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Abstract. We investigate the filtration law of incompressible viscous Newtonian fluids in rigid non-inertial porous media, for example, rotating porous media. The filtration law is obtained by upscaling the flow at the pore scale. We use the method of multiple scale expansions which gives rigorously the macroscopic behaviour without any prerequisite on the form of the macroscopic equations. For finite Ekman numbers the filtration law is shown to resemble a Darcy's law, but with a non-symmetric permeability tensor which depends on the angular velocity of the porous matrix. We obtain the filtration analog of the Hall effect. For large Ekman numbers the filtration law is a small correction to the classical Darcy's law. The corrector is antisymmetric. In this case we recover a structure of law which is similar to phenomenological laws introduced in the literature, but with a dissimilar effective coefficient.

Key words: rotating porous media, filtration law, Ekman, homogenisation.

Nomenclature

Roman Letters

$\mathbf{A}$	dimensionless ratio of rotational to translational. convective inertia.
Ek	Ekman number.
$\mathbf{G}$	macroscopic driving force.
H	second order tensor.
$\mathbf{k}$, $\mathbf{k}^0$, $\mathbf{k}^1$	microscopic velocity fields.
$\mathbf{K}$, $\mathbf{K}^0$, $\mathbf{K}^1$	tensorial permeabilities.
l	characteristic size of the pores.
L	macroscopic characteristic length.
p	pressure.
$\overline{p}^*$	characteristic pressure difference over sample.
Q	dimensionless ratio of pressure to viscous forces.
R	dimensionless ratio of translational convective inertia to viscous force.
$\mathbf{u}$	vector in $\mathcal{W}$.
$\mathbf{v}$	fluid velocity.

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v^*	characteristic velocity.
$\mathcal{W}$	Hilbert space.
$\mathbf{x}$	dimensionless macroscopic space variable.
$\mathbf{y}$	dimensionless microscopic space variable.

Greek Letters

ε	small parameter of scale separation.
ϵ_{ijk}	permutation symbol.
$\boldsymbol{\gamma}_r, \boldsymbol{\gamma}_e, \boldsymbol{\gamma}_c$	relative, convective and Coriolis accelerations. respectively.
Γ	pore surface.
$\boldsymbol{\omega}$	angular velocity.
μ	viscosity.
Ω	periodic cell.
Ω_p	pore volume in the periodic cell.
ϕ	porosity.
ρ	fluid density.

1. Introduction

Fluid flow in non-galilean porous matrix is concerned with numerous practical applications going from geological applications to industry. A good review of applications is given in Bear *et al.* (1984), Vadasz (1997). The filtration law in use is generally Darcy's law (Bear *et al.*, 1984; Jou and Liaw, 1987; Arulanandan *et al.*, 1988; Mitchell, 1994; Lord, 1999). However, fluid flow through a rotating porous media is submitted to a Coriolis force that deviates the flow from its direction. One can find in the literature a few experimental works on cylindrical pipes, for example Bentom and Boyer (1966), Hart (1971), Johnston *et al.* (1972), Lezius and Johnston (1976) and some numerical works on porous media, (Vadasz, 1993, 1997), that account for this phenomenon. The Coriolis force is considered in the filtration law in a phenomenological way (Chakrabarti and Gupta, 1981; Patil and Vaidyanathan, 1983; Palm and Tyvand, 1984; Rudraiah *et al.*, 1986; Vadasz, 1993) by the direct introduction in the seepage law of the Coriolis inertia term that appears in Navier–Stokes equations, see Equations (4) and (7) below. In Vadasz (1993), for example, the following dimensionless isotropic filtration law is introduced

$$\mathbf{q} = -k(\boldsymbol{\nabla} p + Ek^{-1}\mathbf{e}_\omega \times \mathbf{q}), \tag{1}$$

where $\mathbf{q}$ is the flow rate vector, k is the permeability, $\boldsymbol{\nabla} p$ is the pressure gradient, $\mathbf{e}_\omega$ is a unit vector in the direction of the rotational velocity and Ek is the Ekman number, that is the ratio of viscous term to Coriolis term in Navier–Stokes equations. On the other hand, to our knowledge there is no theoretical work on the modelling of fluid flow in non-galilean porous matrix.

The aim of this paper is to investigate the tensorial filtration law in non-galilean rigid porous matrices for steady-state slow flow of an incompressible viscous Newtonian fluid that is outlined in Auriault *et al.* (2000). Extensions to deformable

porous media, to compressible fluids and to transient flows are evoked in the concluding remark section. We use an upscaling technique, that is the method of multiple scale expansions to determine the macroscopic flow from its description at the pore scale. Heterogeneous system as, for example, porous media may be modelled by an equivalent macroscopic continuous system if the condition of separation of scales is verified (Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980).

$$\varepsilon = \frac{l}{L} \ll 1, \tag{2}$$

where l and L are the characteristic lengths of the heterogeneities and of the macroscopic sample or excitation, respectively. The macroscopic equivalent model is obtained from the description at the heterogeneity scale by (Auriault, 1991):

(i) assuming the medium to be periodic, without loss of generality;
(ii) writing the local description in a dimensionless form;
(iii) evaluating the dimensionless numbers with respect to the scale ratio ε;
(iv) looking for the unknown fields in the form of asymptotic expansions in powers of ε;
(v) solving the successive boundary-value problems that are obtained after introducing these expansions in the local dimensionless description.

