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To cite this version:
Arnaud de La Pradelle, D Feyel. NON UNICITY FOR ENRICHED DIFFERENTIAL EQUATIONS. 2018. hal-01931152

HAL Id: hal-01931152
https://hal.science/hal-01931152
Preprint submitted on 22 Nov 2018

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NON UNICITY FOR ENRICHED DIFFERENTIAL EQUATIONS(*)

by D. Feyel and A. de La Pradelle.

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Key words: Hölder continuity, fractional Brownian motion, rough path, enriched path.

AMS Subject classification (2000): 60G15, 60H05, 60H07.

INTRODUCTION. For $\alpha > 1/3$ and then for $\alpha > 1/4$ we consider solutions $Y = (y, y^{(2)}, \ldots)$ of differential equations driven by an $\alpha$-enriched path $X = (x, x^{(2)}, \ldots)$ which is not necessarily geometric. Under suitable regularity conditions, we prove the unicity for the first component $y$, and we give a counter example of unicity for the other components. The existence of $Y$ for $\alpha > 1/4$ shall be given in a forthcoming paper.

WARNING. All the functions and variables are finite dimensional, so that every product is in fact a contracted tensor product which shall be denoted with a $\bullet$ and which can differ from an equation to another possibly with the order of factors. It is very trying to write multiindices all along the text. For example we shall write

\[
  w = \int \varphi(t) \bullet u(t) \bullet dx(t) = \int u \bullet dv,
\]

with $v = \int \varphi(t) \bullet dx(t)$

in place of (Einstein convention)

\[
  w^i = \int \varphi^i_{jk}(t) u^k(t) dx^j(t) = \int u^k(t) dv^i_k,
\]

with $v^i_k = \int \varphi^i_{jk}(t) dx^j(t)$

Let $\beta$ be a positive number, the notation $u = O(\beta)$ means that $|u(a,b)| \leq \text{Cst} \ |b-a|^\beta$ for every $a, b$. 

(*) Written in 2010
Recalls

1°. The sewing lemma ([*]).
Let \( \mu \) be a function defined on \([0, T]^2\) with finite dimensional values. For every \( c \in [a, b] \subset [0, T] \), put

\[
\delta \mu(a, b, c) = \mu(a, b) - \mu(a, c) - \mu(c, b)
\]

Suppose that \( |\delta \mu(a, b, c)| \leq k|b - a|^{1+\varepsilon} \) where \( k \) and \( \varepsilon \) are two positive constants. Then there exists a unique function \( \varphi(t) \) such that

\[
|\varphi(b) - \varphi(a) - \mu(a, b)| \leq \text{Cst} |b - a|^{1+\varepsilon}
\]

The least constant is majorized by \( k.2^{\varepsilon}/(2^{\varepsilon} - 1) \).

Example, the Young integral.
Let \( \alpha, \beta \) be two positive numbers such that \( \alpha + \beta > 1 \). Denote \( C^\alpha \) the space of \( \alpha \)-Hölder real continuous functions defined on \([0, T]\). If \( x \in C^\alpha \) and \( y \in C^\beta \), put \( \mu(a, b) = y_a(x_b - x_a) \). It is easily seen that \( |\delta \mu(a, b, c)| \leq \text{Cst} |b - a|^{\alpha+\beta} \), so that the sewing lemma applies. The function \( \varphi \) is called the Young integral

\[
\varphi(b) - \varphi(a) = \int_a^b y_t \, dx_t
\]

2°. Notations.
If \( u(a, b) \) is a function defined on \([0, T]^2\), we put \( u_{ab} = u(a, b) \). If \( \varphi \) is a function of one variable, we put \( \varphi_{ab} = \varphi(b) - \varphi(a) \).

3°. Enriched paths (\( \alpha > 1/3 \)) ([*]).
Let \( x \) be an element of \( C^\alpha([0, T], B) \). If \( z_{ab} \) is another function, we suppose that \( x_{ab}^{(2)} = z_{ab} - x_a \bullet x_{ab} = O(2\alpha) \). We shall say that \((x, x^{(2)})\) is an \( \alpha \)-enriched path†.
Let \( y \) be another \( C^\alpha \) function, put

\[
A_{ab} = \begin{bmatrix} x_{ab} & 0 \\ y_{ab} & 0 \end{bmatrix}, \quad A_{ab}^{(2)} = \begin{bmatrix} x_{ab}^{(2)} & 0 \\ y_{ab}^{(2)} & 0 \end{bmatrix}
\]

where \( y_{ab}^{(2)} \) is another function. In order to the couple \((A, A^{(2)})\) be an \( \alpha \)-enriched path, it is necessary and sufficient that

\[
y_{ab}^{(2)} = O(2\alpha) \quad \text{and} \quad \delta(y^{(2)})(a, b, c) = y_{ac} \bullet x_{cb}
\]

