Filtration law for power-law fluids in anisotropic porous media

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Abstract

This work is concerned with modelling steady-state slow flow of incompressible power-law fluids in porous media. The macroscopic filtration law is derived by upscaling the pore–scale description. The upscaling technique in use is the homogenisation method of multiple scales. Then, the filtration law is investigated by means of the theory of representation of isotropic tensor function of tensor arguments. The general form of the filtration law is given for isotropic, transversely isotropic, orthotropic and fully anisotropic systems.

Keywords: Power-law fluid; Porous media; Filtration law; Upscaling

1. Introduction

Knowledge of the filtration law for the flow of power-law fluids through porous media is of first importance in numerous engineering areas, e.g., petroleum engineering, paper manufacturing, composite manufacturing, etc. However, there have been very few theoretical investigations that focused on the general form of this law. Indeed, most of them aimed at proposing one-dimensional modified versions of Darcy's law in non-algebraic forms that are obtained on the basis of phenomenological considerations. Wissler [1] used a geometric similarity condition and the homogeneity condition (see (7) below) and obtained the following one-dimensional filtration law:

$$q = k'\left(\frac{\Delta p}{KL}\right)^{1/n}, \tag{1}$$

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where q is the Darcy flux, Δp is the pressure drop across a porous sample of length L; k' and K are two parameters that are dependent on the exponent n. Larson [2] obtained the same filtration law (1) by assuming that the streamlines are independent of the volumetric flow rate. This latter property is actually a direct consequence of the homogeneity condition (7), see below. Slattery [3] derived the three-dimensional isotropic law by volume averaging, considering the homogeneity condition (7) and using a dimensional analysis (Buckingham–Pi theorem). The resulting filtration law can be expressed as follows:

$$\nabla p + K\mathbf{v} = 0, \quad K = K_1 |\nabla p|^{(n-1)/n}, \tag{2}$$

where K_1 depends on n, and on a characteristic length of the porous system. Both filtration laws (1) and (2) have similar structures. When $n = 1$, they reduce to Darcy's law. More recently, there have been several works that aimed at obtaining the filtration law by upscaling the pore–scale behaviour. Getachew et al. [4] applied the method of volume averaging to derive the macroscopic isotropic filtration law. Shah and Yortsos [5] and Bourgeat and Mikelić [6] used the homogenisation method of multiple scales. In [5], the filtration law is obtained in a formal way and its structure is investigated by using physical considerations at the macro-scale. In [6] the existence of the filtration law and the convergence of the expansions are rigorously demonstrated. Other references concerning the flow of a power-law fluid in a porous medium and its applications can be found in [7].

The present work is aimed towards rigorously deriving the most general form of the filtration law in systems of different types of anisotropy. The macroscopic filtration law is obtained by upscaling the pore–scale physical description, by using the homogenisation method of multiple scales.

Let us assume the medium to be locally characterised by a representative elementary volume (REV) whose size is $O(l)$ (Fig. 1). Let us consider the physical description at the pore scale, i.e., within this REV. It consists of the momentum balance (3), the fluid constitutive equations (4), the incompressibility condition (5) in the pores Ω_p and of the no-slip condition (6) over the pore surface Γ:

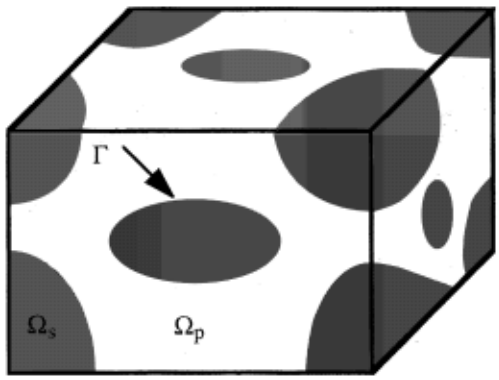

Fig. 1. Periodic cell of the porous medium.