The macroscopic equivalent model is obtained from compatibility conditions which are the necessary conditions for the existence of solutions to the boundary-value problems. The main advantages of the method rely upon the possibility of:

(a) avoiding prerequisites at the macroscopic scale;
(b) modelling finite size macroscopic samples;
(c) modelling macroscopically non-homogeneous media or phenomena;
(d) modelling problems with several separations of scales;
(e) modelling several simultaneous phenomena;
(f) determining whether the system 'medium + phenomena' is homogenisable or not;
(g) providing the domains of validity of the macroscopic models.

Coriolis effects are measured by the inverse of Ekman's number $Ek^{-1} = 2\rho\omega l^2/\mu$, where μ and ρ are the dynamic viscosity and the density of the liquid, respectively, ω is the angular velocity of the porous matrix and l is the pore characteristic size. For example, consider a geotechnical centrifuge with an angular velocity $\omega \sim 300$ rpm, and filtrating water with $\mu \sim 10^{-3}$ Pa.s and $\rho \sim 10^3$ kg/m^3. Then, Ek^{-1} number goes from 62 for $l \sim 10^{-3}$ m, for sand, to 6.2×10^{-5} for $l \sim 10^{-6}$ m, for clay. Earth's rotation may also cause Coriolis suction, but in much larger pores: $Ek^{-1} \sim 1$ in fracture networks of fracture width $l \sim 10$ cm. A typical angular velocity for air rotating multipore distributors in experimental fluidized bed reactors is about 2 s^{-1} (Kvasha, 1985). If we assume $l \sim 1$ mm, we obtain $Ek^{-1} \sim 0.4$. Rotation of remelted alloy ingots in semi-solid state is used to avoid segregation during

solidification (Kou *et al.*, 1978). Typical angular velocity is 76 rpm. Considering, for exmple, liquid steel with $\mu \sim 10^{-3}$ Pa.s and $\rho \sim 7.3 \times 10^3$ kg/m^3 and pore size $l \sim 100\,\mu$m, we have $Ek^{-1} \sim 1.2$. In the paper we firstly investigate Ek^{-1} numbers of O(1), that is, $\varepsilon \ll Ek^{-1} \ll \varepsilon^{-1}$, and secondly, we focus on small Ek^{-1} numbers, $\varepsilon \ll Ek^{-1} \ll 1$.

In Section 2, we give the description of the flow at the pore scale relatively to the moving matrix frame. This description is made dimensionless and we evaluate the different dimensionless numbers with respect to the scale ratio ε. Section 3 is devoted to the upscaling for steady-state slow flow with Ekman number O(1). The filtration law resembles a Darcy's law. However, it is shown in Section 4 that the permeability tensor is non-symmetric and depends on the angular velocity of the matrix frame. The case of large Ekman numbers is investigated in Section 5. The filtration law that is obtained shows a similar structure to the phenomenological law (1), but with a different effective coefficient in front of the vector product. Finally, the case of parallel plane fissures is investigated in Section 6.

2. Local Flow Description and Estimations

Consider the flow of an incompressible liquid through a porous medium. The porous medium is spatially periodic and consists of repeated unit cells (parallelepipeds), see Figure 1. There are two characteristic length scales in this problem: the characteristic microscopic length scale l of the pores and of the unit cell, and the macroscopic length scale that may be represented by either the macroscopic pressure drop scale or by the sample size scale. For simplicity, we assume both macroscopic length scales to be of similar order of magnitude, O(L). Moreover we assume that the two length scales l and L are well separated

$$l \ll L. \tag{3}$$

The unit cell is denoted by Ω and is bounded by $\partial\Omega$, the fluid part of the unit cell is denoted by Ω_p, and the fluid–solid interface inside the unit cell is Γ. Relatively

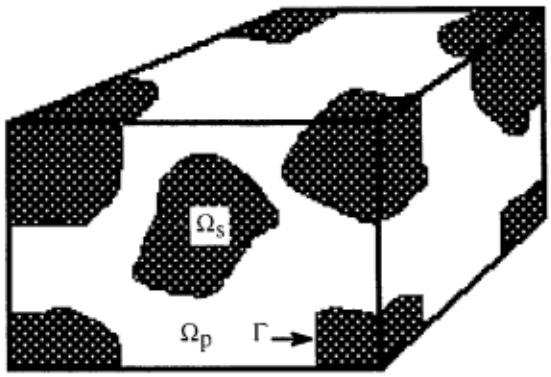

Figure 1. The porous medium.

to the moving porous matrix frame, the momentum balance for the incompressible viscous Newtonian liquid is

$$\mu\nabla^2\mathbf{v} - \boldsymbol{\nabla} p = \rho(\boldsymbol{\gamma}_r + \boldsymbol{\gamma}_e + \boldsymbol{\gamma}_c) \text{ in } \Omega_p, \tag{4}$$

where $\mathbf{v}$ is the velocity vector relative to the matrix frame, p is the pressure, ρ is the density and μ is the viscosity. Gravitational acceleration is included in the pressure term. γ_r, $\boldsymbol{\gamma}_e$ and $\boldsymbol{\gamma}_c$ are the acceleration relative to the matrix frame, the convective and the Coriolis accelerations, respectively

$$\boldsymbol{\gamma}_r = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{5}$$

$$\boldsymbol{\gamma}_e = \boldsymbol{\gamma}(\mathrm{O}) + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{dt}} \times \mathrm{OM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathrm{OM}), \tag{6}$$

$$\boldsymbol{\gamma}_c = 2\boldsymbol{\omega} \times \mathbf{v}, \tag{7}$$

where $\boldsymbol{\omega}$ is the angular velocity of the porous matrix, O is a fixed point of the porous matrix in the investigated period and M is a current point in Ω_p. Equation (4) is completed by the incompressibility condition and the adherence condition on Γ

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega_p, \tag{8}$$

$$\mathbf{v} = 0 \quad \text{on} \quad \Gamma. \tag{9}$$

Since the ratio between microscopic and macroscopic length scales is small, the fundamental perturbation parameter ε is chosen to be

$$\varepsilon = \frac{l}{L}, \qquad \varepsilon \ll 1. \tag{10}$$

The independent dimensionless numbers which characterize the liquid flow problem may be related to the magnitude of ε.