† In fact, this is inspired from the definition of rough path [*].
We shall say that \((y, y^{(2)})\) is an enriched path over \((x, x^{(2)})\), and we shall put
\[
y_{ab}^{(2)} = \int_{a}^{b} y_{at} \cdot dx_t
\]

If \(f\) is a map defined on the \(y\)-space
\[
\mu(a, b) = f(y_a) \cdot x_{ab} + \int_{a}^{b} [\nabla f(y_a) y_{at}] \cdot dx_t
\]

If \(\nabla^2 f\) exists and is Lipschitz, then \(|\delta \mu(a, b, c)| \leq \text{Cst} |b - a|^{3\alpha}\), so that the sewing lemma applies, and the produced function \(\varphi\) is called the integral with respect to \((x, y, y^{(2)})\)
\[
\varphi_{ab} = \int_{a}^{b} f(y_t) \cdot dx_t
\]

4°. Enriched paths \((\alpha > 1/4) (\ast)\).
Let \((x, x^{(2)}, x^{(3)})\) and \((y, y^{(2)}, y^{(3)})\) with \(x, y \in C^\alpha\), \(x_{ab}^{(2)}, y_{ab}^{(2)} = O(2\alpha), x_{ab}^{(3)}, y_{ab}^{(3)} = O(3\alpha)\). Put
\[
A_{ab} = \begin{bmatrix} x_{ab} & 0 \\ y_{ab} & 0 \end{bmatrix}, \quad A_{ab}^{(2)} = \begin{bmatrix} x_{ab}^{(2)} & 0 \\ y_{ab}^{(2)} & 0 \end{bmatrix}, \quad A_{ab}^{(3)} = \begin{bmatrix} x_{ab}^{(3)} & 0 \\ y_{ab}^{(3)} & 0 \end{bmatrix}
\]

We suppose that
\[
\delta(A^{(2)}(a, b, c)) = A_{ac} \cdot A_{cb}
\]
\[
\delta(A^{(3)}(a, b, c)) = A_{ac}^{(2)} \cdot A_{cb} + A_{ac} \cdot A_{cb}^{(2)}
\]

As above we put
\[
y_{ab}^{(2)} = \int_{a}^{b} y_{at} \cdot dx_t, \quad y_{ab}^{(3)} = \int_{a}^{b} y_{at}^{(2)} \cdot dx_t
\]

Besides, suppose that there exists two functions \(\xi_{ab} = O(3\alpha)\) and \(\eta_{ab} = O(3\alpha)\) such that
\[
\delta \xi(a, b, c) = (x_{ac})^{(2)} \cdot x_{cb} + \int_{c}^{b} x_{ac} \cdot x_{ct} \cdot dx_t + \int_{c}^{b} x_{ct} \cdot x_{ac} \cdot dx_t
\]
\[
\delta \eta(a, b, c) = (y_{ac})^{(2)} \cdot x_{cb} + \int_{c}^{b} y_{ac} \cdot y_{ct} \cdot dx_t + \int_{c}^{b} y_{ct} \cdot y_{ac} \cdot dx_t
\]

We put
\[
\xi_{ab} = \int_{a}^{b} (x_{at})^{2} \cdot dx_t, \quad \eta_{ab} = \int_{a}^{b} (y_{at})^{2} \cdot dx_t
\]
The quadruple \( Y = (y, y^{(2)}, y^{(3)}, \eta) \) is an enriched path over \( X = (x, x^{(2)}, x^{(3)}, \xi) \).

If \( f \) is a map, put

\[
\mu(a, b) = f(y_a) \cdot x_{ab} + \int_a^b \nabla f(y_t) y_{at} \cdot dx_t + \frac{1}{2} \int_a^b \nabla^2 f(y_a) y_{at}^2 \cdot dx_t
\]

If \( \nabla^2 f \) is Lipschitz, then (cf. [*]) we get an inequality

\[
|\delta \mu(a, b, c)| \leq \text{Cst} |b - a|^{4\alpha}
\]

Hence the function \( \varphi \) associated with \( \mu \) by the sewing lemma is called the integral of \( f \) with respect to \( X \) and writes

\[
\varphi_{ab} = \int_a^b f(y_t) \cdot dx_t
\]

**Differential equations (unicity).**

For \( \alpha > 1/2 \), resp. \( \alpha > 1/3 \), resp. \( \alpha > 1/4 \) and an enriched path \( X = (x, etc..) \), consider a path \( Y = (y, etc..) \) over \( X \), which satifies the equation

\[
y_{ab} = \int_a^b f(y_t) \cdot dx_t, \quad y_0 \text{ being given} \quad (1)
\]

where \( \cdot \) is a contracted tensor product.

