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MODELLING AND NUMERICAL APPROXIMATION FOR THE NONCONSERVATIVE BITEMPERATURE EULER MODEL

D. Aregba-Driollet[1], J. Breil[2], S. Brull[3], B. Dubroca[4] and E. Estibals[5]

Abstract. This paper is devoted to the study of the nonconservative bitemperature Euler system. We firstly introduce an underlying two species kinetic model coupled with the Poisson equation. The bitemperature Euler system is then established from this kinetic model according to an hydrodynamic limit. A dissipative entropy is proved to exist and a solution is defined to be admissible if it satisfies the related dissipation property. Next, four different numerical methods are presented. Firstly, the kinetic model gives rise to kinetic schemes for the fluid system. The second approach belongs to the family of the discrete BGK schemes introduced by Aregba-Driollet and Natalini. Finally, a quasi-linear relaxation approach and a Lagrange-remap scheme are considered.

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1. Introduction

The present paper is devoted to the approximation of a nonconservative bitemperature compressible Euler system by different numerical schemes.

This fluid model consists of two conservation equations for mass and momentum and two nonconservative equations, that is to say, one for each energy. Physically, this model describes the interaction of a mixture of one species of ions and one species of electrons in thermal nonequilibrium. The pressure of each species is supposed to satisfy a gamma-law with its own γ constant.

Solving nonconservative hyperbolic systems is a delicate problem because the definition of weak admissible solutions remains unclear. In order to define nonconservative products, Dal Maso, Le Floch and Murat proposed in [18] a new theory based on the definition of family of paths. However, it is shown, in [1], that even if the correct path is known, the numerical solution can be far from the expected solution. In [6], the authors consider an hyperbolic system having $n-p$ equations in a conservative form, the remaining p equations being nonconservative. In [30], path-conservative schemes are defined by using the concepts developped in [18]. In [31], the authors use a Roe solver, and an HLLC solver which neglects the nonconservative part of the system. They validate the approach by comparing their results to theoretical temperatures/pressures curves ([35], [34]).

Keywords and phrases: Relaxation method, nonconservative hyperbolic system, kinetic schemes, BGK models, hydrodynamic limit, entropy dissipation

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In [17], the authors assume that the electronic entropy is conserved by all weak solutions including shocks and the system is approached by a system of conservation laws.

The bitemperature Euler system can be constructed from an underlying kinetic model which consists of a BGK model coupled with Poisson equation in the quasi-neutral regime. This BGK model possesses different interspecies collision frequencies in order to take into account discrepancies in the particle masses. This point has been in particular mentionned in [23], where a conservative formulation is proposed. Hence, following this idea we obtain such a model satisfying a H theorem. Next, performing an hydrodynamic limit, we get the nonconservative Euler system. To take into account the different γ constants, we should consider polyatomic kinetic models. We do not make so in order to avoid heavy technical proofs. However, the ideas are analogous to the monoatomic case. We show that the fluid model owns a dissipative strictly convex entropy which can be recovered from the Boltzmann entropy. In this paper, we consider the entropic solutions of the nonconservative Euler system which are the hydrodynamic limits of solutions of the kinetic system. Using the Boltzmann entropy, we prove that the hydrodynamic limits are entropy-dissipative. Moreover this approach leads at the discrete point of view to numerical schemes which are consistent with the physics.

In another way, the present bitemperature Euler system is approached by using numerical methods inspired from the approximation of the Euler system. The first one is a kinetic scheme ([32]) which is based on the underlying kinetic model. The second method is based on discrete BGK schemes introduced in [4]. One important point of this paper is their generalization to the present nonconservative setting and the fact that those models provide entropy-dissipative solutions. We also prove discrete entropy inequalities for the related numerical schemes. The third method is based on a Suliciu relaxation approach for the pressure variables ([16]). The interest of such an approach is that all the characteristic fields of the related homogeneous system are linearly degenerate and hence the Riemann solver is easier to handle. Finally, the last method of the paper is a Lagrangian method ([26]) based on Lagrangian and projection steps. This method is used in the multi-physics CHIC code ([10]) to solve a bitemperature model in the field of inertial confinement fusion (ICF) and high energy density physics (HEDP). Lagrangian formalism is well suited to deal with the multi-material flow encountered in the field of ICF and numerous papers devoted to laser plasma interaction use this code ([29]).

This paper is organised as follows. The second section is dedicated to the physical models that are involved in this paper. Firstly Euler bitemperature macroscopic model is given and we consider the Vlasov-BGK model from which the Euler system is derived. Starting from an ad-hoc scaling the construction of the Euler system is performed. Next, we study the entropy properties of this system. In section 3, we design the different numerical schemes and in section 4, some numerical results are presented. The goal of those first tests is to compare the schemes on 1D examples. It is out of the scope of this paper to propose higher order improvments or multi-dimensionnal computations. Finally, section 5 deals with some conclusions and perspectives to this paper.

2. The physical models

This part is devoted to the presentation of the different physical models that are involved in this paper. Firstly, we give the Euler bitemperature system, the underlying kinetic model and next we derive the fluid model from an hydrodynamic limit. For the sake of simplicity we consider a new BGK model to represent the interaction between ions and electrons coupled with Poisson equation considered at the quasi-neutral regime. Then we prove the existence of a strictly convex dissipative entropy for the fluid model.

There are many BGK models devoted to gas mixtures ([2], [13], [11], [21], [24], [27]). However except ([2], [13], [11]), there are few BGK models for gas mixtures enjoying fundamental properties (H theorem, equilibrium states, nonegativity of the distribution function, ...) inherited from the Boltzmann operator ([3]). In order to take into account disparate masses, we consider in this paper the situation where the interspecies collision frequencies are different. This point has been in particular mentionned in ([23]) where a conservative formulation is proposed. Here we are able to prove an H-theorem and to make the link with the macroscopic entropy. Moreover, one originality of this work is to take into account the electric field in the rescaled kinetic model.

2.1. The Euler bitemperature model

The nonconservative two species Euler equations are given by the hyperbolic system:

$$\left\{\begin{array}{ccc} \partial_t\rho+\partial_x(\rho u) & = & 0, \\ \partial_t(\rho u)+\partial_x(\rho u^2+p_e+p_i) & = & 0, \\ \partial_t(\rho_e\varepsilon_e+\frac{1}{2}\rho_e u^2)+\partial_x(u(\rho_e\varepsilon_e+\frac{1}{2}\rho_e u^2+p_e))-u(c_i\partial_x p_e-c_e\partial_x p_i) & = & \nu_{ei}(T_i-T_e), \\ \partial_t(\rho_i\varepsilon_i+\frac{1}{2}\rho_i u^2)+\partial_x(u(\rho_i\varepsilon_i+\frac{1}{2}\rho_i u^2+p_i))+u(c_i\partial_x p_e-c_e\partial_x p_i) & = & -\nu_{ei}(T_i-T_e), \end{array}\right. \tag{1}$$

where $\rho=\rho_e+\rho_i\geq 0$ is the total density of the plasma, u is the average velocity of the plasma. T_e and T_i represent the temperatures of electrons and ions. $\rho_e=n_e m_e$, $\rho_i=n_i m_i$ are the density of the electrons and ions, where the concentrations of electrons n_e and of ions n_i are related by the average ionization number $Z=n_e/n_i\geq 1$. Z will be consider here constant and as a physical property of the ions. m_e and m_i are the mass of the electrons and ions particles. Hence the mass fractions

$$c_\alpha=\frac{\rho_\alpha}{\rho},\ \alpha=e,i \tag{2}$$

are also constant and c_e and c_i write

$$c_e=\frac{Zm_e}{m_i+Zm_e},\quad c_i=1-c_e. \tag{3}$$

The electronic and ionic pressures and temperatures are related by

$$p_e=n_e k_B T_e,\quad p_i=n_i k_B T_i,$$

where k_B is the Boltzmann constant. The internal energies are given by

$$\varepsilon_e=\frac{k_B T_e}{m_e(\gamma_e-1)},\quad \varepsilon_i=\frac{k_B T_i}{m_i(\gamma_i-1)}, \tag{4}$$

where γ_e,γ_i are constant numbers belonging to the interval $[1,3]$ and $\nu_{ei}\geq 0$ is the frequency exchange between temperatures.

Along this article we denote $\mathcal{E}_\alpha=\rho_\alpha\varepsilon_\alpha+\frac{1}{2}\rho_\alpha u^2$ for $\alpha=e,i$, and

$$\mathcal{U}=(\rho,\rho u,\mathcal{E}_e,\mathcal{E}_i). \tag{5}$$

We also use in this article the expression of the bitemperature model in Lagrangian formalism, which writes as

$$\left\{\begin{array}{ccc} \rho d_t\tau-\partial_x u & = & 0, \\ \rho d_t u+\partial_x(p_e+p_i) & = & 0, \\ \rho_e d_t\varepsilon_e+p_e\rho d_t\tau & = & \nu_{ei}(T_i-T_e), \\ \rho_i d_t\varepsilon_i+p_i\rho d_t\tau & = & -\nu_{ei}(T_i-T_e), \end{array}\right. \tag{6}$$

where $d_t=\partial_t+u\partial_x$ represents the material derivative and $\tau=\frac{1}{\rho}$ is the specific volume.

2.2. The kinetic model

2.2.1. *Notations*

Kinetic models are described by the distribution function f_α of each species depending on the time variable $t\in\mathbb{R}_+$, on the position $x\in\mathbb{R}^3$ and on the velocity $v\in\mathbb{R}^3$. The macroscopic quantities can be obtained by

extracting moments on these distribution functions w.r.t the velocity variable. Indeed density, velocity and total energy of the species α can be defined as

$$n_\alpha = \int_{\mathbb{R}^3} f_\alpha dv, \quad u_\alpha = \frac{1}{n_\alpha}\int_{\mathbb{R}^3} v f_\alpha dv, \quad \mathcal{E}_\alpha = \frac{3}{2}\rho_\alpha \frac{k_B}{m_\alpha} T_\alpha + \frac{1}{2}\rho_\alpha u_\alpha^2 = \int_{\mathbb{R}^3} m_\alpha \frac{v^2}{2} f_\alpha dv, \tag{7}$$

where m_α is the mass particle, $\rho_\alpha = m_\alpha n_\alpha$, and T_α the temperature of species α.

The internal specific energy of species α can be defined as

$$\varepsilon_\alpha = \frac{3}{2m_\alpha} k_B T_\alpha.$$

Those definitions are consistent with (4) for $\gamma_\alpha = \frac{5}{3}$. As already pointed out in introduction, this value corresponds to the monoatomic case. This is not a conceptual restriction because the general case could be obtained with the same approach by considering polyatomic models. This generalization is postponed to a forthcoming paper. In the following we shall use the moment operator P_α defined by

$$P_\alpha(f_\alpha) = m_\alpha \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{v^2}{2} \end{pmatrix} f_\alpha dv. \tag{8}$$

We denote $P_\alpha(f_\alpha) = U_\alpha$:

$$U_\alpha = \begin{pmatrix} \rho_\alpha \\ \rho_\alpha u_\alpha \\ \mathcal{E}_\alpha \end{pmatrix}. \tag{9}$$

Usually the velocity and the temperature of the mixture are defined by

$$u = \frac{\rho_e u_e + \rho_i u_i}{\rho_e + \rho_i}, \quad n k_B T = \sum_\alpha (\frac{1}{3}\rho_\alpha (u_\alpha^2 - u^2)) + \sum_\alpha (n_\alpha k_B T_\alpha), \tag{10}$$

where $n = n_e + n_i$. Finally we define the entropy of the mixture by

$$\mathcal{H}(f_e, f_i) = \mathcal{H}_s(f_e) + \mathcal{H}_s(f_i), \quad \text{with} \quad \mathcal{H}_s(f) = \int_{\mathbb{R}^3} (f\, ln(f) - f) dv. \tag{11}$$

The related entropy flux is then

$$\Phi(f_e, f_i) = \Phi_s(f_e) + \Phi_s(f_i), \quad \text{with} \quad \Phi_s(f) = \int_{\mathbb{R}^3} v(f\, ln(f) - f) dv. \tag{12}$$

2.2.2. *Description of the BGK model*

In this section, we present the kinetic model and we show the fundamental properties of the BGK model describing the plasma interacting with an electric field $E \in \mathbb{R}^3$. This model writes

$$\partial_t f_\alpha + v\nabla_x f_\alpha + \frac{q_\alpha}{m_\alpha} E.\nabla_v f_\alpha = \frac{1}{\tau_\alpha}(\mathcal{M}_\alpha(f_\alpha) - f_\alpha) + \frac{1}{\tau_{\alpha\beta}}(\overline{\mathcal{M}_\alpha}(f_e, f_i) - f_\alpha), \quad \{\alpha, \beta\} \in \{e, i\},\ \alpha \neq \beta, \tag{13}$$

with $\tau_\alpha > 0$, $\tau_{\alpha\beta} > 0$. q_α is the charge of the species α.

The relaxation term $\frac{1}{\tau_\alpha}$ represents the collision frequency for the interaction between α particles and $\frac{1}{\tau_{\alpha\beta}}$ corresponds to the collision frequency for the interaction between ions and electrons.

In this paper, contrarily to classical BGK models, we consider as in ([23]) that $\tau_{ei} \neq \tau_{ie}$, due to the discrepancy of ion and electron masses. In particular, $\frac{\tau_{ie}}{\tau_{ei}}$ is of the same order as $\frac{c_i}{c_e}$ and can reach important

values. Moreover this quantity is kept fixed in the physical model presented in this paper.
The model (13) is coupled to the Maxwell-Ampère and the Poisson equations through the electric field E as follows

$$\partial_t E = -\frac{j}{\varepsilon_0}, \tag{14}$$

$$\nabla_x \cdot E = \frac{\overline{\rho}}{\varepsilon_0}. \tag{15}$$

j represents the current in the plasma, $\overline{\rho}$ the total charge, and ε_0 is the vacuum permitivity. j and $\overline{\rho}$ are defined by

$$\overline{\rho} = \int_{\mathbb{R}^3} (q_e f_e + q_i f_i)\, dv = n_e q_e + n_i q_i, \quad j = \int_{\mathbb{R}^3} v(q_e f_e + q_i f_i)\, dv = n_e q_e u_e + n_i q_i u_i. \tag{16}$$

$\mathcal{M}_\alpha$ and $\overline{\mathcal{M}_\alpha}$ are the two Maxwellian distribution functions

$$\mathcal{M}_\alpha(f_\alpha) = \frac{n_\alpha}{(2\pi k_B T_\alpha/m_\alpha)^{3/2}} exp(-\frac{|v-u_\alpha|^2}{2k_B T_\alpha/m_\alpha}),\ \alpha = e, i, \tag{17}$$

$$\overline{\mathcal{M}_\alpha}(f_e, f_i) = \frac{n_\alpha}{(2\pi k_B T^\#/m_\alpha)^{3/2}} exp(-\frac{|v-u^\#|^2}{2k_B T^\#/m_\alpha}),\ \alpha = e, i, \tag{18}$$

where

$$u^\# = \frac{\frac{1}{\tau_{ei}} m_e n_e u_e + \frac{1}{\tau_{ie}} m_i n_i u_i}{\frac{1}{\tau_{ei}} m_e n_e + \frac{1}{\tau_{ie}} m_i n_i}, \tag{19}$$

$$T^\# = \frac{\frac{3}{2}\frac{1}{\tau_{ei}} k_B n_e T_e + \frac{3}{2}\frac{1}{\tau_{ie}} k_B n_i T_i + \frac{1}{\tau_{ei}}(\frac{1}{2}\rho_e (u_e)^2 - \frac{1}{2}\rho_e (u^\#)^2) + \frac{1}{\tau_{ie}}(\frac{1}{2}\rho_i (u_i)^2 - \frac{1}{2}\rho_i (u^\#)^2)}{\frac{3}{2} k_B \frac{1}{\tau_{ei}} n_e + \frac{3}{2} k_B \frac{1}{\tau_{ie}} n_i}. \tag{20}$$

In (19, 20), $u^\#$ and $T^\#$ are fictitious quantities and are defined in such a way that the BGK model is conservative. In the equilibrium case, the relations $u_e = u_i = u$ and $T_e = T_i = T$ correspond to $u^\# = u$ and $T^\# = T$. Moreover, $u^\#$ being defined in (19) according to a convex combinaison of u_e and u_i, it holds that

$$(\frac{1}{\tau_{ei}}\rho_e + \frac{1}{\tau_{ie}}\rho_i)(u^\#)^2 \leq \frac{1}{\tau_{ei}}\rho_e u_e^2 + \frac{1}{\tau_{ie}}\rho_i u_i^2.$$

Therefore $T^\#$ defined by relation (20) is positive. In the situation where $\tau_{ei} = \tau_{ie}$, we recover $u^\# = u$ and $T^\# = T$, where u and T are given by (10).