We use the local length scale of a pore l as the characteristic length scale for the variations of the differential operators: we apply the so-called microscopic point of view (Auriault, 1991). To fix the ideas we consider a centrifuge of radius r at constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{e}_\omega$, ω = constant. We consider a steady-state fluid flow which is slow relatively to the porous matrix. Therefore, the relative acceleration $\boldsymbol{\gamma}_r$ in Equation (4) is negligible . We also assume the matrix movement to be steady-state. Then, there remain four dimensionless numbers: the ratio Q of pressure to viscous forces, the ratio R of the translational convective inertia to the viscous force, the ratio A of the rotational to the translational convective inertia and the ratio Ek (the Ekman number) of the viscous force to the Coriolis inertia. We have $A = \mathrm{O}(\omega^2\, l/\omega^2\, r) = \mathrm{O}(\varepsilon)$ and we assume $R = \mathrm{O}(1)$. The estimation of R is the consequence of the hypothesis of separation of scales. Higher values would yield non-homogenisable problems, that is, problems for which equivalent

macroscopic description do not exist. For evaluating the two latter dimensionless numbers Q and Ek, we introduce characteristic values shown by a (*). In particular, $\overline{p}^*$ is the characteristic pressure increment in the sample. We obtain

$$Q = \frac{\overline{p}^* l}{\mu v^*}, \tag{11}$$

$$Ek = \frac{\mu}{2\rho\,\omega\,l^2}. \tag{12}$$

The estimate for Q comes from a phenomenological argument, that is, the viscous flow is locally driven by a macroscopic pressure gradient

$$\frac{\mu v^*}{l^2} = \mathrm{O}\left(\frac{\overline{p}^*}{L}\right), \tag{13}$$

and thus

$$Q = \mathrm{O}(\varepsilon^{-1}). \tag{14}$$

The estimate of Ek will be introduced below. For simplicity we use the same names for the dimensionless variables as for the original variables. The formal dimensionless set that describes the flow is in the form

$$\mu\nabla^2\mathbf{v} - \varepsilon^{-1}\boldsymbol{\nabla} p = \rho(\boldsymbol{\gamma}(\mathrm{O}) + Ek^{-1}2\boldsymbol{\omega}\times\mathbf{v} + \varepsilon\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathrm{OM})) \quad \text{in} \quad \Omega_\mathrm{p}, \tag{15}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega_\mathrm{p}, \tag{16}$$

$$\mathbf{v} = 0 \quad \text{on} \quad \Gamma. \tag{17}$$

3. Homogenisation

The next step is to introduce multiple scale coordinates (Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980). The macroscopic space variable $\mathbf{x}$ is related to the microscopic space variable $\mathbf{y}$ by

$$\mathbf{x} = \varepsilon\mathbf{y}, \tag{18}$$

and the derivative operator becomes

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_y + \varepsilon\boldsymbol{\nabla}_x, \tag{19}$$

where the subscripts x, and y denote the derivatives with respect to the variables $\mathbf{x}$ and $\mathbf{y}$, respectively. Clearly, the flow behaviour depends strongly on the evaluation of Ek as function of the perturbation parameter ε. At this stage we assume that

$$\varepsilon \ll Ek \ll \varepsilon^{-1}, \qquad Ek = \mathrm{O}(1). \tag{20}$$

Following the multiple scale expansion technique (Bensoussan *et al.*, 1978; Sanchez-Palencia, 1980), the velocity $\mathbf{v}$ and the pressure fluctuation p are looked for in the form of asymptotic expansions of powers of ε

$$\mathbf{v} = \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon\, \mathbf{v}^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2\, \mathbf{v}^{(2)}(\mathbf{x}, \mathbf{y}) + \ldots, \tag{21}$$

$$p = p^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon\, p^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2\, p^{(2)}(\mathbf{x}, \mathbf{y}) + \ldots. \tag{22}$$

Substituting these expansions in the set (15–16–17) gives, by identification of the like powers of ε, successive boundary value problems to be investigated. The lowest order approximation of the pressure verifies

$$\frac{\partial p^{(0)}}{\partial y_i} = 0, \qquad p^{(0)} = p^{(0)}(\mathbf{x}). \tag{23}$$

The first order approximation of the velocity $\mathbf{v}^{(0)}$ and the second order approximation of the pressure $p^{(1)}$ are determined by the following set

$$\mu \frac{\partial^2 v_i^{(0)}}{\partial y_j \partial y_j} - G_i - \frac{\partial p^{(1)}}{\partial y_i} = 2\, \rho\, \epsilon_{ijk}\, \omega_j\, v_k^{(0)}, \quad \text{in} \quad \Omega_{\mathrm{p}}, \tag{24}$$

$$\frac{\partial v_i^{(0)}}{\partial y_i} = 0 \quad \text{in} \quad \Omega_{\mathrm{p}}, \tag{25}$$

$$v_i^{(0)} = 0 \ \text{ on } \Gamma, \tag{26}$$

where $\mathbf{v}^{(0)}$ and $p^{(1)}$ are Ω-periodic. ϵ_{ijk} is the permutation symbol. $\mathbf{G}$ is the macroscopic driving force which is independent of $\mathbf{y}$ and which is defined by

$$\mathbf{G} = \boldsymbol{\nabla}_x p^{(0)} + \rho \boldsymbol{\gamma}(\mathrm{O}). \tag{27}$$

