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NON UNICITY FOR ENRICHED DIFFERENTIAL EQUATIONS(*)

by D. Feyel and A. de La Pradelle.

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INTRODUCTION. For $\alpha > 1/3$ and then for $\alpha > 1/4$ we consider solutions $Y = (y, y^{(2)}, ...)$ of differential equations driven by an α-enriched path $X = (x, x^{(2)}, ...)$ which is not necessarily geometric. Under suitable regularity conditions, we prove the unicity for the first component y, and we give a counter example of unicity for the other components. The existence of Y for $\alpha > 1/4$ shall be given in a forthcoming paper.

WARNING. All the functions and variables are finite dimensional, so that every product is in fact a contracted tensor product which shall be denoted with a $\bullet$ and which can differ from an equation to another possibly with the order of factors. It is very trying to write multiindices all along the text. For example we shall write

$$w = \int \varphi(t) \bullet u(t) \bullet dx(t) = \int u \bullet dv, \quad \text{with} \quad v = \int \varphi(t) \bullet dx(t)$$

in place of (Einstein convention)

$$w^i = \int \varphi^i_{jk}(t) u^k(t)\, dx^j(t) = \int u^k(t)\, dv^i_k, \quad \text{with} \quad v^i_k = \int \varphi^i_{jk}(t)\, dx^j(t)$$

Let β be a positive number, the notation $u = O(\beta)$ means that $|u(a,b)| \leq \mathrm{Cst}\, |b-a|^\beta$ for every a, b.

(*) *Written in 2010*

Recalls

1°. **The sewing lemma** ([*]).

Let μ be a function defined on $[0,T]^2$ with finite dimensional values. For every $c \in [a,b] \subset [0,T]$, put

$$\delta\mu(a,b,c) = \mu(a,b) - \mu(a,c) - \mu(c,b)$$

Suppose that $|\delta\mu(a,b,c)| \leq k|b-a|^{1+\varepsilon}$ where k and ε are two positive constants. Then there exists a unique function $\varphi(t)$ such that

$$|\varphi(b) - \varphi(a) - \mu(a,b)| \leq \mathrm{Cst}\,|b-a|^{1+\varepsilon}$$

The least constant is majorized by $k.2^{\varepsilon}/(2^{\varepsilon}-1)$.

Example, the Young integral.

Let α, β be two positive numbers such that $\alpha + \beta > 1$. Denote $\mathcal{C}^{\alpha}$ the space of α-Hölder real continuous functions defined on $[0,T]$. If $x \in \mathcal{C}^{\alpha}$ and $y \in \mathcal{C}^{\beta}$, put $\mu(a,b) = y_a(x_b - x_a)$. It is easily seen that $|\delta\mu(a,b,c)| \leq \mathrm{Cst}\,|b-a|^{\alpha+\beta}$, so that the sewing lemma applies. The function φ is called the Young integral

$$\varphi(b) - \varphi(a) = \int_a^b y_t\, dx_t$$

2°. Notations.

If $u(a,b)$ is a function defined on $[0,T]^2$, we put $u_{ab} = u(a,b)$. If φ is a function of one variable, we put $\varphi_{ab} = \varphi(b) - \varphi(a)$.

3°. **Enriched paths** ($\alpha > 1/3$) ([*]).

Let x be an element of $\mathcal{C}^{\alpha}([0,T],B)$. If z_{ab} is another function, we suppose that $x^{(2)}_{ab} = z_{ab} - x_a \bullet x_{ab} = O(2\alpha)$. We shall say that $(x, x^{(2)})$ is an α-enriched path†. Let y be another $\mathcal{C}^{\alpha}$ function, put

$$A_{ab} = \begin{bmatrix} x_{ab} & 0 \\ y_{ab} & 0 \end{bmatrix}, \qquad A^{(2)}_{ab} = \begin{bmatrix} x^{(2)}_{ab} & 0 \\ y^{(2)}_{ab} & 0 \end{bmatrix}$$

where $y^{(2)}_{ab}$ is another function. In order to the couple $(A, A^{(2)})$ be an α-enriched path, it is necessary and sufficiant that

$$y^{(2)}_{ab} = O(2\alpha) \quad \text{and} \quad \delta(y^{(2)})(a,b,c) = y_{ac} \bullet x_{cb}$$

† In fact, this is inspired from the definition of rough path [*].