Remark that in [2], [13], [11], the authors consider only one global BGK operator per species in order to reproduce the interaction between the given species with the other species.

The following properties are well known:

Property 2.1. *For $\alpha = e, i$, let F_α be the flux function of Euler equations and $\eta_\alpha(U_\alpha) = \overline{\eta}_\alpha(\rho_\alpha, \varepsilon_\alpha)$, $Q_\alpha = u\eta_\alpha$ the usual entropy-flux pair for the Euler system with the γ-law (4):*

$$\overline{\eta}_\alpha(\rho_\alpha, \varepsilon_\alpha) = -\frac{\rho_\alpha}{m_\alpha(\gamma_\alpha - 1)}\left[\ln\left(\frac{(\gamma_\alpha - 1)\rho_\alpha \varepsilon_\alpha}{\rho_\alpha^{\gamma_\alpha}}\right) + C\right]. \tag{21}$$

Here C is a nonnegative constant. If $\gamma_e = \gamma_i = 5/3$, then for $\alpha = e, i$:

$$P_\alpha(\mathcal{M}_\alpha(f_\alpha)) = U_\alpha, \quad P_\alpha(v\mathcal{M}_\alpha(f_\alpha)) = F_\alpha(U_\alpha), \tag{22}$$

and

$$\mathcal{H}_s(\mathcal{M}_\alpha(f_\alpha)) = \eta_\alpha(U_\alpha), \quad \Phi_s(\mathcal{M}_\alpha(f_\alpha)) = Q_\alpha(U_\alpha). \tag{23}$$

Proposition 2.1. *The model (13, 17, 18, 19, 20) conserves the mass per species, the total momentum and the total energy.*

Proof. The two Maxwellian distributions (17, 18) satisfy the constraints

$$\int_{\mathbb{R}^3}(\mathcal{M}_\alpha - f_\alpha)\left[\begin{array}{c} m_\alpha \\ m_\alpha v \\ m_\alpha \frac{v^2}{2}\end{array}\right]dv = 0, \quad \int_{\mathbb{R}^3}(\overline{\mathcal{M}_\alpha} - f_\alpha)dv = 0, \ \alpha = e, i. \tag{24}$$

Straightforward computations give

$$\begin{aligned}
\int_{\mathbb{R}^3}\frac{1}{\tau_{ei}}(\overline{\mathcal{M}_e} - f_e)m_e v dv + \int_{\mathbb{R}^3}\frac{1}{\tau_{ie}}(\overline{\mathcal{M}_i} - f_i)m_i v dv &= \frac{1}{\tau_{ei}}m_e n_e(u^\# - u_e) + \frac{1}{\tau_{ie}}m_i n_i(u^\# - u_i) \\
\int_{\mathbb{R}^3}\frac{1}{\tau_{ei}}(\overline{\mathcal{M}_e} - f_e)\frac{1}{2}m_e v^2 dv + \int_{\mathbb{R}^3}\frac{1}{\tau_{ie}}(\overline{\mathcal{M}_i} - f_i)\frac{1}{2}m_i v^2 dv &= \frac{3}{2}k_B(\frac{1}{\tau_{ei}}n_e(T^\# - T_e) + \frac{1}{\tau_{ie}}n_i(T^\# - T_i)) \\
&+ \frac{1}{2\tau_{ei}}(\rho_e(u^\#)^2 - \rho_e u_e^2) + \frac{1}{2\tau_{ie}}(\rho_i(u^\#)^2 - \rho_i u_i^2).
\end{aligned}$$

Then the definition of $u^\#$ and $T^\#$ given by (19) and (20) leads to

$$\frac{1}{\tau_{ei}}\int_{\mathbb{R}^3}((\overline{\mathcal{M}_e} - f_e)\left[\begin{array}{c} m_e v \\ m_e \frac{v^2}{2}\end{array}\right] + \frac{1}{\tau_{ie}}(\overline{\mathcal{M}_i} - f_i)\left[\begin{array}{c} m_i v \\ m_i \frac{v^2}{2}\end{array}\right])dv = 0. \tag{25}$$

Therefore the conservation properties follow. □

Theorem 2.1. *The model (13, 17, 18) satisfies the H theorem.*

(1) *The model satisfies the entropy inequality*

$$\begin{aligned}
\frac{1}{\tau_e}\int_{\mathbb{R}^3}(\mathcal{M}_e - f_e)ln(f_e)dv + \frac{1}{\tau_i}\int_{\mathbb{R}^3}(\mathcal{M}_i - f_i)ln(f_i)dv + \frac{1}{\tau_{ei}}\int_{\mathbb{R}^3}(\overline{\mathcal{M}_e} - f_e)ln(f_e)dv \\
+\frac{1}{\tau_{ie}}\int_{\mathbb{R}^3}(\overline{\mathcal{M}_i} - f_i)ln(f_i)dv \leq 0.
\end{aligned} \tag{26}$$

(2) *The equality holds in the above equation if and only if there exists* $(n_\alpha, u, T) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+$ *such that*

$$f_\alpha = \frac{n_\alpha}{(2\pi k_B T/m_\alpha)^{3/2}}\exp(-\frac{|v-u|^2}{2k_B T/m_\alpha}), \ \alpha = e, i. \tag{27}$$

Proof. For each $\alpha = e, i$, by using the conservation properties (24, 25), it holds that

$$\int_{\mathbb{R}^3}(\mathcal{M}_\alpha - f_\alpha)ln(f_\alpha)dv = \int_{\mathbb{R}^3}(\mathcal{M}_\alpha - f_\alpha)ln(\frac{f_\alpha}{\mathcal{M}_\alpha})dv \tag{28}$$

and

$$\int_{\mathbb{R}^3}(\frac{1}{\tau_{ei}}(\overline{\mathcal{M}_e} - f_e)ln(f_e) + \frac{1}{\tau_{ie}}(\overline{\mathcal{M}_i} - f_i)ln(f_i))dv = \int_{\mathbb{R}^3}(\frac{1}{\tau_{ei}}(\overline{\mathcal{M}_e} - f_e)ln(\frac{f_e}{\overline{\mathcal{M}_e}}) + \frac{1}{\tau_{ie}}(\overline{\mathcal{M}_i} - f_i)ln(\frac{f_i}{\overline{\mathcal{M}_i}}))dv.$$

Therefore the convexity of the function $x \to x\ln x$ yields the first part of the theorem. Moreover if the equality holds in (26), it comes that

$$f_\alpha = \mathcal{M}_\alpha, \quad \overline{\mathcal{M}_\alpha} = \mathcal{M}_\alpha, \ \alpha = e, i.$$

Next, by using the expressions (17, 18, 19, 20) of $\mathcal{M}_\alpha$ and $\overline{\mathcal{M}_\alpha}$, it comes that $u_e = u_i = u = u^\#$, $T_i = T_e = T = T^\#$ and we get (27). Therefore we recover the same equilibrium states as for the Boltzmann operator for a two component gas ([3]). □

2.3. **Hydrodynamic limit**

In this section we derive as an hydrodynamic limit, a nonconservative Euler system. We firstly rescale the model and next perform the hydrodynamic limit.

2.3.1. *Scaling on the one dimensionnal BGK model*

For the sake of clarity, we assume that the system (13-20) is even in $(v_2,\, v_3)$. Such property being preserved by the kinetic equation (13-20), the distribution function f_α of the species α depends on the time variable $t \in \mathbb{R}_+$, the space variable $x \in \mathbb{R}$ and the velocity variable $v\,(v_1, v_2, v_3) \in \mathbb{R}^3$. In the following, the macroscopic velocity $u(t,x)$, the current $j(t,x)$ belong to $\mathbb{R}$ and (7, 12, 16) are to be read with

$$u_\alpha = \frac{1}{n_\alpha}\int_{\mathbb{R}^3} v_1 f_\alpha dv, \quad \Phi_s(f) = \int_{\mathbb{R}^3} v_1(f\, ln(f) - f)dv, \quad j = \int_{\mathbb{R}^3} v_1(q_e f_e + q_i f_i)\, dv.$$

The electric field belongs to $\mathbb{R}$, and $P_\alpha(f_\alpha) = U_\alpha$ belongs to $\mathbb{R}^3$:

$$P_\alpha(f_\alpha) = m_\alpha \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v_1 \\ \frac{v^2}{2} \end{pmatrix} f_\alpha dv. \tag{29}$$

The property (22) reads then as

$$P_\alpha(\mathcal{M}_\alpha(f_\alpha)) = U_\alpha, \quad P_\alpha(v_1\mathcal{M}_\alpha(f_\alpha)) = F_\alpha(U_\alpha). \tag{30}$$

In order to obtain the quasi-neutral limit, the system (13-15) is rescaled in the following way

$$\left\{ \begin{array}{rcl} \partial_t f_\alpha + v_1\partial_x f_\alpha + \frac{q_\alpha}{m_\alpha}E\partial_{v_1} f_\alpha & = & \frac{1}{\varepsilon}(\mathcal{M}_\alpha - f_\alpha) + \frac{1}{\tau_{\alpha\beta}}(\overline{\mathcal{M}_\alpha} - f_\alpha), \quad \alpha \neq \beta \\ \partial_t E & = & -\frac{j}{\varepsilon^2}, \\ \partial_x E & = & \frac{\overline{\rho}}{\varepsilon^2}, \end{array} \right. \tag{31}$$

where ε is a positive parameter proportional to the Knudsen number. In particular the Maxwellian distributions given in (17, 18) write

$$\mathcal{M}_\alpha(f_\alpha) = \frac{n_\alpha}{(2\pi k_B T_\alpha/m_\alpha)^{3/2}} exp(-\frac{(v_1 - u_\alpha)^2 + v_2^2 + v_3^2}{2k_B T_\alpha/m_\alpha}),\ \alpha = e, i \tag{32}$$

and

$$\overline{\mathcal{M}_\alpha}(f_e, f_i) = \frac{n_\alpha}{(2\pi k_B T^\# m_\alpha)^{3/2}} exp(-\frac{(v_1 - u^\#)^2 + v_2^2 + v_3^2}{2k_B T^\#/m_\alpha}),\ \alpha = e, i, \tag{33}$$

where $u^\#$ and $T^\#$ are defined by (19) and (20).

Remark 2.1. *In ([19], [20]), the authors perform formally hydrodynamic limits toward nonconservative Euler systems in the context of mixtures of ions and electrons. In particular they consider for their scaling the mass ratio between ions and electrons as a small parameter. In the present case, the scaling proposed in (31) is different. Our aim is to provide a kinetic approximation of the system (31) in the spirit of ([33]). The comparison between the different scalings at the physical point of view has to be investigated and is postponed to a future paper.*

2.3.2. *Derivation of the Euler equations*

The two species Euler system is obtained starting from (31, 32, 33) and performing an hydrodamic limit.

Proposition 2.2. *The system (31, 32, 33) formally converges to the nonconservative two species Euler equations* (1) *where E is given according to the Ohm's law*

$$\frac{1}{\rho_e}\partial_x p_e - \frac{1}{\rho_i}\partial_x p_i = (\frac{n_e q_e}{\rho_e} - \frac{n_i q_i}{\rho_i})E = \frac{\rho}{\rho_e\rho_i} n_e q_e E = -\frac{\rho}{\rho_e\rho_i} n_i q_i E \tag{34}$$

and

$$\nu_{ei} = \frac{3k_B n_e n_i}{2(\tau_{ie} n_e + \tau_{ei} n_i)}. \tag{35}$$

Remark that the three dimensional version of (1) can be obtained by considering the system (13-15) with the same scaling as in (31).

Proof. For this proof, for any g belonging to $L^1_2 = \{g \in L^1/(1+v^2)f \in L^1\}$, we use the notation

$$\forall g \in L^1_2, \quad \langle g \rangle = \int_{\mathbb{R}^3} g\, dv.$$

To obtain the Euler system (1), we perform a Chapman-Enskog expansion for each distribution function

$$f_\alpha = f^0_\alpha + \varepsilon f^1_\alpha, \quad \alpha \in \{e, i\},$$

with the constraints

$$\int_{\mathbb{R}^3} f_\alpha dv = \int_{\mathbb{R}^3} f^0_\alpha dv, \ \int_{\mathbb{R}^3} f^1_\alpha dv = 0, \tag{36}$$

$$\int_{\mathbb{R}^3} v_1(m_e f_e + m_i f_i) dv = \int_{\mathbb{R}^3} v_1(m_e\, f^0_e + m_i\, f^0_i) dv, \ \int_{\mathbb{R}^3} v_1(m_e f^1_e + m_i f^1_i) dv = 0, \tag{37}$$

$$\int_{\mathbb{R}^3} v^2 m_\alpha f_\alpha\, dv = \int_{\mathbb{R}^3} v^2\, m_\alpha f^0_\alpha\, dv, \ \int_{\mathbb{R}^3} v^2 m_\alpha f^1_\alpha\, dv = 0. \tag{38}$$

By identifying the terms of (31) of order ε^{-1}, we get

$$f^0_\alpha = \mathcal{M}_\alpha, \quad \alpha \in \{e, i\},$$

where $\mathcal{M}_\alpha$ is given by (32). Letting ε tend to 0, it comes that $\overline{\rho} = 0$ and $j = 0$. Next, we use the relations

$$q_e = -e, \quad q_i = Ze \tag{39}$$

where Z represents the ionisation rate and e the elementary charge. Then we deduce

$$n_e = Zn_i, \quad u_e = u_i = u. \tag{40}$$

Therefore,

$$f^0_\alpha = \mathcal{M}_\alpha, \quad \text{with} \quad u_\alpha = u. \tag{41}$$

Moreover, since $u_e = u_i$, the relation (19) implies $u^\# = u = u_e = u_i$. The terms of order ε^0 of (31) are given by

$$\partial_t f^0_\alpha + v_1 \partial_x f^0_\alpha + \frac{q_\alpha}{m_\alpha} E \partial_{v_1} f^0_\alpha = \frac{1}{\tau_{\alpha\beta}}(\overline{\mathcal{M}}_\alpha - f^0_\alpha) - f^1_\alpha, \quad \alpha, \beta \in \{e, i\}, \quad \alpha \neq \beta. \tag{42}$$

Multiply (42) by m_α, integrate w.r.t v, use that $\langle \partial_{v_1} f^0_\alpha \rangle = 0$ and $\langle f^1 \rangle = 0$ leads to

$$\partial_t \langle m_\alpha f^0_\alpha \rangle + \partial_x \langle m_\alpha\, v_1 f^0_\alpha \rangle = 0.$$

Hence we obtain the mass conservation equation for species α

$$\partial_t \rho_\alpha + \partial_x(\rho_\alpha u) = 0, \qquad \alpha \in \{e, i\}.$$

By summing on α this equation, we obtain the conservation of total mass. In order to obtain the conservation of momentum, multiply (42) by $m_\alpha v_1$ and integrate w.r.t v. Hence we get

$$\partial_t \langle m_\alpha v_1 f_\alpha^0\rangle + \partial_x \langle m_\alpha v_1^2 f_\alpha^0\rangle + q_\alpha E \langle v_1 \partial_{v_1} f_\alpha^0\rangle = \frac{1}{\tau_{\alpha\beta}} \langle m_\alpha v_1 (\overline{\mathcal{M}_\alpha} - f_\alpha^0)\rangle - \langle m_\alpha v_1 f_\alpha^1\rangle, \quad \alpha \neq \beta. \tag{43}$$

