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► To cite this version:

Martin Bracke, Jean-Marie Lantau. Mathematical modelling of dynamical systems and implementation at school. CERME 10, Feb 2017, Dublin, Ireland. hal-01933489

HAL Id: hal-01933489

<https://hal.science/hal-01933489>

Submitted on 23 Nov 2018

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Mathematical modelling of dynamical systems and implementation at school

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Dynamical systems like a segway can be described by linear systems of ordinary differential equations of the form $\dot{x} = Ax + Bu$. Obviously, modelling of dynamical systems at school is a huge mathematical challenge for students and teachers. Therefore, in a pilot study with a math course consisting of 12 students of grade 12, it was analysed if, and up to which depth, modelling projects of dynamical systems (using the example of a segway) can be implemented at school. The interdisciplinary project was based on the modelling cycle of Blum and Leiss (2007). First of all, we describe the implementation and results of the pilot study. Then, we outline the design of a follow up study that will be carried out during school year 2016/2017. In this part we formulate our research questions, mainly the design of teacher training with a special focus on teachers' attitudes, which are driven by the findings from the pilot.

Keywords: Dynamical systems, interdisciplinary project, mathematical modelling.

Introduction

In a pilot project the modelling of dynamical systems at school (with a math course of twelve grade 12 students) was carried out by the example of a segway. The project was designed as an interdisciplinary project and started with the students independent modelling that lead to a physical description of acting forces of a segway. After a review on the appropriate linearisation of nonlinear terms, a linear time-invariant system for the segway was proposed. This introductory phase took five lessons. Next, we had three full project days where the students worked in groups of up to four students to concentrate on one of the several interdisciplinary aspects of the project. Namely they worked on 1. How to construct and control a segway and the basics of the Lego Mindstorms® EV3-brick, 2. How to simulate a segway by the help of suitable software and 3. The mathematical model of a segway, including finding of a mathematical solution for the theoretical stabilisation. The project was completed by a final presentation of the students for a mixed audience of teachers as well as university staff members. Obviously, the theoretical and practical issues are highly challenging for students and teachers and therefore the research issue of the pilot study was if an interdisciplinary modelling project of this complexity can be realised at school. The results of the pilot project were impressive, because the students managed to construct and control a Lego Mindstorms® segway as well as acquiring mathematical knowledge about systems of ordinary differential equations and their solutions and eigenvalues to assess the stability of solutions of a system of linear ordinary differential equations. Furthermore, they applied a theoretical and a practical control of a segway by using a proportional derivative control (PD-control). Based on the outstanding results of the pilot study, but with concern that the data set of 12 students is not representative for a quantitative analysis, the following main research questions are raised for a follow up study: Which form of modelling supervision (*open* = research-based, *fine-structured* = content-based, or *semi-structured* = connecting the two approaches) promotes the best development of mathematical competencies for the students during this kind of interdisciplinary projects? Secondly, it will be discussed how teacher training with respect to modelling projects for dynamical systems has to be designed. Altogether there will be a short introductory meeting at the beginning followed by a two-day teacher training focussing on possible implementations in class (with a focus on different aspects of the whole STEM project). The third part is going to be the implementation of the project by the teachers participating in our study. Accompanying the whole study, the attitude of teachers (regarding self-efficacy and prior experience) as well as self-assessment (regarding

opportunities and difficulties for themselves and their students towards interdisciplinary modelling projects of dynamical systems) will be evaluated by a series of pre and post surveys (and interviews). Furthermore, it will be researched, which competencies can be promoted through an interdisciplinary modelling project. The study shall be conducted with 10 to 15 math teachers and 100 to 150 students from grade 10 to 12.

The pilot project

The pilot project (cf. Lantau, 2016) will be described in detail since due to the complexity this is necessary to understand the generation of our research questions as well as the design of the upcoming study.