For \( \alpha > 1/3 \), the equation (1) has solutions for sufficiently regular \( f \), as is well known ([***]). For \( \alpha > 1/4 \) a solution shall be given in a forthcoming paper.

1°. **Unicity. Case \( \alpha > 1/2 \).**

Suppose \( f \) is \( C^{1,1} \) with values in the convenient space. There exists \( F \) Lipschitz such that \( f(y) - f(\tilde{y}) = F(y, \tilde{y}) \cdot (y - \tilde{y}) \).

Let \( y \) and \( \tilde{y} \) be two paths which are solution of (1). Put \( u_t = y_t - \tilde{y}_t \), we get \( u_0 = 0 \) and

\[
u_{ab} = \int_a^b \varphi_t \cdot u_t \cdot dx_t = \int_a^b u_t \cdot dv_t, \quad \text{with} \quad v_t = \int_0^t \varphi_s \cdot dx_s
\]

where \( \varphi_t = F(y_t, \tilde{y}_t) \) and \( v_t \) are \( C^\alpha \). Then we get in the Young sense

\[
u_{ab} = \int_a^b u_t \cdot dv_t
\]
By iteration we put \( u_{ab}^1 = u_{ab} \),

\[
u_{ab}^{(n+1)} = \int_a^b u_{t}^{(n)} \cdot dv_t
\]

Applying theorem (*) of [†], we get

\[|u_{ab}| \leq Kc^n|b - a|^{n\alpha}/(n!)^\beta\]

where \( 0 < \beta < \alpha \), and \( c, K > 0 \). Hence \( u = 0 \) and \( y = \tilde{y} \).

2°. Unicity. Case \( \alpha > 1/3 \).

Suppose \( f \) is \( C^{2,1} \) with values in the convenient space. There exists \( F \in C^{1,1} \) such that \( f(y) - f(\tilde{y}) = F(y, \tilde{y}) \cdot (y - \tilde{y}) \).

Put \( u_t = y_t - \tilde{y}_t \), and

\[
\varphi_t = F(y_t, \tilde{y}_t), \quad \text{and} \quad \varphi_{ab}^{(2)} = \int_a^b [F(y_t, \tilde{y}_t) - F(y_a, \tilde{y}_a)] \cdot dx_t
\]

Put

\[
v_{ab} = \int_a^b \varphi_t \cdot dx_t = \varphi_a \cdot x_{ab} + \varphi_{ab}^{(2)}
\]

and

\[
\mu(a, b) = \varphi_a \int_a^b x_{at} \cdot \varphi_t \cdot dx_t
\]

We get

\[
\delta \mu(a, b, c) = \varphi_a \cdot x_{ac} \cdot v_{cb} + O(3\alpha) = v_{ac} \cdot v_{cb} + O(3\alpha) = \delta(v_a \cdot v_{ab}) + O(3\alpha)
\]

Hence by the sewing lemma, there exists a function \( \psi(t) \) such that

\[
\mu(a, b) + v_a \cdot v_{ab} - \psi_{ab} = O(3\alpha)
\]

Put \( \hat{v}_{ab}^{(2)} = \psi_{ab} - v_a \cdot v_{ab} = O(2\alpha) \), then the couple \((v, \hat{v}_{ab}^{(2)})\) is an enriched path.

1 Proposition :

\[
u_{ab} = \int_a^b u_t \cdot dv_t
\]

where the integral is taken with respect to the enriched path \((v, \hat{v}_{ab}^{(2)})\).

Démonstration : Put \( \mu(a, b) = u_{ab} - u_a \cdot v_{ab} \). We see that \( \mu(a, b) = O(2\alpha) \). Moreover

\[
\delta \mu(a, b, c) = u_{ac} \cdot v_{cb}
\]
so that \( \hat{u}_{ab}^{(2)} = \mu(a, b) \) is an enrichment of \( u \) with respect to \( (v, \tilde{v}^{(2)}) \). Hence the integral is worth \( u_a \bullet v_{ab} + \hat{u}_{ab}^{(2)} \) which is \( u_{ab} \).

