . Thus, it is also tangent to \mathcal{F}^n . Hence, $\partial \mathcal{F}^n(w) = 0$.

Now, suppose that $\partial \mathcal{F}^n(w) = 0$. For a covering $\{U_i\}_{i \in I}$ of a neighborhood of $\pi^{-1}(U)$, there exists a family of projectable basic vector fields $\{v_i\}_{i \in I}$, $v_i \in \Theta_{X^n}(U_i)$ such that each v_i projects on w and such that $v_i - v_j$ is tangent to \mathcal{F}^n . In the local coordinates given by Lemma 3, the vector field v_i is written

$$v_i = \sum_{i=1}^{m-p-n} a_i(x_1, \dots, x_{m-p-n}, \tau) \frac{\partial}{\partial x_i} + \sum_{i=m-p-n+1}^{m-p} a_i(x, \tau) \frac{\partial}{\partial x_i} + w.$$

Since

$$\left[v_i - \sum_{i=m-p-n+1}^{m-p} a_i(x, \tau) \frac{\partial}{\partial x_i}, \partial_{x_k} \right] = 0$$

for $k = m-p-n+1, \dots, m-p$, the family of vector fields $\{\partial_{x_{m-p-n+1}}, \dots, \partial_{x_{m-p}}, v_i\}$ is involutive and defines a local regular integrating foliation \mathcal{G}_i of dimension $n+1$. The foliation \mathcal{G}_i depends only on the basic part of v_i and $v_i - v_j$ is tangent to \mathcal{F}^n , thus the family $\{\mathcal{G}_i\}_{i \in I}$ can be glued in a global foliation that integrates the deformation over U . \square

Now, we can prove the main result of this article.

Theorem. *Let $(X^m, \mathcal{F}^n) \xrightarrow{\pi} B^p$ be a deformation of foliation. Then, there exists a unique singular analytic foliation \mathcal{D} on B^p of maximal dimension among those that integrate the deformation.*

Proof. Let us consider the sub-sheaf \mathcal{D} of Θ_{B^p} defined by $v \in \mathcal{D}$ if and only the foliation defined by v integrates the deformation. According to Proposition 8, \mathcal{D} is a foliation. By construction, if \mathcal{D} has the property of the theorem, it will be of maximal dimension for that property. Let d be its dimension and consider a point p in its regular locus. The sheaf \mathcal{D} is locally around p spanned by a family of exactly d non-vanishing vector fields $\mathcal{D}(U) = \langle w_1, \dots, w_d \rangle$, $U \ni p$. Now, for some covering $\{U_i\}_{i \in I}$ of a neighborhood of $\pi^{-1}(U)$, there exists a family of projectable basic vector fields $\{v_{i,k}\}_{i \in I, k=1 \dots d}$, $v_{i,k} \in \Theta_{X^n}(U_i)$ such that $v_{i,k}$ projects on w_k and $v_{i,k} - v_{j,k}$ is tangent to \mathcal{F}^n . In the local coordinates given by Lemma 3, the family of vector fields

$$\{\partial_{x_{m-p-n+1}}, \dots, \partial_{x_{m-p}}, v_{i,1}, \dots, v_{i,d}\}$$

is involutive and defines a family of local integrating foliations $\{\mathcal{G}_i\}_{i \in I}$ of dimension $n+d$ that can be glued. Finally, we obtain a global regular integrating foliation that integrates the deformation over U . \square

Example 9. *Linear foliations on complex tori of dimension 2.*

A complex torus of dimension 2 is given by a lattice Λ written

$$\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$$

where $\left\{ e_i = \begin{pmatrix} e_{i1} \\ e_{i2} \end{pmatrix} \right\}_{i=1 \dots 4}$ is a \mathbb{R} -free family of four vectors in \mathbb{C}^2 . The complex torus \mathbb{C}^2/Λ is endowed with a linear foliation given by the closed 1-form

$$\omega_\alpha = dx + \alpha dy$$

with $\alpha \in \mathbb{P}^1$. The linear foliations on complex tori of dimension 2 form a 9-dimensional family of foliations whose space of parameters is $\mathcal{U} \times \mathbb{P}^1$. Here, \mathcal{U} is the real Zariski open set in \mathbb{C}^8 given by the equation

$$\det \begin{pmatrix} \Re e_1 & \Re e_2 & \Re e_3 & \Re e_4 \\ \Im e_1 & \Im e_2 & \Im e_3 & \Im e_4 \end{pmatrix} \neq 0.$$