$$\frac{\partial \sigma_{ij}}{\partial X_j} = 0, \tag{3}$$

$$\sigma_{ij} = -pI_{ij} + \tau_{ij}, \quad \tau_{ij} = \mu\left(\frac{1}{2}D_{pq}D_{pq}\right)^{(n-1)/2} D_{ij}, \tag{4}$$

$$\frac{\partial v_i}{\partial X_i} = 0 \quad \text{in } \Omega_p, \tag{5}$$

$$v_i = 0 \quad \text{on } \Gamma. \tag{6}$$

In these equations, p is the pressure, $\mathbf{v}$ is the velocity, $\mu > 0$ and $n > 0$ are two constants. $\mathbf{D}$ is the rate-of-deformation tensor

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right).$$

As it can be seen from (4), the stress deviator $\boldsymbol{\tau}$ is a homogeneous function of degree n of the velocity $\mathbf{v}$:

$$\tau_{ij}(\lambda \mathbf{v}) = \lambda^n \tau_{ij}(\mathbf{v}). \tag{7}$$

The purpose of our work is to upscale the pore flow description (3)–(6). We use a rigorous upscaling process to obtain the macroscopic law without any prerequisite on the form of the macroscopic behaviour. Section 2 is devoted to a short presentation of the upscaling technique, the homogenisation method of multiple scales. The upscaling is conducted in Section 3. It yields a vectorial relationship between the Darcy velocity and the macroscopic pressure gradient. We present a demonstration which is different from that carried out in [6]. This new formulation, see (26), is of practical interest since it is suitable for numerical calculations. The purpose of Section 4 is to analyse this filtration law by means of the theory of representation of isotropic tensor functions of tensor arguments and the homogeneity condition (7). Different cases of anisotropy are investigated: isotropy, transverse isotropy, orthotropy and general anisotropy.

2. Upscaling technique

The essence of homogenisation method is to determine an equivalent macroscopic behaviour by upscaling the local description. The fundamental assumption behind homogenisation theory is that the scales must be separated:

$$l \ll L, \tag{8}$$

where l and L are the characteristic lengths at the heterogeneity scale and at the macroscopic scale, respectively. As this definition conjures up a geometrical separation of scales, we shall draw attention to the fact that this fundamental condition must also be satisfied regarding the

phenomenon. For example, in the case of wave propagation, the microscopic characteristic length, l, must also be small compared to the wavelength.

In this study, we use the method of homogenisation for periodic structures – also called method of multiple scales – introduced by Bensoussan [8] and Sanchez-Palencia [9]. The key parameter of the method is the small parameter

$$\varepsilon = \frac{l}{L} \ll 1, \tag{9}$$

in which L is the macroscopic characteristic length and, depending on the problem under consideration, is either geometrical (i.e., the sample size) or related to the excitation (e.g. wavelength). The pore size l_p can vary in the REV and can be different in different directions. Thus the porous medium can show macroscopic anisotropy. However, we assume l_p is of the order of magnitude of l, $l_p = O(l)$ or equivalently $\varepsilon \ll l_p/l \ll \varepsilon^{-1}$.

We also assume the medium to be periodic. This assumption is actually not a restriction, see [10] for details. In the context of a periodic medium, the REV is simply the period.

In this study, we use the approach suggested in [10], by which the problem is tackled in a physical rather than mathematical manner. Indeed, it offers the additional benefit that the conditions under which homogenisation does apply are expressly stated. This formulation of the method is on the basis of definition and estimation of the non-dimensional numbers arising from the local description under consideration. This fundamental step is called normalisation and is aimed at specifying all cases that can be homogenised.

As a result of the separation of scales, two non-dimensional space variables may be defined:

$$\mathbf{y} = \frac{\mathbf{X}}{l},$$

$$\mathbf{x} = \frac{\mathbf{X}}{L},$$

where $\mathbf{X}$ is the physical space variable.

If the condition of separation of scales is verified, then $\mathbf{y}$ and $\mathbf{x}$ appear as two independent space variables: $\mathbf{y}$ is the microscopic variable and describes the heterogeneity scale whereas $\mathbf{x}$ is the macroscopic variable.

As a consequence, the physical variables of the problem, p and $\mathbf{v}$, are a priori functions of $\mathbf{y}$ and $\mathbf{x}$:

$$p = p(\mathbf{y}, \mathbf{x}),$$

$$\mathbf{v} = \mathbf{v}(\mathbf{y}, \mathbf{x}).$$

Moreover, the partial derivative with respect to the physical space variable $\mathbf{X}$ can be written as

$$\frac{\partial}{\partial X_i} = \frac{1}{l}\frac{\partial}{\partial y_i} + \frac{1}{L}\frac{\partial}{\partial x_i}. \tag{10}$$

When using l as the reference characteristic length, the non-dimensional gradient operator is given by

$$\nabla_y + \varepsilon \nabla_x, \tag{11}$$

where ∇_y and ∇_x are the gradient operators with respect to variables $\mathbf{y}$ and $\mathbf{x}$, respectively.

