HAL
open science

Relationship Between Granger Non-Causality and Network Graph of State-Space Representations

Monika Jozsa, Mihaly Petreczky, M. Kanat Camlibel

To cite this version:

Monika Jozsa, Mihaly Petreczky, M. Kanat Camlibel. Relationship Between Granger Non-Causality and Network Graph of State-Space Representations. IEEE Transactions on Automatic Control, 2018, 10.1109/TAC.2018.2832469 . hal-01931151

HAL Id: hal-01931151

https://hal.science/hal-01931151v1

Submitted on 2 Sep 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Relationship between Granger non-causality and network graph of state-space representations

Mónika Józsa, Mihály Petreczky and M. Kanat Camlibel

***Abstract*—The goal of this paper is to explore the relationship between the network graph of a state-space representation of an observed process and the causal relations among the components of that process. We will show that the existence of a linear time-invariant state-space representation, with its network graph being the star graph, is equivalent to (conditional) Granger non-causal relations among the components of the output process. Granger non-causality is a statistical concept, which applies to arbitrary processes and does not depend on the representation of the process. That is, we relate intrinsic properties of a process with the network graph of its state-space representations.**

I. Introduction

Complex dynamical systems, arising as networks of several subsystems, appear in fields ranging from cyber physical systems to systems biology and neuroscience. However, the relationship between the network topology and the observed behavior of the system is not straightforward. By the network topology, or network graph, we mean the graph whose nodes correspond to the subsystems of the network, and whose edges correspond to interactions among subsystems. More specifically, there is an edge from one node to another, if the state and noise process of the subsystem corresponding to the source node serve as an input to the subsystem which corresponds to the target node. While the existence of a certain interconnection in the network is expected to show up in the observed behavior, in principle, different systems with completely different network topology can exhibit the same observed behavior. The results of this paper show that for a specific class of dynamical systems, the network structure and certain observed behaviors are equivalent. The results include algorithms for the construction of these dynamical systems based on the observed output process.

Contribution: We consider *linear time-invariant state-space* (*LTI-SS*) representations. Within LTI-SS representations we focus on stochastic representations where the observed process is stochastic. For this reason, unless stated otherwise, *the term LTI-SS representation means stochastic LTI-SS representation*. Our focus is on LTI-SS representations with the *star graph* as their network graph. A star graph is a tree graph which has precisely one root, and all its other nodes are leaves. We then associate the existence of an LTI-SS representation having the star graph as its network graph to non-causal relations among the components of its output process. For this, we use the well established notion of Granger (non-)causality.

M. Józsa is with IMT Lille Douai, Univ. Lille, Unité de Recherche Informatique Automatique, F-59000 Lille, France and Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AK Groningen, The Netherlands (e-mail: monika.jozsa@imt-lille-douai.fr)

M. Petreczky is with Univ. Lille, CNRS, Centrale Lille, UMR 9189 - CRIStAL - Centre de Recherche en Informatique Signal et Automatique de Lille, F-59000 Lille, France, (e-mail: mihaly.petreczky@centralelille.fr)

M. K. Camlibel is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AK Groningen, The Netherlands (e-mail: m.k.camlibel@rug.nl)

This work was partially funded by ESTIREZ project of the region Hauts-de-France, France.

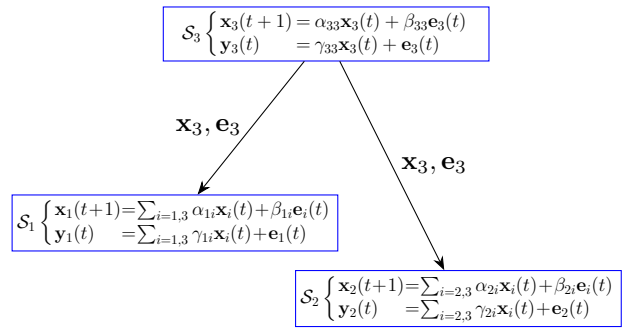

Fig. 1: LTI-SS representation of $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$ in coordinated form. Subsystem $\mathcal{S}_3$ represents the coordinator while $\mathcal{S}_1$ and $\mathcal{S}_2$ represent the agents.

More precisely, given a partition $\mathbf{y} = \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T\right]^T$ of a stochastic process $\mathbf{y}$ and an LTI-SS representation $\mathcal{S}$ of $\mathbf{y}$, we consider the network of subsystems $\mathcal{S}_i$, $i = 1, \ldots, n$ of the representation $\mathcal{S}$, where $\mathcal{S}_i$ generates the component $\mathbf{y}_i$. The network graph has nodes $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ and there is edge from $\mathcal{S}_i$ to $\mathcal{S}_j$, if the state and noise process of $\mathcal{S}_i$ serve as an input of $\mathcal{S}_j$. In fact, if the system matrices of $\mathcal{S}$ are A, B, C and D (see Definition 2 in Section II), then an edge $(\mathcal{S}_i, \mathcal{S}_j)$ in the network graph corresponds to non-zero ji blocks, whereas the lack of this edge corresponds to zero ji blocks in the matrices A, B, C, D with appropriate block dimensions. As we mentioned before, we restrict ourselves to network graphs having star graph structure. Intuitively, one can think of the subsystem which corresponds to the root node as representing a coordinator, while the other nodes represent agents. Information can flow from the coordinator to the agents, but there is no information exchange among the agents themselves. For this reason, we call such LTI-SS representations *LTI-SS representations in coordinated form*. Fig. 1 illustrates the case when $n = 3$. In the case of $n = 2$, this is just a cascade interconnection of two subsystems.

Conditional Granger non-causality is a general form of the classical notion, Granger non-causality [1], originating in econometrics: informally a stochastic process $\mathbf{y}_1$ *conditionally does not Granger cause* $\mathbf{y}_2$ with respect to $\mathbf{y}_3$ if the knowledge of the past values of $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ does not yield a more accurate prediction of the future values of $\mathbf{y}_2$ than the knowledge of the past values of only $\mathbf{y}_2$ and $\mathbf{y}_3$.

The main result of the paper shows that a process $\mathbf{y} = \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T\right]^T$ admits a specific LTI-SS representation in coordinated form with the coordinator being the subsystem representing $\mathbf{y}_n$, if and only if $\mathbf{y}_i$ does not Granger cause $\mathbf{y}_n$ and $\mathbf{y}_i$ conditionally does not Granger cause $\mathbf{y}_j$ with respect to $\mathbf{y}_n$ for all $i, j = 1, \ldots, n-1$, $i \neq j$. For $n = 2$, this result gives a characterization of Granger non-causality in terms of the existence of an LTI-SS representation. Besides this formal equivalence, if the appropriate (conditional) Granger causalities hold in a process $\mathbf{y}$, we provide algorithms to compute the LTI-SS representation of $\mathbf{y}$ in coordinated form.

Motivation: These contributions are useful for reverse engineering of the network graph of dynamical systems. In addition, they can be relevant for distributed estimation/control, and for structure preserving model reduction.

Reverse engineering of the network topology: By reverse engineering of the network topology we mean finding out how various subsystems of a dynamical system interact with each other based on observed data. This problem arises in several domains such as systems biology, neuroscience, smart grids, etc. [2]–[10]. To solve this problem, we need to understand when the observed behavior can be realized by a system with a specific network topology.

This is important for applications in neuroscience [7], [8], [11], [12], where the goal is to reverse engineer the interactions between brain regions using fMRI, EEG, MEG, etc... data. For this purpose, both Granger causality based methods [12] and state-space based methods [11] were used. In the former case, the presence of an interaction was identified with the presence of Granger causality between the outputs associated with various brain regions. In the latter case, the presence of an interaction was interpreted as the presence of an edge in the network graph of a state-space representation, whose parameters were identified from data. However, the formal relationship between these methods was not always clear. This has lead to a lively debate regarding the advantages/disadvantages of both methods [8], [13], [14]. The results of this paper imply that the network graph of a specific LTI-SS representation defines causal relations in the output process and the causal properties of the output process restricts the network graph of a potential LTI-SS representation. In fact, considering conditional Granger non-causality and LTI-SS representations in coordinated form, the two approaches are formally equivalent and produce the same outcome. This opens up the possibility of reconciling both approaches.

