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Choosing a Mathematical Working Space in a modelling task: The influence of teaching

Elizabeth Montoya Delgadillo¹, Fernanda Viola² and Laurent Vivier³

¹Pontificia Universitaria Católica de Valparaíso, Valparaíso, Chile; elizabeth.montoya@pucv.cl

²Universidad Nacional de Córdoba, Córdoba, Argentina; fviola@famaf.unc.edu.ar

³Université Paris Diderot, Paris, France; laurent.vivier@univ-paris-diderot.fr

In this paper we report on students' work in Argentina and in France when performing a modelling task. The problem given, named the "gutter", is quite a classic in university calculus courses. Analyses have been realized by using the Mathematic Working Space (MWS) research model in relation to mathematization of a modelling process. We are mainly concerned with the influence of the teaching in students' productions. Since the modelling process is an important issue of teaching, we have chosen the population of the study among pre-service teachers.

Keywords: Mathematical Working Space, paradigms, modelling task.

Introduction

Nowadays, modelling is fostered in secondary school and at the university level, and one expects teachers and engineers to be trained to achieve the modelling competence. For instance, modelling constitutes a specific competence that has to be developed by civil engineers in Latin America, according to Tuning project (Guerrero, 2013). But, what modelling problems have to be considered in upper secondary school and in university? Are the modelling problems used in high school, and in the university, standardized by the institutions? Do they refer to real situations? We think that mathematical objects, different representations used, and properties involved in the modelling process give a richer sense for students in term of the knowledge they can construct. Teachers have of course an important role to play that highlights the importance of the teachers training. In that way, the population of this study was mainly chosen in secondary teacher training.

In this study we present a relation between the Mathematical Working Space research model (Kuzniak & Richard, 2014; Kuzniak, Tanguay & Elia, 2016) and the modelling cycle (Blum & Borromeo-Ferri, 2009) that could be understood as a *first cycle* for the resolution of the modelling task. Mathematical Working Space (MWS) is a model that is used in research in mathematics education, first developed in the field of geometry. When a student starts with the given situation, one assumes that he/she begins a horizontal mathematization process which is a foundation for bringing the situation problem into a mathematical domain. Then a vertical mathematization process takes place where the MWS framework and the modelling cycle can interact with each other. That is our approach in this study.

We first present how to use the MWS model to analyse a modelling activity. Then, we study a modelling task given to pre-service teachers in Argentina and in France. Our aim is not to make a comparative study, since both populations are different, but rather to identify the personal MWS of the students when solving this modelling task. The focus is mainly on three levels of teachers training in Argentina. We also looked at primary teachers training in France in order to have an idea of no scientific students' answers.

MWS and Modelling

We consider a MWS that depends on a specific mathematical field (Kuzniak, Tanguay & Elia, 2016) such as, among others, analysis, geometry, algebra or statistics. Paradigms in an MWS, depending of the domain, serve to characterize the work according to a community or an institution. We develop the paradigms for analysis (Montoya Delgadillo & Vivier, 2016).

The MWS model

Three types of MWS may be distinguished: (i) **MWS of reference**, which is defined according to the relation to knowledge, ideally under mathematical criteria; (ii) **suitable MWS**, which depends on the institution involved, and is defined according to the way that this knowledge is supposed to be taught in the institution; (iii) **personal MWS**, which depends on the individual and is defined by the way in which the individual handles a mathematical problem with his or her own knowledge and cognitive capacities.

MWS is an environment in which reflection results from the interaction between an individual and a problem in a mathematical domain. It is an environment organized for an expert of this domain, by means of two interconnecting planes: the epistemological and the cognitive planes (Kuzniak & Richard, 2014; Kuzniak, Tanguay & Elia, 2016).

The *epistemological plane* is composed of three poles (Figure 1a), namely *referential* (properties, theorems, definitions...), *representamen* (semiotic signs), and *artefacts* (material or symbolic). The *cognitive plane* consists of the processes of *visualization*, *construction* and *proof*. The functioning of a MWS must not be understood as a union of single components lying on the epistemological and cognitive planes, but rather as links activated by two or three geneses, *semiotic*, *instrumental* and *discursive geneses*, that articulate the two planes.