The homogenisation method of multiple scales is based on the fundamental statement that if the scales are well separated, then all physical variables can be looked for in the form of asymptotic expansions in powers of ε:

$$p = p^0(\mathbf{y}, \mathbf{x}) + \varepsilon p^1(\mathbf{y}, \mathbf{x}) + \cdots, \tag{12}$$

$$\mathbf{v} = \mathbf{v}^0(\mathbf{y}, \mathbf{x}) + \varepsilon \mathbf{v}^1(\mathbf{y}, \mathbf{x}) + \cdots, \tag{13}$$

in which the functions p^i and $\mathbf{v}^i$ are $\mathbf{y}$-periodic.

The method consists in incorporating expansions (12) and (13) in the dimensionless form of the local description (3)–(6). Solving the boundary-value problems arising at the successive orders of ε leads to the macroscopic description. The method allows determination of the macroscopic behaviour without any prerequisite on the form of the macroscopic equations.

3. Derivation of the macroscopic filtration law

Firstly, we shall cast the set (3)–(6) in a dimensionless form. It turns out that only one dimensionless number, Q, may be defined from this set of equations

$$Q = \left| \frac{\partial p}{\partial X_i} \right| \Big/ \left| \frac{\partial \tau_{ij}}{\partial X_j} \right|. \tag{14}$$

The fluid is driven through the pores by a macroscopic pressure gradient which is of the order of $\delta p/L$, where δp is the macroscopic increment of pressure. This macroscopic pressure gradient is balanced by the viscous forces which act at the pore level

$$\frac{\partial \tau_{yij}}{\partial y_j} = \frac{\partial}{\partial y_j} \left(\mu \left(\frac{1}{2} D_{ypq} D_{ypq} \right)^{(n-1)/2} D_{yij} \right),$$

where the subscript y designates the partial derivative with respect to $\mathbf{y}$. Let us arbitrarily choose l as the reference characteristic length. Therefore we have

$$Q_l = \frac{\delta p / l}{|\partial \tau_{yij} / \partial y_j|} = \frac{L}{l} = \varepsilon^{-1}. \tag{15}$$

For the sake of simplicity we use similar notations for dimensional and dimensionless quantities, with the exception of the space variables. Thus, when cast in a dimensionless form the set (3)–(6) is written as follows:

$$-\frac{\partial p}{\partial y_i}+\varepsilon\frac{\partial}{\partial y_j}\left(\mu\left(\frac{1}{2}D_{ypq}D_{ypq}\right)^{(n-1)/2}D_{yij}\right)=0, \tag{16}$$

$$\frac{\partial v_i}{\partial y_i}=0 \quad \text{in } \Omega_p, \tag{17}$$

$$v_i=0 \quad \text{on } \Gamma. \tag{18}$$

Let us now incorporate expansions (12) and (13) in the set (16)–(18). At the first order, (16) gives

$$\frac{\partial p^0}{\partial y_i}=0, \quad p^0=p^0(\mathbf{x}). \tag{19}$$

The next order in Eqs. (16)–(18) yields a boundary-value problem for $\mathbf{v}^0$ and p^1:

$$-\frac{\partial p^1}{\partial y_i}-\frac{\partial p^0}{\partial x_i}+\frac{\partial}{\partial y_j}\left(\mu\left(\frac{1}{2}D^0_{ypq}D^0_{ypq}\right)^{(n-1)/2}D^0_{yij}\right)=0, \tag{20}$$

$$D^0_{yij}=\frac{1}{2}\left(\frac{\partial v^0_i}{\partial y_j}+\frac{\partial v^0_j}{\partial y_i}\right),$$

$$\frac{\partial v^0_i}{\partial y_i}=0 \quad \text{in } \Omega_p, \tag{21}$$

$$v^0_i=0 \quad \text{on } \Gamma. \tag{22}$$

Let us introduce the vector space W of free-divergence periodic vectors that are zero-valued over Γ, and which is equipped with the scalar product

$$(\mathbf{v},\mathbf{w})_W=\int_{\Omega_p}\frac{\partial u_i}{\partial y_j}\frac{\partial w_i}{\partial y_j}\,\mathrm{d}\Omega.$$