The cited applications [2]–[8], [11] often use nonlinear state-space representations with inputs. For those state-space representations, there exist no easy methods for checking Granger non-causality. Hence, our general goal is to translate Granger non-causality to properties of nonlinear state-space representations with inputs. For this, first autonomous linear state-space representations has to be completely understood. This paper accomplishes this first step.

Distributed estimation/control: For the design of interconnected systems, choosing alternative network graphs realizing the same functionality can be beneficial. For example, for deterministic coordinated LTI-SS systems with inputs [15], [16], several control problems, such as stabilization can be solved in a distributed manner: in order to stabilize the coordinator, no knowledge of the state of the agents is required, and in order to stabilize each agent, only the state of this agent and of the coordinator is needed. In the case of autonomous LTI-SS systems in coordinated form, their network graph allows for distributed estimation of their states. Moreover, the proposed algorithms of this paper open up the possibility of distributed parameter estimation; for calculating the subsystem that generates the coordinator only the observed process of the coordinator is needed and for calculating the subsystem that generates an agent, only the observed coordinator and that agent is used. That is, the results of this paper provide LTI-SS representations that are suitable for distributed estimation and, if the results are extended to non-autonomous systems (which remains a topic of future work), then for distributed control.

Structure preserving model reduction: The results of the paper could also be of interest for structure preserving model reduction, where the goal is to replace an interconnected model of the system by another, smaller dimensional (in terms of the dimension of states) interconnected model which has the same or similar network graph as the original model, see [17], [18] and [19]. By the methods of this paper, for stochastic LTI-SS representations in coordinated form, one can reduce the order of a subsystem generating the coordinator or generating an agent, preserving the coordinated structure.

Related work: The need to understand the relationship between the observed behavior and the network topology of linear systems is an active research area, see for example [2]–[4], [20]. In [2], [3], [21] the network topology of a deterministic system was defined on components of the observed output, by using the notion of the so-called dynamic structure function. The dynamic structure function is determined by the properties of a class of deterministic LTI-SS representations of that system. That is, contrary to the current paper, the cited papers did not aim at relating network topology defined directly on the output with network topology defined based on state-space representations; rather, they tried to combine these two approaches. Moreover, they dealt with deterministic systems. In [20] the relationship between dynamic structure function from [2], [3] and Granger non-causality was investigated. In contrast to the current paper, [20] does not aim at establishing a relationship between the state-space defined network topology and Granger non-causality.

The study of causal relations between stochastic processes is an established research topic in econometrics, neuroscience and control theory. This relationship can be characterized in terms of the network graph of transfer function representations (see [22]–[26] and the references therein). If there is one agent, then our results can be viewed as a counterpart of the cited papers for LTI-SS representations. Granger causality for LTI-SS representations was studied by using a transfer function approach in [27]. In contrast to [27] we give a characterization for Granger non-causality by constructing an LTI-SS representation in a certain form.

The papers [28]–[30] are the closest ones to the present paper. They provide necessary and sufficient conditions for the existence of LTI-SS representations in the so-called conditional orthogonal form. Conditionally orthogonal LTI-SS

representations form a specific subclass of LTI-SS representations in coordinated form where the noise process is assumed to have block diagonal covariance matrix. The conditions of [28]–[30] for the existence of such systems are much stronger than the conditions proposed in this paper. Note that [28]–[30] did not provide algorithms to calculate the representations and did not deal with the so-called non-coercive processes.

Deterministic LTI-SS representations in coordinated form were studied in [16], [31], [32]. In [16] and [31], a general method was presented to transform a system into coordinated form. In [16] and [33], Gaussian coordinated systems were studied and their LQG control. However, the cited papers did not relate the coordinated system structure to properties of the observed process whereas we formally relate linear stochastic (not necessarily Gaussian) systems in coordinated form to causal properties of the output process.

Certain results of the current paper were presented in [34]. More precisely, in [34] the equivalence (iii) $\Leftrightarrow$ (iv) of Theorem 2 in the present paper is stated, without proof. That equivalence essentially states that LTI-SS representations in coordinated form can be characterized by (conditional) Granger non-causal conditions on the output process. All the other results of the present paper are new to the best of our knowledge and were not published before. Furthermore, here we include detailed proofs, explicit algorithms and a numerical example. In [35], the problem of relating transfer functions and Granger causality was studied. Accordingly, [35] does not deal with state-space representations. Moreover, [35] restricts attention to coercive processes. Finally, the conference paper [36] deals with LTI-SS representations having transitive acyclic network graphs. It shows that non-causal properties imply the existence of an LTI-SS representation with transitive acyclic network graph, thus only shows one-directional implication and not the reverse. Also, it does not cope with the minimality of the representations and does not include detailed proofs. The results of [36] rely on the present paper.

The structure of the paper is as follows: In Section II, we introduce the notation and terminology and discuss some background material on linear stochastic realization theory. In Section III, we present the results for the simple case when besides the coordinator there is one agent. The more general coordinated systems, when besides the coordinator there are multiple agents, is discussed in Section IV. In Section V, the results are illustrated by an example. Finally, after concluding our paper we provide the proofs of the results in Appendix.

II. Preliminaries

In this paper we consider discrete-time processes, whose values are vectors with real entries. The discrete-time axis is the set of integers $\mathbb{Z}$. The random variable of a process $\mathbf{z}$ at time t is denoted by $\mathbf{z}(t)$. If $\mathbf{z}(t)$ is k-dimensional (for all $t \in \mathbb{Z}$), then we write $\mathbf{z} \in \mathbb{R}^k$ and we call $k = \dim(\mathbf{z})$ the dimension of $\mathbf{z}$. The $n \times n$ identity matrix is denoted by I_n or by I when its dimension is clear from the context.

A. Hilbert spaces of stochastic processes

We denote by $\mathcal{H}$ the Hilbert space of zero-mean square-integrable random variables, where the inner product between two random variables $\mathbf{y}, \mathbf{z}$ is the covariance matrix $E[\mathbf{y}\mathbf{z}^T]$. The Hilbert space generated by a set $U \subset \mathcal{H}$ is the smallest (w.r.t. set inclusion) closed subspace of $\mathcal{H}$ which contains U. Consider a zero-mean square-integrable process $\mathbf{z} \in \mathbb{R}^k$. Then $\mathcal{H}^{\mathbf{z}}_{t-}$, $\mathcal{H}^{\mathbf{z}}_{t+}$, $\mathcal{H}^{\mathbf{z}}_{t}$, $t \in \mathbb{Z}$ are the Hilbert spaces generated by the sets $\{\ell^T\mathbf{z}(s) \mid s \in \mathbb{Z}, s < t, \ell \in \mathbb{R}^k\}$, $\{\ell^T\mathbf{z}(s) \mid s \in \mathbb{Z}, s \geq t, \ell \in \mathbb{R}^k\}$, and $\{\ell^T\mathbf{z}(t) | \ell \in \mathbb{R}^k\}$, respectively. If $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are vector valued processes, then $\mathbf{z} = \left[\mathbf{z}_1^T, \ldots, \mathbf{z}_n^T\right]^T$ denotes the process defined by $\mathbf{z}(t) = \left[\mathbf{z}_1^T(t), \ldots, \mathbf{z}_n^T(t)\right]^T$, $t \in \mathbb{Z}$.