The scaling of the second equation of (31) gives

$$\int_{\mathbb{R}^3} v_1 (q_e f_e^1 + q_i f_i^1)\, dv = 0. \tag{44}$$

Hence combining (44) with (37) leads to $\langle m_\alpha v_1 f_\alpha^1\rangle = 0$. Moreover, as $\langle m_\alpha v_1 f_\alpha^0\rangle = \rho_\alpha u$, it comes that $\langle m_\alpha v^2 f_\alpha^0\rangle = \rho_\alpha u^2 + p_\alpha$, where p_α is the pressure of species $\alpha = e, i$. As $v_1 \partial_{v_1} f_\alpha^0 = \partial_{v_1}(v_1 f_\alpha^0) - f_\alpha^0$, we get $\langle v_1 \partial_{v_1} f_\alpha^0\rangle = -n_\alpha$. Moreover, using that $\langle m_\alpha v_1 (\overline{\mathcal{M}_\alpha} - f_\alpha^0)\rangle = 0$, we obtain

$$\partial_t(\rho_\alpha u) + \partial_x(\rho_\alpha u^2 + p_\alpha) - q_\alpha n_\alpha E = 0. \tag{45}$$

By summing term by term (45) and using (25), we get the second equation of the system (1). Moreover (45) can be rewritten on the form

$$\partial_t u + u \partial_x u + \frac{1}{\rho_\alpha} \partial_x p_\alpha - \frac{q_\alpha n_\alpha}{\rho_\alpha} E = 0. \tag{46}$$

By substracting equation (46) for electrons and ions, we get Ohm's law (34). In order to obtain one equation on each internal energy for each species, we multiply equation (31) by $m_\alpha v^2/2$ and integrate w.r.t v

$$\partial_t \langle m_\alpha \frac{v^2}{2} f_\alpha^0\rangle + \partial_x \langle m_\alpha \frac{v^2}{2} f_\alpha^0 v_1\rangle + q_\alpha E \langle \frac{v^2}{2} \partial_{v_1} f_\alpha^0\rangle = \frac{1}{\tau_{\alpha\beta}} \langle m_\alpha \frac{v^2}{2} (\overline{\mathcal{M}_\alpha} - f_\alpha^0)\rangle - \langle m_\alpha \frac{v^2}{2} f_\alpha^1\rangle, \quad \alpha, \beta \in \{e, i\}, \alpha \neq \beta. \tag{47}$$

Firstly, we compute

$$\langle m_\alpha \frac{v^2}{2} f_\alpha^0\rangle = \rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha u^2, \quad \langle m_\alpha \frac{v^2}{2} f_\alpha^0 v_1\rangle = u(\rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha u^2 + p_\alpha).$$

For the last term of the left-hand side of (47), the relation $\frac{v^2}{2}\partial_{v_1} f_\alpha^0 = \partial_{v_1}(\frac{v^2}{2} f_\alpha^0) - v_1 f_\alpha^0$ leads to

$$q_\alpha E \langle \frac{v^2}{2} \partial_{v_1} f_\alpha^0\rangle = -n_\alpha q_\alpha E u.$$

Let $\alpha \neq \beta \in \{e, i\}$. Then (34) reads

$$\frac{1}{\rho_\alpha} \partial_x p_\alpha - \frac{1}{\rho_\beta} \partial_x p_\beta = \frac{\rho}{\rho_\alpha \rho_\beta}. \tag{48}$$

Next for the right-hand side of (47), a direct computation yields

$$\langle m_\alpha \frac{v^2}{2} (\overline{\mathcal{M}_\alpha} - f_\alpha^0)\rangle = \frac{3}{2} n_\alpha k_B (T^\# - T_\alpha).$$

Moreover, the relation (20) implies that

$$T^\# - T_\alpha = \frac{\frac{1}{\tau_{\beta\alpha}} n_\beta}{\frac{1}{\tau_{ei}} n_e + \frac{1}{\tau_{ie}} n_i} (T_\beta - T_\alpha), \;\; \beta \neq \alpha, \; \{\alpha, \beta\} \in \{e, i\}, \tag{49}$$

because $u^{\#} = u = u_e = u_i$. So

$$\langle m_\alpha \frac{v^2}{2}(\overline{\mathcal{M}_\alpha} - f_\alpha^0)\rangle = \frac{3}{2}\frac{n_e n_i}{\tau_{ie} n_e + \tau_{ei} n_i} k_B (T_\beta - T_\alpha), \quad \beta \neq \alpha,\ \alpha \in \{e, i\}. \tag{50}$$

Hence, using that $\langle m_\alpha \frac{v^2}{2} f_\alpha^1\rangle = 0$ and (50), (47) leads to the 2 last equations of (1). □

Remark 2.2. *The relation (48) shows that the force term of the kinetic equations leads as in ([33]) to the nonconservative terms of the bitemperature Euler system (1).*

2.4. **Entropy dissipation**

This section is devoted to the entropy study for the system (56) and is one of the crucial part of the paper. We firstly prove the existence of an entropy-entropy flux pair directly on the fluid system. Next, the entropy dissipation property is obtained through the kinetic point of view. It is shown that the two obtained entropy dissipation properties are compatible.

2.4.1. *Existence of a dissipative entropy for the Euler bitemperature model*

In this part we aim to identify an entropy for the system (1). More precisely, we are looking for a dissipative entropy-entropy flux pair (η, Q), defined as follows:

Definition 2.1. *Consider a hyperbolic quasilinear system*

$$\partial_t \mathcal{U} + A(\mathcal{U})\partial_x \mathcal{U} = G(\mathcal{U}). \tag{51}$$

Let η be a strictly convex real valued function defined on an open domain Ω of $\mathbb{R}^n$. Let Q be a real valued function defined on Ω. (η, Q) is a dissipative entropy-entropy flux pair for the system (51) if for all $\mathcal{U} \in \Omega$:

$$\eta'(\mathcal{U})A(\mathcal{U}) = Q'(\mathcal{U}) \tag{52}$$

and

$$\eta'(\mathcal{U})G(\mathcal{U}) \leq 0. \tag{53}$$

Let us perform the change of variable $\mathcal{U} = \phi(\mathcal{V})$. We denote

$$\tilde{A}(\mathcal{V}) = (\phi'(\mathcal{V}))^{-1} A(\phi(\mathcal{V}))\phi'(\mathcal{V}), \quad \tilde{\eta}(\mathcal{V}) = \eta(\phi(\mathcal{V})), \quad \tilde{Q}(\mathcal{V}) = Q(\phi(\mathcal{V})).$$

It is an easy matter to prove that $\mathcal{U}$ is a solution of system (51) if and only if $\mathcal{V}$ is a solution of

$$\partial_t \mathcal{V} + \tilde{A}(\mathcal{V})\partial_x \mathcal{V} = (\phi'(\mathcal{V}))^{-1} G(\phi(\mathcal{V})), \tag{54}$$

and that condition (52) is satisfied if and only if

$$\tilde{\eta}'(\mathcal{V})\tilde{A}(\mathcal{V}) = \tilde{Q}'(\mathcal{V}). \tag{55}$$

In our case, the primal variable $\mathcal{U} = (\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i)$ can be changed into $\mathcal{V} = (\rho, u, \varepsilon_e, \varepsilon_i)$. A straightforward calculation leads to the following system for $\mathcal{V}$:

$$\begin{cases} \partial_t \rho + u\partial_x \rho + \rho \partial_x u = 0, \\ \partial_t u + u\partial_x u + \rho^{-1}\partial_x(p_e + p_i) = 0, \\ \partial_t \varepsilon_e + u\partial_x \varepsilon_e + \rho_e^{-1} p_e \partial_x u = \rho_e^{-1}\nu_{ei}(T_i - T_e), \\ \partial_t \varepsilon_i + u\partial_x \varepsilon_i + \rho_i^{-1} p_i \partial_x u = \rho_i^{-1}\nu_{ei}(T_e - T_i). \end{cases} \tag{56}$$

Lemma 2.2. *For $\alpha = e, i$, let $\eta_\alpha(\rho_\alpha, \rho_\alpha u, \mathcal{E}_\alpha) = \overline{\eta}_\alpha(\rho_\alpha, \varepsilon_\alpha)$ be as defined in (21), and $\overline{Q}_\alpha = u\overline{\eta}_\alpha$. We define $(\tilde{\eta}, \tilde{Q})$ for system (56) as*

$$\begin{aligned} \tilde{\eta}(\rho, u, \varepsilon_e, \varepsilon_i) &= \mu_e \overline{\eta}_e(\rho c_e, \varepsilon_e) + \mu_i \overline{\eta}_i(\rho c_i, \varepsilon_i), \\ \tilde{Q}(\rho, u, \varepsilon_e, \varepsilon_i) &= \mu_e \overline{Q}_e(\rho c_e, u, \varepsilon_e) + \mu_i \overline{Q}_i(\rho c_i, u, \varepsilon_i) \end{aligned} \tag{57}$$

where μ_e and μ_i are constant. Hence relation (55) is satisfied.

We skip the proof of this lemma: it is a direct calculation where one uses the fact that the entropy $\overline{\eta}_\alpha$ does not depend on the variable u.

As a consequence, if we define $\eta(\mathcal{U}) = \tilde{\eta}(\rho, u, \varepsilon_e, \varepsilon_i)$ and the fonction $\mathbf{U}_\alpha$ as

$$\mathbf{U}_\alpha(\mathcal{U}) = (c_\alpha \rho, c_\alpha \rho u, \mathcal{E}_\alpha), \quad \alpha = e, i, \tag{58}$$

equality (52) is satisfied with

$$\eta(\mathcal{U}) = \mu_e \eta_e(\mathbf{U}_e(\mathcal{U})) + \mu_i \eta_i(\mathbf{U}_i(\mathcal{U})), \quad Q(\mathcal{U}) = u\eta(\mathcal{U}). \tag{59}$$

We can now prove the existence of a strictly convex entropy for system (1).

Theorem 2.3. *Let (η, Q) be defined by (59), with η_e, η_i defined in lemma 2.2 and $\mathbf{U}_e$, $\mathbf{U}_i$ defined in (58). Let us set*

$$\mu_e = \mu_i = 1. \tag{60}$$

Then η is a strictly convex dissipative entropy for system (1) and Q is the related entropy flux. More precisely, any smooth solution of the system satisfies the following equality:

$$\partial_t \eta(\mathcal{U}) + \partial_x Q(\mathcal{U}) = -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2. \tag{61}$$

Proof. It remains to prove convexity and dissipation by the source-term. Consider $\mathcal{U} \neq \mathcal{W}$. Then $\mathbf{U}_\alpha(\mathcal{U}) \neq \mathbf{U}_\alpha(\mathcal{W})$ for at least one α. As η_e and η_i are strictly convex and μ_e and μ_i are positive:

$$\eta(\mathcal{W}) > \eta(\mathcal{U}) + \sum_{\alpha=e,i} \mu_\alpha \eta'_\alpha(\mathbf{U}_\alpha(\mathcal{U}))(\mathbf{U}_\alpha(\mathcal{W}) - \mathbf{U}_\alpha(\mathcal{U})).$$

As the $\mathbf{U}_\alpha$ are linear, $\mathbf{U}_\alpha(\mathcal{W}) - \mathbf{U}_\alpha(\mathcal{U}) = \mathbf{U}'_\alpha(\mathcal{U})(\mathcal{W} - \mathcal{U})$, and therefore

$$\eta(\mathcal{W}) > \eta(\mathcal{U}) + \eta'(\mathcal{U})(\mathcal{W} - \mathcal{U}),$$

which proves that η is strictly convex.

Then by using (4) one has

$$\left(\mu_e \frac{\partial(\eta_e \circ \mathbf{U}_e)}{\partial \mathcal{E}_e}(\mathcal{U}) - \mu_i \frac{\partial(\eta_i \circ \mathbf{U}_i)}{\partial \mathcal{E}_i}(\mathcal{U})\right)(T_i - T_e) = -\frac{1}{k_B}(T_i - T_e)\left(\frac{\mu_e}{T_e} - \frac{\mu_i}{T_i}\right)$$

and (61) follows. □

2.4.2. *Entropy inequality for hydrodynamic limits*

Theorem 2.3 is valid for any value of γ_e, γ_i. In the present case, the kinetic model (31) being monoatomic only the values of γ_e and γ_i equal to 5/3 can be considered. However, recall that the problem can be overcome by introducing a polyatomic kinetic model. The existence of a strictly convex dissipative entropy allows us to precise which solutions are admissible. In that goal, we study the entropy dissipation induced by the kinetic model.

Proposition 2.3. *Let $\mathcal{U}$ be a solution of (1) with $\gamma_e = \gamma_i = 5/3$. If $\mathcal{U}$ is a limit of a solution to the kinetic model (31-33) then the following inequality holds:*

$$\partial_t \eta(\mathcal{U}) + \partial_x Q(\mathcal{U}) \leq -\frac{\nu_{ei}}{k_B T_i T_e}(T_i - T_e)^2. \tag{62}$$

Proof. Multiply equation for f_α of (31) by $\ln(f_\alpha)$ and integrate w.r.t v gives

$$\partial_t \left(\int_{\mathbb{R}^3} f_\alpha \ln(f_\alpha) - f_\alpha dv \right) + \partial_x \left(\int_{\mathbb{R}^3} v_1 (f_\alpha \ln(f_\alpha) - f_\alpha) dv \right) + \frac{q_\alpha}{m_\alpha} \int_{\mathbb{R}^3} E\, \partial_{v_1} (f_\alpha \ln(f_\alpha) - f_\alpha) dv$$
$$= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} (\mathcal{M}_\alpha - f_\alpha) \ln(f_\alpha) dv + \frac{1}{\tau_{ei}} \int_{\mathbb{R}^3} (\overline{\mathcal{M}}_\alpha - f_\alpha) \ln(f_\alpha)\, dv.$$

The third term of the left-hand side is zero. By using (28) and notations (11,12):

$$\partial_t \mathcal{H}_s(f_\alpha) + \partial_x \Phi_s(f_\alpha) \leq \frac{1}{\tau_{\alpha\beta}} \int_{\mathbb{R}^3} (\overline{\mathcal{M}}_\alpha - f_\alpha) \ln(f_\alpha)\, dv, \quad \alpha \in \{e; i\},\ \beta \neq \alpha. \tag{63}$$

Now we sum over $\alpha = e, i$ and formally pass to the limit $\varepsilon = 0$. We obtain $f_\alpha = \mathcal{M}_\alpha$, with $\overline{\rho} = 0$ and $j = 0$. So $P_\alpha(f_\alpha) = \mathbf{U}_\alpha(\mathcal{U})$. The left-hand-side tends to the left-hand-side of inequality (62) as a direct consequence of (23). Moreover we get for $\alpha \in \{e, i\}$:

$$\int_{\mathbb{R}^3} (\overline{\mathcal{M}}_\alpha - \mathcal{M}_\alpha) \ln(\mathcal{M}_\alpha)\, dv = \int_{\mathbb{R}^3} (\overline{\mathcal{M}}_\alpha - \mathcal{M}_\alpha) \ln\left(\frac{n_\alpha}{(2\pi k_B \frac{T_\alpha}{m_\alpha})^{\frac{3}{2}}}\right) dv - \int_{\mathbb{R}^3} (\overline{\mathcal{M}}_\alpha - \mathcal{M}_\alpha) \frac{(v-u)^2}{2k_B \frac{T_\alpha}{m_\alpha}} dv.$$

The first term of the right-hand-side is equal to zero by (24). For the second, a straightforward computation gives:

$$\int_{\mathbb{R}^3} (\overline{\mathcal{M}}_\alpha - \mathcal{M}_\alpha) \frac{(v-u)^2}{2k_B \frac{T_\alpha}{m_\alpha}} dv = \frac{3}{2} \frac{n_\alpha}{T_\alpha} (T^\# - T_\alpha).$$

Therefore the right-hand side of (63) reads

$$\mathcal{A} = -\frac{3}{2} \left(\frac{1}{\tau_{ei}} \frac{n_e}{T_e} (T^\# - T_e) + \frac{1}{\tau_{ie}} \frac{n_i}{T_i} (T^\# - T_i) \right).$$

We conclude by using (49) and (35). □

This result is our motivation to define the admissibility of a solution of the bitemperature Euler model even in the general case $\gamma_e \neq \gamma_i$.

Definition 2.2. *A solution $\mathcal{U}$ of system (1) is said to be admissible if it satisfies inequality (62).*

3. Numerical Approximation

The spatial discretization is defined by a step Δx and discretization cells $C_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$. We consider that Δx is constant, except in the Lagrangian formalism below. The time step can be variable in the applications and is denoted Δt: $t_0 = 0$, $t_{n+1} = t_n + \Delta t$.