2 Théorème (unicity) : \( y = \tilde{y} \).

Démonstration : By iteration, we get

\[
\hat{u}_{0t}^{(n)} = \int_0^t \hat{u}_{0s}^{(n-1)} \bullet dv_s = u_t \quad \text{integral with respect to } (v, \tilde{v}^{(2)})
\]

By theorem [*] of [*], one has for some \( \beta < \alpha, K, c \) and every \( n \)

\[
|u_t| \leq K c^{n} t^{n\alpha} / (n!)^\beta
\]

so that \( u = 0 \) and \( y = \tilde{y} \).

3°. Unicity. Case \( \alpha > 1/4 \).

Suppose \( f \) is \( C^{3,1} \) with values in the convenient space. There exists \( F \in C^{2,1} \) such that \( f(y) - f(\tilde{y}) = F(y, \tilde{y}) \bullet (y - \tilde{y}) \).

Put \( u_t = y_t - \tilde{y}_t \), and thanks to [*]

\[
\varphi_t = F(y_t, \tilde{y}_t), \quad \text{and} \quad \varphi_{ab}^{(2)} = \int_a^b [F(y_t, \tilde{y}_t) - F(y_a, \tilde{y}_a)] \bullet dx_t
\]

\[
\varphi_{ab}^{(3)} = \int_a^b \varphi_{at}^{(2)} \bullet dx_t
\]

Put

\[
v_{ab} = \int_a^b \varphi_t \bullet dx_t = \varphi_a \bullet x_{ab} + \varphi_{ab}^{(2)}
\]

We want to define \( \hat{v}^{(2)}, \hat{v}^{(3)} \) and \( w \) such that \( V = (v, \hat{v}^{(2)}, \hat{v}^{(3)}, w) \) is an enriched path. First, put

\[
\mu(a, b) = \varphi_a \bullet \int_a^b x_{at} \bullet \varphi_t \bullet dx_t + \int_a^b \varphi_{at}^{(2)} \bullet \varphi_a \bullet dx_t = A_{ab} + B_{ab}
\]

Observe that all the written integrals make sense (theorem 3.2 of [*]). We get

\[
\delta A(a, b, c) = \varphi_a \bullet x_{ac} \bullet v_{cb} - \varphi_{ac} \bullet \varphi_{cb} \bullet \varphi_c + O(4\alpha)
\]

\[
\delta B(a, b, c) = \varphi_{ac} \bullet \varphi_c \bullet x_{cb} + \varphi_{ac} \bullet \varphi_{cb} \bullet \varphi_c + O(4\alpha)
\]

then

\[
\delta \mu = \delta A + \delta B = \varphi_a \bullet x_{ac} \bullet v_{cb} + \varphi_{ac} \bullet \varphi_c \bullet x_{cb} + O(4\alpha)
\]
δµ = (ϕ_a • x_{ac} + \varphi_a^{(2)}) • v_{cb} + O(4\alpha) = v_{ac} • v_{cb} + O(4\alpha)

As above, there exists by the sewing lemma a function Φ(t) such that µ(a, b) + v_a • v_{cb} - Φ_{ab} = O(4\alpha). Then we put

\hat{v}_{ab}^{(2)} = Φ_{ab} - v_a • v_{ab} = O(2\alpha)

Next, put

ν(a, b) = \varphi_a • \int_a^b \left[ \int_a^t x_{as} • ϕ_a • dx_s \right] • ϕ_a • dx_t

This iterated integral makes sense (definition of x^{(2)}).

δν(a, b, c) = \varphi_a • x_{ac} • ϕ_c • x_{cb} + \varphi_a • x_{ac} • ϕ_c • x_{cb}^{(2)} • ϕ_c + O(4\alpha)

Noting that \hat{v}_{ab}^{(2)} = ϕ_a • x_{ab} • ϕ_a + O(3\alpha) we get

δν(a, b, c) = \hat{v}_{ab}^{(2)} • v_{cb} + v_{ac} • \hat{v}_{cb}^{(2)} + O(4\alpha)

δ(\nu • \hat{v}_{ab}^{(2)} + Φ_{0a} • v_{ab})(a, b, c) = -\hat{v}_{ac}^{(2)} • v_{cb} - v_{ac} • \hat{v}_{cb}

Hence by the sewing lemma, there exists a function Ψ(t) such that

ν(a, b) + (v_a • \hat{v}_{ab}^{(2)} + Φ_{0a} • v_{ab}) - Ψ_{ab} = O(4\alpha)