If $\mathbf{z}(t) \in \mathcal{H}$ is a random variable and U is a closed subspace in $\mathcal{H}$, then we denote by $E_l[\mathbf{z}(t)\,|\,U]$ the orthogonal projection of $\mathbf{z}(t)$ onto U. The orthogonal projection onto U of a random variable $\mathbf{z}(t) = [\mathbf{z}_1(t), \ldots, \mathbf{z}_k(t)]^T$ taking values in $\mathbb{R}^k$ is denoted by $E_l[\mathbf{z}(t)|U]$ and defined element-wise as $E_l[\mathbf{z}(t)|U] := [\hat{\mathbf{z}}_1(t), \ldots, \hat{\mathbf{z}}_k(t)]^T$, where $\hat{\mathbf{z}}_i(t) = E_l[\mathbf{z}_i(t)|U]$, $i = 1, \ldots, k$. That is, $E_l[\mathbf{z}(t)|U]$ is the random variable with values in $\mathbb{R}^k$ obtained by projecting the coordinates of $\mathbf{z}(t)$ onto U. Accordingly, the orthogonality of a multidimensional random variable to a closed subspace in $\mathcal{H}$ is meant element-wise. The orthogonal projection of a closed subspace $U \subseteq \mathcal{H}$ onto a closed subspace $V \subseteq \mathcal{H}$ is written by $E_l[U|V] := \{E_l[u|V], u \in U\}$. For jointly Gaussian processes $\mathbf{y}$ and $\mathbf{z}$ the orthogonal projection $E_l[\mathbf{y}(t)|\mathcal{H}^{\mathbf{z}}_t]$ is equivalent to the conditional expectation of $\mathbf{y}(t)$ given $\mathbf{z}(t)$.

Below we define the class of processes we will work with.

Definition 1 (ZMSIR)**.** A stochastic process is called zero-mean square-integrable with rational spectrum (abbreviated by ZMSIR) if it is weakly-stationary, square-integrable, zero-mean, full rank, purely non-deterministic, and its spectral density is a proper rational function.

B. Review of stochastic realization theory

The results of this paper are based on linear stochastic realization theory, therefore, we present a brief overview of basic results in the field (see [37]). To begin with, we define the term LTI-SS representation for the class of ZMSIR processes.

Definition 2 (LTI-SS representation)**.** A stochastic LTI-SS representation is a stochastic dynamical system of the form

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + B\mathbf{v}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{v}(t) \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ for $n \geq 0$, $m, p > 0$ and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^p$, $\mathbf{v} \in \mathbb{R}^m$ are ZMSIR processes. The processes $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{v}$ are called state, output and noise process, respectively. Furthermore, we require that A is stable (all its eigenvalues are inside the open unit circle) and that for any $t, k \in \mathbb{Z}$, $k \geq 0$, $E[\mathbf{v}(t)\mathbf{v}^T(t-k-1)] = 0$, $E[\mathbf{v}(t)\mathbf{x}^T(t-k)] = 0$, i.e., $\mathbf{v}(t)$ is white noise and uncorrelated with $\mathbf{x}(t-k)$. An *LTI-SS representation of a given process* $\mathbf{y}$ is an LTI-SS representation with output process $\mathbf{y}$.

In (1) the state process $\mathbf{x}$ is uniquely determined by the noise process $\mathbf{v}$ and the system matrices A and B so that $\mathbf{x}(t) = \sum_{k=0}^{\infty} A^k B\mathbf{v}(t-k)$, where the convergence of the infinite sum is understood in the mean square sense. On this basis (1) is referred to as LTI-SS representation

$(A, B, C, D, \mathbf{v}, \mathbf{y})$ or LTI-SS representation $(A, B, C, D, \mathbf{v})$ of $\mathbf{y}$. Following the classical terminology, we call the dimension of the state process the *dimension* of (1). Also, an LTI-SS representation $(A, B, C, D, \mathbf{v})$ of $\mathbf{y}$ is called *minimal* if it has minimal dimension among all the LTI-SS representations of $\mathbf{y}$. Notice that in Definition 2 we allow (1) to have zero dimension which corresponds to representations of white noise processes ($\mathbf{y} = D\mathbf{v}$). Whenever we say that $(A, B, C, D, \mathbf{v})$ is a minimal LTI-SS representation of a white noise process it means that A, B, C are absent (or they are zero by zero empty matrices). Zero-dimensional representations are considered to be minimal, observable and controllable.

Stochastic LTI-SS representations of a given process $\mathbf{y}$ are strongly related to deterministic LTI-SS realizations of the covariance sequence $\{\Lambda_k^{\mathbf{y}} := E[\mathbf{y}(t+k)\mathbf{y}^T(t)]\}_{k=0}^{\infty}$, see [37, Section 6]. Below we briefly sketch this relationship, as it plays an important role in deriving the results of the paper. Consider an LTI-SS representation $(A, B, C, D, \mathbf{v})$ of $\mathbf{y}$ and denote the (time-independent[1]) noise variance matrix by $\Lambda_0^{\mathbf{v}} = E[\mathbf{v}(t)\mathbf{v}^T(t)]$. Then, the variance matrix $\Lambda_0^{\mathbf{x}} = E[\mathbf{x}(t)\mathbf{x}^T(t)]$ of the state process $\mathbf{x}$ of $(A, B, C, D, \mathbf{v})$ is the unique symmetric solution of the Lyapunov equation $\Sigma = A\Sigma A^T + B\Lambda_0^{\mathbf{v}} B^T$ and the covariance $G := E[\mathbf{y}(t)\mathbf{x}^T(t+1)]$ satisfies

$$G = C\Lambda_0^{\mathbf{x}} A^T + D\Lambda_0^{\mathbf{v}} B^T. \tag{2}$$

In light of this, the Markov parameters of the deterministic LTI-SS system $(A, G^T, C, \Lambda_0^{\mathbf{v}})$ are equal to the covariances $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$. More precisely,

$$\Lambda_k^{\mathbf{y}} = CA^{k-1}G^T \quad k > 0. \tag{3}$$

Therefore, LTI-SS representations of $\mathbf{y}$ yield deterministic LTI-SS systems whose Markov parameters are the covariances $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$ of $\mathbf{y}$. Conversely, deterministic LTI-SS systems whose Markov parameters are the covariances $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$ yield LTI-SS representations of $\mathbf{y}$. To this end, we use the following terminology: Recall that $\mathcal{H}_{t-}^{\mathbf{y}}$ denotes the Hilbert space generated by $\mathbf{y}(t-k)$, $k > 0$. We call the process

$$\mathbf{e}(t) := \mathbf{y}(t) - E_l[\mathbf{y}(t)|\mathcal{H}_{t-}^{\mathbf{y}}], \ \forall t \in \mathbb{Z}$$

the *(forward) innovation process* of $\mathbf{y}$. Assume now that $(A, G^T, C, \Lambda_0^{\mathbf{y}})$ is a stable minimal deterministic LTI-SS system whose Markov parameters are the covariances of $\mathbf{y}$, i.e., (3) holds. Let $\Sigma_{\mathbf{x}}$ be the minimal symmetric solution[2] of the algebraic Riccati equation

$$\Sigma = A\Sigma A^T + (G^T - A\Sigma C^T)(\Delta(\Sigma))^{-1}(G^T - A\Sigma C^T)^T, \tag{4}$$

where $\Delta(\Sigma) = (\Lambda_0^{\mathbf{y}} - C\Sigma C^T)$ and set K as

$$K := (G^T - A\Sigma_{\mathbf{x}} C^T)(\Delta(\Sigma_{\mathbf{x}})^{-1}). \tag{5}$$

Proposition 1. *[38, Section 7.7] Let K be as in* (5) *and* $\mathbf{e}$ *be the innovation process of* $\mathbf{y}$*. Then the following LTI-SS representation of* $\mathbf{y}$ *is minimal:*

$$(A, K, C, I, \mathbf{e}). \tag{6}$$

Note that if $\mathbf{x}$ is the state of $(A, K, C, I, \mathbf{e})$, then $\Sigma_{\mathbf{x}} = E[\mathbf{x}(t)\mathbf{x}^T(t)]$, $\Delta(\Sigma_{\mathbf{x}}) = E[\mathbf{e}(t)\mathbf{e}^T(t)]$ and $K = E[\mathbf{x}(t+1)\mathbf{e}^T(t)]E[\mathbf{e}(t)\mathbf{e}^T(t)]^{-1}$ is the gain of the steady-state Kalman filter [37, Section 6.9]. This motivates the following definition:

Definition 3. Let $\mathbf{e}, \mathbf{y} \in \mathbb{R}^p$ be ZMSIR processes and $A \in \mathbb{R}^{n\times n}, K \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times p}$. An LTI-SS representation $(A, K, C, D, \mathbf{e}, \mathbf{y})$ is called *Kalman representation* if $\mathbf{e}$ is the innovation process of $\mathbf{y}$ and $D = I_p$.