An explicit computation shows that the Schlesinger foliation of this deformation is given by the closed system

$$(3.1) \quad \mathcal{H} : \begin{cases} d \left(\frac{e_{11} + e_{12}\alpha}{e_{21} + e_{22}\alpha} \right) = 0 \\ d \left(\frac{e_{11} + e_{12}\alpha}{e_{31} + e_{32}\alpha} \right) = 0 \\ d \left(\frac{e_{11} + e_{12}\alpha}{e_{41} + e_{42}\alpha} \right) = 0 \end{cases} .$$

It is regular on the space of parameters $\mathcal{U} \times \mathbb{P}^1$. This result can also be seen as follows: the representation of monodromy of ω_α on \mathbb{C}^2/Λ computed on the transversal line $x = 0$ is written

$$\pi_1(\mathbb{C}^2/\Lambda) \simeq \mathbb{Z}^4 \rightarrow \text{Aut}(\mathbb{CP}^1) : \gamma_i \rightarrow (y \rightarrow y + e_{i1} + e_{i2}\alpha), \quad i = 1, 2, 3, 4$$

A change of global coordinates $y \rightarrow ay + b$ on the transversal line $x = 0$ acts on the monodromy the following way

$$y \rightarrow y + a(e_{i1} + e_{i2}\alpha), \quad i = 1, 2, 3, 4$$

Hence, the conjugacy class of the representation of monodromy is constant along a deformation if and only if the quotients $\frac{e_{11} + e_{12}\alpha}{e_{i1} + e_{i2}\alpha}$, $i = 2, 3, 4$ are constant, which is precisely the condition 3.1. Besides that, in this example, any leaf of the Schlesinger foliation parametrizes the maximal locus of integrability for any point in the space of parameters.

The foliation \mathcal{H} can be compactified as an algebraic foliation $\overline{\mathcal{H}}$ on $\mathbb{CP}^8 \times \mathbb{CP}^1$ and the fibration

$$\begin{array}{ccc} \mathbb{CP}^8 \times \mathbb{CP}^1 & \rightarrow & \mathbb{CP}^6 \\ (\{e_{ij}\}_{i,j}, \alpha) & \rightarrow & (e_{11} + e_{12}\alpha, e_{12}, e_{22}, e_{32}, e_{41}, e_{42}) \end{array}$$

is transverse to $\overline{\mathcal{H}}$ except over the critical locus $S = \{(e_{11} + e_{12}\alpha)e_{42} = 0\} \subset \mathbb{CP}^6$. This fibration has compact fibers. Thus, according to the Ehresmann's theorem, the foliation $\overline{\mathcal{H}}$ has the *Painlevé property* with respect to that fibration as defined in [8]. Since, the Painlevé VI has also the same property, it is natural to address the following problem:

Consider an algebraic deformation of an algebraic foliation. Has the Schlesinger foliation the Painlevé property for an adapted fibration?

REFERENCES

- [1] Philip Boalch. Brief introduction to Painlevé VI. In *Groupes de Galois arithmétiques et différentiels*, volume 13 of *Sémin. Congr.*, pages 69–77. Soc. Math. France, Paris, 2006.
- [2] Dominique Cerveau. Distributions involutives singulières. *Ann. Inst. Fourier (Grenoble)*, 29(3):xii, 261–294, 1979.
- [3] H. Grauert and R. Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.
- [4] Kazuhiro Kiso. Local properties of intransitive infinite Lie algebra sheaves. *Japan. J. Math. (N.S.)*, 5(1):101–155, 1979.
- [5] K. Kodaira and D. C. Spencer. On deformations of complex analytic structures. I, II. *Ann. of Math. (2)*, 67:328–466, 1958.

- [6] Kunihiko Kodaira. *Complex manifolds and deformation of complex structures*, volume 283 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1986. Translated from the Japanese by Kazuo Akao, With an appendix by Daisuke Fujiwara.
- [7] Yoshiki Mitera and Junya Yoshizaki. The local analytical triviality of a complex analytic singular foliation. *Hokkaido Math. J.*, 33(2):275–297, 2004.
- [8] Kazuo Okamoto. Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé. *Japan. J. Math. (N.S.)*, 5(1):1–79, 1979.
- [9] Claude Sabbah. *Isomonodromic deformations and Frobenius manifolds*. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, french edition, 2007. An introduction.
- [10] L. Schlesinger. Über eine klasse von ordnung differentialsystemen beliebiger avec festen kritischen punkten. *J. für Math.*, (141):96–145, 1912.

Y. Genzmer

Institut de Mathématiques de Toulouse

Université Paul Sabatier

118 route de Narbonne

31062 Toulouse cedex 9, France.

yohann.genzmer@math.univ-toulouse.fr