To investigate the above set, we introduce the Hilbert space $\mathcal{W}$ of Ω-periodic, divergence free vectors, where the vectors vanish on Γ, with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{W}} = \int_{\Omega_p} \frac{\partial u_i}{\partial y_j} \frac{\partial v_i}{\partial y_j}\, \mathrm{d}y. \tag{28}$$

Now, let us multiply Equation (24) by $\mathbf{u} \in \mathcal{W}$ and integrate over Ω_p. By using integration by parts, the divergence theorem, periodicity, and the boundary condition on Γ, one obtains

$$\forall\, \mathbf{u} \in \mathcal{W}, \quad \int_{\Omega_{\mathrm{p}}} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^{(0)}}{\partial y_j}\, \mathrm{d}y + \int_{\Omega_{\mathrm{p}}} 2\, \rho\, \epsilon_{ijk}\, \omega_j\, v_k^{(0)}\, u_i\, \mathrm{d}y = -\int_{\Omega_{\mathrm{p}}} u_i G_i\, \mathrm{d}y. \tag{29}$$

For any $\mathbf{u}, \mathbf{v}$ in $\mathcal{W}$ we have

$$\left| \int_{\Omega_p} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i}{\partial y_j} \, \mathrm{d}y + \int_{\Omega_p} 2\, \rho\, \epsilon_{ijk}\, \omega_j\, v_k u_i \, \mathrm{d}y \right| \leqslant c_1 \|\mathbf{u}\| \, \|\mathbf{v}\|,$$

$$\left| \int_{\Omega_p} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial u_i}{\partial y_j} \, \mathrm{d}y + \int_{\Omega_p} 2\, \rho\, \epsilon_{ijk}\, \omega_j\, u_k u_i \, \mathrm{d}y \right| \geqslant c_2 \|\mathbf{u}\|^2,$$

where c_1 and c_2 are positive constants. Therefore, the formulation (29) is strongly elliptic, (John, 1970), and there exists a unique $\mathbf{v}^{(0)}$ which is a linear vector function of $\mathbf{G}$

$$v_i^{(0)} = -k_{ij}\, G_j, \tag{30}$$

where the tensor field $\mathbf{k}$ depends on $\boldsymbol{\omega}$ and $\mathbf{y}$.

Finally, the volume balance (16) gives at the second order

$$\frac{\partial v_i^{(1)}}{\partial y_i} + \frac{\partial v_i^{(0)}}{\partial x_i} = 0, \qquad \text{in} \quad \Omega_p. \tag{31}$$

By integrating over Ω_p, we obtain

$$\frac{\partial \langle v_i^{(0)} \rangle}{\partial x_i} = 0, \qquad \langle v_i^{(0)} \rangle = -K_{ij}\, G_j, \qquad K_{ij} = \frac{1}{\Omega} \int_{\Omega_p} k_{ij} \, \mathrm{d}y. \tag{32}$$

Remarks are as follows

- The filtration tensor $\mathbf{K}$ depends on the rotational velocity $\boldsymbol{\omega}$.
- Model (32) is an approximation since $\mathbf{v}^{(0)} = \mathbf{v} + \mathrm{O}(\varepsilon)$.
- Model (32) is valid in the range $\varepsilon \ll Ek \ll \varepsilon^{-1}$ which can be quite large if the separation of scales is good.
- Filtration law $(32)_2$ is not likely to reduce to phenomenological law (1).

An example of $\mathbf{K}(\omega)$ is given in Section 6.

4. Properties of the Permeability $\mathbf{K}(\boldsymbol{\omega})$

Letting $\mathbf{u} = \mathbf{v}^{(0)}$ in formulation (29) gives

$$\frac{1}{\Omega_p} \int_{\Omega_p} \mu \frac{\partial v_i^{(0)}}{\partial y_j} \frac{\partial v_i^{(0)}}{\partial y_j} \, \mathrm{d}y = -\frac{1}{\Omega_p} \int_{\Omega_p} v_i^{(0)} G_i \, \mathrm{d}y = K_{ij} G_j G_i. \tag{33}$$

The left-hand member is positive. We deduce that $\mathbf{K}$ is a positive tensor.