A Kalman representation with output process $\mathbf{y}$ is called *Kalman representation of* $\mathbf{y}$. A Kalman representation is minimal, called *minimal Kalman representation*, if it is a minimal LTI-SS representation. The representation in Proposition 1 is a minimal Kalman representation, thus we conclude that

Proposition 2. *Every ZMSIR process* $\mathbf{y}$ *has a minimal Kalman representation.*

In view of the foregoing, we have the following algorithms:

Algorithm 1 Minimal Kalman representation based on output covariances

Input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$: Markov parameters of $\mathbf{y}$

Output $\{A, K, C, \Lambda_0^{\mathbf{e}}\}$: system matrices of (6) and variance of the innovation process of $\mathbf{y}$

Step 1 Define the Hankel and the shifted Hankel matrices

$$H_0 = \begin{bmatrix} \Lambda_1^{\mathbf{y}} & \Lambda_2^{\mathbf{y}} & \cdots & \Lambda_N^{\mathbf{y}} \\ \Lambda_2^{\mathbf{y}} & \Lambda_3^{\mathbf{y}} & \cdots & \Lambda_{N+1}^{\mathbf{y}} \\ \vdots & \vdots & & \vdots \\ \Lambda_N^{\mathbf{y}} & \Lambda_{N+1}^{\mathbf{y}} & \cdots & \Lambda_{2N-1}^{\mathbf{y}} \end{bmatrix} \quad H_1 = \begin{bmatrix} \Lambda_2^{\mathbf{y}} & \Lambda_3^{\mathbf{y}} & \cdots & \Lambda_{N+1}^{\mathbf{y}} \\ \Lambda_3^{\mathbf{y}} & \Lambda_4^{\mathbf{y}} & \cdots & \Lambda_{N+2}^{\mathbf{y}} \\ \vdots & \vdots & & \vdots \\ \Lambda_{N+1}^{\mathbf{y}} & \Lambda_{N+2}^{\mathbf{y}} & \cdots & \Lambda_{2N}^{\mathbf{y}} \end{bmatrix}.$$

Step 2 Calculate the SVD of $H_0 = USV^T$.

Step 3 Let m be such that $\Lambda_0^{\mathbf{y}} \in \mathbb{R}^{m\times m}$ and denote the first m rows of a matrix by $(.)_{1:m}$. Define

$$A := S^{-1/2}U^T H_1 V S^{-1/2}$$
$$C := (US^{1/2})_{1:m} \quad G := (VS^{1/2})_{1:m}$$

Step 4 Find the minimal symmetric solution $\Sigma_{\mathbf{x}}$ of the Riccati equation (4) (see *e.g.*, [38, Section 7.4.2]).

Step 5 Set K as in (5) and define $\Lambda_0^{\mathbf{e}} := \Lambda_0^{\mathbf{y}} - C\Sigma_{\mathbf{x}}C^T$.

Algorithm 2 Minimal Kalman representation based on LTI-SS representation

Input $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \Lambda_0^{\mathbf{v}}\}$: $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ is an LTI-SS representation of $\mathbf{y}$ and $\Lambda_0^{\mathbf{v}} = E[\mathbf{v}(t)\mathbf{v}^T(t)]$

Output $\{A, K, C, \Lambda_0^{\mathbf{e}}\}$: system matrices of (6) and variance of the innovation process of $\mathbf{y}$

Step 1 Find the solution $\Sigma_{\mathbf{x}}$ of the Lyapunov equation $\Sigma = \bar{A}\Sigma\bar{A}^T + \bar{B}\Lambda_0^{\mathbf{v}}\bar{B}^T$.

Step 2 Define $G := \bar{C}\Sigma_{\mathbf{x}}\bar{A}^T + \bar{D}\Lambda_0^{\mathbf{v}}\bar{B}^T$ and calculate the output covariance matrices $\Lambda_k^{\mathbf{y}} := \bar{C}\bar{A}^{k-1}G^T$ for $k = 0, \ldots, 2n$, where n is such that $\bar{A} \in \mathbb{R}^{n\times n}$.

Step 3 Apply Algorithm 1 with input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2n}$ and denote the output by $\{A, K, C, \Lambda_0^{\mathbf{e}}\}$.

Note that Steps 1–3 of Algorithm 1 calculate a minimal deterministic LTI-SS system (A, G^T, C, Λ_0) such that (3) holds

[1] stationarity implies that the (co)variance matrices are time-independent

[2] for any other symmetric solution $\tilde{\Sigma}$, the matrix $\tilde{\Sigma} - \Sigma$ is positive definite

using the classical Kalman-Ho realization algorithm.

Remark 1 (Correctness of Algorithms 1–2). Consider a ZMSIR process $\mathbf{y}$ with covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$ and an LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ of $\mathbf{y}$. Let $\mathbf{e}$ be the innovation process of $\mathbf{y}$ and N be larger than or equal to the dimension of a minimal LTI-SS representation of $\mathbf{y}$. Then it follows from [38, Lemma 7.9, Section 7.7] that if $\{A, K, C, \Lambda_0^{\mathbf{e}}\}$ is the output of Algorithm 1 with input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$, then $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$ and $\Lambda_0^{\mathbf{e}} = E[\mathbf{e}(t)\mathbf{e}^T(t)]$. Likewise, if $\{A, K, C, \Lambda_0^{\mathbf{e}}\}$ is the output of Algorithm 2 with input $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, E[\mathbf{v}(t)\mathbf{v}^T(t)]\}$, then $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$ and $\Lambda_0^{\mathbf{e}} = E[\mathbf{e}(t)\mathbf{e}^T(t)]$.

Remark 2. Algorithms 1 and 2 involve matrix multiplication, inversion, SVD, solving Ricatti and Lyapunov equations. The computational complexity of all involved matrix operations is polynomial in the sizes of the matrices [39]. Also, solving Ricatti and Lyapunov equations is polynomial in the size of the solution matrix [40]. For Algorithm 1, the sizes of the matrices involved are polynomial in the number $2N+1$ and the size p of the output covariances, hence its complexity is polynomial in N and p. By similar reasoning, Algorithm 2 has polynomial complexity in the dimensions of the state, output, and noise processes of the input LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$.

All algorithms in this paper are based on Algorithms 1–2 and, under certain conditions, calculate minimal Kalman representations which have the following useful properties:

Proposition 3. *A Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is minimal if and only if (A, K) is controllable and (A, C) is observable.*

Proposition 3 shows that minimality of a Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$ can be characterized by minimality of the deterministic system (A, K, C, I). In general, the characterization of minimality in LTI-SS representations is more involved, and it is related to the minimality of the deterministic LTI-SS system $(A, G^T, C, \Lambda_0^{\mathbf{y}})$ associated with the stochastic LTI-SS representation (see [37, Corollary 6.5.5]). The next Proposition shows that minimal Kalman representations are *isomorphic*. Isomorphism is defined as follows: two Kalman representations $(A, K, C, I, \mathbf{e})$ and $(\tilde{A}, \tilde{K}, \tilde{C}, I, \mathbf{e})$ of a process $\mathbf{y}$ are isomorphic if there exists a non-singular matrix T such that $A = T\tilde{A}T^{-1}, K = T\tilde{K}$ and $\tilde{C} = CT^{-1}$. Again, in general, the result does not apply for LTI-SS representations. The statement and its proof can be found in [37, Theorem 6.6.1, Section 6.6] with the modification that here the noise process is not normalized.