Then, the set (20)–(22) is equivalent to the following variational formulation:

$$\forall\boldsymbol{\alpha}\in W, \quad \int_{\Omega_l}\tau_{ij}(\mathbf{v}^0)D_{ij}(\boldsymbol{\alpha})\,\mathrm{d}\Omega=-\int_{\Omega_l}\frac{\partial p^0}{\partial x_i}\alpha_i\,\mathrm{d}\Omega, \tag{23}$$

$$\tau_{ij}(\mathbf{v}^0)=\mu\left(\frac{1}{2}D_{ypq}(\mathbf{v}^0)D_{ypq}(\mathbf{v}^0)\right)^{(n-1)/2}D_{yij}(\mathbf{v}^0).$$

From (23) it is clear that, if $\mathbf{v}^0$ does exist, then it is a homogeneous function of degree $1/n$ of the macroscopic gradient $\mathbf{V}_x p^0=\mathbf{G}$. The stress deviator $\boldsymbol{\tau}$ is the gradient of a potential function

$$\tau_{ij} = \frac{\partial \psi}{\partial D_{ij}}, \quad \psi = \frac{2\mu}{n+1}\left(\frac{1}{2}D_{pq}D_{pq}\right)^{(n+1)/2} = \frac{1}{n+1}\tau_{ij}D_{ij},$$

and ψ is convex

$$\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in W, \quad \psi(\boldsymbol{\alpha}) - \psi(\boldsymbol{\beta}) - \frac{\partial \psi}{\partial D_{ij}}(\boldsymbol{\beta})(D_{ij}(\boldsymbol{\alpha}) - D_{ij}(\boldsymbol{\beta})) \geqslant 0. \tag{24}$$

Letting $\boldsymbol{\alpha} = \mathbf{v}^0$ in (23) and using the convexity property (24) yield the following variational inequality:

$$\forall \boldsymbol{\alpha} \in W, \quad \int_{\Omega_f} (\psi(\boldsymbol{\alpha}) - \psi(\mathbf{v}^0) - \mathbf{G}(\boldsymbol{\alpha} - \mathbf{v}^0))\, \mathrm{d}\Omega \geqslant 0. \tag{25}$$

This variational inequality can be rewritten as

$$\forall \boldsymbol{\alpha} \in W, \quad J(\boldsymbol{\alpha}) - J(\mathbf{v}^0) \geqslant 0, \tag{26}$$

in which

$$J(\boldsymbol{\alpha}) = \int_{\Omega_f} (\psi(\boldsymbol{\alpha}) - \mathbf{G}\boldsymbol{\alpha})\, \mathrm{d}\Omega$$

is convex. The problem is then to find $\boldsymbol{\alpha} \in W$ that minimises $J(\boldsymbol{\alpha})$. Therefore, there exists a unique $\mathbf{v}^0$:

$$\mathbf{v}^0 = \boldsymbol{E}(\mathbf{G}, \mu, n), \tag{27}$$

where $\boldsymbol{E}$ is a homogeneous vector function of degree $1/n$ of its argument $\mathbf{G}$. Clearly, $\boldsymbol{E}$ also depends on the fluid properties μ and n. We refer to [6] for the demonstration of the convergence of expansions (12) and (13).

The next orders of the volume balance (17) and of the no-slip condition (18) are written

$$\frac{\partial v_i^1}{\partial y_i} + \frac{\partial v_i^0}{\partial x_i} = 0 \quad \text{in } \Omega_p, \qquad v_i^1 = 0 \quad \text{on } \Gamma.$$

Integrating this volume balance on Ω_p, using the divergence theorem, the periodicity of $\mathbf{v}^1$ and the adherence condition yield the macroscopic volume balance