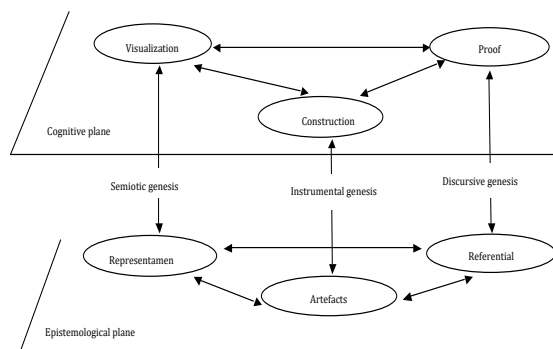


Figure 1a: The Mathematics Working Space, geneses and vertical planes (Kuzniak & Richard, 2014)

Paradigms of analysis

The situation we propose (see following section) is an optimization task. Hence, using (mathematical) analysis is quite natural, even if the problem can be solved in various ways. In order to identify the paradigms of analysis in the answers of the students, we present the three working paradigms of analysis identified by Montoya Delgadillo and Vivier (2016):

- **Arithmetic/Geometric Analysis (GA):** it enables interpretations with implicit assumptions based on geometry, arithmetic calculations or the real world.

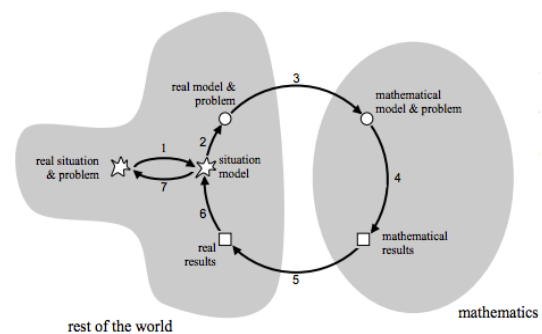


Figure 1b: The modelling cycle (Bloom & Borromeo-Ferri, 2009)

- **Calculation Analysis (CA):** the rules of calculation are defined more or less explicitly and are applied independently of reflection on the existence and nature of the objects.
- **Real Analysis (RA):** it is characterized by work involving approximation and neighbourhoods; definition and properties are set theoretically; an “ ϵ work”.

Modelling

The development of the modelling skills (Blum & Borromeo Ferri, 2009) mobilizes notions and mathematical objects of different mathematical domains, where the knowledge that the students can learn is grounded on arguments that belong to different domains (analysis, probabilities, geometry, etc.). This gives rise to different MWS and paradigms.

Recently, two doctoral theses in probability in Paris Diderot university proposed a use of MWS in modelling processes¹. The whole modeling cycle (Figure 1b) is not taken into its totality: the focus is on phases 3 to 5 of the cycle and our aim is to analyse, with the MWS model, the mathematization process when students solve a modelling task. We expect to identify, in students’ work, the solving mathematical domain and mathematical objects, representations or signs, artefacts, mathematical knowledge, and the working paradigm. We show below how the two frameworks may be used together and their complementarity.

In the following, we do not consider all the modelling process with the lens of the MWS model, but rather how is it possible to analyse mathematical activity. In particular, we set the following questions: Given a specific task, what domain and (personal) MWS will a student choose during the modelling process? Is it possible to see an influence of a suitable MWS?

Experimental study

The "gutter" situation (see below) was given to three groups of future secondary mathematics teachers in Argentina: two groups of a private training institute, namely 1PC of 1st year and 2PC of 2nd year, and one group of 4th university year, 4PF, of the Universidad Nacional de Córdoba – unfortunately, a 3rd level group was not available. The study focuses on these pre-service teachers, but the task was also posed in France, in an examination of 3rd year future primary teachers, at the Université Paris Diderot. This extra population helps to understand what kind of solving processes one can expect from non-scientific students. Students of the study are named L1 to L28, 1PC1 to 1PC24, 2PC1 to 2PC15, and 4PF1 to 4PF12 accordingly.

As announced above, it is an optimization task, quite classic except for modelling aspects. Here, we make the hypothesis that the majority of students who followed a calculus course that provided a method for solving a class of optimization problems (derivation or optimum of quadratic functions) will work in paradigm CA, and that they will resort to functions.

The aforementioned was the case for 4PF students trained at the university, since these students had courses on mathematical analysis. But CA paradigm was not expected for 1PC students and 2PC students, since in Argentina the first calculus course takes place during the second year and the experimentation was at the very beginning of the academic year. We then expected more GA methods (see below).