We adopt the finite volume viewpoint: for an unknown $V(x,t)$, we look for approximations V_j^n of the average of V at time t_n on cells C_j.

3.1. **A kinetic scheme**

In this section we design a numerical approximation of the bitemperature Euler model (1) by using the kinetic approximation (31-33). We use the well-known transport-projection method that can be described as follows.

An initial data $\mathcal{U}^0$ being given, for $\alpha = e, i$, we set $\rho_\alpha^0 = \rho^0 c_\alpha$, $n_\alpha^0 = \rho_\alpha^0/m_\alpha$, and $u_\alpha^0 = u^0$.

Suppose that at time t_n an approximate solution $\mathcal{U}^n = (\rho^n, (\rho u)^n, \mathcal{E}_e^n, \mathcal{E}_i^n)$ is known and that we have been able to define U_e^n and U_i^n such that

$$\rho_\alpha^n = \rho^n c_\alpha, \quad u_\alpha^n = u^n, \quad \alpha = e, i. \tag{64}$$

As a consequence, $n_\alpha^n = \rho_\alpha^n/m_\alpha$ is also well defined for $\alpha = e, i$ and we are able to compute the microscopic quantities $f_e^n(v)$, $f_i^n(v)$ by

$$f_\alpha^n(v) = \mathcal{M}_\alpha(U_\alpha^n), \quad \alpha = e,\, i. \tag{65}$$

We recall that the moment operator P_α is defined by (29) and satisfies (30). We obtain:

$$P_\alpha(f_\alpha^n) = U_\alpha^n, \quad P_\alpha(v_1 \mathcal{M}_\alpha(U_\alpha^n)) = F_\alpha(U_\alpha^n), \tag{66}$$

where F_α is the flux function of Euler equations with $\gamma = 5/3$.

Now, at the microscopic level, we approximate the equations

$$\partial_t f_\alpha + v_1 \partial_x f_\alpha + \frac{q_\alpha}{m_\alpha} E \partial_{v_1} f_\alpha = \frac{1}{\tau_{\alpha\beta}} (\overline{\mathcal{M}}_\alpha - f_\alpha),\ \alpha = e, i,\ \beta \neq \alpha. \tag{67}$$

Definition 3.1. *For each value of v we define a numerical flux $h_{\alpha,j+\frac{1}{2}}(v) = h_\alpha(f_{\alpha,j}(v), f_{\alpha,j+1}(v), v)$ such that for all v, $h_\alpha(.,.,v)$ is Lipschitz continuous and for all f:*

$$h_\alpha(f, f, v) = v_1 f. \tag{68}$$

For $\alpha \in \{e, i\}$ with $\alpha \neq \beta$, we define $f_{\alpha,j}^{n+\frac{1}{2}}$ and U_α^{n+1} as

$$f_{\alpha,j}^{n+\frac{1}{2}} = f_{\alpha,j}^n - \frac{\Delta t}{\Delta x}(h_{\alpha,j+\frac{1}{2}}^n(v) - h_{\alpha,j-\frac{1}{2}}^n(v)) - \Delta t \frac{q_\alpha}{m_\alpha} E_j^{n+1} \partial_{v_1} f_{\alpha,j}^{n+\frac{1}{2}} + \frac{\Delta t}{\tau_{\alpha\beta}}(\overline{\mathcal{M}_\alpha}(f_{e,j}^{n+\frac{1}{2}}, f_{i,j}^{n+\frac{1}{2}}) - f_{\alpha,j}^{n+\frac{1}{2}}), \tag{69}$$

and

$$U_{\alpha,j}^{n+1} = P_\alpha(f_{\alpha,j}^{n+\frac{1}{2}}). \tag{70}$$

E_j^{n+1} not being defined, we impose the two following relations, which are the relaxed limit of the last two equations of (31):

$$\begin{cases} \dfrac{q_e}{m_e}\rho_{e,j}^{n+1} + \dfrac{q_i}{m_i}\rho_{i,j}^{n+1} = 0, \\ \dfrac{q_e}{m_e}\rho_{e,j}^{n+1} u_{e,j}^{n+1} + \dfrac{q_i}{m_i}\rho_{i,j}^{n+1} u_{i,j}^{n+1} = 0. \end{cases} \tag{71}$$

Therefore $u_{i,j}^{n+1} = u_{e,j}^{n+1}$ and we define $\mathcal{U}_j^{n+1}$ by setting

$$\rho_j^{n+1} = \rho_{e,j}^{n+1} + \rho_{i,j}^{n+1}, \quad u_j^{n+1} = u_{i,j}^{n+1} = u_{e,j}^{n+1}.$$

Consequently, relations (64) are satisfied at level $n+1$. For $\alpha = e, i$, we set

$$F_{\alpha,j+\frac{1}{2}} = \mathcal{F}_\alpha(U_{\alpha,j}, U_{\alpha,j+1}) \tag{72}$$

with

$$\mathcal{F}_\alpha(U_\alpha, V_\alpha) = P_\alpha(h_\alpha(\mathcal{M}_\alpha(U_\alpha), \mathcal{M}_\alpha(V_\alpha), \cdot)). \tag{73}$$

This is a consistent numerical flux for Euler system of species α:

$$\mathcal{F}_\alpha(U_\alpha, U_\alpha) = P_\alpha(v_1 \mathcal{M}_\alpha(U_\alpha)) = F_\alpha(U_\alpha).$$

Now we proceed as in section 2.3.2. First we note again that

$$P_\alpha(\partial_{v_1} f_\alpha) = (0, -\rho_\alpha, -\rho_\alpha u).$$

We thus obtain the following relations:

$$\begin{aligned} c_e \rho_j^{n+1} u_j^{n+1} &= c_e \rho_j^n u_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{e,j+\frac{1}{2},2} - F^n_{e,j-\frac{1}{2},2}\right) + \frac{\Delta t\, q_e}{m_e} E_j^{n+1} \rho_{e,j}^{n+1}, \\ c_i \rho_j^{n+1} u_j^{n+1} &= c_i \rho_j^n u_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{i,j+\frac{1}{2},2} - F^n_{i,j-\frac{1}{2},2}\right) + \frac{\Delta t\, q_i}{m_i} E_j^{n+1} \rho_{i,j}^{n+1}. \end{aligned}$$

Hence, denoting

$$\delta^n_{j+\frac{1}{2}} = -c_i F^n_{e,j+\frac{1}{2},2} + c_e F^n_{i,j+\frac{1}{2},2}, \tag{74}$$

we have

$$c_i c_e \rho_j^{n+1} E_j^{n+1}\left(-\frac{q_e}{m_e} + \frac{q_i}{m_i}\right) = \frac{1}{\Delta x}\left(\delta^n_{j+\frac{1}{2}} - \delta^n_{j-\frac{1}{2}}\right).$$

Therefore by (3) and (39):

$$-\frac{q_e}{m_e} E_j^{n+1} \rho_{e,j}^{n+1} = \frac{q_i}{m_i} E_j^{n+1} \rho_{i,j}^{n+1} = \frac{1}{\Delta x}\left(\delta^n_{j+\frac{1}{2}} - \delta^n_{j-\frac{1}{2}}\right).$$

It is clear that $\delta^n_{j+\frac{1}{2}}$ is consistent with $-c_i p_e + c_e p_i$:

$$\delta^n_{j+\frac{1}{2}} = \delta(\mathcal{U}_j^n, \mathcal{U}_{j+1}^n), \quad \delta(\mathcal{U}, \mathcal{U}) = -c_i p_e + c_e p_i.$$

Finally, if we define $F_{j+\frac{1}{2}}$ as

$$F_{j+\frac{1}{2}} = \begin{pmatrix} F_{e,j+\frac{1}{2},1} + F_{i,j+\frac{1}{2},1} \\ F_{e,j+\frac{1}{2},2} + F_{i,j+\frac{1}{2},2} \\ F_{e,j+\frac{1}{2},3} \\ F_{i,j+\frac{1}{2},3} \end{pmatrix} \tag{75}$$

we obtain the following consistent scheme.

Proposition 3.1. *A consistent scheme for Euler bitemperature system (1) is constructed as follows. For all $n \geq 0$ if $\mathcal{U}^n = \{\mathcal{U}_j^n\}_{j\in\mathbb{Z}}$ is the approximate solution of system (1) at time t_n, we set*

$$U^n_{\alpha,j} = (c_\alpha \rho_j^n, c_\alpha \rho_j^n u_j^n, \mathcal{E}_\alpha), \quad \alpha = e, i. \tag{76}$$

A kinetic flux h_α is chosen as in definition 3.1. We then define the numerical fluxes $F_{\alpha,j+\frac{1}{2}}$, $F_{j+\frac{1}{2}}$ and $\delta_{j+\frac{1}{2}}$ by (72), (73), (74), (75). The approximate solution at time t_{n+1} is defined by the implicit system:

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \dfrac{\Delta t}{\Delta x}\left(F^n_{j+\frac{1}{2},1} - F^n_{j-\frac{1}{2},1}\right), \\ \rho_j^{n+1} u_j^{n+1} = \rho_j^n u_j^n - \dfrac{\Delta t}{\Delta x}\left(F^n_{j+\frac{1}{2},2} - F^n_{j-\frac{1}{2},2}\right), \\ \mathcal{E}_{e,j}^{n+1} = \mathcal{E}_{e,j}^n - \dfrac{\Delta t}{\Delta x}\left(F^n_{e,j+\frac{1}{2},3} - F^n_{e,j-\frac{1}{2},3}\right) - u_j^{n+1}\dfrac{\Delta t}{\Delta x}\left(\delta^n_{j+\frac{1}{2}} - \delta^n_{j-\frac{1}{2}}\right) + \Delta t \nu_{ei}(T_{i,j}^{n+1} - T_{e,j}^{n+1}), \\ \mathcal{E}_{i,j}^{n+1} = \mathcal{E}_{i,j}^n - \dfrac{\Delta t}{\Delta x}\left(F^n_{i,j+\frac{1}{2},3} - F^n_{i,j-\frac{1}{2},3}\right) + u_j^{n+1}\dfrac{\Delta t}{\Delta x}\left(\delta^n_{j+\frac{1}{2}} - \delta^n_{j-\frac{1}{2}}\right) - \Delta t \nu_{ei}(T_{i,j}^{n+1} - T_{e,j}^{n+1}). \end{cases} \tag{77}$$

At the implementation level, the values of ρ and u are computed explicitly. It just remains to handle the source terms, and this happens to be a linear problem.

Remark 3.1. *The relations (77) lead to a conservative numerical scheme for the conservative system satisfied by ρ, ρu and $\mathcal{E} = \mathcal{E}_e + \mathcal{E}_i$.*

Remark 3.2. (77) *gives rise to a numerical scheme as soon as the fluxes coming from a discretisation of the conservative Euler system. In particular, starting from* (77)*, the numerical fluxes do not need to come from a kinetic process. For example, HLLC fluxes for Euler can be chosen. This point needs further investigations.*

As an example, suppose that

$$h_\alpha(f,g,v) = \frac{v_1}{2}(f+g) - \frac{|v_{\max}|}{2}(g-f). \tag{78}$$

Then

$$\mathcal{F}_\alpha(U_\alpha, V_\alpha) = \frac{1}{2}P_\alpha(v_1\mathcal{M}_\alpha(U_\alpha) + v_1\mathcal{M}_\alpha(V_\alpha) - |v_{\max}|(\mathcal{M}_\alpha(V_\alpha) - \mathcal{M}_\alpha(U_\alpha))),$$

that is

$$\mathcal{F}_\alpha(U_\alpha, V_\alpha) = \frac{1}{2}\left(F_\alpha(U_\alpha) + F_\alpha(V_\alpha)\right) - \frac{|v_{\max}|}{2}\left(V_\alpha - U_\alpha\right). \tag{79}$$

Consequently:

$$F_{j+\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} \rho_j u_j + \rho_{j+1}u_{j+1} - |v_{\max}|(\rho_{j+1} - \rho_j) \\ \rho_j u_j^2 + \rho_{j+1}u_{j+1}^2 + p_{e,j} + p_{e,j+1} + p_{i,j} + p_{i,j+1} - |v_{\max}|(\rho_{j+1}u_{j+1} - \rho_j u_j) \\ u_{j+1}(\mathcal{E}_{e,j+1} + p_{e,j+1}) + u_j(\mathcal{E}_{e,j} + p_{e,j}) - |v_{\max}|(\mathcal{E}_{e,j+1} - \mathcal{E}_{e,j}) \\ u_{j+1}(\mathcal{E}_{i,j+1} + p_{i,j+1}) + u_j(\mathcal{E}_{i,j} + p_{i,j}) - |v_{\max}|(\mathcal{E}_{i,j+1} - \mathcal{E}_{i,j}) \end{pmatrix}.$$

The nonconservative part is approximated with

$$\delta_{j+\frac{1}{2}} = -\frac{c_i}{2}(p_{e,j} + p_{e,j+1}) + \frac{c_e}{2}(p_{i,j} + p_{i,j+1}).$$

Finally, denoting $\eta_e = 1$, $\eta_i = -1$ the scheme reads as:

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{2\Delta x}\left(\rho_{j+1}^n u_{j+1}^n - \rho_{j-1}^n u_{j-1}^n\right) + \frac{|v_{\max}|\Delta t}{2\Delta x}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right), \tag{80}$$

$$\begin{aligned} \rho_j^{n+1}u_j^{n+1} = {} & \rho_j^n u_j^n - \frac{\Delta t}{2\Delta x}\left(\rho_{j+1}^n (u_{j+1}^n)^2 + p_{e,j+1}^n + p_{i,j+1}^n - \rho_{j-1}^n (u_{j-1}^n)^2 - p_{e,j-1}^n - p_{i,j-1}^n\right) \\ & + \frac{|v_{\max}|\Delta t}{2\Delta x}\left(\rho_{j+1}^n u_{j+1}^n - 2\rho_j^n u_j^n + \rho_{j-1}^n u_{j-1}^n\right), \end{aligned} \tag{81}$$

$$\begin{aligned} \mathcal{E}_{\alpha,j}^{n+1} = {} & \mathcal{E}_{\alpha,j}^n - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n(\mathcal{E}_{\alpha,j+1}^n + p_{\alpha,j+1}^n) - u_{j-1}^n(\mathcal{E}_{\alpha,j-1}^n + p_{\alpha,j-1}^n)\right) \\ & + \frac{|v_{\max}|\Delta t}{2\Delta x}\left(\mathcal{E}_{\alpha,j+1}^n - 2\mathcal{E}_{\alpha,j}^n + \mathcal{E}_{\alpha,j-1}^n\right) + \eta_\alpha\,\Delta t\,\nu_{ei}(T_{i,j}^{n+1} - T_{e,j}^{n+1}) \\ & - \eta_\alpha u_j^{n+1}\frac{\Delta t}{2\Delta x}\left(-c_i(p_{e,j+1}^n - p_{e,j-1}^n) + c_e(p_{i,j+1}^n - p_{i,j-1}^n)\right), \qquad \alpha = e, i. \end{aligned} \tag{82}$$

For the numerical tests, we used the more general HLL flux, which is obtained by the choice

$$h_\alpha(f,g,v) = v_1\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}f - \frac{\lambda_1}{\lambda_3 - \lambda_1}g\right) - \frac{\lambda_1\lambda_3}{\lambda_3 - \lambda_1}(g-f). \tag{83}$$

Here λ_1 and λ_3 are constants to be fixed. The fact that this scheme satisfies discrete entropy inequalities is not clear at this point because the transport scheme with flux (78) or (83) is not monotone for all v, but it is actually true, as it will be proved in section 3.2. In the situation where for any v_1, one has $|v_1| \leq |v_{max}|$, a discrete entropy inequality can be proved. This condition is fullfilled, when the velocity grid that is used is compactly supported. This situation is classical in kinetic theory for gas dynamics, where the authors consider velocity grids equal to $[u - 4\sqrt{RT}, u + 4\sqrt{RT}]$ (see [12]). In particular, the Maxwellian distributions are truncated beyond this domain. There is another way to proceed which consists in replacing the Maxwellian by compactly supported distributions shearing the same macroscopic quantities as the associated Maxwellian.

3.2. **Discrete BGK schemes**

In this section we start from the discrete BGK approach of [28], [4] to construct another family of schemes for system (1). An important feature of those schemes is that they satisfy discrete entropy inequalities. As a particular case, we recover the scheme (80-82), which is therefore entropic.