Proposition 4. *[37, Theorem 6.6.1] If $(A, K, C, I, \mathbf{e})$ and $(\tilde{A}, \tilde{K}, \tilde{C}, I, \mathbf{e})$ are minimal Kalman representations of a process $\mathbf{y}$, then they are isomorphic.*

Lastly, in this paper we use the property of *coercivity*: Recall from [37, Definition 9.4.1] that $\mathbf{y}$ is *coercive* if its spectrum is strictly positive definite on the unit disk. Coercivity of a process $\mathbf{y}$ is further equivalent to the invertibility of any Kalman representation $(A, K, C, I, \mathbf{e})$ of $\mathbf{y}$, i.e., with the existence of $(A - KC)^{-1}$, see [37, Theorem 9.4.2]. It also implies that $\mathbf{e}(t) = \mathbf{y}(t) - \sum_{k=0}^{\infty} C(A-KC)^k K\mathbf{y}(t-k-1)$.

III. Characterization of Granger non-causality

In this section we show that Granger non-causality from $\mathbf{y}_1$ to $\mathbf{y}_2$ is equivalent to the existence of a Kalman representation of the joint process $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ in the so-called *block triangular form*. Besides, we provide an algorithm which calculates this Kalman representation with block triangular matrices in the presence of the above-mentioned Granger non-causality. *Throughout this section, we assume that $\mathbf{y}$ is a ZMSIR process and it admits a partitioning $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$, such that $\mathbf{y}_i \in \mathbb{R}^{r_i}$ for $r_i > 0$, $i = 1, 2$.*

A. Kalman representation in block triangular form

Definition 4. A Kalman representation $(A, K, C, I, \mathbf{e} = [\mathbf{e}_1^T, \mathbf{e}_2^T]^T, \mathbf{y})$, where $\mathbf{e}_i \in \mathbb{R}^{r_i}$, $i = 1, 2$, is called a *Kalman representation in block triangular form*, if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad K = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \quad C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}, \tag{7}$$

where $A_{ij} \in \mathbb{R}^{p_i \times p_j}, K_{ij} \in \mathbb{R}^{p_i \times r_j}, C_{ij} \in \mathbb{R}^{r_i \times p_j}$ and $p_i \geq 0$ for $i, j = 1, 2$. If, in addition, $(A_{22}, K_{22}, C_{22}, I_{r_2}, \mathbf{e}_2)$ is a minimal Kalman representation of $\mathbf{y}_2$, then $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is called a *Kalman representation in causal block triangular form*.

Remark 3. If $p_2 = 0$ in Definition 4, then the block matrices $A_{12}, A_{22}, K_{22}, C_{12}$ and C_{22} are absent, whereas if $p_1 = 0$, then $A_{11}, A_{12}, K_{11}, K_{12}$ and C_{11} are absent. Furthermore, if $p_2 = 0$, then $\mathbf{y}_2 = \mathbf{e}_2$ is a white noise process and if $p_1 = p_2 = 0$, then $\mathbf{y} = \mathbf{e}$ is a white noise process. In both cases, block triangular form implies causal block triangular form. A minimal Kalman representation $(A, K, C, I, \mathbf{e})$ of a white noise process $\mathbf{y}$ has zero dimension (A, K, C are absent) and it is the trivial equation $\mathbf{y} = \mathbf{e}$.

If $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is a Kalman representation in causal block triangular form satisfying (7), then $\mathbf{e}_2$ is the innovation process of $\mathbf{y}_2$. Moreover, the dimensions of the block matrices A_{ij}, K_{ij}, C_{ij}, $i, j = 1, 2$ are uniquely determined by $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$. Indeed the matrices $A_{22} \in \mathbb{R}^{p_2 \times p_2}$, $K_{22} \in \mathbb{R}^{p_2 \times r_2}$, $C_{22} \in \mathbb{R}^{r_2 \times p_2}$, where p_2 is the dimension of a minimal LTI-SS representation of $\mathbf{y}_2$, hence p_2 is determined by $\mathbf{y}_2$. That is, the dimension of a minimal LTI-SS representation of $\mathbf{y}_2$ and $\mathbf{y}$ and the dimensions of $\mathbf{y}_1$ and $\mathbf{y}_2$ determine the dimensions of A_{ij}, K_{ij}, C_{ij}, $i, j = 1, 2$.

Kalman representations in block triangular form can be viewed as a cascade interconnection of two subsystems, see Fig. 2. More precisely, let $(A, K, C, I, \mathbf{e})$ be a Kalman representation of $\mathbf{y}$ in block triangular form satisfying (7) and let $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$ be its state process where $\mathbf{x}_i \in \mathbb{R}^{p_i}$, $i = 1, 2$. Then we can define the dynamical systems $\mathcal{S}_c$ and $\mathcal{S}_a$ below.

$$\mathcal{S}_c \begin{cases} \mathbf{x}_2(t+1) = A_{22}\mathbf{x}_2(t) + K_{22}\mathbf{e}_2(t) \\ \mathbf{y}_2(t) = C_{22}\mathbf{x}_2(t) + \mathbf{e}_2(t) \end{cases} \tag{8}$$

$$\mathcal{S}_a \begin{cases} \mathbf{x}_1(t+1) = \sum_{i=1}^{2}(A_{1i}\mathbf{x}_i(t) + K_{1i}\mathbf{e}_i(t)) \\ \mathbf{y}_1(t) = \sum_{i=1}^{2} C_{1i}\mathbf{x}_i(t)) + \mathbf{e}_2(t) \end{cases} \tag{9}$$

The subsystem $\mathcal{S}_c$, which generates $\mathbf{y}_2$, will be called *coordinator*, and the subsystem $\mathcal{S}_a$, which generates $\mathbf{y}_1$, will be called *agent*. The coordinator sends its state $\mathbf{x}_2$ and noise $\mathbf{e}_2$ to the agent while the agent does not send information to the coordinator. Accordingly, the network graph of $(A, K, C, I, \mathbf{e}, \mathbf{y})$

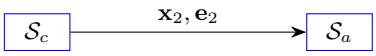

Fig. 2: Network graph of a Kalman representation in block triangular form: $\mathcal{S}_c$ is the coordinator (8), $\mathcal{S}_a$ is the agent (9).

is the two-node star graph with $\mathcal{S}_c$ being the root node and $\mathcal{S}_a$ being the leave.

Motivation for Kalman representations in block triangular form: If we considered an LTI-SS representation of $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ without requiring it to be a Kalman representation, then in general the subsystem $\mathcal{S}_1$ with output $\mathbf{y}_1$ and $\mathcal{S}_2$ with output $\mathbf{y}_2$ could change information in both direction via the noise process. The fact that we require the LTI-SS representation to be a Kalman representation in block triangular form implies that this cannot be the case: If $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$, then the second noise component $\mathbf{e}_2$ is the innovation process of $\mathbf{y}_2$ and thus $\mathbf{e}_2$ and $\mathbf{x}_2$ depend only on the past and present values of $\mathbf{y}_2$. In contrast, the first noise and state components $\mathbf{e}_1$ and $\mathbf{x}_1$ depend on the past and present values of both $\mathbf{y}_1$ and $\mathbf{y}_2$. This ensures that indeed, if $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$, then a Kalman representation in block triangular form means that there is no communication from the agent to the coordinator.

Kalman representations in causal block triangular form guarantee *the subsystem which corresponds to the coordinator to be minimal* and thus unique up to isomorphism (see Proposition 4). An advantage of minimality is that it implies observability, and hence the state $\mathbf{x}_2$ can be estimated from $\mathbf{y}_2$. This opens up the perspective of distributed estimation, and possibly, with the future inclusions of inputs, of distributed control. An example for not requiring the subsystem of the coordinator to be minimal: Let us consider $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and choose the coordinator to be the subsystem with system matrices $A_{22} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.7 \end{bmatrix}$, $K_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $C_{22} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. In this case we obtain a Kalman representation in block triangular form, but not causal block triangular form. What happens is that the coordinator contains dynamics which could also be made part of the agent. For this example, it is not enough to know $\mathbf{y}_2$ to estimate the state of the coordinator, for that we have to use the output $\mathbf{y}_1$ of the agent. Imposing minimality on the coordinator avoids such degenerate cases.