Then we put

\hat{v}_{ab}^{(3)} = Ψ_{ab} - (v_a • \hat{v}_{ab}^{(2)} + Φ_{0a} • v_{ab}) = O(3\alpha)

Finally put

ρ(a, b) = \varphi_a • \int_a^b x_{at} • ϕ_a • x_{at} • ϕ_a • dx_t

δρ(a, b, c) = (ϕ_a • x_{ac})^2 • v_{cb} + ϕ_a • x_{ac} • ϕ_c • x_{cb} • ϕ_c + ϕ_c • x_{cb}^{(2)} • ϕ_c • x_{ac} • ϕ_c

δρ(a, b, c) = v_{ac}^2 • v_{cb} + v_{ac} • \hat{v}_{cb}^{(2)} + \int_b^c S(v_{ac} • v_{cs} • dv_s) + O(4\alpha)

Where

S(u • v • w) = v • u • w

Then we have

δρ(a, b, c) = v_{ac}^2 • v_{cb} + v_{ac} • \hat{v}_{cb}^{(2)} + \int_c^b S(v_{ac} • v_{cs} • dv_s) + O(4\alpha)

Now

δ[(v_a)^2 • v_{ab} - \int_a^b v_t • v_a • dv_t - \int_a^b v_a • v_t • dv_t](a, b, c) =
\[ v_{ac}^2 \cdot v_{cb} + v_{ac} \cdot \hat{v}_{cb}^{(2)} + \int_c^b S(v_{ac} \cdot v_{cs} \cdot dv_s) \]

Then there exists a function \( \Theta \) such that

\[ \rho(a, b) - (v_a)^2 \cdot v_{ab} + \int_a^b v_t \cdot v_a \cdot dv_t + \int_a^b v_a \cdot v_t \cdot dv_t - \Theta_{ab} = O(4\alpha) \]

Put

\[ w_{ab} = \Theta - (v_a)^2 \cdot v_{ab} - \int_a^b v_t \cdot v_a \cdot dv_t - \int_a^b v_a \cdot v_t \cdot dv_t \]

Symbolically \( w_{ab} = \int_a^b v_{at} \cdot v_{at} \cdot dv_t \) And we obtain an enriched path

\[ V = (v, \hat{v}_{ab}, \hat{v}_{ab}^{(3)}, w_{ab}) \]

3 Proposition :

\[ u_{ab} = \int_a^b u_t \cdot dv_t \]

where the integral is taken with respect to the enriched path \( V \).

Démonstration : Put \( \hat{u}^{(2)}(a, b) = u_{ab} - u_a \cdot v_{ab} \). We see that \( \hat{u}^{(2)}(a, b) = O(2\alpha) \).

Put \( \hat{u}^{(3)}(a, b) = \hat{u}^{(2)}(a, b) - u_a \hat{v}_{ab}^{(2)} \). We see that \( \hat{u}^{(3)}(a, b) = O(3\alpha) \). Put

\[ A_{ab} = \begin{bmatrix} v_{ab} & 0 \\ u_{ab} & 0 \end{bmatrix}, \quad A_{ab}^{(2)} = \begin{bmatrix} \hat{v}_{ab}^{(2)} & 0 \\ \hat{u}_{ab}^{(2)} & 0 \end{bmatrix}, \quad A_{ab}^{(3)} = \begin{bmatrix} \hat{v}_{ab}^{(3)} & 0 \\ \hat{u}_{ab}^{(3)} & 0 \end{bmatrix} \]

Then we get a matrix enriched path. As in the case \( \alpha > 1/3 \) we define \( A^{(n)} \) and deduce the inequalities by \([*]\]

\[ |\hat{u}^{(n)}| \leq K e^{n \cdot t^{3\alpha} / (n!)^3} \]

By induction, we also get \( \hat{u}_{0t}^{(n)} = u_{0t}, \) so that \( u = 0 \).

Conclusion

To the end, we give an example for \( 1 > \alpha > 1/3 \) where there is unicity for \( y \) but not for \( (y, y^2) \). Take \( x_t = t, \ x_{ab}^{(2)} = (b - a)^2 / 2, \) and the equation

\[ y_t = y_0 + \int_0^t 0. dt \]  \hspace{1cm} (18)

If \( \varphi \) is a function belonging to \( C^{(2\alpha)} \), put \( y_{ab}^{(2)} = \varphi(b) - \varphi(a) \). Then \( (y, y^{(2)}) \) is an enriched path over \( (x, x^{(2)}) \), and is a solution of (18). As \( \varphi \) is arbitrary, there is no unicity.
REFERENCES