The model developped in section 2 deals with a monoatomic setting. That is why the values $\gamma_i = \gamma_e = \frac{5}{3}$ are imposed. Here the situation $\gamma_i \neq \gamma_e$ is allowed.

3.2.1. *Models*

We take a discrete BGK model with L velocities ([4]) for Euler equations with $\gamma = \gamma_\alpha$, $\alpha = e, i$: denoting $F_\alpha(U)$ the Euler flux function, there exists L functions $M_{\alpha,l} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\lambda_{1,\alpha}, \ldots, \lambda_{L,\alpha} \in \mathbb{R}$ such that

$$\forall U \in \mathbb{R}^3, \quad \sum_{l=1}^{L} M_{\alpha,l}(U) = U, \quad \sum_{l=1}^{L} \lambda_{\alpha,l} M_{\alpha,l}(U) = F_\alpha(U). \tag{84}$$

We denote $M_\alpha = (M_{\alpha,l})_{1\leq l\leq L}$, $\Lambda_\alpha = diag(\lambda_{\alpha,1} I_3, \ldots, \lambda_{\alpha,L} I_3)$ where I_3 is the 3×3 identity matrix, $P = (I_3, \ldots, I_3) \in \mathcal{M}_{3,3L}(\mathbb{R})$, so that relations (84) may be written as follows:

$$\forall U \in \mathbb{R}^3, \quad P M_\alpha(U) = U, \quad P \Lambda_\alpha M_\alpha(U) = F_\alpha(U). \tag{85}$$

Moreover we denote

$$N_\alpha = \frac{q_\alpha}{m_\alpha} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{N}_\alpha = diag(N_\alpha, \ldots, N_\alpha) \in \mathcal{M}_{3L}(\mathbb{R}).$$

For all $f = (f_1, \ldots, f_L) \in \mathbb{R}^{3L}$, the macroscopic variables ρ, u, $\mathcal{E}$ and ε are defined by:

$$Pf = U = \begin{pmatrix} \rho \\ \rho u \\ \mathcal{E} = \rho\varepsilon + \frac{1}{2}\rho u^2 \end{pmatrix}.$$

For α, $\beta \in \{e, i\}$, $\alpha \neq \beta$, we define

$$\tau_{\alpha,l} = \nu_{ei} \frac{m_\alpha}{k_B} (\gamma_\alpha - 1) \left(\frac{1}{\rho_\alpha} f_{\alpha,3,l} - \frac{\rho_\alpha u_\alpha}{2\rho_\alpha^2} f_{\alpha,2,l} \right), \quad B_{\alpha\beta,l}(f_e, f_i) = (0, 0, \tau_{\beta,l} - \tau_{\alpha,l}), \quad B_{\alpha\beta} = (B_{\alpha\beta,l})_{1\leq l\leq L}$$

and we set

$$\begin{cases} \partial_t f_e^\varepsilon + \Lambda_e \partial_x f_e^\varepsilon - E^\varepsilon(x,t)\mathcal{N}_e f_e^\varepsilon = \dfrac{1}{\varepsilon}(M_e(U_e^\varepsilon) - f_e^\varepsilon) + B_{ei}(f_e^\varepsilon, f_i^\varepsilon), \\ \partial_t f_i^\varepsilon + \Lambda_i \partial_x f_i^\varepsilon - E^\varepsilon(x,t)\mathcal{N}_i f_i^\varepsilon = \dfrac{1}{\varepsilon}(M_i(U_i^\varepsilon) - f_i^\varepsilon) + B_{ie}(f_e^\varepsilon, f_i^\varepsilon), \\ \partial_t E^\varepsilon = -\dfrac{1}{\varepsilon^2}\left(\dfrac{q_e}{m_e}\rho_e^\varepsilon u_e^\varepsilon + \dfrac{q_i}{m_i}\rho_i^\varepsilon u_i^\varepsilon\right), \\ \partial_x E^\varepsilon = \dfrac{1}{\varepsilon^2}\left(\dfrac{q_e}{m_e}\rho_e^\varepsilon + \dfrac{q_i}{m_i}\rho_i^\varepsilon\right), \end{cases} \tag{86}$$

with U_α^ε defined by $U_\alpha^\varepsilon = P f_\alpha^\varepsilon$, $\alpha = e, i$.

If $f_\alpha^\varepsilon \to f_\alpha$ when ε tends to 0 ($\alpha = e, i$), then passing to the limit in the last two equations of (86) we obtain that

$$u_e = u_i = u, \quad \frac{q_e}{m_e}\rho_e + \frac{q_i}{m_i}\rho_i = 0\,. \tag{87}$$

Moreover $q_e = -e$ and $q_i = Ze$, so that $m_i\rho_e = Zm_e\rho_i$. As c_e and c_i satisfy (3), we obtain that $\rho_\alpha = c_\alpha\rho$, $\alpha = e, i$.

We have also for $\alpha = e, i$:

$$M_\alpha(U_\alpha) = f_\alpha.$$

We apply P on the two first equations of (86) and we obtain the relaxed system:

$$\begin{cases} \partial_t \rho_\alpha + \partial_x(\rho_\alpha u) = 0, & \alpha = e, i, \\ \partial_t(\rho_\alpha u) + \partial_x(\rho_\alpha u^2 + p_\alpha) - \dfrac{q_\alpha}{m_\alpha} E\rho_\alpha = 0, & \alpha = e, i, \\ \partial_t(\rho_e\varepsilon_e + \dfrac{1}{2}\rho_e u^2) + \partial_x(u(\rho_e\varepsilon_e + \dfrac{1}{2}\rho_e u^2 + p_e)) - \dfrac{q_e}{m_e}E\rho_e u = \nu_{ei}(T_i - T_e), \\ \partial_t(\rho_i\varepsilon_i + \dfrac{1}{2}\rho_i u^2) + \partial_x(u(\rho_i\varepsilon_i + \dfrac{1}{2}\rho_i u^2 + p_i)) - \dfrac{q_i}{m_i}E\rho_i u = -\nu_{ei}(T_i - T_e). \end{cases} \tag{88}$$

We then proceed as in the proof of Proposition 2.2: we retrieve Ohm's law and $\mathcal{U}$ is a solution of the bitemperature Euler system.

3.2.2. *Compatibility of microscopic entropies*

In this paragraph we suppose that the Maxwellian functions are of the following form:

$$M_{\alpha,l}(U_\alpha) = \xi_{\alpha,l}U_\alpha + \zeta_{\alpha,l}F_\alpha(U_\alpha), \quad 1 \le l \le L, \quad \alpha = e, i. \tag{89}$$

Here $\xi_{\alpha,l}$, $\zeta_{\alpha,l}$ are real constants. For instance, this is the case for the model (100-101) below. We also suppose that those functions are one-to-one and that for all U_α under consideration:

$$\sigma(M'_{\alpha,l}(U_\alpha)) \subset]0, +\infty[, \quad \alpha = e, i. \tag{90}$$

This condition is related to the well-known Liu's subcharacteristic condition, see [25], [4]. The Euler entropy being fixed as in property 2.1, we define

$$G_{\alpha,l}(U) = \xi_{\alpha,l}\eta_\alpha(U) + \zeta_{\alpha,l}Q_\alpha(U), \quad H_{\alpha,l}(f_l) = G_{\alpha,l}(M_{\alpha,l}^{-1}(f_{\alpha,l})), \quad 1 \le l \le L, \quad \alpha = e, i.$$

Then F. Bouchut proved that the following properties hold for $\alpha = e, i$ ([8]):

- for $l = 1, \dots, L$, $H_{\alpha,l}$ is convex, (E0)
- $\displaystyle\sum_{l=1}^{L} H_{\alpha,l}(M_{\alpha,l}(U)) = \eta_\alpha(U)$, (E1).

- for all f, if $U_f = Pf$, one has $\displaystyle\sum_{l=1}^{L} H_{\alpha,l}(M_{\alpha,l}(U_f)) \leq \sum_{l=1}^{L} H_{\alpha,l}(f_l)$. (E2)

Those properties are similar to the ones of the Boltzmann entropy for the physical Maxwellian distributions of paragraph 2 and the same formalism could be used for both of them. Using moreover the fact that

$$\eta'(U_\alpha)N_\alpha U_\alpha = 0, \qquad \alpha \in \{e, i\} \tag{91}$$

we can prove straightforwardly that our relaxation model preserves the entropy properties of the solutions:

Proposition 3.2. *If $\mathcal{U}$ is a solution of system (1) obtained as a limit of the discrete BGK model (86) then $\mathcal{U}$ is an admissible solution.*

The counterpart of this result is that the related numerical scheme is entropic, as it is stated in the following paragraph.

3.2.3. *An entropic numerical scheme*

We proceed as in section 3.1. Here h_α is a numerical flux for the linear system of transport equations

$$\partial_t f_\alpha^\varepsilon + \Lambda_\alpha \partial_x f_\alpha^\varepsilon = 0,$$

that is

Definition 3.2. *For $\alpha = e, i$ we define a numerical flux $h_\alpha = (h_{\alpha,l})_{1\leq l\leq L}$:*

$$\forall l \in \{1, \dots, L\}, \quad h_{\alpha,j+\frac{1}{2},l} = h_{\alpha,l}(f_{\alpha,j,l}, f_{\alpha,j+1,l}) :$$

h_α is Lipschitz continuous and satisfies the following consistency property

$$(\forall\, f \in \mathbb{R}^{3L}) \quad h_\alpha(f, f) = \Lambda_\alpha f. \tag{92}$$

An approximate solution $\mathcal{U}^n$ being known and U_e^n, U_i^n having been defined, we set

$$f_{\alpha,l}^n = M_{\alpha,l}(U_\alpha^n). \tag{93}$$

For $\alpha, \beta \in e, i$, with $\beta \neq \alpha$, we define $(f_{\alpha,j}^{n+\frac{1}{2}})_{\alpha=e,i}$ as

$$f_{\alpha,j}^{n+\frac{1}{2}} = f_{\alpha,j}^n - \frac{\Delta t}{\Delta x}(h_{\alpha,j+\frac{1}{2}}^n - h_{\alpha,j-\frac{1}{2}}^n) + \Delta t E_j^{n+1} \mathcal{N}_\alpha f_{\alpha,j}^{n+\frac{1}{2}} + \Delta t B_{\alpha\beta}(f_{\alpha,j}^{n+\frac{1}{2}}, f_{\beta,j}^{n+\frac{1}{2}}). \tag{94}$$

We define U_α^{n+1}, $\alpha = e, i$, as

$$U_{\alpha,j}^{n+1} = P(f_{\alpha,j}^{n+\frac{1}{2}}) \tag{95}$$

and we impose (71). The remaining of the method follows the lines of paragraph 3.1, simply replacing the numerical flux (73) by

$$\mathcal{F}_\alpha(U_\alpha, V_\alpha) = P(h_\alpha(M_\alpha(U_\alpha), M_\alpha(V_\alpha))). \tag{96}$$

It is easy to see that this is a consistent numerical flux for Euler system of the species α. As a final result, we obtain the following consistent scheme.

Proposition 3.3. *A consistent scheme for Euler bitemperature system (1) is constructed as follows. For all $n \geq 0$ if $\mathcal{U}^n = \{\mathcal{U}_j^n\}_{j\in\mathbb{Z}}$ is the approximate solution of system (1) at time t_n, $U_{e,j}^n$ and $U_{i,j}^n$ are defined by (76). A kinetic flux h_α is chosen as in definition 3.2. We then define the numerical fluxes $F_{\alpha,j+\frac{1}{2}}$, $F_{j+\frac{1}{2}}$ and $\delta_{j+\frac{1}{2}}$ by (72), (96), (74), (75). The approximate solution at time t_{n+1} is defined by the implicit system (77).*

In all the sequel, we choose h_α as the upwind flux: denoting for $\lambda \in \mathbb{R}$

$$\lambda^- = \max(0, -\lambda), \quad \lambda^+ = \max(0, \lambda), \tag{97}$$

we define

$$\forall f, g \in \mathbb{R}^{3L}, \quad h_{\alpha,l}(f_l, g_l) = \lambda^+_{\alpha,l} f_l - \lambda^-_{\alpha,l} g_l. \tag{98}$$

We then find

$$\mathcal{F}_\alpha(U_\alpha, V_\alpha) = \sum_{l=1}^{L} \left(-\lambda^-_{\alpha,l} M_{\alpha,l}(V_\alpha) + \lambda^+_{\alpha,l} M_{\alpha,l}(U_\alpha) \right). \tag{99}$$

A first choice of parameters is as follows, for both $\alpha = e$, $\alpha = i$. As the parameters are the same, we omit to mention α in the notations. The characteristic velocities are:

$$\lambda_1 < \lambda_3, \quad \lambda_2 = \frac{1}{2}(\lambda_1 + \lambda_3) \tag{100}$$

while the Maxwellian functions depend on a parameter $\kappa \geq 0$:

$$M_1(U) = \frac{(\lambda_3 - \kappa)U - F(U)}{\lambda_3 - \lambda_1}, \qquad M_2(U) = \frac{2\kappa U}{\lambda_3 - \lambda_1}, \qquad M_3(U) = \frac{-(\lambda_1 + \kappa)U + F(U)}{\lambda_3 - \lambda_1}. \tag{101}$$

The subcharacteristic condition

$$\forall \lambda \in \sigma(F'(U)), \quad \lambda_1 + \kappa < \lambda < \lambda_3 - \kappa$$

ensures that if $\kappa > 0$ then condition (90) is satisfied for $l = 1, 2, 3$. If $\kappa = 0$, then $M_2 = 0$, the model reduces to a 2×2 one and the condition (90) is satisfied.

The related numerical flux reads as an upwind flux if $\lambda_1 \lambda_3 \geq 0$. Otherwise we obtain, setting again α:

$$\begin{aligned} \mathcal{F}_\alpha(U_\alpha, V_\alpha) = &- \frac{\lambda_1}{\lambda_3 - \lambda_1} F_\alpha(V_\alpha) + \frac{\lambda_3}{\lambda_3 - \lambda_1} F_\alpha(U_\alpha) \\ &+ \frac{\lambda_1 \lambda_3}{\lambda_3 - \lambda_1}(V_\alpha - U_\alpha) + \frac{\kappa}{\lambda_3 - \lambda_1}((-\lambda_1 - (\lambda_1 + \lambda_3)^-)V_\alpha + (-\lambda_3 + (\lambda_1 + \lambda_3)^+)U_\alpha). \end{aligned} \tag{102}$$

If $\kappa = 0$, one retrieves the HLL flux given by (83). If $\kappa = 0$ and $\lambda_1 = -|v_{max}| = -\lambda_3$, one retrieves (79).

The entropic behaviour of our schemes is proved by the same method as in [5] because (91) holds. We denote

$$\tilde{f}^{n+\frac{1}{2}}_{\alpha,j,l} = f^n_{\alpha,j,l} - \frac{\Delta t}{\Delta x} \left(h_{\alpha,l}(f^n_{\alpha,j,l}, f^n_{\alpha,j+1,l}) - h_{\alpha,l}(f^n_{\alpha,j-1,l}, f^n_{\alpha,j,l}) \right), \quad 1 \leq l \leq L. \tag{103}$$

Theorem 3.1. *Suppose that the Maxwellian functions are of form (89) and that condition (90) is satisfied. Suppose moreover that for $\alpha = e, i$ and for all $l \in \{1, \ldots, L\}$ there exists a numerical entropy flux $\mathcal{G}^n_{\alpha,j+\frac{1}{2},l} = \mathcal{G}_{\alpha,l}(f^n_{\alpha,j,l}, f^n_{\alpha,j+1,l})$ such that*

$$\frac{H_{\alpha,l}(\tilde{f}^{n+\frac{1}{2}}_{\alpha,j}) - H_{\alpha,l}(f^n_{\alpha,j})}{\Delta t} + \frac{\mathcal{G}^n_{\alpha,j+\frac{1}{2},l} - \mathcal{G}^n_{\alpha,j-\frac{1}{2},l}}{\Delta x} \leq 0. \tag{104}$$

Then the following discrete entropy inequality holds for the numerical scheme defined in proposition 3.3:

$$\frac{\eta(\mathcal{U}^{n+1}_j) - \eta(\mathcal{U}^n_j)}{\Delta t} + \frac{\mathcal{Q}^n_{j+\frac{1}{2}} - \mathcal{Q}^n_{j-\frac{1}{2}}}{\Delta x} \leq - \frac{\nu_{ei}}{k_B T^{n+1}_{i,j} T^{n+1}_{e,j}} (T^{n+1}_{i,j} - T^{n+1}_{e,j})^2, \tag{105}$$

where the numerical entropy flux $\mathcal{Q}^n_{j+\frac{1}{2}} = \mathcal{Q}(\mathcal{U}^n_j, \mathcal{U}^n_{j+1})$ *is defined by*

$$\mathcal{Q}(\mathcal{U}, \mathcal{V}) = \sum_{\alpha=e,i} \sum_{l=1}^{L} \mathcal{G}_{\alpha,l}(M_{\alpha,l}(U_\alpha), M_{\alpha,l}(V_\alpha)) \ .$$

In the applications, as we choose the upwind flux for the transport equations, the numerical entropy fluxes $\mathcal{G}_{\alpha,l}$ exist under the CFL condition

$$\max\{|\lambda_{\alpha,l}|\frac{\Delta t}{\Delta x},\ 1 \le l \le L,\ \alpha = e, i\} \le 1. \tag{106}$$

They read as

$$\mathcal{G}_{\alpha,l}(f, g) = \lambda^+_{\alpha,l} H_{\alpha,l}(f) - \lambda^-_{\alpha,l} H_{\alpha,l}(g).$$

This is different from the kinetic scheme of section 3.1 where the CFL condition cannot be satisfied for all v, so that the scheme with flux (78) is not entropic.