A short mathematical background of modelling a dynamical system, such as a Segway

The mathematical analysis of a dynamical system is based on the comprehension of system and control theory. A real dynamical object, in this case a segway, is at first described by physical equations to model the acting radial- and horizontal forces. After using suitable simplifications for nonlinear terms in the equations of forces (by using linear Taylor approximations) the physical model is transformed in a mathematical model by the introduction of variables like angle and position and their derivatives into a state-space vector x . Considering the force u of the regulating motor, after linearisation of nonlinear terms one can model a dynamical system by a linear time-invariant system $\dot{x} = Ax + Bu$ of ordinary differential equations. In the case of a segway, the state-space vector x includes the variables angle, angular velocity, (horizontal) position and (horizontal) velocity. This linear time-invariant system can be shifted into a linear system of ordinary differential equations by feedback-regulation $u = Fx$, to get $\dot{x} = (A + BF)x$. Now, the theorem of Wonham (Sontag, 1985) yields that *if and only if the linear time-invariant system is controllable (which means that every possible state \tilde{x} can be reached due to a regulation of the motorforce u ; that holds true for the segway), for every real monic polynomial P there exists a matrix F , such that P is the characteristic polynomial of the matrix $A + BF$* . Since the eigenvalues of the matrix $A + BF$ are the roots of the polynomial P , the goal is now, to place all eigenvalues of the matrix $A + BF$ into the open left half complex plane \mathbb{C}_- , such that the segway stabilizes itself to the rest position $x = 0$. Another approach to control a segway can be realised by a proportional-derivative control. The basic idea of a proportional-derivative control, e.g. for the control of the angle φ , is to choose the motor force $u = -\alpha \cdot \varphi - \beta \cdot \dot{\varphi}$, to stabilise the segway to its rest position. To implement this type of control, some conditions on the parameters α and β have to be fulfilled.

Realisation of the pilot project

The pilot study was inspired by the modelling cycle of Blum and Leiss (2007) consisting of the seven steps Constructing (1), Simplifying/Structuring (2), Mathematising (3), Working mathematically (4), Interpreting (5), Validating (6) and Exposing (7). In what follows, the seven steps of the modelling cycle will be connected to corresponding parts of the pilot study. During the project the teacher played different roles (research guide/advisor, leader of phases with questions and development) that will be explained in detail in what follows.

1. Constructing:

The introduction into the project was given by means of a video showing the popular German entertainer Stefan Raab unsuccessfully trying to control a segway (URL: <https://www.youtube.com/watch?v=m3YBSQYGUw>). After this motivating start, the students were asked to create their own model of a segway. This task, carried out through a group work of four students per group, lasted 30 minutes and was done as an independent work. The construction of three fitting models for a segway has successfully been realised by the students (cf. Figure 1) who were experienced in modelling real-life situations, since many modelling tasks have been

established in this class. But in general it is not necessary to be experienced in mathematical modelling to participate in teachers trainings or as a student in the project.

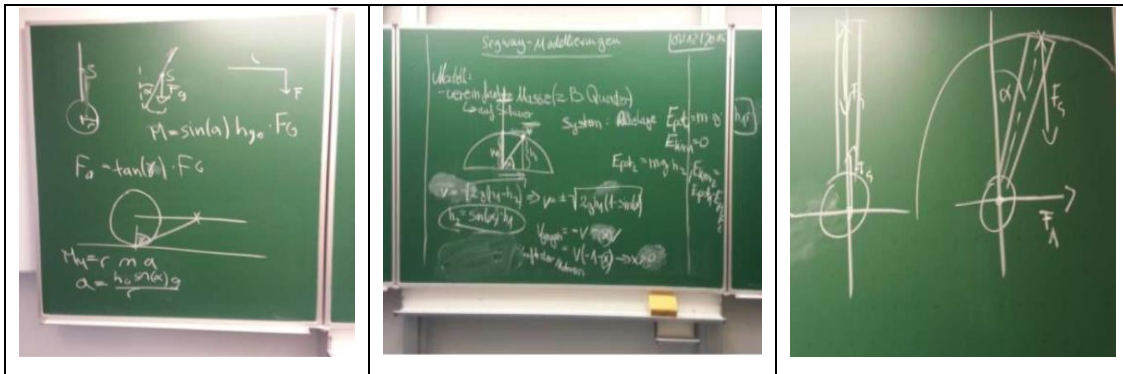


Figure 1: The independent modelling of a segway – Three different approaches

2. Simplifying/Structuring

Figure 1 shows, that the students recognized that the description of acting forces is a necessary part to obtain a mathematical model of a segway. In the next step the three different approaches to model a segway were summarised in one model, to describe the acting forces of a segway.