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{28}$$

$$\mathbf{u} = \frac{1}{\Omega}\int_{\Omega_f} \mathbf{v}^0\, \mathrm{d}\Omega = \boldsymbol{F}(\mathbf{G}, \mu, n), \tag{29}$$

where $\boldsymbol{F}$ is a homogeneous vector function of degree $1/n$ of its argument $\mathbf{G}$. As for $\boldsymbol{E},\boldsymbol{F}$ also depends on μ and n. Eqs. (28) and (29) represent the macroscopic model of the fluid flow. Relation (29) stands for the macroscopic filtration law. Letting $\boldsymbol{\alpha} = \mathbf{v}^0$ in (23) gives

$$\mathbf{Gu} = -\frac{1}{\Omega}\int_{\Omega_l} \tau_{ij}(\mathbf{v}^0)D_{ij}(\mathbf{v}^0)\,\mathrm{d}\Omega < 0, \tag{30}$$

which means that the macroscopic dissipation density is equal to the pore dissipation average.

We shall now examine the vector function $\boldsymbol{F}$ by means of the theory of representation of isotropic tensor function of tensor arguments.

4. The anisotropic filtration law

4.1. General representation

According to Lokhine and Sedov [11], Boehler [12–14], and Liu [15] any anisotropic function can be represented by an isotropic function after introducing additional arguments that are related to the anisotropy. By using this isotropic extension method, the filtration law (29) can be rewritten as

$$\mathbf{u} = \boldsymbol{F}(\mathbf{G},\mu,n) = \boldsymbol{F}^*(\mathbf{G},\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_r,\mu,n), \tag{31}$$

where $\boldsymbol{F}^*$ is an isotropic function of all its arguments, and an anisotropic function of $\mathbf{G}$. Remember that $\boldsymbol{F}^*$ also depends on μ and n. The additional arguments $\mathbf{a}_i$, $i = 1,2,\ldots,r$ are structural tensors that characterise the anisotropy. Let S be the material symmetry group of the filtration law (29). The additional arguments are chosen so as to possess S as material symmetry group. In the following, the additional arguments are vectors $\mathbf{w}_l$, $l = 1,2,\ldots,m$, or symmetrical second order tensors $\mathbf{M}_p$, $p = 1,2,\ldots,t$.

As shown in [16,17], the isotropy property enables $\boldsymbol{F}^*$ to be resolved into components on a finite functional basis. In the present case this basis is constituted of vectors $\mathbf{H}_q$ that represent an irreducible functional basis of the arguments of $\boldsymbol{F}^*$. We have

$$\boldsymbol{F}^* = \gamma_q\mathbf{H}_q, \quad \gamma_q = \gamma_q(I_1,I_2,I_3,\ldots,I_s). \tag{32}$$

The γ_q's are arbitrary scalar functions of the invariants I_j, $j = 1,2,\ldots,s$, of the functional basis. In the present case, the $\mathbf{H}_q$'s are the following [16,17]:

$$\begin{cases} \mathbf{G}, \quad \mathbf{w}_l, \quad \mathbf{M}_p\mathbf{G}, \quad \mathbf{M}_p\mathbf{w}_l, \quad \mathbf{M}_p^2\mathbf{G}, \quad \mathbf{M}_p^2\mathbf{w}_l, \\ (\mathbf{M}_p\mathbf{M}_q - \mathbf{M}_q\mathbf{M}_p)\mathbf{G}, \quad (\mathbf{M}_p\mathbf{M}_q - \mathbf{M}_q\mathbf{M}_p)\mathbf{w}_l, \end{cases} \tag{33}$$