We shall say that $(y, y^{(2)})$ is an enriched path over $(x, x^{(2)})$, and we shall put

$$y^{(2)}_{ab} = \int_a^b y_{at} \bullet dx_t$$

If f is a map defined on the y-space

$$\mu(a,b) = f(y_a) \bullet x_{ab} + \int_a^b [\nabla f(y_a) y_{at}] \bullet dx_t$$

If $\nabla^2 f$ exists and is Lipschitz, then $|\delta\mu(a,b,c)| \leq \mathrm{Cst}\, |b-a|^{3\alpha}$, so that the sewing lemma applies, and the produced function φ is called the integral with respect to $(x, y, y^{(2)})$

$$\varphi_{ab} = \int_a^b f(y_t) \bullet dx_t$$

4°. **Enriched paths ($\alpha > 1/4$)** ([*]).

Let $(x, x^{(2)}, x^{(3)})$ and $(y, y^{(2)}, y^{(3)})$ with $x, y \in \mathcal{C}^\alpha$, $x^{(2)}_{ab}, y^{(2)}_{ab} = O(2\alpha)$, $x^{(3)}_{ab}, y^{(3)}_{ab} = O(3\alpha)$. Put

$$A_{ab} = \begin{bmatrix} x_{ab} & 0 \\ y_{ab} & 0 \end{bmatrix}, \quad A^{(2)}_{ab} = \begin{bmatrix} x^{(2)}_{ab} & 0 \\ y^{(2)}_{ab} & 0 \end{bmatrix}, \quad A^{(3)}_{ab} = \begin{bmatrix} x^{(3)}_{ab} & 0 \\ y^{(3)}_{ab} & 0 \end{bmatrix}$$

We suppose that

$$\delta(A^{(2)}(a,b,c)) = A_{ac} \bullet A_{cb}$$

$$\delta(A^{(3)}(a,b,c)) = A^{(2)}_{ac} \bullet A_{cb} + A_{ac} \bullet A^{(2)}_{cb}$$

As above we put

$$y^{(2)}_{ab} = \int_a^b y_{at} \bullet dx_t, \qquad y^{(3)}_{ab} = \int_a^b y^{(2)}_{at} \bullet dx_t$$

Besides, suppose that there exists two functions $\xi_{ab} = O(3\alpha)$ and $\eta_{ab} = O(3\alpha)$ such that

$$\delta\xi(a,b,c) = (x_{ac})^{\bullet 2} \bullet x_{cb} + \int_c^b x_{ac} \bullet x_{ct} \bullet dx_t + \int_c^b x_{ct} \bullet x_{ac} \bullet dx_t$$

$$\delta\eta(a,b,c) = (y_{ac})^{\bullet 2} \bullet x_{cb} + \int_c^b y_{ac} \bullet y_{ct} \bullet dx_t + \int_c^b y_{ct} \bullet y_{ac} \bullet dx_t$$

We put

$$\xi_{ab} = \int_a^b (x_{at})^{\bullet 2} \bullet dx_t, \qquad \eta_{ab} = \int_a^b (y_{at})^{\bullet 2} \bullet dx_t$$

The quadruple $Y = (y, y^{(2)}, y^{(3)}, \eta)$ is an *enriched path over* $X = (x, x^{(2)}, x^{(3)}, \xi)$.

If f is a map, put

$$\mu(a,b) = f(y_a) \bullet x_{ab} + \int_a^b \nabla f(y_a) y_{at} \bullet dx_t + \frac{1}{2} \int_a^b \nabla^2 f(y_a).y_{at}^{\bullet 2} \bullet dx_t$$

If $\nabla^2 f$ is Lipschitz, then (*cf.* [*]) we get an inequality

$$|\delta\mu(a,b,c)| \leq \mathrm{Cst}\, |b-a|^{4\alpha}$$

Hence the function φ associated with μ by the sewing lemma is called the integral of f with respect to X and writes

$$\varphi_{ab} = \int_a^b f(y_t) \bullet dx_t$$

Differential equations (unicity).

For $\alpha > 1/2$, resp. $\alpha > 1/3$, resp. $\alpha > 1/4$ and an enriched path $X = (x,\ etc..\ )$, consider a path $Y = (y,\ etc..\ )$ over X, which satifies the equation

$$y_{ab} = \int_a^b f(y_t) \bullet dx_t, \qquad , y_0 \text{ being given} \tag{1}$$

where $\bullet$ is a contracted tensor product.

For $\alpha > 1/3$, the equation (1) has solutions for sufficiantly regular f, as is well known ([***]). For $\alpha > 1/4$ a solution shall be given in a forthcoming paper.

1°. Unicity. Case $\alpha > 1/2$.