In the present case we have

$$\mathcal{G}_{\alpha,l}(M_{\alpha,l}(U_\alpha), M_{\alpha,l}(V_\alpha)) = \xi_{\alpha,l}\left(\lambda^+_{\alpha,l}\eta_\alpha(U_\alpha) - \lambda^-_{\alpha,l}\eta_\alpha(V_\alpha)\right) + \zeta_{\alpha,l}\left(\lambda^+_{\alpha,l}Q_\alpha(U_\alpha) - \lambda^-_{\alpha,l}Q_\alpha(V_\alpha)\right).$$

3.3. **Relaxation approach (Suliciu)**

3.3.1. *Relaxed system and relations of Suliciu*

We consider a relaxation approach to solve the bitemperature Euler equations (1). As for the Euler system described in [9] for one species, the partial pressures p_e and p_i are replaced by passive scalar variables π_e and π_i. These new variables are relaxed towards partial pressures p_e and p_i.

We consider the system under the form (56). The Suliciu relaxation system can be written as :

$$\begin{cases} \partial_t \rho + u\partial_x \rho + \rho\partial_x u = 0, \\ \partial_t u + u\partial_x u + \dfrac{1}{\rho}\partial_x(\pi_e + \pi_i) = 0, \\ \partial_t \varepsilon_e + u\partial_x \varepsilon_e + \dfrac{\pi_e}{\rho_e}\partial_x u = \dfrac{\nu_{ei}}{\rho_e}(T_i - T_e), \\ \partial_t \varepsilon_i + u\partial_x \varepsilon_i + \dfrac{\pi_i}{\rho_i}\partial_x u = -\dfrac{\nu_{ei}}{\rho_i}(T_i - T_e), \\ \partial_t \pi_e + u\partial_x \pi_e + \dfrac{a^2 c_e}{\rho}\partial_x u = \dfrac{1}{\tau}(p_e - \pi_e), \\ \partial_t \pi_i + u\partial_x \pi_i + \dfrac{a^2 c_i}{\rho}\partial_x u = \dfrac{1}{\tau}(p_i - \pi_i). \end{cases} \tag{107}$$

When the positive relaxation parameter τ tends to zero one has

$$\pi_\alpha = p_\alpha + \delta\tau + O(\tau^2),\ \alpha = e, i. \tag{108}$$

From equation (108) and system (107), we can deduce that :

$$\delta = \frac{c_\alpha}{\rho}(\rho^2 a_\alpha^2 - a^2)\partial_x u + O(\tau), \quad a_\alpha = \sqrt{\frac{\gamma_\alpha p_\alpha}{\rho_\alpha}}\,.$$

Hence the stability condition needed by the parameter a is :

$$a^2 \ge \rho^2 \max(a_e^2, a_i^2)\,. \tag{109}$$

It is necessary to diminish the numerical diffusion induced by the increasing of parameter a to insure stability. Hence, a new equation about a is introduced:

$$\partial_t a + u\partial_x a = 0. \tag{110}$$

Finally, with respect to the primal variable, system (107, 110) reads as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi_e + \pi_i) = 0, \\ \partial_t \mathcal{E}_e + \partial_x(u(\mathcal{E}_e + \pi_e)) - u\partial_x(c_i\pi_e - c_e\pi_i) = \nu_{ei}(T_i - T_e), \\ \partial_t \mathcal{E}_i + \partial_x(u(\mathcal{E}_i + \pi_i)) + u\partial_x(c_i\pi_e - c_e\pi_i) = \nu_{ei}(T_e - T_i), \\ \partial_t(\frac{\rho_e\pi_e}{(ac_e)^2}) + \partial_x(u(\frac{\rho_e\pi_e}{(ac_e)^2} + 1)) = \frac{\rho_e}{(ac_e)^2\tau}(p_e - \pi_e), \\ \partial_t(\frac{\rho_i\pi_i}{(ac_i)^2}) + \partial_x(u(\frac{\rho_i\pi_i}{(ac_i)^2} + 1)) = \frac{\rho_i}{(ac_i)^2\tau}(p_i - \pi_i), \\ \partial_t(\rho a) + \partial_x(\rho a u) = 0. \end{cases} \tag{111}$$

The Suliciu relaxation scheme for system (1) consists in three steps. The approximate solution $\mathcal{U}^n$ of (1) being known, we define a^n such that condition (109) is satisfied and we project the pressures onto equilibrium:

$$\pi_\alpha^n = p_\alpha^n, \quad \alpha = e, i.$$

Then we use an exact Riemann solver for the homogeneous system related to (111):

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \pi_e + \pi_i) = 0, \\ \partial_t \mathcal{E}_e + \partial_x(u(\mathcal{E}_e + \pi_e)) - u\partial_x(c_i\pi_e - c_e\pi_i) = 0, \\ \partial_t \mathcal{E}_i + \partial_x(u(\mathcal{E}_i + \pi_i)) + u\partial_x(c_i\pi_e - c_e\pi_i) = 0, \\ \partial_t(\frac{\rho_e\pi_e}{(ac_e)^2}) + \partial_x(u(\frac{\rho_e\pi_e}{(ac_e)^2} + 1)) = 0, \\ \partial_t(\frac{\rho_i\pi_i}{(ac_i)^2}) + \partial_x(u(\frac{\rho_i\pi_i}{(ac_i)^2} + 1)) = 0, \\ \partial_t(\rho a) + \partial_x(\rho a u) = 0. \end{cases} \tag{112}$$

Remark that the nonconservative product $u\partial_x(c_i\pi_e - c_e\pi_i)$ makes difficult the definition of a weak solution for this system. However, we show here below that the solution of the Riemann problem is well defined and this is the only property we need. Let $\overline{\mathcal{U}}^{n+\frac{1}{2}}$ be the obtained solution. The third step takes the temperatures interaction into account implicitly: the approximate solution of system (1) at time t^{n+1} is defined by

$$\rho^{n+1} = \overline{\rho}^{n+\frac{1}{2}}, \quad u^{n+1} = \overline{u}^{n+\frac{1}{2}}$$

and

$$\begin{cases} \mathcal{E}_e^{n+1} = \overline{\mathcal{E}}_e^{n+\frac{1}{2}} + \Delta t\, \nu_{ei}(T_i^{n+1} - T_e^{n+1}), \\ \mathcal{E}_i^{n+1} = \overline{\mathcal{E}}_i^{n+\frac{1}{2}} + \Delta t\, \nu_{ei}(T_e^{n+1} - T_i^{n+1}). \end{cases}$$

This system is linear and owns an explicit solution.

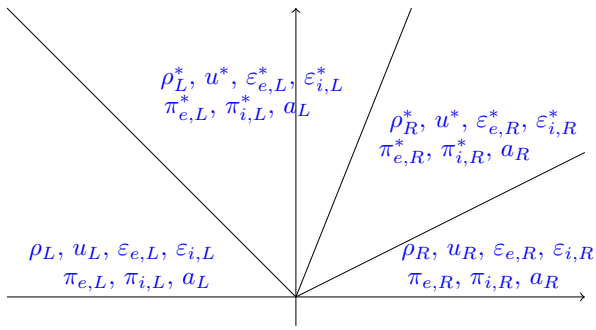

FIGURE 1. Structure of the Riemann problem.

3.3.2. *Riemann solver for the homogeneous system (112)*

By using the variable $U = (\rho, u, \varepsilon_e, \varepsilon_i, \pi_e, \pi_i, a)$, one can easily compute the eigenvalues of system (112). They read as $\{u - a/\rho, u, u + a/\rho\}$ where u is an eigenvalue of order 5. All the fields are linearly degenerated.

Let $\overline{\mathcal{U}}_L$, $\overline{\mathcal{U}}_R$ be a Riemann data for system (112). We look for a weak solution as a superposition of three contact discontinuities propagating with velocities $u_L - \frac{a_L}{\rho_L}$, u^* (to be determined), $u_R + \frac{a_R}{\rho_R}$ with

$$u_L - \frac{a_L}{\rho_L} \leq u^* \leq u_R + \frac{a_R}{\rho_R}, \tag{113}$$

Riemann invariants can be given by :

$$\begin{array}{lcllllll}
\text{Field } (u - \frac{a}{\rho}) & : & u - \frac{a}{\rho}, & a, & \pi_e + \frac{a^2 c_e}{\rho}, & \pi_i + \frac{a^2 c_i}{\rho}, & \varepsilon_e - \frac{\pi_e^2}{2(c_e a)^2}, & \varepsilon_i - \frac{\pi_i^2}{2(c_i a)^2}. \\
\text{Field } (u) & : & u, & \pi_e + \pi_i\,. & & & & \\
\text{Field } (u + \frac{a}{\rho}) & : & u + \frac{a}{\rho}, & a, & \pi_e + \frac{a^2 c_e}{\rho}, & \pi_i + \frac{a^2 c_i}{\rho}, & \varepsilon_e - \frac{\pi_e^2}{2(c_e a)^2}, & \varepsilon_i - \frac{\pi_i^2}{2(c_i a)^2}.
\end{array}$$

As a consequence, $c_i\pi_e - c_e\pi_i$ is a Riemann invariant for both extreme eigenvalues. This means that this quantity remains constant through the related contact discontinuities, so that $u\partial_x(c_i\pi_e - c_e\pi_i) = 0$ there. For the central discontinuity, u is constant so that $u\partial_x(c_i\pi_e - c_e\pi_i) = \partial_x(u(c_i\pi_e - c_e\pi_i))$. Hence, this product is also well defined in the usual weak sense. Hence the superposition of the three discontinuities is actually a weak solution of system (112). The Riemann invariants lead to the following system:

$$\begin{cases}
u_L - \dfrac{a_L}{\rho_L} = u^* - \dfrac{a_L}{\rho_L^*}, & u_R + \dfrac{a_R}{\rho_R} = u^* + \dfrac{a_R}{\rho_R^*}, \\
\pi_{e,L} + \dfrac{a_L^2 c_e}{\rho_L} = \pi_{e,L}^* + \dfrac{a_L^2 c_e}{\rho_L^*}, & \pi_{e,R} + \dfrac{a_R^2 c_e}{\rho_R} = \pi_{e,R}^* + \dfrac{a_R^2 c_e}{\rho_R^*}, \\
\pi_{i,L} + \dfrac{a_L^2 c_i}{\rho_L} = \pi_{i,L}^* + \dfrac{a_L^2 c_i}{\rho_L^*}, & \pi_{i,R} + \dfrac{a_R^2 c_i}{\rho_R} = \pi_{i,R}^* + \dfrac{a_R^2 c_i}{\rho_R^*}, \\
\varepsilon_{e,L} - \dfrac{\pi_{e,L}^2}{2(c_e a_L)^2} = \varepsilon_{e,L}^* - \dfrac{(\pi_{e,L}^*)^2}{2(c_e a_L)^2}, & \varepsilon_{e,R} - \dfrac{\pi_{e,R}^2}{2(c_e a_R)^2} = \varepsilon_{e,R}^* - \dfrac{(\pi_{e,R}^*)^2}{2(c_e a_R)^2}, \\
\varepsilon_{i,L} - \dfrac{\pi_{i,L}^2}{2(c_i a_L)^2} = \varepsilon_{i,L}^* - \dfrac{(\pi_{i,L}^*)^2}{2(c_i a_L)^2}, & \varepsilon_{i,R} - \dfrac{\pi_{i,R}^2}{2(c_i a_R)^2} = \varepsilon_{i,R}^* - \dfrac{(\pi_{i,R}^*)^2}{2(c_i a_R)^2}, \\
\pi_{e,L}^* + \pi_{i,L}^* = \pi_{e,R}^* + \pi_{i,R}^*. &
\end{cases}$$

Denoting $\pi = \pi_e + \pi_i$:

$$\begin{cases} \dfrac{1}{\rho_L^*} = \dfrac{1}{\rho_L} + \dfrac{a_R(u_R - u_L) + \pi_L - \pi_R}{a_L(a_L + a_R)}, \\ \dfrac{1}{\rho_R^*} = \dfrac{1}{\rho_R} + \dfrac{a_L(u_R - u_L) + \pi_R - \pi_L}{a_R(a_L + a_R)}, \end{cases}$$

$$u^* = \frac{a_R u_R + a_L u_L + \pi_L - \pi_R}{a_L + a_R}, \tag{114}$$

$$\begin{cases} \pi_{e,L}^* = \pi_{e,L} + a_L c_e \dfrac{\pi_R - \pi_L - a_R(u_R - u_L)}{a_L + a_R}, \\ \pi_{i,L}^* = \pi_{i,L} + a_L c_i \dfrac{\pi_R - \pi_L - a_R(u_R - u_L)}{a_L + a_R}, \\ \pi_{e,R}^* = \pi_{e,R} + a_R c_e \dfrac{\pi_L - \pi_R - a_L(u_R - u_L)}{a_L + a_R}, \\ \pi_{i,R}^* = \pi_{i,R} + a_R c_i \dfrac{\pi_L - \pi_R - a_L(u_R - u_L)}{a_L + a_R}, \end{cases} \qquad \begin{cases} \varepsilon_{e,L}^* = \varepsilon_{e,L} + \dfrac{1}{2(c_e a_L)^2}((\pi_{e,L}^*)^2 - \pi_{e,L}^2), \\ \varepsilon_{i,L}^* = \varepsilon_{i,L} + \dfrac{1}{2(c_i a_L)^2}((\pi_{i,L}^*)^2 - \pi_{i,L}^2), \\ \varepsilon_{e,R}^* = \varepsilon_{e,R} + \dfrac{1}{2(c_e a_R)^2}((\pi_{e,R}^*)^2 - \pi_{e,R}^2), \\ \varepsilon_{i,R}^* = \varepsilon_{i,R} + \dfrac{1}{2(c_i a_R)^2}((\pi_{i,R}^*)^2 - \pi_{i,R}^2). \end{cases}$$

We must now provide some sufficient conditions on a_L and a_R in order to satisfy (113) and the realisability of the intermediate states, that is positivity of ρ_L^*, ρ_R^*, $\varepsilon_{e,L}^*$, $\varepsilon_{i,L}^*$, $\varepsilon_{e,R}^*$ and $\varepsilon_{i,R}^*$. First we can note that (113) implies the positivity of ρ_L^* and ρ_R^*. On the other hand, from a straightforward calculation, $\varepsilon_{e,L}^*$, $\varepsilon_{i,L}^*$, $\varepsilon_{e,R}^*$ and $\varepsilon_{i,R}^*$ are positive if condition (109) is fulfilled.

Finally we can deduce that a must satisfy

$$\begin{cases} a_L(a_L + a_R) \geq \rho_L(p_{e,R} + p_{i,R} - p_{e,L} - p_{i,L}), \\ a_R(a_L + a_R) \geq -\rho_R(p_{e,R} + p_{i,R} - p_{e,L} - p_{i,L}), \\ a_L \geq \rho_L c_L, \\ a_R \geq \rho_R c_R, \end{cases}$$

where the two first conditions are set to satisfy (113).