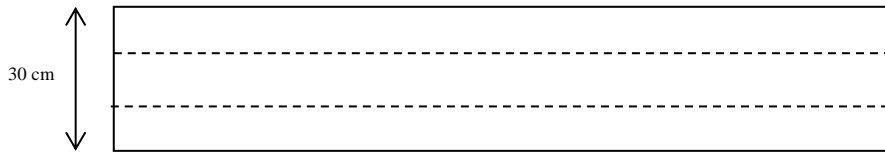
¹ See also the poster “Modelling tasks and mathematical work” in TSG6 of CERME 10.

On the other hand, French students were in a multidisciplinary third university year after the validation of two disciplinary university years (mathematics, or biology, or history, or English...). Students' profiles were very diverse and a few of them studied mathematics, and specifically calculus. We make the hypothesis that, spontaneously, most of L-students work in GA paradigm, whether or not using functions.

The gutter situation: a priori analysis

First, we give the statement of "the gutter", then we make an a priori analysis.

We have a rectangular metal sheet of 30 cm width and of big length. We fold up perpendicularly edges on each side to make a gutter (see dotted lines on the figure below). For obvious reasons, both side edges of the gutter will have the same size.



How should we fold up the metal sheet in order to obtain a gutter with a maximum flow?

This is a partially modelled task, since many parameters are fixed and, moreover, the geometrical mathematization is given. The choice to fold perpendicularly allows simplifying the task by eliminating other cases. There is a lack of parameters to study the flow, among them, the slope of the gutter (that we shall not discuss) and the length of the sheet.

The latter is not necessary because we can replace the study of the flow by the study of the area of a cross-section of the gutter supposing that the flow through this section is constant. Nevertheless, it is simpler to work with the volume than with the flow which is a quotient magnitude. Thus, some students may choose a length to make a calculation of volume. Is this length considered as a parameter or either as a numerical value added to the statement? This length is used in calculations or is it only a useful intermediary to think of the situation?

Let us note that there may be problems of understanding of the proposed geometrical modelling. An inadequate understanding of the situation corresponds to the addition of an idea or a belief, which is not correct regarding the situation. We expect in particular to have equitable answers with 10 cm for each of the three edges, as well as the assertion that the area, and thus the flow, does not vary because of the constant 30 cm.

On the other hand, there is no indication on the mathematization allowing to make calculations for solving. We focus our analyses on this phase of mathematization, which we interpret as the choice of a MWS by students, and the phase of solving. We then look at students' answers with an adequate understanding of the modelling situation: good geometrical shape (specified in the statement), and considering the area of a cross-section, or a volume, to study the various values (the variations) of the flow. We also look at a schema of the gutter: cross-section, in 2D or in 3D.

In the following, we present the types of answers expected for solving the modelling task. Before that, let us notice that the task can be solved in the geometry field of the statement, with magnitude and a knowledge on the areas of rectangles of given perimeter: By taking two "gutters", one forms a rectangular pipe the section of which has a perimeter of 60 cm. Since the area is maximal for a square,

the solution is obtained for a 60 cm/4 side, that is, 15 cm. So, the basis has 15 cm of length and one has to fold in $15 \text{ cm}/2 = 7.5 \text{ cm}$.

The resolution can be made in a numerical MWS, that is, by doing several calculations of the area or the volume. Several levels are possible: choice of some values of the length of the edge, calculations, and then decision-making. Numbers to be considered are essentially whole or decimal numbers. The solution being 7.5 cm, considering only integers multiples of cm does not allow to find the optimal value (unless changing the unit of length). These values can be grouped in a table or not, obtained in an organized way or not (for example, with all integers from 0 to 15, or by an oriented search), with a sign (as the letter x) to denote the length of the edge or not. A formula allows to automate calculations and, possibly, the implementation in an instrument of calculation. Obtained values can be also put in a graph.

The production of a formula can allow to change of MWS to a functional MWS, in calculus. One expects the introduction of a quadratic function²:

- Using only algebra, with the expression of a global variational principle: $f(x) \leq f(a)$ where a is the candidate value, that has to be find, to be the optimal value;
- With recognition of a quadratic function, its properties, vertices or symmetry axis of a parabola, allow to solve the problem. This is a work in CA paradigm.
- Using the derivative of the function. Is the theorem quoted? Is the change of sign of the derivative evoked or forgotten? This is mainly a work in CA paradigm.
- Setting a values table or a graphic, recognizing or not a parabola, in GA paradigm. The difference with numerical MWS mentioned above may be difficult to identify.