We now investigate the symmetries of tensor $\mathbf{K}$. k_{ip} is the solution of (29) for $G_l = -1$ if $l = p$, $G_l = 0$ if $l \neq p$. Consider now formulation (29) successively with $v_i^{(0)} = k_{ip}, u_i = k_{iq}$ and $v_i^{(0)} = k_{iq}), u_i = k_{ip}$. We obtain

$$\int_{\Omega_p} \mu \frac{\partial k_{iq}}{\partial y_j} \frac{\partial k_{ip}}{\partial y_j} \, \mathrm{d}y - \int_{\Omega_p} 2\, \rho\, \epsilon_{ijk}\, \omega_j\, k_{kp}\, k_{iq} \, \mathrm{d}y$$

$$= \int_{\Omega_p} k_{pq}\, \mathrm{d}y = \Omega K_{pq}, \tag{34}$$

$$\int_{\Omega_p} \mu \frac{\partial k_{ip}}{\partial y_j} \frac{\partial k_{iq}}{\partial y_j}\, \mathrm{d}y + \int_{\Omega_p} 2\,\rho\,\epsilon_{ijk}\,\omega_j\,k_{kq}\,k_{ip}\,\mathrm{d}y$$

$$= \int_{\Omega_p} k_{qp}\, \mathrm{d}y = \Omega K_{qp}. \tag{35}$$

Subtracting member to member yields

$$\Omega(K_{pq} - K_{qp}) = 4\,\rho\,\epsilon_{ijk}\,\omega_j \int_{\Omega_p} k_{kp}\,k_{iq}\,\mathrm{d}y, \tag{36}$$

which generally does not cancel out when $\boldsymbol{\omega} \neq 0$. Therefore, unlike the classical permeability tensor $\mathbf{K}(0)$, $\mathbf{K}(\boldsymbol{\omega})$ is not a symmetric tensor. The divergence operator in the balance $(32)_1$ kills the antisymmetric part of $\mathbf{K}$. However, this antisymmetric part is of importance for porous media that are heterogeneous at the macroscopic scale or in case of flux boundary conditions. By following a similar route, it is possible to show that

$$K_{pq}(\boldsymbol{\omega}) = K_{qp}(-\boldsymbol{\omega}),$$

which stands for the filtration analog of the Hall effect.

5. Large Ekman Number

We now consider the case of large Ekman numbers, $\eta = Ek^{-1} \ll 1$, which is satisfied in many practical applications (Vadasz, 1993). We look for $\mathbf{v}^{(0)}$ and $p^{(1)}$ in the form of the expansions

$$\mathbf{v}^{(0)} = \mathbf{v}^0 + \eta\,\mathbf{v}^1 + \eta^2\,\mathbf{v}^2 + \ldots, \tag{37}$$

$$p^{(1)} = p^0 + \eta\,p^1 + \eta^2\,p^2 + \ldots. \tag{38}$$

The substitution of expansions (37) and (38) into expansions (21)–(22) shows expansions (37) and (38) are valid up to the term in η^n provided that

$$\varepsilon^{1/n} \ll \eta \ll 1. \tag{39}$$

Introducing the above expansion in the formulation (29) and identifying like powers of η yields, to the zeroth order in η

$$\forall\,\mathbf{u} \in \mathcal{W},\quad \int_{\Omega_p} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^0}{\partial y_j}\, \mathrm{d}y = -\int_{\Omega_p} u_i G_i\, \mathrm{d}y. \tag{40}$$

As expected, this is the linear Darcy formulation (Sanchez-Palencia, 1980). Formulation (40) shows that $\mathbf{v}^0$ is a linear vectorial function of $\mathbf{G}$. The Lax–Milgram theorem ensures a unique solution. Let k_{ij}^0 be the particular solution on v_j^0 when $G_k = -\delta_{kj}$. We have

$$v_i^0 = -k_{ij}^0 G_j, \tag{41}$$

which by averaging yields Darcy's law

$$\langle v_i^0 \rangle = -K_{ij}^0 G_j, \tag{42}$$

$$\mathbf{K}^0 = \langle \mathbf{k}^0 \rangle = \frac{1}{\Omega} \int_{\Omega_\mathrm{p}} \mathbf{k}^0 \mathrm{d}y = \mathbf{K}(0). \tag{43}$$

The second rank permeability tensor $\mathbf{K}^0$ is symmetric and positive (Sanchez-Palencia, 1980).

The first order problem in η is the rotation correction problem

$$\forall\, \mathbf{u} \in \mathcal{W}, \quad \int_{\Omega_\mathrm{p}} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^1}{\partial y_j}\, \mathrm{d}y + \int_{\Omega_\mathrm{p}} 2\, \rho\, \epsilon_{ijk}\, \omega_j\, v_k^0\, u_i\, \mathrm{d}y$$

$$= \int_{\Omega_\mathrm{p}} \mu \frac{\partial u_i}{\partial y_j} \frac{\partial v_i^1}{\partial y_j}\, \mathrm{d}y - \int_{\Omega_\mathrm{p}} 2\, \rho\, \epsilon_{ijk}\, \omega_j\, k_{kl}^0 G_l\, u_i\, \mathrm{d}y = 0, \tag{44}$$

which is also a linear problem with respect to $\mathbf{G}$ that has a unique solution

$$v_i^1 = -k_{ij}^1 G_j, \tag{45}$$

where $\mathbf{k}^1$ itself is a linear function of $\boldsymbol{\omega}$. By averaging we obtain

$$\langle v_i^1 \rangle = -K_{ij}^1 G_j, \qquad K_{ij}^1 = \langle k_{ij}^1 \rangle. \tag{46}$$

Tensor $\mathbf{K}^1$ depends linearly on $\boldsymbol{\omega}$, like $\mathbf{k}^1$. Finally, up to the two first terms the expansion (37) gives

$$\langle v_i^{(0)} \rangle = \langle v_i^0 + Ek^{-1}\, v_i^1 \rangle = -(K_{ij}^0 + Ek^{-1} K_{ij}^1)\, G_i, \tag{47}$$

which is valid for $\varepsilon \ll Ek^{-1} \ll 1$. Relation (47) represents an approximation of the filtration law at large Ekman numbers.