B. Characterization of Granger non-causality by Kalman representations in block triangular form

In this section we show that the existence of a Kalman representation in causal block triangular form characterizes Granger non-causality. The next definition is a particular case of the concept of causality between stochastic processes defined in [1], if the latter is applied to ZMSIR processes, and if, using the terminology of [1], there is no external process.

Definition 5 (Granger non-causality)**.** Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$. We say that $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$ if for all $t, k \in \mathbb{Z}$, $k \geq 0$

$$E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2}] = E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}}].$$

Otherwise, we say that $\mathbf{y}_1$ Granger causes $\mathbf{y}_2$.

Informally, $\mathbf{y}_1$ *does not Granger cause* $\mathbf{y}_2$, if for all $k \geq 0$, the best k-step linear prediction of $\mathbf{y}_2$ based on the past values of $\mathbf{y}_2$ is the same as that of based on the past values of $\mathbf{y}$.

Remark 4 (Related work). If $\mathbf{y}$ is coercive, Granger non-causality from $\mathbf{y}_1$ to $\mathbf{y}_2$ is further equivalent to, (see [27])

$$\forall k \geq 0 : (C(A - KC)^k K)_{21} = 0, \tag{10}$$

where $(.)_{21}$ denotes the $r_2 \times r_1$ left lower block of a matrix and $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$. The system matrices of a minimal Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$ in block triangular form naturally satisfy (10). Moreover, they imply block triangular Wold decomposition, i.e., block triangular transfer matrix between the innovation process $\mathbf{e}$ and $\mathbf{y}$ [24], [25], [27]. In contrast to the cited papers, Theorem 1 below covers the case when $\mathbf{y}$ is non-coercive and in its proof the LTI-SS representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$ that characterizes Granger non-causality in $\mathbf{y}$ is constructed.

Theorem 1. *Consider the following statements for a ZMSIR process* $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$:

(i) $\mathbf{y}_1$ *does not Granger cause* $\mathbf{y}_2$*;*
(ii) there exists a minimal Kalman representation of $\mathbf{y}$ *in causal block triangular form;*
(iii) there exists a minimal Kalman representation of $\mathbf{y}$ *in block triangular form;*
(iv) there exists a Kalman representation of $\mathbf{y}$ *in block triangular form;*

Then (i) $\iff$ *(ii). If* $\mathbf{y}$ *is coercive, then (i)* $\iff$ *(ii)* $\iff$ *(iii)* $\iff$ *(iv).*

The proof can be found in Appendix. Intuitively, Granger non-causality in Theorem 1 means that for predicting $\mathbf{y}_2$, there is nothing to be gained from the knowledge of the past of $\mathbf{y}_1$. In parallel, considering a Kalman representation (7) of $\mathbf{y}$ in causal block triangular form the subsystem $(A_{22}, K_{22}, C_{22}, I_{r_2}, \mathbf{e}_2)$ of $\mathbf{y}_2$ depends only on $\mathbf{e}_2$ and thus only on the past values of $\mathbf{y}_2$. On the other hand, the subsystem which generates $\mathbf{y}_1$ depends on the entire history of $\mathbf{y}$. Theorem 1 can also be interpreted as follows: Granger non-causality from $\mathbf{y}_1$ to $\mathbf{y}_2$ is equivalent to the process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ admitting a minimal Kalman representation with network graph depicted on Fig. 2.

Theorem 1 relates Granger non-causality with existence of *minimal* Kalman representations in block triangular form. Since all minimal Kalman representations of $\mathbf{y}$ are isomorphic, Theorem 1 not only guarantees that Granger non-causality translates into existence of a Kalman representation with a suitable network graph, but that any minimal Kalman representation of $\mathbf{y}$ is isomorphic to this particular one. Hence, most of the interesting dynamical properties of this Kalman representation are also valid for any other minimal Kalman representation. Furthermore, any minimal Kalman representation can be brought to this specific one via a linear state-space transformation. Since black-box identification algorithms, for example subspace methods, yield minimal Kalman representations, Theorem 1 is also interesting for deriving and interpreting network graphs based on data.

C. Computing Kalman representation in block triangular form

Next, we present a procedure for constructing a minimal Kalman representation in causal block triangular form. Let $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ be an LTI-SS representation of $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$. Then it can be transformed into a minimal Kalman representation $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e})$ of $\mathbf{y}$ using Algorithm 2. Now take the partition $\hat{C} = \begin{bmatrix}\hat{C}_1^T & \hat{C}_2^T\end{bmatrix}^T$ such that the number of rows of $\hat{C}_i$ equals $r_i = \dim(\mathbf{y}_i)$ for $i = 1, 2$. Furthermore, define a non-singular matrix T such that $(T\hat{A}T^{-1}, \hat{C}_2 T^{-1})$ is in the form (see, e.g., [41])

$$T\hat{A}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \ \hat{C}_2 T^{-1} = \begin{bmatrix} 0 & C_{22} \end{bmatrix}, \qquad (11)$$

where (A_{22}, C_{22}) is observable and $A_{11} \in \mathbb{R}^{p_1 \times p_1}$, $A_{22} \in \mathbb{R}^{p_2 \times p_2}$ such that p_2 is the rank of the observability matrix of the pair $(\hat{C}_2, \hat{A})$. Note that if $(\hat{A}, \hat{C}_2)$ is observable, then $p_1 = 0$ and A_{11}, A_{12} are absent in (11). In addition, if the observability matrix of $(\hat{A}, \hat{C}_2)$ has zero rank, then $p_2 = 0$, in which case A_{12}, A_{22} and C_{22} are absent in (11). Define

$$A := T\hat{A}T^{-1}, \quad K := T\hat{K}, \quad C := \hat{C}T^{-1} \qquad (12)$$

and consider the partition

$$C = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \ K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \qquad (13)$$

where $C_{1i} \in \mathbb{R}^{(r_1+r_2)\times p_i}, K_{ij} \in \mathbb{R}^{p_i \times r_j}$, $i, j = 1, 2$. Based on Theorem 1 we can state the following result:

Corollary 1. *The following statements hold:*

- *If $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$, then either K_{21} is absent or $K_{21} = 0$. Furthermore, $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ in causal block triangular form with A_{ij}, K_{ij}, C_{ij}, $i, j = 1, 2$ defined by* (11)*,* (12) *and* (13)*.*
- *If $\mathbf{y}$ is coercive, then the absence of K_{21} or $K_{21} = 0$ implies that $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$.*

The proof can be found in Appendix. Corollary 1 yields a method to calculate a minimal Kalman representation in causal block triangular form in the absence of Granger causality. This idea is elaborated in Algorithms 3–4 that rely on Algorithms 1–2. Algorithm 3 takes an LTI-SS representation as its input and transforms it into a minimal Kalman representation in causal block triangular form. Algorithm 4 calculates the same representation from output covariances. Hence, by using empirical covariances, it can be applied to data.

Remark 5 (Correctness of Algorithms 3–4). Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ with covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$ and an LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ of $\mathbf{y}$. Let $\mathbf{e}$ be the innovation process of $\mathbf{y}$ and N be any number larger than or equal to the dimension of a minimal LTI-SS representation of $\mathbf{y}$. Assume that $\mathbf{y}$ satisfies condition (i) of Theorem 1 and note that Algorithms 1–2 calculate a minimal Kalman representation (Remark 1). Then from Corollary 1 it follows that if $\{A, K, C\}$ is the output of Algorithm 3 with input $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \Lambda_0^{\mathbf{y}} = E[\mathbf{v}(t)\mathbf{v}^T(t)]\}$, then $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation in causal block triangular form. Similarly, from Corollary 1 it follows that if $\{A, K, C\}$ is the output of Algorithm 4 with input $\{\Lambda\}_{k=0}^{2N}$, then $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation in causal block triangular form.