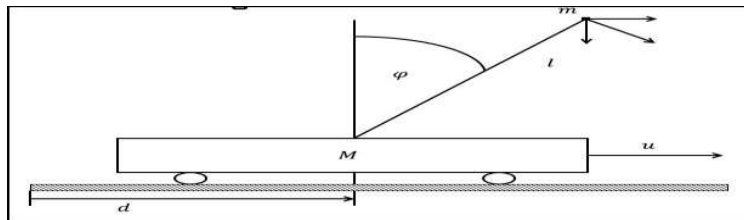


Figure 2: Model of a segway based on acting forces

The part of simplifying and structuring was supported by a matching task, where the students should match acting forces, such as radial forces and horizontal forces, to equivalent terms. In the sequel the different forces were collected to describe the horizontal forces and the radial forces in two equations:

$$u = (M + m) \cdot \ddot{d} + m \cdot l \cdot \cos(\varphi) \cdot \ddot{\varphi} - m \cdot l \cdot \dot{\varphi}^2 \cdot \sin(\varphi) \quad (\text{horizontal forces})$$

$$0 = m \cdot l \cdot \ddot{\varphi} + m \cdot \cos(\varphi) \cdot \ddot{d} - m \cdot g \cdot \sin(\varphi) \quad (\text{radial forces})$$

At this point it has to be mentioned, that during the project three different approaches to model a segway were proposed to the students. The first one, presented in this paper, leads to a control of a segway using the variables *angle* and *position* of the segway as well as their derivatives. Another possibility to model a segway, is by modelling acting horizontal forces for the variables *position of the wheels and the centre of gravity* of the segway (and their derivatives). The third approach is to model the segway as an inverted pendulum by describing radial forces. This leads to a proportional-derivative control for the *angle* and the *angular velocity* of a segway. Due to the fact, that in the modelling approaches 2 and 3, only the variables position (of wheels and centre of gravity) and angle, respectively, are analysed, the number of physical terms to describe the segway is less in comparison to the first approach. However, it has to be clarified, that the physical comprehension of acting forces is essential to promote a mathematical model. And this motivates some deep mathematical concepts: (linear) ordinary differential equations and their solutions as well as a stability analysis through the concept of eigenvalues/-vectors.

3. Mathematising

The transfer from the physical model to a mathematical model is achieved by a linear Taylor-approximation around the rest position taking into account the nonlinear terms in the equations of

forces. The linearisation for the functions sine, cosine and quadratic function were discussed geometrically on the blackboard but the general concept of Taylor-approximation has not been introduced to the students. After linearisation, the two equations can be summarised into a linear time-invariant system introducing a so-called state-space vector. This phase of the project was designed by lessons with question and development. The introductory phase of the study preparing the three project days was concluded by the formulation of the linear time-invariant system that was jointly developed on the blackboard.

Signum - Differentialgleichungen & Eigenwerte

$\Rightarrow \frac{U}{(M+m)} = \frac{g}{4} \quad f = \frac{U \cdot (M+m) \cdot g \cdot f}{m \cdot l} = 0$

$\Rightarrow \ddot{S} = \frac{4}{(M+m)} \cdot g$

$\ddot{S} = \frac{4}{M+m} \Rightarrow \ddot{x} = \begin{pmatrix} \ddot{S} \\ \ddot{S} \\ \ddot{S} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S \\ \dot{S} \\ \ddot{S} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot 4$

$+ \ddot{S} = g$

$\ddot{S} = \frac{4}{M+m} + \frac{g}{2}$

$\ddot{x} = A \cdot x + B \cdot u$

\Rightarrow System.