To investigate $\mathbf{K}^1$ and to get more insight in the filtration law (47), consider the formulation (40) with $v_i^0 = k_{ip}^0$ and $u_i = k_{iq}^1$

$$\int_{\Omega_\mathrm{p}} \mu \frac{\partial k_{iq}^1}{\partial y_j} \frac{\partial k_{ip}^0}{\partial y_j}\, \mathrm{d}y = \int_{\Omega_\mathrm{p}} k_{pq}^1\, \mathrm{d}y = \Omega K_{pq}^1, \tag{48}$$

and formulation (44) with $v_i^1 = k_{iq}^1$ and $u_i = k_{ip}^0$

$$\int_{\Omega_p} \mu \frac{\partial k^0_{ip}}{\partial y_j} \frac{\partial k^1_{iq}}{\partial y_j} \, \mathrm{d}y + \int_{\Omega_p} 2\,\rho\,\epsilon_{ijk}\,\omega_j\,k^0_{kq}k^0_{ip}\,\mathrm{d}y = 0. \tag{49}$$

By subtracting member to member these two relations we deduce

$$K^1_{pq} = -\frac{1}{\Omega}\int_{\Omega_p} 2\,\rho\,\epsilon_{ijk}\,\omega_j\,k^0_{kq}k^0_{ip}\,\mathrm{d}y. \tag{50}$$

Remark the resemblance between the expressions of $\mathbf{K}^1$ and the antisymmetric part of $\mathbf{K}(\boldsymbol{\omega})$ in (36). Let us show that $\mathbf{K}^1$ is antisymmetric. We have

$$\begin{aligned} K^1_{qp} &= -\frac{1}{\Omega}\int_{\Omega_p} 2\,\rho\,\epsilon_{ijk}\,\omega_j\,k^0_{kp}k^0_{iq}\,\mathrm{d}y, \\ &= -\frac{1}{\Omega}\int_{\Omega_p} 2\,\rho\,\epsilon_{kji}\,\omega_j\,k^0_{ip}k^0_{kq}\,\mathrm{d}y, \\ &= +\frac{1}{\Omega}\int_{\Omega_p} 2\,\rho\,\epsilon_{ijk}\,\omega_j\,k^0_{ip}k^0_{kq}\,\mathrm{d}y = -K^1_{pq}. \end{aligned} \tag{51}$$

Tensor $\mathbf{K}^1$ is antisymmetric. Therefore, $\mathbf{K}^1$ is associated to a vector. Because $\mathbf{K}^1$ is linear with respect to $\boldsymbol{\omega}$, this vector is in the form $\mathbf{H}\cdot\boldsymbol{\omega}$ where $\mathbf{H}$ is a second order tensor and we have

$$\mathbf{K}^1\cdot\mathbf{G} = \mathbf{H}\cdot\boldsymbol{\omega}\times\mathbf{G}. \tag{52}$$

Incorporating expression (50) for $\mathbf{K}^1$ into Equation (52) and identifying the factors of $\omega_j G_q$ yields

$$\epsilon_{pmq}H_{mj} = -2\,\rho\,\epsilon_{ijk}\,\frac{1}{\Omega}\int_{\Omega_p} k^0_{kq}k^0_{ip}\,\mathrm{d}y,$$

which, by multiplying both members by ϵ_{plq} and by using the identity $\epsilon_{plq}\,\epsilon_{pmq} = 2I_{ml}$ gives

$$H_{lj} = -\rho\,\epsilon_{plq}\,\epsilon_{ijk}\,\frac{1}{\Omega}\int_{\Omega_p} k^0_{kq}k^0_{ip}\,\mathrm{d}y. \tag{53}$$

Finally the filtration law (47) for large Ekman numbers is in dimensional form and at the second order of approximation in the form

$$\langle\mathbf{v}\rangle = -(\mathbf{K}^0\cdot + \mathbf{H}\cdot\boldsymbol{\omega}\times)\mathbf{G}. \tag{54}$$

$(\mathbf{K}^0\cdot + \mathbf{H}\cdot\boldsymbol{\omega}\times)$ stands for an approximation of $\mathbf{K}(\omega)$ at large Ekman numbers. Tensor $(\mathbf{K}^0\cdot + \mathbf{H}\cdot\boldsymbol{\omega}\times)$ is not symmetrical. Remark that both $\mathbf{K}^0$ and $\mathbf{H}$ are obtained by averaging and that we need only the local Darcy's velocity field $\mathbf{k}^0$ for their determination.

To compare the filtration law (54) to phenomenogical law (1), let us first assume $\mathbf{K}^0$ and $\mathbf{H}$ to be isotropic, $\mathbf{K}^0 = K^0\mathbf{I}$, $\mathbf{H} = H\mathbf{I}$. The filtration law (54) becomes

$$\langle \mathbf{v} \rangle = -(K^0 \mathbf{G} + H\boldsymbol{\omega} \times \mathbf{G}). \tag{55}$$

Performing a vector product by $\boldsymbol{\omega}$ to both members in the above equality and taking notice of the smallness of the term in H yields

$$\boldsymbol{\omega} \times \langle \mathbf{v} \rangle = -K^0 \boldsymbol{\omega} \times \mathbf{G} - \vec{\boldsymbol{\omega}} \times (H\boldsymbol{\omega} \times \mathbf{G}),$$

$$H\boldsymbol{\omega} \times \mathbf{G} = -\frac{H}{K^0}\boldsymbol{\omega} \times < \mathbf{v} > + \frac{H^2}{K^0}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{G}) \approx -\frac{H}{K^0}\boldsymbol{\omega} \times \langle \mathbf{v} \rangle.$$