Suppose f is $\mathcal{C}^{1,1}$ with values in the convenient space. There exists F Lipschitz such that $f(y) - f(\widetilde{y}) = F(y, \widetilde{y}) \bullet (y - \widetilde{y})$.

Let y and $\widetilde{y}$ be two paths which are solution of (1). Put $u_t = y_t - \widetilde{y}_t$, we get $u_0 = 0$ and

$$u_{ab} = \int_a^b \varphi_t \bullet u_t \bullet dx_t = \int_a^b u_t \bullet dv_t, \quad \text{with} \quad v_t = \int_0^t \varphi_s \bullet dx_s$$

where $\varphi_t = F(y_t, \widetilde{y}_t)$ and v_t are $\mathcal{C}^\alpha$. Then we get in the Young sense

$$u_{ab} = \int_a^b u_t \bullet dv_t$$

By iteration we put $u^1_{ab} = u_{ab}$,

$$u^{(n+1)}_{ab} = \int_a^b u^{(n)}_t \bullet dv_t$$

Applying theorem (*) of [*], we get

$$|u_{ab}| \leq Kc^n |b-a|^{n\alpha}/(n!)^\beta$$

where $0 < \beta < \alpha$, and $c, K > 0$. Hence $u = 0$ and $y = \widetilde{y}$.

2°. Unicity. Case $\alpha > 1/3$.

Suppose f is $\mathcal{C}^{2,1}$ with values in the convenient space. There exists $F \in \mathcal{C}^{1,1}$ such that $f(y) - f(\widetilde{y}) = F(y, \widetilde{y}) \bullet (y - \widetilde{y})$.
Put $u_t = y_t - \widetilde{y}_t$, and

$$\varphi_t = F(y_t, \widetilde{y}_t), \quad \text{and} \quad \varphi^{(2)}_{ab} = \int_a^b [F(y_t, \widetilde{y}_t) - F(y_a, \widetilde{y}_a)] \bullet dx_t$$

Put

$$v_{ab} = \int_a^b \varphi_t \bullet dx_t = \varphi_a \bullet x_{ab} + \varphi^{(2)}_{ab}$$

and

$$\mu(a,b) = \varphi_a \int_a^b x_{at} \bullet \varphi_t \bullet dx_t$$

We get

$$\delta\mu(a,b,c) = \varphi_a \bullet x_{ac} \bullet v_{cb} + O(3\alpha) = v_{ac} \bullet v_{cb} + O(3\alpha) = \delta(v_a \bullet v_{ab}) + O(3\alpha)$$

Hence by the sewing lemma, there exists a function $\psi(t)$ such that

$$\mu(a,b) + v_a \bullet v_{ab} - \psi_{ab} = O(3\alpha)$$

Put $\widehat{v}^{(2)}_{ab} = \psi_{ab} - v_a \bullet v_{ab} = O(2\alpha)$, then the couple $(v, \widehat{v}^{(2)})$ is an enriched path.

1 Proposition :

$$u_{ab} = \int_a^b u_t \bullet dv_t$$

where the integral is taken with respect to the enriched path $(v, \widehat{v}^{(2)})$.
Démonstration : Put $\mu(a,b) = u_{ab} - u_a \bullet v_{ab}$. We see that $\mu(a,b) = O(2\alpha)$. Moreover

$$\delta\mu(a,b,c) = u_{ac} \bullet v_{cb}$$

so that $\widehat{u}^{(2)}_{ab} = \mu(a,b)$ is an enrichment of u with respect to $(v, \widehat{v}^{(2)})$. Hence the integral is worth $u_a \bullet v_{ab} + \widehat{u}^{(2)}_{ab}$ which is u_{ab}.

2 Théorème (unicity) : $y = \widetilde{y}$.
Démonstration : By iteration, we get

$$\widehat{u}^{(n)}_{0t} = \int_0^t \widehat{u}^{(n-1)}_{0s} \bullet dv_s = u_t \quad \text{integral with respect to } (v, \widehat{v}^{(2)})$$

By theorem [*] of [*], one has for some $\beta < \alpha$, K, c and every n

$$|u_t| \leq K c^n t^{n\alpha}/(n!)^\beta$$

so that $u = 0$ and $y = \widetilde{y}$.

3°. Unicity. Case $\alpha > 1/4$.