Results

Let us note that 12 L-students do not answer, and utterances from 5 students of 1PC are not classified. Three of these 1PC-students gave the solution 7.5 cm without any justification: possibly an exchange of information between students or, for at most two students, the divisions of 30 cm by 4 like for a square³. There is also the atypical 1PC24's production, which will be mentioned later on.

In Table 1 there is a summary of the results with four groups of indicators:

1. Representamen: cross-section of the gutter (CS), letter for the edges (Ledg), letter (L) or value (V) for the fixed length of the gutter (Par), table (T) and graph (G);
2. Objects used: function (Fu) or formula (Fo);
3. Knowledge: derivation (Der), vertex of a parabola or quadratic function (Ver) in a MWS of function, CA paradigm; numerical calculations (Num) in a numerical MWS;
4. Non adequate modelling: 10 cm for each edge (10), no variation (noV) and also extreme folding (ExtFold) for “the edge must be the lower” or “the greater”.

² A quadratic function limits the technical difficulties. The task was also chosen for this reason.

³ That method is not expected here: since $(a,b) \rightarrow ab$ is bilinear, optimizing the area of rectangles with fixed perimeter p is the same than optimizing the area of rectangles of sizes a and b with $a + b + a = p$.

Table 1: Students' answers to the "gutter task"

	CS	Ledg	Par	T/G	Fu/Fo	Der	Ver	Num	10	noV	ExtFold
4PF (12)	8	12	6L,0V	0T, 3G	12Fu, 0Fo	7	5	0	0	0	0
2PC (15)	3	2	0L,1V	2T, 0G	0Fu, 1Fo	0	1	5	4	5	2
1PC (24)	7	2	1L,2V	0T, 0G	0Fu, 0Fo	0	1	11	5	2	0
L (16)	0	7	2L,3V	2T, 2G	4Fu,2Fo	2	0	5	2	0	4

There is a lot of 2D schema, close to the statement schema, but only four 3D schema. On the other hand, a cross-cutting (see Table 1) seems characteristic of 4PF-students, more advanced mathematically. It is the same with formalization, introduction of letters (variables and parameters) or using functions: the 12 4PF-students used a modelling with functions. It is also the same with the use of knowledge: 4PF-students work in CA paradigm, either by using derivation or the formula giving the vertex of a parabola (rarely justifying that it is a maximum; only two students justified it, by calculating the second derivative). There were a few students that used this knowledge in the three other groups, or in an empirical way by working in GA paradigm (by numerical calculations, more or less organized, or graphs).

Although the work seems largely guided by the knowledge the 4PF-students have, with few variations, modelling was adequate, what is not the case in other groups. For students who do not well understand the situation, either no MWS rises from modelling, or a very poor MWS appears, not allowing a substantial mathematical work. Answers of inadequate modelling are:

- Equitable length of the three sides: 10 cm each (5 students);
- There is no variation, all is constant (4 students);
- An extreme folding, the littlest lateral side possible (5 students), but sometimes with the (real) constraint that 0 is forbidden, so that the water remains in the gutter;
- L27: a pyramid (to have a pipe?) without understanding the geometry of the statement;
- 1PC24 drew a roof explaining: "if there is a few water that does not go strongly", and "if there is a lot of water and with great strength".

Although the productions of the group 1PC are very poor, with not much formalization, in the groups 2PC and L we find a various types of rich productions: trying values with calculations, sometimes leading to the accurate solution when the student thinks out of the integers domain (some students remain in the field of the arithmetic of integers, sometimes with answers with two solutions, 7 and 8, and even the constraints to have a basis of even length in order to be able to divide by two), graphs and tabulations which can lead to the awareness of a symmetry (L6, L28 below).

L14 proposes a formula with a letter x for the length of the edge and $l=10$ for the length of the gutter, to calculate what he calls Dmax (figure 2). He uses this formula to find a values table for a whole number between 1 and 15. He writes: "we notice that the maximal flow is achieved for an edge between 7 and 8 cm". Other students stop at this stage, while L14 continues with another table for the values of x between 7 and 8 with a step of 0,1. This allows him to conclude correctly, in GA paradigm, that "to have a maximal flow, edges have to be 7.5 cm each".

Student L6, after the introduction of the function $f(x) = (30-2x)x$, establishes a valuable table for integers and draws the graph of f (figure 3). She concludes with visualization on the graph to determine the solution by drawing what seems to be the axis of symmetry of the curve.