Figure 3: Development of a linear time-invariant system on the blackboard

4. Working mathematically

To emphasize the interdisciplinary character of the project the three project days (each lasting from 8 am to 2 pm) started by the formation of three groups. The task of the first group was to construct a segway by using a Lego Mindstorms® set. Afterwards, the aim was to control the segway, specifically regulating the segway into the rest position, using the Lego Mindstorms® Software or using the Java-based software lejosEV3 – in each case with the help of a gyro-sensor. The students were highly experienced in working with Lego Mindstorms®, since they have managed several projects in the past, where the use of Lego Mindstorms® was necessary. For the second group four exercise sheets were designed in which the students were introduced to new mathematical concepts, namely: One-dimensional ordinary differential equations and their solutions, linear systems of ordinary differential equations, the matrix exponential function as a solution for linear systems of ordinary differential equations, stability theory for solutions of linear ordinary differential equations, eigenvalues in a non-geometric concept, proportional-derivative control and at last, feedback-regulation for linear time-invariant systems. The task of the third group was to carry out computer simulations according to the mathematical concepts worked on by the second group. To this end, a single exercise sheet including six tasks was designed to guide these students. All exercise sheets can be found as an appendix to the master's thesis of Lantau (2016). During the three project days the students worked nearly autonomously in class on their tasks and the teacher supervised the work of the different groups following the concept of minimal help.

5. Interpreting and 6. Validating

The interpretation and validation of the mathematical model also took place during the three project days and was motivated by the interdisciplinary character in a very natural way. In particular, this can be seen in the application of the proportional-derivative control: While groups 2 and 3 used the inverted pendulum to model the segway in order to use the proportional-derivative control for the angle and angular velocity, the first group used the information about the theoretical restrictions for the proportional- and derivative parameters α and β to practically stabilise the Lego segway for about 10 seconds. While two groups established a theoretical concept to model and control a dynamical system, including the development of fitting constraints for the control (*interpretation*), the third group used the results of the other groups to practically stabilise the segway (*validation*).

Considering steps 2, 5 and 6 of the modelling cycle, we observe that a successful modelling of dynamical systems, such as a segway, includes several disciplines like physical comprehension of acting forces, engineers' competencies to construct and control a Lego Mindstorms® segway, scientific programming for the control and finally, mathematical competencies to acquire a theoretical comprehension for the control of dynamical systems. The students also observed that several disciplines must be considered in a highly connected sense to promote the success of the modelling task.

7. Exposing

In order to collect and structure the theoretical and practical results of the project the students were asked to create a final presentation. This promoted the mathematical learning success in a holistic sense because the students who worked more practically on the project got a deeper insight into theoretical results through the explanations of their classmates and the connection of the theoretical results to practical results. The same also holds vice versa. During the creating of a final presentation the students needed to prepare their newly acquired knowledge properly to present it for a mixed audience of math teachers, students, schoolmates, parents and math professors. The exploration of the pilot study shows that the phase of presenting (*exposing*) the results of the modelling project promotes many mathematical competencies. Hence, in our view, this step is essential when modelling dynamical systems at school.

Main results of the pilot study

Considering the project's realisation it can be observed that a modelling project of a dynamical system sets a high demand on physical-, engineering-, computer science- and mathematical competencies for both students and teachers. Therefore, the main question of the pilot study was if this kind of interdisciplinary projects can be realised at school. The pilot gives a positive answer, and it also shows that fundamental mathematical competencies, as proposed by the German Education Minister Conference (KMK, 2012), are highly promoted by the students during the project. Next, the six fundamental mathematical competencies are listed and connected to processes that promoted the corresponding competency during the interdisciplinary project.