The so defined solver is entropic for the case of Euler one temperature conservative system (see [7] paragraph 4.2). By analogy, one can expect that this property remains true for the bitemperature system, but this needs to be proved.

3.3.3. *Numerical Scheme*

In this paragraph we detail the second step of the scheme, namely the use of the Riemann solver to define the approximate solution $\overline{\mathcal{U}}^{n+\frac{1}{2}} = \{\overline{\mathcal{U}}_j^{n+\frac{1}{2}}\}_{j\in\mathbb{Z}}$. At each interface $\{x_{j+\frac{1}{2}}\} \times [t_n, t_{n+1}]$ we use the Riemann solver with $\overline{\mathcal{U}}_L = \overline{\mathcal{U}}_j^n$, $\overline{\mathcal{U}}_R = \overline{\mathcal{U}}_{j+1}^n$. The speed of the different waves are

$$\lambda_{1,j+\frac{1}{2}} = u_j^n - \frac{a_j^n}{\rho_j^n}, \qquad \lambda_{2,j+\frac{1}{2}} = u_{j+\frac{1}{2}}^*, \qquad \lambda_{3,j+\frac{1}{2}} = u_{j+1}^n + \frac{a_{j+1}^n}{\rho_{j+1}^n}.$$

We define $\overline{\mathcal{U}}(x,t)$ on $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t_n, t_{n+1}]$ as the superposition of the Riemann solutions (see fig 2). In order to avoid overlapping Riemann problems on $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, the following CFL condition is used :

$$\lambda_{max}\frac{\Delta t}{\Delta x} \leq \frac{1}{2}, \quad \lambda_{max} = \max_{j\in\mathbb{Z}}(\max_{1\leq k\leq 3}(\lambda_{k,j-\frac{1}{2}}^+, \lambda_{k,j+\frac{1}{2}}^-)).$$

We then define $\overline{\mathcal{U}}_j^{n+\frac{1}{2}}$ as

$$\overline{\mathcal{U}}_j^{n+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \overline{\mathcal{U}}(x, \Delta t)dx.$$

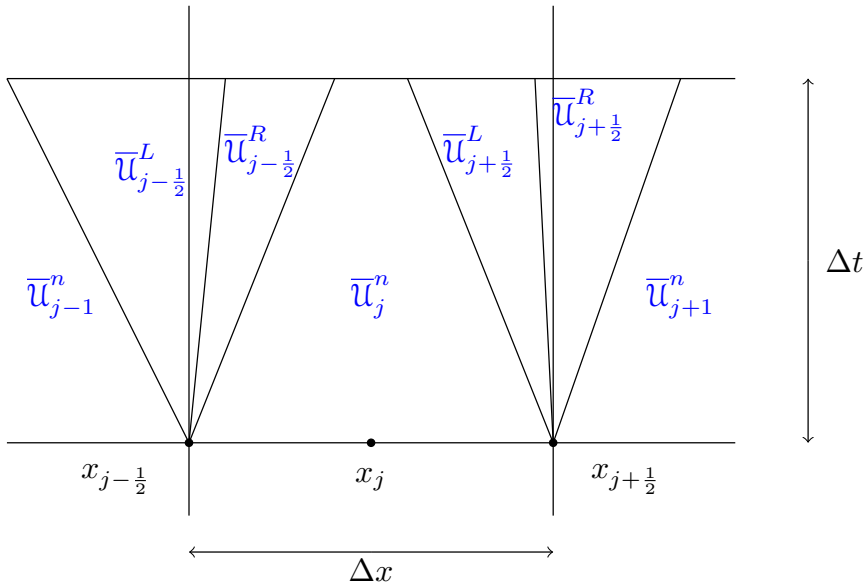

FIGURE 2. Structure of Riemann solutions on $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t_n, t_{n+1}]$.

We set :

$$\overline{\mathcal{U}}_j^- = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_j} \overline{\mathcal{U}}(x,\Delta t)dx, \quad \overline{\mathcal{U}}_j^+ = \frac{1}{\Delta x}\int_{x_j}^{x_{j+\frac{1}{2}}} \overline{\mathcal{U}}(x,\Delta t)dx.$$

Finally, using notation (97) we obtain

$$\begin{cases} \overline{\mathcal{U}}_j^- = \frac{1}{2}\overline{\mathcal{U}}_j^n - \frac{\Delta t}{\Delta x}\left(\lambda^+_{3,j-\frac{1}{2}}(\overline{\mathcal{U}}_j^n - \overline{\mathcal{U}}^R_{j-\frac{1}{2}}) + \lambda^+_{2,j-\frac{1}{2}}(\overline{\mathcal{U}}^R_{j-\frac{1}{2}} - \overline{\mathcal{U}}^L_{j-\frac{1}{2}}) + \lambda^+_{1,j-\frac{1}{2}}(\overline{\mathcal{U}}^L_{j-\frac{1}{2}} - \overline{\mathcal{U}}^n_{j-1})\right), \\ \overline{\mathcal{U}}_j^+ = \frac{1}{2}\overline{\mathcal{U}}_j^n + \frac{\Delta t}{\Delta x}\left(\lambda^-_{3,j+\frac{1}{2}}(\overline{\mathcal{U}}_{j+1}^n - \overline{\mathcal{U}}^R_{j+\frac{1}{2}}) + \lambda^-_{2,j+\frac{1}{2}}(\overline{\mathcal{U}}^R_{j+\frac{1}{2}} - \overline{\mathcal{U}}^L_{j+\frac{1}{2}}) + \lambda^-_{1,j+\frac{1}{2}}(\overline{\mathcal{U}}^L_{j+\frac{1}{2}} - \overline{\mathcal{U}}^n_{j})\right), \\ \overline{\mathcal{U}}_j^{n+\frac{1}{2}} = \overline{\mathcal{U}}_j^- + \overline{\mathcal{U}}_j^+. \end{cases}$$

3.4. **Lagrangian approximation**

The system (6) is solved by using a splitting. The gas dynamic is first solved using Lagrangian formalism. Then the computation of the new ion and electron internal energies is done in two steps. First, after the Lagrangian step we have a new total internal energy which is the sum of ion and electron internal energy. To compute the electron internal energy we make the assumption that the entropy deposition due to the shock is done on the ion internal energy [36]. Thus, electrons follow an isentropic evolution. Then the exchange of energy between ion and electron is computed. Finally, the Lagrangian solution is conservatively interpolated on the initial grid. This approach is also called Lagrange-projection scheme ([10], [14], [15]).

3.4.1. *Lagrangian phase*

In Lagrangian hydrodynamic methods, a computational cell moves with the flow velocity its mass being time-invariant. Thus, Lagrangian methods can capture contact discontinuities sharply in multi-material fluid flows. Here the Lagrangian scheme is based on a cell-centered discretization of Lagrangian hydrodynamics equations.

This scheme is written in total energy form. Here, e is the total specific energy.

$$\left\{\begin{array}{rcl} \rho d_t\tau - \partial_x u &=& 0, \\ \rho d_t u + \partial_x p &=& 0, \\ \rho d_t e + \partial_x pu &=& 0, \end{array}\right. \tag{115}$$

where $p = p_i + p_e$ and e are respectively the total pressure and the total energy. From total energy we can get internal energy from $\varepsilon = e - \frac{1}{2}u^2$. As the mesh is moving we have to add the local kinetic equation : $d_t x = u$.

3.4.2. *Numerical scheme*

In order to find an approximation of the flux of (115) we use the Godunov Riemann solver and we denote p^*, u^* the intermediate quantities. By using the Riemann invariants (u^*, p^*) are given by

$$\left\{\begin{array}{l} u^* = \dfrac{p_l - p_r + a_r u_r + a_l u_l}{a_r + a_l}, \\ p^* = \dfrac{a_r p_l + a_l p_r + a_l a_r (u_l - u_r)}{a_r + a_l}. \end{array}\right. \tag{116}$$

The problem is solved on the interval $[0; L]$ which is discretized in N elements $C_1, \ldots, C_N$. Furthermore, we can define the jth cell center as $x_j = \frac{1}{2}(x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}})$.
We perform first the integration in space on $\Delta x_j = [x_{j-1/2}, x_{j+1/2}]$:

$$\left\{\begin{array}{l} \rho \dfrac{d}{dt}\displaystyle\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tau(x,t)\,\mathrm{d}x - \Big[u(x_{j+\frac{1}{2}},t) - u(x_{j-\frac{1}{2}},t)\Big] = 0, \\ \rho \dfrac{d}{dt}\displaystyle\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)\,\mathrm{d}x + \Big[p(x_{j+\frac{1}{2}},t) - p(x_{j-\frac{1}{2}},t)\Big] = 0, \\ \rho \dfrac{d}{dt}\displaystyle\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} e(x,t)\,\mathrm{d}x + \Big[pu(x_{j+\frac{1}{2}},t) - pu(x_{j-\frac{1}{2}},t)\Big] = 0. \end{array}\right. \tag{117}$$

Let be $\varphi_j = \dfrac{1}{\Delta x_j}\displaystyle\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \varphi(x,t)\,\mathrm{d}x$, now we integrate in time :

$$\left\{\begin{array}{l} \rho\Delta x_j \displaystyle\int_{t_n}^{t_{n+1}} \frac{d\tau_j}{dt}\,\mathrm{d}t - \int_{t_n}^{t_{n+1}} \Big[u(x_{j+\frac{1}{2}},t) - u(x_{j-\frac{1}{2}},t)\Big]\,\mathrm{d}t = 0, \\ \rho\Delta x_j \displaystyle\int_{t_n}^{t_{n+1}} \frac{du_j}{dt}\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \Big[p(x_{j+\frac{1}{2}},t) - p(x_{j-\frac{1}{2}},t)\Big]\,\mathrm{d}t = 0, \\ \rho\Delta x_j \displaystyle\int_{t_n}^{t_{n+1}} \frac{de_j}{dt}\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \Big[pu(x_{j+\frac{1}{2}},t) - pu(x_{j-\frac{1}{2}},t)\Big]\,\mathrm{d}t = 0. \end{array}\right. \tag{118}$$

Now with $\varphi^n = \varphi(t_n)$ and $\varphi^*_{j+\frac{1}{2}} = \dfrac{1}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}} \varphi(x_{j+\frac{1}{2}},t)\,\mathrm{d}t$, the system becomes :

$$\left\{\begin{array}{l} m_j(\tau_j^{n+1} - \tau_j^n) - \Delta t(u^*_{j+\frac{1}{2}} - u^*_{j-\frac{1}{2}}) = 0, \\ m_j(u_j^{n+1} - u_j^n) + \Delta t(p^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}}) = 0, \\ m_j(e_j^{n+1} - e_j^n) + \Delta t(p^*_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}) = 0, \end{array}\right. \tag{119}$$

with $m_j = \rho_j \Delta x_j$, $(pu)^* = p^* u^*$ and where (p^*, u^*) are given by (116).

Thanks to (119) we can compute a discrete form for internal energy using the difference between the total energy and the kinetic energy.

Next remark that the scheme (119) is entropic. More precisely, considering a macroscopic expression s of the entropy (see [22]), the second law of thermodynamics writes

$$T\frac{ds}{dt} = \frac{d\varepsilon}{dt} + p\frac{d\tau}{dt}.$$

In that case, it is shown in ([26]), that the numerical scheme defined by (119) satisfies

$$m_j T_j \frac{ds_j}{dt} = \rho_j c_j \left(u^*_{j+\frac{1}{2}} - u_j\right)^2 + \rho_{j+1} c_{j+1} \left(u^*_{j-\frac{1}{2}} - u_j\right)^2 \geq 0, \tag{120}$$

corresponding to a semi-discrete entropy inequality.

3.4.3. *Bitemperature model*

Based on total energy conservation, the Lagrangian scheme must deal with the exchange between the kinetic energy and the internal energy of ion and electron. The system we aim to solve for the internal energy of ion and electron is

$$\rho\left(\frac{d\varepsilon_e}{dt} + p_e\frac{d\tau}{dt}\right) = \nu_{ei}(T_i - T_e),$$
$$\rho\left(\frac{d\varepsilon_i}{dt} + p_i\frac{d\tau}{dt}\right) = \nu_{ei}(T_e - T_i).$$

As the scheme is written in total energy formulation after the Lagrangian hydrodynamic step, we need to solve the exchange of energy in the system. The new ion and electron internal energy are computed in two steps. First, after the Lagrangian step we have a new internal energy which is the sum of ion and electron internal energy. To compute the electron internal energy we make the assumption that the entropy deposition is done on the ion internal energy [36]. Thus, electrons follow the isentropic evolution

$$m\frac{d\,\varepsilon_e}{dt} + p_e\frac{d\,V}{dt} = 0,$$

where $m = \rho\ V$ is the mass, V the volume and $\frac{d\,V}{dt}$ is the change of volume during the Lagrangian step. We can deduce ion internal energy from $\varepsilon_i = \varepsilon - \varepsilon_e$, where $\varepsilon = e - \frac{1}{2}u^2$ is the mean total internal energy. Then, to solve the energy transfer between ion and electron we solve

$$\rho\frac{d\varepsilon_{\mathrm{e}}}{dt} = \nu_{ei}(T_i - T_e),$$
$$\rho\frac{d\varepsilon_i}{dt} = \nu_{ei}(T_e - T_i).$$

3.4.4. *Remap step*

As we deal here with a Lagrange-projection like scheme the remapping phase corresponds here to a conservative interpolation of the physical variables from the Lagrangian grid at time t^{n+1} onto the initial grid. Since we are using a Lagrangian scheme where in the placement of the variables is cell-centered, we are developing a cell-centered remapping phase. In what follows, all quantities related to the initial grid are denoted here with the tilde accent. Let ψ be a physical variable of the flow defined on a cell by its piecewise constant representation

$\bar{\psi}_j$. Being given $\psi(x,t)$, on the Lagrangian grid we want to compute

$$\tilde{\psi}_j = \frac{1}{\tilde{V}_j} \int_{\tilde{x}_{j-\frac{1}{2}}}^{\tilde{x}_{j+\frac{1}{2}}} \psi(x,t)\, dx. \tag{121}$$

That is, knowing the mean value of $\psi(x,t) = \bar{\psi}$ over each cell of the Lagrangian grid, we want to compute its mean value over each cell of the initial grid. After the remap of all the conservative quantities we can reconstruct our solution.

4. Numerical results

In this part, some comparisons of the numerical schemes designed in Section 3 are performed. When $\gamma_e = \gamma_i$, the total density, total momentum and total energy $\mathcal{E} = \mathcal{E}_e + \mathcal{E}_i$ are solutions to the conservative 3×3 Euler system. Therefore exact solutions of the Riemann problem are known for those quantities.

Firstly, we consider a robustness test case dedicated to a double rarefaction wave whose parameters are

$$k_B = 1, \quad \gamma_e = \gamma_i = \gamma = \frac{5}{3}, \quad Z = 1, \quad m_e = 1, \quad m_i = 1.$$

Next we present two other test cases in order to check the validity of the source term approximation and to study the behaviour of shocks. In order to compare our results to more realistic physical situations ([36]), the physical parameters are set as follows:

$$k_B = 1, \quad \gamma_e = \gamma_i = \gamma = \frac{5}{3}, \quad Z = 1, \quad m_e = 10^{-3}, \quad m_i = 1.$$

This choice allows us to isolate the small m_e/m_i mass ratio feature.

The characteristic velocities σ_i of system (1) are u (double), $u \pm a$, with

$$a = \sqrt{\frac{\gamma_e p_e + \gamma_i p_i}{\rho}}.$$

In all the test cases that are presented, the time step Δt^n satisfies the CFL condition over all the cells:

$$\max_{1 \le i \le 4} |\sigma_i| \frac{\Delta t^n}{\Delta x} \le 0.25.$$

4.1. **Double rarefaction wave**

The double rarefaction waves test case does not involve shocks. The goal of this test case is to check the robustness of the different schemes.