[3]the dimensions $r_i = \dim(\mathbf{y}_i)$, $i = 1, 2$ are fixed in the whole section

Algorithm 3 Minimal Kalman representation in causal block triangular form based on LTI-SS representation

Input	$\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \Lambda_0^{\mathbf{v}}\}$: $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ is an LTI-SS representation of $\mathbf{y}$ and $\Lambda_0^{\mathbf{v}} = E[\mathbf{v}(t)\mathbf{v}^T(t)]$
Output	$\{A, K, C\}$: system matrices of (7)

Step 1 Apply Algorithm 2 with input $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \Lambda_0^{\mathbf{v}}\}$ and denote its output by $\{\hat{A}, \hat{K}, \hat{C}\}$.
Step 2 Let $\hat{C} = \begin{bmatrix}\hat{C}_1^T \hat{C}_2^T\end{bmatrix}^T$ be such that $\hat{C}_i \in \mathbb{R}^{r_i \times n}$. Calculate a non-singular matrix T such that (11) holds and (A_{22}, C_{22}) is observable.[3]
Step 3 Set $A := T\hat{A}T^{-1}$, $K := T\hat{K}$, $C := \hat{C}T^{-1}$.

Algorithm 4 Minimal Kalman representation in causal block triangular form based on output covariances

Input	$\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$: Markov parameters of $\mathbf{y} = \begin{bmatrix}\mathbf{y}_1^T, \mathbf{y}_2^T\end{bmatrix}^T$
Output	$\{A, K, C\}$: system matrices of (7)

Step 1 Apply Algorithm 1 with input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$ and denote its output by $\{\hat{A}, \hat{K}, \hat{C}\}$.
Step 2 Step 2–3 of Algorithm 3.

Remark 6. In a similar fashion as in Remark 2, Algorithms 4 and 3 have polynomial complexity. Algorithm 3 is polynomial in the dimension of the state, output and noise processes of the LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$. Algorithm 4 is polynomial in the number and size of the output covariances.

Remark 7 (Checking Granger non-causality). Algorithm 3 and 4 can be used to check Granger non-causality by looking weather the left lower block of matrix K is zero ($K_{21} = 0$) in the partition (13), where $\{A, K, C\}$ are the matrices returned by Algorithm 3 or 4. If $K_{21} \neq 0$, then $\mathbf{y}_1$ Granger causes $\mathbf{y}_2$. If $\mathbf{y}_2$ is coercive and K_{21} is absent or $K_{21} = 0$, then $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$ in the view of Corollary 1. If $\mathbf{y}$ is non-coercive, then it should be checked if $(A_{22}, K_{22}, C_{22}, I_{r_2}, \mathbf{e}_2)$ is a minimal Kalman representation of $\mathbf{y}_2$. This can be done by computing a minimal Kalman representation $(\tilde{A}_{22}, \tilde{K}_{22}, \tilde{C}_{22}, I, \tilde{\mathbf{e}}_2)$ of $\mathbf{y}_2$ using Algorithm 1 or 2. If the noise variance $E[\mathbf{e}_2(t)\mathbf{e}_2^T(t)]$ is equal to the new noise variance $E[\tilde{\mathbf{e}}_2(t)\tilde{\mathbf{e}}_2^T(t)]$, then from [23, Proposition 2.3] we know that $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$ and $\mathbf{e}_2 = \tilde{\mathbf{e}}_2$.

IV. Coordinated representation

In this section we introduce a class of LTI-SS representations having the star graph as their network graph, called Kalman representations in coordinated form. We show that a process has a Kalman representation in coordinated form if and only if the components of this process satisfy certain (conditional) Granger causality conditions. We also present algorithms for constructing a Kalman representation in coordinated form. *Throughout this section we assume that $\mathbf{y}$ is a ZMSIR*

processes and it admits a partitioning $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$ *for some* $n \geq 2$*, such that* $\mathbf{y}_i \in \mathbb{R}^{r_i}$*,* $r_i > 0$ *for* $i = 1, \dots, n$*.*

A. Kalman representation in coordinated form

Definition 6. A Kalman representation $(A, K, C, I, \mathbf{e} = [\mathbf{e}_1^T, \dots, \mathbf{e}_n^T]^T, \mathbf{y})$, where $\mathbf{e}_i \in \mathbb{R}^{r_i}$, $i = 1, \dots, n$, is called a *Kalman representation in coordinated form*, if

$$A = \left[\begin{array}{cccc|c} A_{11} & 0 & \dots & 0 & A_{1n} \\ 0 & A_{22} & \dots & 0 & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_{(n-1)(n-1)} & A_{(n-1)n} \\ \hline 0 & 0 & \dots & 0 & A_{nn} \end{array}\right]$$

$$K = \left[\begin{array}{cccc|c} K_{11} & 0 & \dots & 0 & K_{1n} \\ 0 & K_{22} & \dots & 0 & K_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & K_{(n-1)(n-1)} & K_{(n-1)n} \\ \hline 0 & 0 & \dots & 0 & K_{nn} \end{array}\right] \tag{14}$$

$$C = \left[\begin{array}{cccc|c} C_{11} & 0 & \dots & 0 & C_{1n} \\ 0 & C_{22} & \dots & 0 & C_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C_{(n-1)(n-1)} & C_{(n-1)n} \\ \hline 0 & 0 & \dots & 0 & C_{nn} \end{array}\right]$$

where $A_{ij} \in \mathbb{R}^{p_i \times p_j}, K_{ij} \in \mathbb{R}^{p_i \times r_j}, C_{ij} \in \mathbb{R}^{r_i \times p_j}$ and $p_i \geq 0$ for $i, j = 1, \dots, n$. If, in addition, for each $i = 1, \dots, n-1$

$$\left(\begin{bmatrix} A_{ii} & A_{in} \\ 0 & A_{nn} \end{bmatrix}, \begin{bmatrix} K_{ii} & K_{in} \\ 0 & K_{nn} \end{bmatrix}, \begin{bmatrix} C_{ii} & C_{in} \\ 0 & C_{nn} \end{bmatrix}, I_{r_i + r_n}, \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_n \end{bmatrix} \right) \tag{15}$$

is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in causal block triangular form, then $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is called a *Kalman representation in causal coordinated form*.

If $n = 2$, then Definition 6 coincides with Definition 4. Furthermore, if $(A, K, C, I, \mathbf{e})$ is a Kalman representation in causal coordinated form, then the dimensions of A_{ij}, K_{ij}, C_{ij}, $i, j = 1, \dots, n$ are uniquely determined by $\mathbf{y}$. Indeed, since (15) is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$, $i = 1, \dots, n-1$ in causal block triangular form, it follows from the previous section that the dimensions of A_{ii}, K_{ii}, C_{ii} and A_{in}, K_{in}, C_{in} are uniquely determined by $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$. Therefore, considering (14), all dimensions of the blocks of A, K and C are determined by $\mathbf{y}$. Definition 6 is based on the deterministic terminology ([16], [32]) and on the definition of Gaussian coordinated systems ([16], [33]).

The term *coordinated* is used because the LTI-SS representation at hand can be viewed as consisting of several subsystems; one of which plays the role of a coordinator, and the others play the role of agents. More precisely, let $(A, K, C, I, \mathbf{e}, \mathbf{y})$ be a Kalman representation in coordinated form as in (14) and let $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_n^T]^T$ be its state such that $\mathbf{x}_i \in \mathbb{R}^{p_i}$, $i = 1, \dots, n$. Then, for $i = 1, \dots, n-1$

$$\Sigma_c \begin{cases} \mathbf{x}_n(t+1) = A_{nn}\mathbf{x}_n(t) + K_{nn}\mathbf{e}_n(t) \\ \mathbf{y}_n(t) = C_{nn}\mathbf{x}_i(t) + \mathbf{e}_n(t) \end{cases} \tag{16}$$

$$\Sigma_{a_i} \begin{cases} \mathbf{x}_i(t+1) = \sum_{j=\{i,n\}} A_{ij}\mathbf{x}_j(t) + K_{ij}\mathbf{e}_j(t) \\ \mathbf{y}_i(t) = \sum_{j=\{i,n\}} C_{ij}\mathbf{x}_j(t) + \mathbf{e}_i(t) \ . \end{cases} \tag{17}$$

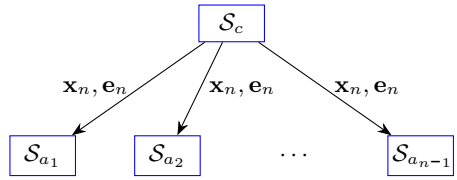

Fig. 3: Network graph of a Kalman representation in coordinated form: $\mathcal{S}_c$ is the coordinator (16) and $\mathcal{S}_{a_i}$, $i = 1, \dots, n-1$ are the agents (17).