and the invariants I_j are [12,14,17],

$$\begin{cases} \mathbf{G}\cdot\mathbf{G}, \quad \mathbf{w}_l\cdot\mathbf{w}_l, \quad \mathbf{G}\cdot\mathbf{w}_l, \quad \mathbf{w}_l\cdot\mathbf{w}_j, \\ \operatorname{tr}\mathbf{M}_p, \quad \operatorname{tr}\mathbf{M}_p^2, \quad \operatorname{tr}\mathbf{M}_p^3, \quad \operatorname{tr}\mathbf{M}_p\mathbf{M}_q, \quad \operatorname{tr}\mathbf{M}_p^2\mathbf{M}_q, \\ \operatorname{tr}\mathbf{M}_p\mathbf{M}_q^2, \quad \operatorname{tr}\mathbf{M}_p^2\mathbf{M}_q^2, \quad \operatorname{tr}\mathbf{M}_p\mathbf{M}_q\mathbf{M}_r, \\ \mathbf{G}\cdot\mathbf{M}_p\mathbf{G}, \quad \mathbf{w}_l\cdot\mathbf{M}_p\mathbf{w}_l, \quad \mathbf{G}\cdot\mathbf{M}_p\mathbf{w}_l, \quad \mathbf{w}_l\cdot\mathbf{M}_p\mathbf{G}, \\ \mathbf{G}\cdot\mathbf{M}_p^2\mathbf{G}, \quad \mathbf{w}_l\cdot\mathbf{M}_p^2\mathbf{w}_l, \quad \mathbf{G}\cdot\mathbf{M}_p^2\mathbf{w}_l, \quad \mathbf{w}_l\cdot\mathbf{M}_p^2\mathbf{G}, \\ \mathbf{G}\cdot\mathbf{M}_p\mathbf{M}_q\mathbf{G}, \quad \mathbf{w}_l\cdot\mathbf{M}_p\mathbf{M}_q\mathbf{w}_l, \\ \mathbf{G}\cdot(\mathbf{M}_p\mathbf{M}_q-\mathbf{M}_q\mathbf{M}_p)\mathbf{w}_l, \quad \mathbf{w}_l\cdot(\mathbf{M}_p\mathbf{M}_q-\mathbf{M}_q\mathbf{M}_p)\mathbf{G}. \end{cases} \tag{34}$$

The purpose of the next paragraphs is to successively examine the properties of the function $\boldsymbol{F}^*$ in the cases of isotropy, transverse isotropy, orthotropy and general anisotropy. In this analysis to follow, the homogeneity property (7) is used, which can be expressed as follows. Let $\beta(x_1,x_2,\ldots,x_q)$ be a homogeneous function of degree m of its arguments. Consider an x_p such that $x_p\neq 0$. Then β can be written as

$$\beta = x_p^m\phi\left(\frac{x_1}{x_p},\frac{x_2}{x_p},\ldots,\frac{x_q}{x_p}\right), \tag{35}$$

where ϕ is an arbitrary function of its $q-1$ arguments $\frac{x_1}{x_p},\frac{x_2}{x_p},\ldots,\frac{x_q}{x_p}$.

4.2. Isotropy

In the case of isotropy, $\boldsymbol{F}^*$ has a single argument $\mathbf{G}$. There is no additional argument. The basis vectors (33) reduce to the single vector $\mathbf{G}$ and the invariants (34) reduce to the single invariant $\mathbf{G}\cdot\mathbf{G}$. The filtration law takes the form

$$\mathbf{u} = \beta(\mathbf{G}\cdot\mathbf{G},\mu,n)\mathbf{G}. \tag{36}$$

β is a scalar function which is homogeneous of degree $(1/n-1)/2$ of its argument $\mathbf{G}\cdot\mathbf{G}$. β also depends on μ and n. Therefore, from (35) we have

$$\beta = -C(\mu,n)(\mathbf{G}\cdot\mathbf{G})^{(1-n)/2n}, \quad \mathbf{u} = -C(\mu,n)(\mathbf{G}\cdot\mathbf{G})^{(1-n)/2n}\mathbf{G}, \tag{37}$$

where $C(\mu,n)$ is independent of $\mathbf{G}$. From (30), the dissipation $-\mathbf{G}\cdot\mathbf{u}$ is positive. The filtration law takes the form

$$\mathbf{u} = -C(\mu,n)|\mathbf{G}|^{(1-n)/n}\mathbf{G}, \tag{38}$$

with $C>0$. We recover in this case a filtration law similar to (1) and (2).