Suppose f is $\mathcal{C}^{3,1}$ with values in the convenient space. There exists $F \in \mathcal{C}^{2,1}$ such that $f(y) - f(\widetilde{y}) = F(y, \widetilde{y}) \bullet (y - \widetilde{y})$.
Put $u_t = y_t - \widetilde{y}_t$, and thanks to [*]

$$\varphi_t = F(y_t, \widetilde{y}_t), \quad \text{and} \quad \varphi^{(2)}_{ab} = \int_a^b [F(y_t, \widetilde{y}_t) - F(y_a, \widetilde{y}_a)] \bullet dx_t$$

$$\varphi^{(3)}_{ab} = \int_a^b \varphi^{(2)}_{at} \bullet dx_t$$

Put

$$v_{ab} = \int_a^b \varphi_t \bullet dx_t = \varphi_a \bullet x_{ab} + \varphi^{(2)}_{ab}$$

We want to define $\widehat{v}^{(2)}$, $\widehat{v}^{(3)}$ and w such that $V = (v, \widehat{v}^{(2)}, \widehat{v}^{(3)}, w)$ is an enriched path. First, put

$$\mu(a,b) = \varphi_a \bullet \int_a^b x_{at} \bullet \varphi_t \bullet dx_t + \int_a^b \varphi^{(2)}_{at} \bullet \varphi_a \bullet dx_t = A_{ab} + B_{ab}$$

Observe that all the written integrals make sense (theorem 3.2 of [*]). We get

$$\delta A(a,b,c) = \varphi_a \bullet x_{ac} \bullet v_{cb} - \varphi_{ac} \bullet \varphi^{(2)}_{cb} \bullet \varphi_c + O(4\alpha)$$

$$\delta B(a,b,c) = \varphi^{(2)}_{ac} \bullet \varphi_c \bullet x_{cb} + \varphi_{ac} \bullet \varphi^{(2)}_{cb} \bullet \varphi_c + O(4\alpha)$$

then

$$\delta\mu = \delta A + \delta B = \varphi_a \bullet x_{ac} \bullet v_{cb} + \varphi^{(2)}_{ac} \bullet \varphi_c \bullet x_{cb} + O(4\alpha)$$

$$\delta\mu = (\varphi_a \bullet x_{ac} + \varphi_{ac}^{(2)}) \bullet v_{cb} + O(4\alpha) = v_{ac} \bullet v_{cb} + O(4\alpha)$$

As above, there exists by the sewing lemma a function $\Phi(t)$ such that $\mu(a,b) + v_a \bullet v_{cb} - \Phi_{ab} = O(4\alpha)$. Then we put

$$\widehat{v}_{ab}^{(2)} = \Phi_{ab} - v_a \bullet v_{ab} = O(2\alpha)$$

Next, put

$$\nu(a,b) = \varphi_a \bullet \int_a^b \left[\int_a^t x_{as} \bullet \varphi_a \bullet dx_s \right] \bullet \varphi_a \bullet dx_t$$

This iterated integral makes sense (definition of $x^{(2)}$).

$$\delta\nu(a,b,c) = \varphi_a \bullet x_{ac}^{(2)} \bullet \varphi_a \bullet \varphi_c \bullet x_{cb} + \varphi_a \bullet x_{ac} \bullet \varphi_c \bullet x_{cb}^{(2)} \bullet \varphi_c + O(4\alpha)$$

Noting that $\widehat{v}_{ab}^{(2)} = \varphi_a \bullet x_{ab}^{(2)} \bullet \varphi_a + O(3\alpha)$ we get

$$\delta\nu(a,b,c) = \widehat{v}_{ac}^{(2)} \bullet v_{cb} + v_{ac} \bullet \widehat{v}_{cb}^{(2)} + O(4\alpha)$$

$$\delta(v_a \bullet \widehat{v}_{ab}^{(2)} + \Phi_{0a} \bullet v_{ab})(a,b,c) = -\widehat{v}_{ac}^{(2)} \bullet v_{cb} - v_{ac} \bullet \widehat{v}_{cb}^{(2)}$$

Hence by the sewing lemma, there exists a function $\Psi(t)$ such that

$$\nu(a,b) + (v_a \bullet \widehat{v}_{ab}^{(2)} + \Phi_{0a} \bullet v_{ab})) - \Psi_{ab} = O(4\alpha)$$

Then we put

$$\widehat{v}_{ab}^{(3)} = \Psi_{ab} - (v_a \bullet \widehat{v}_{ab}^{(2)} + \Phi_{0a} \bullet v_{ab})) = O(3\alpha)$$