1. *K1: to argue mathematically*: This competency was promoted at three different stages of the project. At first it was promoted during the discussion of linearisation for nonlinear terms, secondly during the development of stability criteria for linear systems of ordinary differential equations and thirdly during the discussion of constraints for the parameters of the PD-control.
2. *K2: to solve problems mathematically*: The research issue for the students was how a segway can be theoretically and practically stabilised. This problem has been solved practically by a PD-control based on the Lego Mindstorms® Software and theoretically, by learning the concepts of feedback- and PD-control.
3. *K3: to model mathematically*: The competency of creating mathematical models has been the centre of the pilot and was highly promoted. The students got the insight that mathematical modelling is very useful to solve real-life problems.
4. *K4: to use mathematical forms of representations*: By the preparation of a final presentation the students were requested to illustrate their main results in a mathematically correct way.
5. *K5: to work with technical, symbolic and formal elements of mathematics*: During the development of a linear time-invariant system as a model of a dynamical system the students used technical, formal and symbolic elements. This competency was also highly requested for the two groups that worked out the concepts of eigenvalues, PD-control, feedback-regulation and ODE-theory.
6. *K6: to communicate*: During their creation of a model for a segway and especially during preparation of the final presentation the students communicated frequently to discuss models, mathematical solutions and open questions. Furthermore, this competency was promoted through the final presentation itself.

Corresponding to the development of mathematical competencies during the project, the mathematical learning success of the students has been detected. To investigate the learning success a multiple-choice test was developed containing 10 questions regarding the mathematical concepts that are introduced during the project. The test has been carried out one week after the final presentation and was analysed by counting the correct answers of the content-related questions. More than half of the class reached at least 83 % of the maximum score and two students answered every question completely correct. This shows that the concept of ordinary differential equations, their solution and the stability of solutions, the concept of eigenvalues considering the stability of solutions for a linear system of ordinary differential equations, modelling a segway as a linear time-invariant system and controlling it by means of feedback-control or PD-control can be taught through an interdisciplinary modelling project. The concepts of eigenvalues (embedded in a geometric context of linear mappings and its characteristics) and (one-dimensional) ordinary differential equations and their solutions are part of the curricular standards in mathematic for secondary level in Rhineland-Palatinate (MBWWK, 2015). Both of them can not only be taught through an interdisciplinary modelling project, but also the connection between a classical algebraic concept (eigenvalues) and a concept of analysis (ordinary differential equations) can be established. But not only these two fundamental mathematical concepts for secondary level courses could be transmitted; even concepts that are usually taught in master degree courses for mathematics teachers can be included. From our point of view, the combination of promoted mathematical-, physical-, engineering-, and computer scientific competencies and a huge learning success in advanced mathematical concepts legitimates the implementation of interdisciplinary modelling projects in secondary level courses. Beside the description of the pilot project, the paper focuses on research issues concerning a follow-up study, explained in the next section.

Research issues for a follow-up study

Regarding the results of the pilot study two main research questions are aimed to be analysed within the framework of *Grounded Theory*:

1. How shall teacher trainings for interdisciplinary modelling projects be designed?
2. Which competencies can students acquire, depending on the implementation and the type of supervision of an interdisciplinary modelling project?

As explained in Kaiser (2013) the most advanced approach of including mathematical modelling in school is by an interdisciplinary project which requires each discipline – in our case of modelling dynamical systems: physics, mathematics, engineering and computer science – to share its concepts and information in order to guarantee success of the modelling project. This task is very challenging for math teachers, considering the fact that some teachers neither have the specific mathematical knowledge to understand real problems (here: theory of dynamical systems) and its didactic transposition (Chevallard 1985) nor they are experts in other required fields like physics or computer science. Therefore, one aim is to design appropriate trainings that help teachers to convey the interdisciplinary character of modelling dynamical systems. In our example this includes the adequate physical-didactical preparation of detecting acting forces of a segway as well as an introduction to basic concepts for the control and programming of a Lego Mindstorms® segway. However, in order to promote mathematical competencies and a mathematical learning success, it is also necessary to design trainings for the inner mathematical concepts required by this project. For this purpose it is planned to design learning material for complex mathematical concepts like *stability theory for ordinary differential equations* as well as an *introduction to linear time-invariant systems and their control*. To this end the material designed for the pilot study will be analysed and modified based on the students' and teachers' comments.