Introducing this expression for $H\boldsymbol{\omega} \times \mathbf{G}$ into Equation (55) gives

$$\langle \mathbf{v} \rangle = -K^0 (\mathbf{G} - \frac{H}{(K^0)^2}\boldsymbol{\omega} \times < \mathbf{v} >), \tag{56}$$

which has to be compared to phenomenogical law (1). The structures of the two filtration laws are similar. However, identification of both laws is obtained under the condition that

$$\frac{H}{2\rho\ (K^0)^2} = -1, \tag{57}$$

which is not likely to be verified since we have from (43) and (53)

$$H = H_{11} = 2\rho\ \langle k^0_{32} k^0_{23} - k^0_{22} k^0_{33} \rangle,$$

$$K^0 = K^0_{11} = \langle k^0_{11} \rangle.$$

6. Example

Analytical results are of great interest because they permit us to point out general features concerning effective coefficients. Unfortunately, such results are available for a few pore geometries. We investigate in this section porous media which pore system consists of parallel plane fissures, as shown in Figure 2. Φ is the porosity and the pore thickness is denoted h. The porous medium is rigid and it is submitted to an angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_z$, perpendicular to the pore surfaces. The pore domain under consideration is $-h/2 \leqslant z \leqslant h/2$. The periodicity is arbitrary in the directions x and y. Therefore, $\mathbf{v}$ and p are functions of z only. Consider $\mathbf{G} = G_1 \mathbf{e}_x$. Equations (24–26) become

$$\mu \frac{\mathrm{d}^2 v_1}{\mathrm{d}z^2} - G_1 = -2\omega\rho v_2, \tag{58}$$

$$\mu \frac{\mathrm{d}^2 v_2}{\mathrm{d}z^2} = 2\omega\rho v_1, \tag{59}$$

$$\mu \frac{\mathrm{d}^2 v_3}{\mathrm{d}z^2} - \frac{\mathrm{d}p}{\mathrm{d}z} = 0, \tag{60}$$

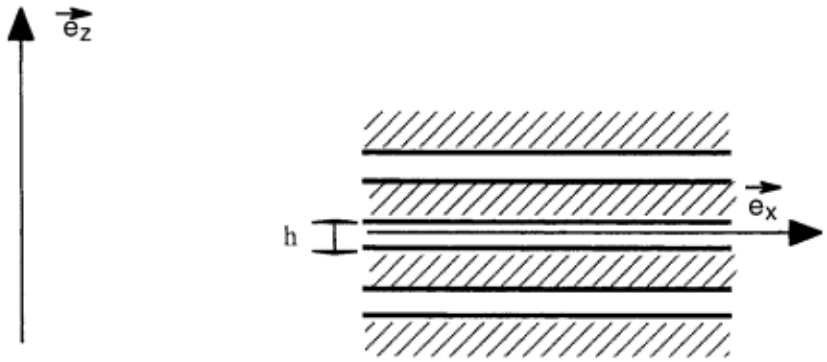

Figure 2. Flat pore.

$$\frac{\mathrm{d}v_3}{\mathrm{d}z} = 0, \qquad \mathbf{v}\left(\pm\frac{h}{2}\right) = 0. \tag{61}$$

The solution is in the form

$$v_1 = \frac{G_1(\cos\beta\cosh\beta\sin\alpha z\sinh\alpha z - \sin\beta\sinh\beta\cos\alpha z\cosh\alpha z)}{2\alpha^2\mu(\cos^2\beta\cosh^2\beta + \sin^2\beta\sinh^2\beta)}, \tag{62}$$

$$v_2 = +\frac{G_1}{2\rho\omega}$$

$$-\frac{G_1(\cos\beta\cosh\beta\cos\alpha z\cosh\alpha z + \sin\beta\sinh\beta\sin\alpha z\sinh\alpha z)}{2\rho\omega(\cos^2\beta\cosh^2\beta + \sin^2\beta\sinh^2\beta)}, \tag{63}$$

$$v_3 = 0, \qquad p = \text{constant}, \tag{64}$$

with

$$\alpha = \sqrt{\frac{\omega\rho}{\mu}}, \qquad \beta = \alpha\frac{h}{2}.$$

By averaging we obtain the macroscopic flux in the form

$$\langle v_1\rangle = \frac{\phi G_1(\cos\beta\sin\beta - \cosh\beta\sinh\beta)}{2\alpha^3\mu h(\cos^2\beta\cosh^2\beta + \sin^2\beta\sinh^2\beta)}, \tag{65}$$

$$\langle v_2\rangle = -\frac{\phi G_1(\cos\beta\sin\beta + \cosh\beta\sinh\beta)}{2\rho\omega\alpha h(\cos^2\beta\cosh^2\beta + \sin^2\beta\sinh^2\beta)} + \frac{\phi G_1}{2\rho\omega}, \tag{66}$$

$$\langle v_3\rangle = 0. \tag{67}$$

The solution for $\mathbf{G} = G_2\mathbf{e}_y$ is obtained by rotating the axes. Finally we get

$$K_{11}(\omega) = K_{22}(\omega) = -\frac{\phi(\cos\beta\sin\beta - \cosh\beta\sinh\beta)}{2\alpha^3\mu h(\cos^2\beta\cosh^2\beta + \sin^2\beta\sinh^2\beta)}, \tag{68}$$