The left and the right states of the Rieman problem are the following

$$\begin{array}{llll} \rho_L = 1, & u_L = -10, & T_{e,L} = 1000, & T_{i,L} = 1, \\ \rho_R = 1, & u_R = 10, & T_{e,R} = 1000, & T_{i,R} = 1. \end{array}$$

The solutions are computed for $x \in [0,1]$ at time $t = 0.05$ for 200 cells. On the pictures, the term AN3V is for the discrete kinetic scheme of subsection 3.2 with the choice $\kappa = (\lambda_3 - \lambda_1)/5$ in (101), while HLL is for the kinetic scheme of subsection 3.1 with the flux (83). In both cases, λ_1 and λ_3 are computed locally, at each cell interface. The term relaxation refers to the scheme presented in subsection 3.3.

The numerical results presented in figures 3 and 4 are in good agreement with the exact solutions for the conserved quantities. The results of the Lagrange remaping method show important descrepancies on the ionic

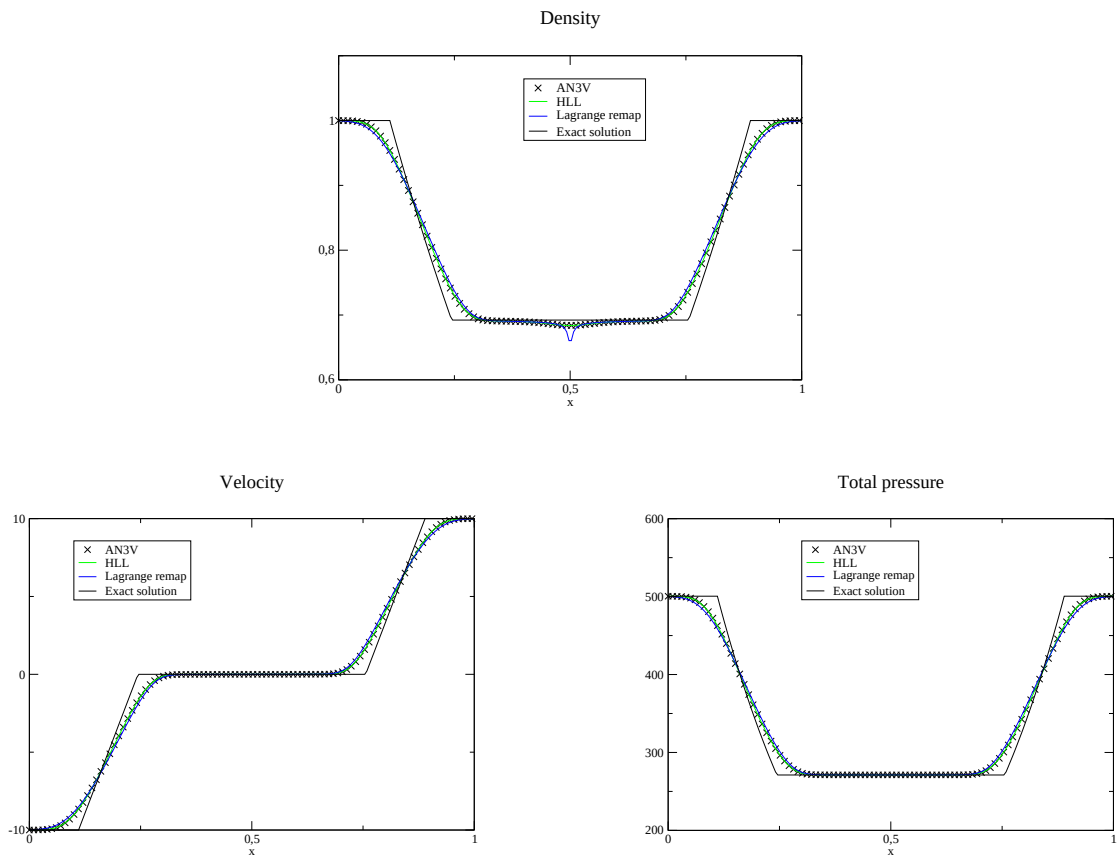

FIGURE 3. Density, velocity and pressure computed with AN3V, HLL and Lagrangian remaping schemes compared with exact solution for the double rarefaction waves problem.

temperature. This fact is well known and is due to the isentropic assumptions on the electrons. Then the entropy of the ions is overestimated in a problem without any shock. We remark also that AN3V and HLL give very close results. Indeed if $\kappa = 0$, HLL and AN3V coincide. A different value of this parameter does not seem to improve the results. For these reasons, in the following we chose to focus on HLL and relaxation schemes. Others schemes based on the approach of subsection 3.2 are to be tested in a forthcoming work.

4.2. **Source-term approximation: an analytical solution**

We take initial data such that

$$\forall x \in \mathbb{R} \quad \rho(x,0) = 1, \quad u(x,0) = 10, \quad T_i(x,0) + T_e(x,0) = \overline{T} = 2.$$

As

$$\rho_e \varepsilon_e + \rho_i \varepsilon_i = \frac{p_e + p_i}{\gamma - 1} = \frac{\rho k_B}{(\gamma - 1)(m_i + m_e)}(T_e + T_i),$$

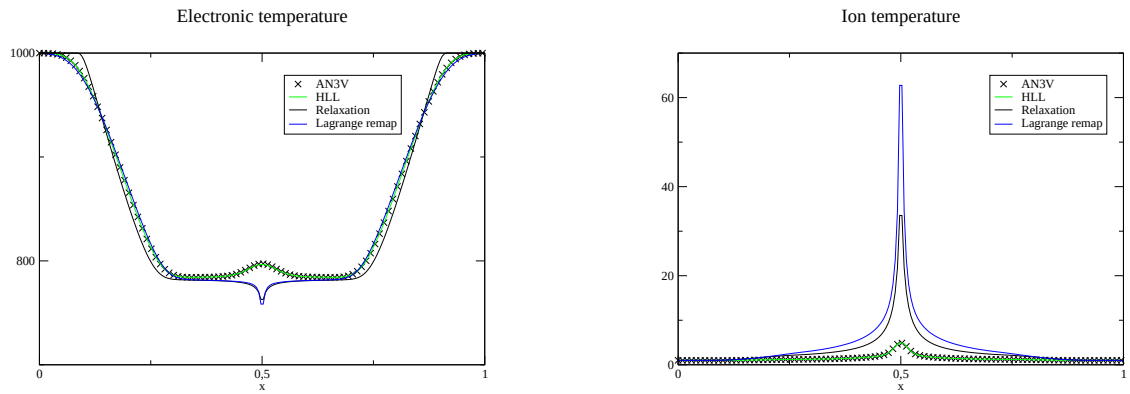

FIGURE 4. Electronic and ionic temperatures computed with AN3V, HLL, Lagrangian remaping and relaxation schemes for the double rarefaction waves problem.

an exact solution of system (1) is available:

$$\forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad \rho(x,t) = 1, \quad u(x,t) = 10, \quad T_\alpha(x,t) = \mathrm{e}^{-2\mu t}\left(T_\alpha(x-ut,0) - \frac{\overline{T}}{2}\right) + \frac{\overline{T}}{2}, \quad \alpha = e, i \quad (122)$$

where $\mu = k_B^{-1}\nu_{ei}(\gamma-1)(m_e+m_i)$. Here we choose $\overline{T} = 2$,

$$T_e(x,0) = 1 \quad \text{if} \quad x < 1/2, \quad T_e(x,0) = 1 + \exp(-200(\mathrm{x}-1/2)^2) \quad \text{else.} \quad (123)$$

We compute the solution on $[0,1]$ at time $t = 0.01$ with 1000 points. The results that are obtained for the HLL and the relaxation scheme displayed in figure 5 are in good agreement with the analytical solution (122, 123).

4.3. **A Riemann problem with a stationary shock**

The left and right states of the Riemann problem are the following:

$$\begin{array}{llll} \rho_L = 1.001, & u_L = 10, & T_{e,L} = 1, & T_{i,L} = 1, \\ \rho_R = 3.640330609, & u_R = 2.749750250, & T_{e,R} = 3., & T_{i,R} = 17.5060240977. \end{array}$$

Then the solution of the 3×3 Euler system is a stationary shock. The solutions are computed for $x \in [0,1]$ at time $t = 0.05$ with 1000 cells. The results are depicted figures 6 and 7.

For $\nu_{ei} = 0$, we observe a contact discontinuity propagating at velocity $u_R = 2.749750250$, see figure 7 left. For the 3×3 Euler equations, u is a single eigenvalue and the left and right values of a contact discontinuity lie on an integral curve of the eigenvector $(-\rho, 0, \varepsilon)$, in (ρ, u, ε) variables. Thus, a contact discontinuity must involve a jump of ρ. For the Euler bi-temperature system, the eigenvalue u is double and the eigenvectors are $r_2 = (0, 0, -(\gamma_i - 1)c_i, (\gamma_e - 1)c_e)$ and $r_3 = (-\rho, 0, \varepsilon_e, \varepsilon_i)$. The observed contact discontinuity lies on an integral curve of r_2: only electronic and ionic temperatures jump. We also observe that the intermediate values of those temperatures differ with respect with the scheme. This is due to the fact that each scheme has a different viscosity, and therefore converges to a different interpretation of the nonconservative products in the equations.

For $\nu_{ei} = 100$, both schemes give similar results. For all x, density, velocity and total energy remain the same and are not depicted again. Electronic and ionic temperatures are represented on figure 7 right. For $x < 0.5$, as $T_i = T_e$, the value of ν_{ei} does not influence the solution. For $x > tu_R$, by finite propagation speed, one can compute the value of T_e and T_i as in subsection 4.2. For $x \in [0, tu_R]$ the result is qualitatively in coherence

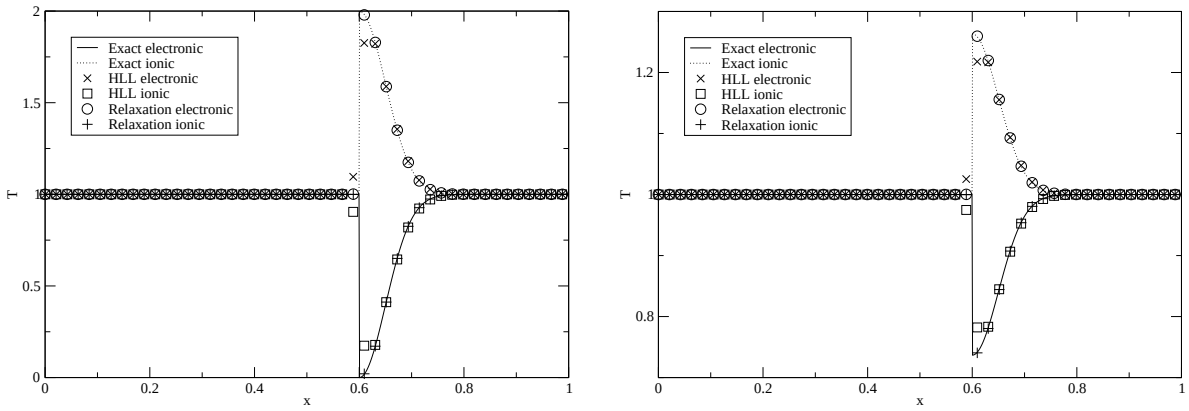

FIGURE 5. Electronic and ionic temperatures computed with HLL and relaxation schemes for the analytical test case. Left : $\nu_{ei} = 0$, right : $\nu_{ei} = 100$.

with the physical behavior of the plasma, as predicted in Zeldovitch ([36]). In particular, we observe a high ionic temperature at the shock, and then a decrease, while the electronic temperature increases.

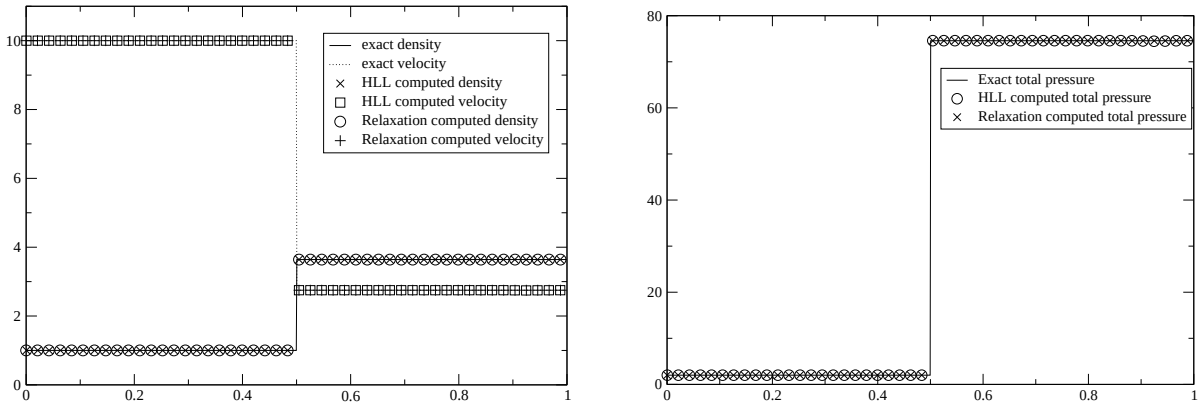

FIGURE 6. Density, velocity and pressure computed with HLL and relaxation schemes compared with exact solution for the stationary shock problem.

5. Conclusion

In this paper we have investigated at the modelling and numerical point of views the bitemperature Euler system.

At the modelling point of view, we have proposed a multicomponent BGK kinetic system enjoying fundamental mathematical properties (positivity of the distribution function, conservation properties, H theorem) coupled with Ampère and Poisson equations. Next, be perfoming an hydrodynamic limit the bitemperature Euler model has been recovered. Moreover, we have shown for this system the existence of an entropy that is consistent with Boltzmann entropy.

At the numerical level, we have investigated and compared four numerical schemes for the bitemperature Euler system. We have firstly presented a kinetic scheme based on a discrete hydrodynamic limit. Next, the numerical method of Aregba-Natalini based on discrete BGK schemes has been generalized to a nonconservative setting. Doing this we show that the previous kinetic scheme is entropic and we recover a HLL scheme for the non-conservative system. A Suliciu relaxation approximation has also been designed. Finally all these schemes are compared to a Lagrangian scheme. Moreover, the kinetic and discrete BGK approaches are consistent with

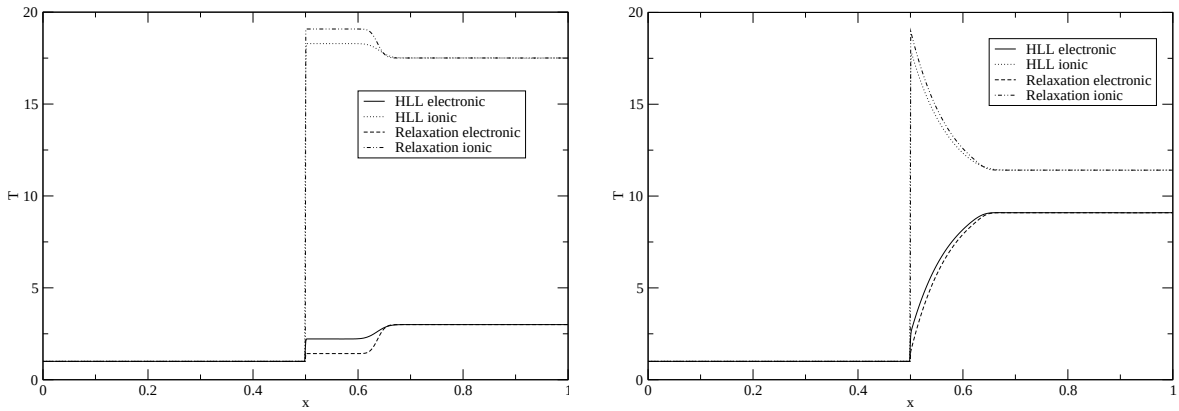

FIGURE 7. Electronic and ionic temperatures computed with HLL and relaxation schemes for the stationary shock problem. Left : $\nu_{ei} = 0$, right : $\nu_{ei} = 100$.

the entropy dissipation properties of the model, which should play a fundamental role for selecting the correct solutions.

We shall address to forthcoming papers the following different points. Firstly, we plan to propose high order and multi-dimensionnal versions of the schemes developped in this paper. Moreover, the derivation of a polyatomic BGK model based on an additional continous internal energy variable has to be performed. Indeed, this model will be devoted to lead to general γ_e and γ_i for the fluid system. The case of Navier-Stokes asymptotics has also to be considered. In another direction, we aim to take into account the magnetic field in the equations with a coupling through the Maxwell system.

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