During the follow-up study the attitude of teachers towards mathematical modelling of dynamical systems, represented by the example of a segway, will be explored and evaluated through a series of

surveys. The attitude of teachers will be measured by analysing three connected questions. Since the mathematical foundation of a dynamical system is located within the scope of *systems and control theory* and the project requires knowledge from several disciplines, one question is which inner and extra mathematical comprehension issues do maths teachers have or expect when they think of modelling a dynamical system. According to this question it will also be analysed how far the teachers are engaged to fill their content-related gaps. Additionally, teachers will be asked for possible implementations of each step of the modelling cycle in secondary level math courses. During the first teacher training (end of 2016) there will be pre and post surveys to answer these questions. Then, the results will be considered to prepare the second teacher training in spring 2017. Moreover, the teachers' expectations on possible obstacles will be assessed before, in between an after a series of trainings regarding the design and supervision of the modelling of a segway. Regarding these questions it is planned to design a survey, based on the theory of planned behaviour (Ajzen, 1991). The survey shall point out to what extend the teachers' attitude to modelling projects, their subjective norms and their perceived behavioural control influence their intention and, later on, the execution of their modelling project of dynamical systems. Additionally, teachers shall comment on the possibilities of developing each of the six mathematical competencies K1–K6 by modelling a dynamical system. Furthermore, they shall assess the potential learning success regarding the two fundamental mathematical concepts of *ordinary differential equations* and *eigenvalues/-vectors* that can be addressed by this modelling project. Finally, we want to analyse the correlation between the teachers' attitudes and the students learning success.

Beside the teacher focussed research, our second research question is *Which mathematical (physical, engineering- and computer scientific-) competencies of the students can be strengthened significantly through an interdisciplinary modelling project of dynamical systems.* It is planned to use video recordings to detect the enhancement of students' mathematical competencies during the project. The video material will be transcribed based on the work of Mayring (2015). To evaluate the enhancement of mathematical knowledge there will be pre-and posttests. Furthermore, the mathematical working techniques preferred by the students will be evaluated. For the analysis of the students' prerequisites surveys will be designed and it is planned to use video recordings to analyse the mathematical working techniques and to detect the phases of the project in which competencies of the students are promoted.

For the teacher trainings, options for different ways to conduct and supervise the project will be designed: The first approach is an open modelling process in which the initial problem of a segway to be stabilised is given to the students without further hints or work sheets. The main idea is to develop the students' independence in modelling activities including the competencies of „developing productive dispositions, flexible strategies, and foster student persistence and independent thinking” (Common Core State Initiative 2010; National Research Council, 2001 as cited in Doerr & Ärlebäck 2015, p.1). As described in the paper of Doerr and Ärlebäck (Brodie 2011; Lobato & Ellis, 2005; Magiera & Zawojewski, 2011 as cited in Doerr & Ärlebäck, 2015, p.1), this type of modelling challenges the teacher „to tackle classroom discussions, to structure group interactions and to provide effective feedback to students.” Regarding the task of modelling a segway we would like to find out how far students can model a segway independently in a way that develops mathematical techniques (e.g. PD-control) to solve the real problem. This question refers to the framework of Wake, Foster and Swan, who proposed that students' competencies of a simplification of the reality and the development of a mathematical structure that represents and simplifies the reality are under-emphasised in school mathematics (Wake et al. 2015, p. 8). Following the theory of Wake et al. it is also planned to create material that promotes a pre-structured and more teacher-controlled modelling of a dynamical system as an alternative. Our aim is to check which approach teachers prefer for their class and to analyse which competencies are

promoted in a modelling project depending on the conduction and supervision in one of the two specified ways.

Acknowledgement

This work was partially supported by the European Social Fund (ESF) project *SchuMaMoMINT* and by *U.EDU* as part of the national and federal project “Qualitätsoffensive Lehrerbildung” (University of Kaiserslautern, Funding Code 01JA1616), funded by the Federal Ministry of Education and Research.

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