C'est chercher le débit maximal de la gouttière.
 $D_{\max} = l \times l \times h$
 $D_{\max} = (30-2x) \times 10 \times x$

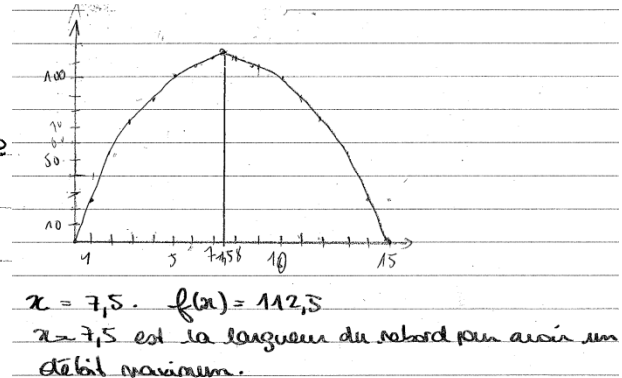


Figure 2 : Answer by student L14

Figure 3 : L6's graph

L28 declares the same function but in a more formal way: " $\forall x \in (0,15) f(x) = (30-2x)x$ ". He then calculates the values for the whole values of x that he associates two by two, which is an organization adapted to the function at stake. "We notice a symmetry in the values of $f(x)$ when x varies between 1 and 15. We deduce that the maximum is in the middle of the values, that is: $f_{\max}(x) = f(7,5) = 112,5 \text{ cm}^2$."

On cherche donc ce volume maximal.
 $V(x) = ax(30-2x)$ et on fixe $a = 200 \text{ cm}$
 donc on a $V(x) = 200x(30-2x)$
 $V(x) = 6000x - 400x^2 = x(6000 - 400x)$
 Avec la calculatrice, on trace cette fonction polynomiale de degré 2. Il semblerait que le maximum soit atteint pour $x = 7,5 \text{ cm}$ et que le volume maximal vaille $V \approx 22500 \text{ cm}^3$. Essayons de trouver la valeur exacte de x .

Figure 4: Production of L10

L10's work is very complete (figure 4). She uses a formalism (she fixes the parameter a to 200 cm), a function, and a graphic calculator to propose a conjecture for the candidate maximum value in a visualization work in GA paradigm. She strengthens her conjecture by considering the middle of the function roots. She continues by calculating, algebraically, in paradigm CA, $V(x) - V(7.5) = -(20x - 150)^2$, what allows her to justify her conjecture and to conclude.

Finally, 2PC4 has an atypical production with a modelling by means of a paper model: he takes two paper strips, which he folds to form a pipe and gives, by a reasoning close to the solution in the MWS_{Geometry} described above, the right solution.

Conclusion

The recognition of a function in the situation, like the introduction of a letter, is an important modelling activity, because a concept is introduced, which a priori has no relation to the situation. But more than a letter – by using it the work can remain in a numerical MWS –, the introduction of a

function switches the work to a MWS of functions with specific techniques (e.g. derivation) and representations (graphic, table value). However, the work done can be very different and the personal MWS involved can also be different from a student to another.

The understanding of the situation and the asked question made each student choose a MWS, that is to say, mathematical objects, knowledge, theorems, signs (letters, graphics,...), artefacts (calculator, spreadsheet,...) and the processes of visualization, construction and proving in various domains – in other words, the MWS... –. Those are complex issues for a teacher to control the various knowledge involved.

Then we wonder whether the modelling process for some students is a routine that has been "normalized" by the institution, mostly for a specific class of problems, or a mathematical activity which has been taught in their institution. It seems that for 4PF group, students' answers are standardized with only a *choice* between two knowledge on functions. This is a first influence of the suitable MWS that we point out. The diversity of the L-students productions reinforce this standardization and we set the problem of the possibility for teachers to be aware and open to alternative answers that their students may have.

MWS model allows us to analyse in depth the concepts and mathematical objects coming from mathematization identified in the modelling process. However, the possibility that different MWS from this process are generated makes an attractive but complex task to control for the teacher, overall for a *second modelling cycle*. Indeed, this study focus on a *first cycle* where one asks to students to solve a modelling task. Then, teachers may use a *second cycle* in order to reorganize and to point out knowledge that arose during the first cycle.

Acknowledgments

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