4.3. Transverse isotropy

Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be three orthogonal unit vectors and $\mathbf{e}_3$ be the axis of transverse isotropy. In this case, see [14], there is only one additional argument to $\boldsymbol{F}^*$, i.e. $\mathbf{M}_3=\mathbf{e}_3\otimes\mathbf{e}_3$. The basis vectors (33) are $\mathbf{G}$ and $G_3\mathbf{e}_3$ and the invariants (34) are $\mathbf{G}\cdot\mathbf{G}$ and G_3^2. The filtration law is written

$$\mathbf{u} = \beta(\mathbf{G}\cdot\mathbf{G}, G_3^2, \mu, n)\mathbf{G} + \beta_3(\mathbf{G}\cdot\mathbf{G}, G_3^2, \mu, n)G_3\mathbf{e}_3, \tag{39}$$

where β and β_3 are homogeneous functions of degree $(1/n-1)/2$ of their arguments. They also depend on μ and n. Then, from (35) we get

$$\mathbf{u} = (\mathbf{G}\cdot\mathbf{G})^{(1-n)/2n}\varphi^*\left(\frac{G_3^2}{\mathbf{G}\cdot\mathbf{G}}, \mu, n\right)\mathbf{G} + (G_3^2)^{(1-n)/2n}\varphi_3^*\left(\frac{G_3^2}{\mathbf{G}\cdot\mathbf{G}}, \mu, n\right)G_3\mathbf{e}_3. \tag{40}$$

The positivity of the dissipation entails

$$\mathbf{u} = -|\mathbf{G}|^{(1-n)/n}\varphi\left(\frac{G_3^2}{\mathbf{G}\cdot\mathbf{G}}, \mu, n\right)\mathbf{G} - |G_3|^{(1-n)/n}\varphi_3\left(\frac{G_3^2}{\mathbf{G}\cdot\mathbf{G}}, \mu, n\right)G_3\mathbf{e}_3, \tag{41}$$

where φ and φ_3 are positive arbitrary functions of $G_3^2/(\mathbf{G}\cdot\mathbf{G})$ that depend also on μ and n. Remark that the transverse isotropic law (41) reduces to the one-dimensional law (2) when the flow direction is a principal direction of the anisotropic medium, only.

4.4. Orthotropy

The axis of orthotropy are chosen to be the coordinate axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The additional arguments that characterise the orthotropy are the three following tensors, [14]:

$$\mathbf{M}_1 = \mathbf{e}_1\otimes\mathbf{e}_1, \quad \mathbf{M}_2 = \mathbf{e}_2\otimes\mathbf{e}_2, \quad \mathbf{M}_3 = \mathbf{e}_3\otimes\mathbf{e}_3. \tag{42}$$

The basis vectors (33) reduce to

$$G_1\mathbf{e}_1, \quad G_2\mathbf{e}_2, \quad G_3\mathbf{e}_3, \tag{43}$$

and the invariants (34) are

$$G_1^2, \quad G_2^2, \quad G_3^2. \tag{44}$$

Therefore the filtration law is expressed as

$$\mathbf{u} = \beta_1 G_1\mathbf{e}_1 + \beta_2 G_2\mathbf{e}_2 + \beta_3 G_3\mathbf{e}_3, \tag{45}$$

where the β_i's are in the form

$$\beta_i = \beta_i\left(G_1^2, G_2^2, G_3^2, \mu, n\right). \tag{46}$$

The β_i's are homogeneous functions of degree $(1/n-1)/2$ of their arguments. Choosing $G_p \neq 0$ gives from (35)

$$\beta_i = ((G_p)^2)^{(1-n)/2n}\phi_i\left(\frac{(G_{p-1})^2}{(G_p)^2}, \frac{(G_{p+1})^2}{(G_p)^2}, \mu, n\right), \tag{47}$$

where the components of $\mathbf{G}$ are arranged in the order $G_1, G_2, G_3, G_1, \ldots$ The positivity of the dissipation then yields

$$\begin{aligned}\mathbf{u} = &-|G_p|^{(1-n)/n}\phi_1\left(\frac{(G_{p-1})^2}{(G_p)^2}, \frac{(G_{p+1})^2}{(G_p)^2}, \mu, n\right)G_1\mathbf{e}_1\\ &-|G_p|^{(1-n)/n}\phi_2\left(\frac{(G_{p-1})^2}{(G_p)^2}, \frac{(G_{p+1})^2}{(G_p)^2}, \mu, n\right)G_2\mathbf{e}_2\\ &-|G_p|^{(1-n)/n}\phi_3\left(\frac{(G_{p-1})^2}{(G_p)^2}, \frac{(G_{p+1})^2}{(G_p)^2}, \mu, n\right)G_3\mathbf{e}_3,\end{aligned} \tag{48}$$

where the ϕ_i's are positive arbitrary functions that depend also on μ and n. As for the transverse isotropic law, orthotropic law (48) reduces to the one-dimensional law (2) when the flow direction is a principal direction of the anisotropic medium, only.