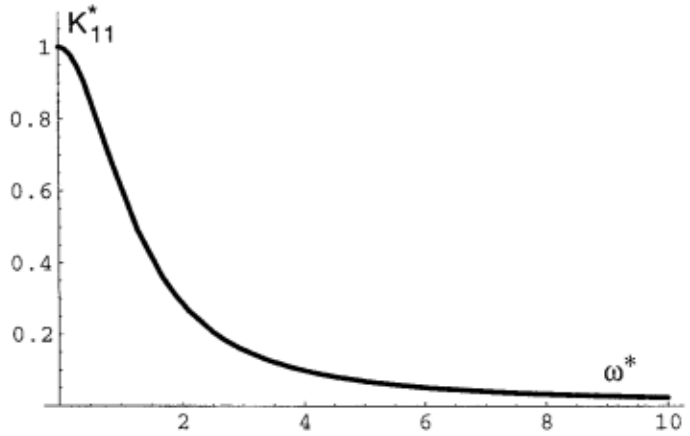

Figure 3. Dimensionless permeability $K_{11}^* = K_{11}(\omega)/K_{11}(0)$ versus dimensionless frequency $\omega^* = \rho h^2 \omega / 4\mu$.

$$K_{21}(\omega) = -K_{12}(\omega),$$
$$= \frac{\phi(\cos\beta\sin\beta + \cosh\beta\sinh\beta)}{2\rho\omega\alpha h(\cos^2\beta\cosh^2\beta + \sin^2\beta\sinh^2\beta)} - \frac{\phi}{2\rho\omega}. \tag{69}$$

In this very particular case, tensor **K** is antisymmetric. The dependence of $K_{11}(\omega)/K_{11}(0)$ and $K_{21}(\omega)/K_{11}(0)$ upon ω is shown in Figures 3 and 4. Both K_{11} and K_{21} are strongly varying with ω. Permeability K_{11} decreases to zero with increasing ω. Permeability K_{21} is negative and shows a minimum for $\omega^* = \rho h^2\omega/4\mu \approx 1$. The rotation gives rise to a filtration analog of the Hall effect. Although these general behaviours may be similar for less simplistic pore geometries, we should remember that the above investigation is valid for Ekman numbers O(1), i.e. for not too large frequencies. Note also that the inplane pore flow perpendicular

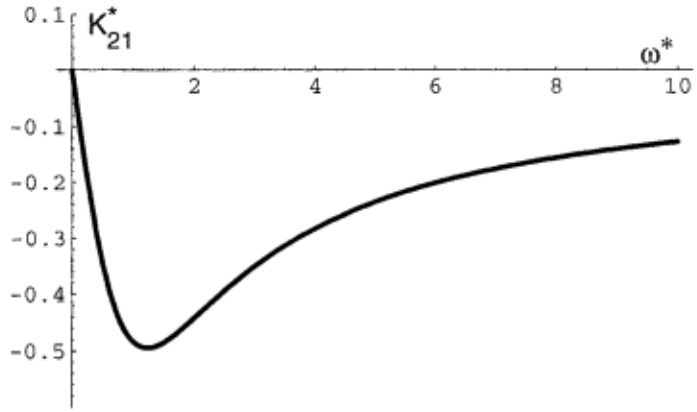

Figure 4. Dimensionless permeability $K_{21}^* = K_{21}(\omega)/K_{11}(0)$ versus dimensionless frequency $\omega^* = \rho h^2 \omega / 4\mu$.

to **G** is not constrained by the surface of the plane fissure. For actual porous media Coriolis effects may be less important.

Large Ekman number permeabilities can be obtained by following two ways. Firstly, by letting ω to be small in Equations (68–69) we obtain, when neglecting higher order terms than ω

$$K_{11}(\omega) \sim \frac{\phi h^2}{12\mu} = K_{11}(0), \qquad K_{21}(\omega) \sim -\frac{\phi\rho\omega h^4}{60\mu^2}. \tag{70}$$

Secondly, these results are also obtained from (53) where we use

$$k^0_{11} = k^0_{22} = \frac{1}{2\mu}\left(\frac{h^2}{4} - z^2\right), \qquad k^0_{ij} = 0 \text{ otherwise.}$$

That yields

$$H_{33} = -\frac{\phi\rho h^4}{60\mu^2}, \qquad H_{ij} = 0 \quad \text{otherwise.}$$

Introducing these values of H_{ij} in Equation (54) yields relations (70).

7. Concluding Remarks

We have used the homogenisation method so as to describe the steady-state slow filtration law of an incompressible liquid in a rotating porous media. The filtration law is shown to strongly depend on the angular velocity of the porous matrix. For a low angular velocity, that is a large Ekman number, the filtration law reduces to the phenomenological law (1), but with a different effective coefficient. Moreover, for both large and O(1) Ekman numbers, the filtration tensor **K** is not a symmetric tensor. The rotation gives rise to a filtration analog of the Hall effect. Although the divergence operator in the flow mass balance $(32)_1$ for macroscopically homogeneous porous media kills the antisymmetric part of **K**, this later one can play an important role in non-homogeneous porous media or in presence of impervious boundary conditions.

As for classical Darcy's law, filtration laws $(32)_2$ and (54) are valid for compressible fluids, since compressible fluids are asymptotically incompressible when the condition of separation of scales (3) is verified, see (Auriault *et al.*, 1990) for details. Only the macroscopic mass balance $(32)_1$ is changed. The above results are also valid when the porous medium is deformable, as it is for classical Darcy's law. As in the present study, the velocity $\langle \mathbf{v} \rangle$ stands for the relative velocity to the porous matrix.

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