Finally put

$$\rho(a,b) = \varphi_a \bullet \int_a^b x_{at} \bullet \varphi_a \bullet x_{at} \bullet \varphi_a \bullet dx_t$$

$$\delta\rho(a,b,c) = (\varphi_a \bullet x_{ac})^{\bullet 2} \bullet v_{cb} + \varphi_a \bullet x_{ac} \bullet \varphi_c \bullet x_{cb}^{(2)} \bullet \varphi_c + \varphi_c \bullet x_{cb}^{(2)} \bullet \varphi_c \bullet x_{ac} \bullet \varphi_c$$

$$\delta\rho(a,b,c) = v_{ac}^{\bullet 2} \bullet v_{cb} + v_{ac} \bullet \hat{v}_{cb}^{(2)} + \int_c^b S(v_{ac} \bullet v_{cs} \bullet dv_s) + O(4\alpha)$$

Where

$$S(u \bullet v \bullet w) = v \bullet u \bullet w$$

Then we have

$$\delta\rho(a,b,c) = v_{ac}^{\bullet 2} \bullet v_{cb} + v_{ac} \bullet \hat{v}_{cb}^{(2)} + \int_c^b S(v_{ac} \bullet v_{cs} \bullet dv_s) + O(4\alpha)$$

Now

$$\delta[(v_a)^{\bullet 2} \bullet v_{ab} - \int_a^b v_t \bullet v_a \bullet dv_t - \int_a^b v_a \bullet v_t \bullet dv_t](a,b,c) =$$

$$v_{ac}^{\bullet 2} \bullet v_{cb} + v_{ac} \bullet \hat{v}_{cb}^{(2)} + \int_c^b S(v_{ac} \bullet v_{cs} \bullet dv_s)$$

Then there exists a function Θ such that

$$\rho(a,b) - (v_a)^{\bullet 2} \bullet v_{ab} + \int_a^b v_t \bullet v_a \bullet dv_t + \int_a^b v_a \bullet v_t \bullet dv_t - \Theta_{ab} = O(4\alpha)$$

Put

$$w_{ab} = \Theta - (v_a)^{\bullet 2} \bullet v_{ab} - \int_a^b v_t \bullet v_a \bullet dv_t - \int_a^b v_a \bullet v_t \bullet dv_t$$

Symbolically $w_{ab} = \int_a^b v_{at} \bullet v_{at} \bullet dv_t$ And we obtain an enriched path

$$V = (v, \hat{v}_{ab}^{(2)}, \hat{v}_{ab}^{(3)}, w_{ab})$$

3 Proposition :

$$u_{ab} = \int_a^b u_t \bullet dv_t$$

where the integral is taken with respect to the enriched path V.

Démonstration : Put $\widehat{u}^{(2)}(a,b) = u_{ab} - u_a \bullet v_{ab}$. We see that $\widehat{u}^{(2)}(a,b) = O(2\alpha)$. Put $\widehat{u}^{(3)}(a,b) = \widehat{u}^{(2)}(a,b) - u_a \widehat{v}_{ab}^{(2)}$. We see that $\widehat{u}^{(3)}(a,b) = O(3\alpha)$. Put

$$A_{ab} = \begin{bmatrix} v_{ab} & 0 \\ u_{ab} & 0 \end{bmatrix}, \qquad A_{ab}^{(2)} = \begin{bmatrix} \widehat{v}_{ab}^{(2)} & 0 \\ \widehat{u}_{ab}^{(2)} & 0 \end{bmatrix}, \qquad A_{ab}^{(3)} = \begin{bmatrix} \widehat{v}_{ab}^{(3)} & 0 \\ \widehat{u}_{ab}^{(3)} & 0 \end{bmatrix}$$

Then we get a matrix enriched path. As in the case $\alpha > 1/3$ we define $A^{(n)}$ and deduce the inequalities by [*]

$$|\widehat{u}^{(n)}| \leq K c^n t^{n\alpha} / (n!)^\beta$$

By induction, we also get $\widehat{u}_{0t}^{(n)} = u_{0t}$, so that $u = 0$.

Conclusion

To the end, we give an example for $1 > \alpha > 1/3$ where there is unicity for y but not for (y, y^2). Take $x_t = t$, $x_{ab}^{(2)} = (b-a)^2/2$, and the equation

$$y_t = y_0 + \int_0^t 0.dt \tag{18}$$

If φ is a function belonging to $\mathcal{C}^{(2\alpha)}$, put $y_{ab}^{(2)} = \varphi(b) - \varphi(a)$. Then $(y, y^{(2)})$ is an enriched path over $(x, x^{(2)})$, and is a solution of (18). As φ is arbitrary, there is no unicity.

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