4.5. General anisotropy

The additional arguments are now vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ [12]. The basis vectors (33) are $\mathbf{G}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the invariants (34) are G_1, G_2 and G_3. The filtration law is written as

$$\mathbf{u} = \beta(G_1, G_2, G_3, \mu, n)\mathbf{G} + \sum_{i=1}^{3}\beta_i(G_1, G_2, G_3, \mu, n)\mathbf{e}_i. \tag{49}$$

β and the β_i's are homogeneous functions of their arguments of degree $(1-n)/n$ and $1/n$, respectively. Therefore we obtain with $G_p \neq 0$,

$$\mathbf{u} = G_p^{(1-n)/n}\varphi^*\left(\frac{G_{p-1}}{G_p}, \frac{G_{p+1}}{G_p}, \mu, n\right)\mathbf{G} + \sum_{i=1}^{3}G_p^{1/n}\varphi_i^*\left(\frac{G_{p-1}}{G_p}, \frac{G_{p+1}}{G_p}, \mu, n\right)\mathbf{e}_i, \tag{50}$$

where the components of $\mathbf{G}$ are arranged in the order $G_1, G_2, G_3, G_1, \ldots$ Using the positivity of the dissipation finally yields

$$\mathbf{u} = -|G_p|^{(1-n)/n}\varphi\left(\frac{G_{p-1}}{G_p}, \frac{G_{p+1}}{G_p}, \mu, n\right)\mathbf{G} - \sum_{i=1}^{3}|G_p|^{1/n}\varphi_i\left(\frac{G_{p-1}}{G_p}, \frac{G_{p+1}}{G_p}, \mu, n\right)\mathbf{e}_i, \tag{51}$$

where the φ's are positive arbitrary functions that depend also on μ and n. Filtration laws (48) and (51) obtained in the cases of orthotropy and general anisotropy, respectively, are quite different. Anyhow, when the fluid is viscous Newtonian ($n = 1$), the ϕ_i's, φ and the φ_i's are constant and, as a consequence, the fully anisotropic law reduces to the orthotropic law in this particular case. Anisotropic law (51) is not reducible to one-dimensional law (2).

5. Conclusions

We have used the homogenisation technique and the theory of representation of isotropic tensor function so as to derive and then to investigate the general form of the filtration law for slow permanent flow of power-law fluids in porous media. This model is valid under quite restrictive conditions as in particular inertial terms were neglected in the momentum balance (3). Anyhow the model is rigorous within its domain of validity. We have examined four cases of structural symmetries: isotropy, transverse isotropy, orthotropy and general anisotropy. We have shown, as expected, that in the very specialised case of linear fluids ($n = 1$), the most general anisotropy reduces to orthotropy. For non-linear fluids, it is worthwhile noticing that the isotropic filtration law (38) is similar to Slattery's law (2). We have demonstrated that for transversely isotropic and orthotropic systems the structure of the one-dimensional law (2) remains valid, but only when the flow direction is parallel to a principal axis. The most important result is the proof that for fully anisotropic systems, one-dimensional flows never verify (2).

The above results are derived for macroscopically homogeneous porous media. They can be directly extended to macroscopically heterogeneous media. In effect, the space variable $\mathbf{x}$ plays the role of a parameter throughout the homogenisation process: integrations are conducted with respect to variable $\mathbf{y}$. Therefore, considering porous media where μ, n as well as the period Ω are $\mathbf{x}$ dependent yields to macroscopic equivalent descriptions whose effective coefficients are also $\mathbf{x}$ dependent, i.e. to macroscopically heterogeneous porous media. These macroscopic descriptions are similar to those presented in the paper.

It would have been of interest to get explicit expressions for the effective coefficients entering the filtration laws. Unfortunately, owing to the nonlinear constitutive equation of state, an analytical solution of the local boundary value problem is not possible, unless the pore geometry is very simple as for plane fissures. Although the results for such simplified pore systems agree with our theoretical results, they are of low interest in case of anisotropy. To check our results, an experimental setup is under construction.

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