Notice that subsystem $\mathcal{S}_{a_i}$ generates $\mathbf{y}_i$ as output, has $\mathbf{x}_i, \mathbf{e}_i$ as its state and noise process and takes $\mathbf{x}_n, \mathbf{e}_n$ as its inputs, thus takes inputs from subsystem $\mathcal{S}_c$. In contrast, $\mathcal{S}_c$ is autonomous, generating $\mathbf{y}_n$ as output and having $\mathbf{x}_n, \mathbf{e}_n$ as its state and noise process but not taking input from subsystems $\mathcal{S}_{a_i}$, $i = 1, \dots, n-1$ (see Fig. 3). We call $\mathcal{S}_c$ the *coordinator*, and $\mathcal{S}_{a_i}$, $i = 1, \dots, n-1$ the *agents*. Intuitively, the agents do not communicate with each other, only the coordinator sends information ($\mathbf{x}_n$ and $\mathbf{e}_n$) to all agents and does not receive information from them.

Motivation for Kalman representations in causal coordinated form: If we considered a general LTI-SS representation with a network graph like on Fig. 3, then the noise process $\mathbf{e}$ could be any process. If $\mathbf{e}$ were not the innovation process of $\mathbf{y}$, then it could happen that the agents communicate with each other in an implicit way through $\mathbf{e}$. However, if we assume that $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is a Kalman representation in causal coordinated form satisfying (14), then $[\mathbf{e}_i^T, \mathbf{e}_n^T]^T$ is the innovation process of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ for $i = 1, \dots, n-1$ and $\mathbf{e}_n$ is the innovation process of $\mathbf{y}_n$. Hence, the values of $\mathbf{e}_i, \mathbf{e}_n$ depend only on the past and present values of $\mathbf{y}_i, \mathbf{y}_n$. Moreover, $\mathbf{x}_n$ depends only on the past values of $\mathbf{y}_n$ and therefore $\mathbf{x}_i$ depends only on the past values of $\mathbf{y}_i, \mathbf{y}_n$. That is, Kalman representations in coordinated form have the property that there is no communication among the agents or from the agents to the coordinator hidden in the noise process. Note that the lack of communication from the agents to the coordinator is ensured by that (15) is a Kalman representation in causal block triangular form.

Kalman representations in causal coordinated form have a number of desirable properties, e.g., *they have the smallest possible coordinator*. By definition, the subsystem (15) of a Kalman representation in causal coordinated form (14) is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in causal block triangular form. This implies that (16) is a minimal Kalman representation of $\mathbf{y}_n$ and thus the coordinator is minimal. It assures observability of the coordinator and enables to *estimate the states of Kalman representations in causal coordinated form using distributed filters*. That is, in order to estimate the state $\mathbf{x}_n$ of the coordinator using a Kalman filter, only the output $\mathbf{y}_n$ of the coordinator is necessary. Since (17) is also minimal and thus observable, in order to estimate the state $\mathbf{x}_i$ of the ith agent using a Kalman filter, only the output $\mathbf{y}_i$ of this agent and the output $\mathbf{y}_n$ of the coordinator are necessary. Furthermore, from Lemma 1 below, Kalman representations in causal coordinated form are isomorphic. Hence, if they represent the same output process, their properties are essentially the same. The proof of Lemma 1 can be found in Appendix.

Lemma 1. *Any two Kalman representations of* $\mathbf{y}$ *in causal coordinated form are isomorphic.*

B. Conditional Granger causality and coordinated systems

Below we show that the existence of a Kalman representation of $\mathbf{y}$ in causal coordinated form can be characterized by conditional Granger non-causalities among the components of $\mathbf{y}$. To this end, we define conditional Granger non-causality: the generalization of Granger non-causality between two components of a process in the presence of a third component. The next definition is a particular case of the concept of causality defined in [1], if the latter is applied to ZMSIR processes, and if, using the terminology of [1], there is one external process.

Definition 7. Consider a ZMSIR process $\mathbf{y}=[\mathbf{y}_1^T,\mathbf{y}_2^T,\mathbf{y}_3^T]^T$. We say that $\mathbf{y}_1$ conditionally does not Granger cause $\mathbf{y}_2$ with respect to $\mathbf{y}_3$, if for all $t,k\in\mathbb{Z}$, $k\geq 0$

$$E_l[\mathbf{y}_2(t+k)\mid\mathcal{H}_{t-}^{\mathbf{y}_2,\mathbf{y}_3}]=E_l[\mathbf{y}_2(t+k)\mid\mathcal{H}_{t-}^{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}].$$

Otherwise, we say that $\mathbf{y}_1$ conditionally Granger causes $\mathbf{y}_2$ with respect to $\mathbf{y}_3$.

If $\mathbf{y}=[\mathbf{y}_1^T,\mathbf{y}_2^T]^T$, then considering $\mathbf{y}_3=\mathbf{y}_2$ Definition 7 coincides with Definition 5. We will be interested in a particular combination of causal dependencies in a process $\mathbf{y}=[\mathbf{y}_1^T,\ldots,\mathbf{y}_n^T]^T$. Namely, when $\mathbf{y}_i$ does not Granger cause $\mathbf{y}_n$ and $\mathbf{y}_i$ does not Granger cause $\mathbf{y}_j$ with respect to $\mathbf{y}_n$ for all $i,j=1,\ldots,n-1$, $i\neq j$. We will show that these causal relations in $\mathbf{y}=[\mathbf{y}_1^T,\ldots,\mathbf{y}_n^T]^T$ hold if and only if $\mathbf{y}$ has a Kalman representation in causal coordinated form whose network graph is as in Fig. 3. For this, we first define conditionally trivial intersection of two subspaces U,V with respect to a closed subspace W, denoted by $U\cap V|W=\{0\}$.

Definition 8. Consider the subspaces $U,V,W\subseteq\mathcal{H}$ such that W is closed. Then U,V have a *conditionally trivial intersection* with respect to W denoted by $U\cap V|W=\{0\}$ if

$$\{u-E_l[u|W]\mid u\in U\}\cap\{v-E_l[v|W]\mid v\in V\}=\{0\},$$

i.e., the intersection of the projections of U and V onto the orthogonal complement of W in $\mathcal{H}$ is the zero subspace.

Theorem 2. *Consider the following statements for a ZMSIR process* $\mathbf{y}=[\mathbf{y}_1^T,\ldots,\mathbf{y}_n^T]^T$*:*

(i) $\mathbf{y}_i$ *does not Granger cause* $\mathbf{y}_n$*,* $i=1,\ldots,n-1$*;*

(ii) $\mathbf{y}_i$ *conditionally does not Granger cause* $\mathbf{y}_j$ *with respect to* $\mathbf{y}_n$*,* $i,j=1,\ldots,n-1$*,* $i\neq j$*;*

(iii) (i) and (ii) hold and for $i,j\in\{1,\ldots,n-1\}, i\neq j$

$$E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_i,\mathbf{y}_n}]\cap E_l[\mathcal{H}_{t+}^{\mathbf{y}_j}|\mathcal{H}_{t-}^{\mathbf{y}_j,\mathbf{y}_n}]\,|\,E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}]=\{0\}\quad(18)$$

(iv) there exists a minimal Kalman representation of $\mathbf{y}$ *in causal coordinated form;*

(v) there exists a Kalman representation of $\mathbf{y}$ *in causal coordinated form;*

(vi) there exists a Kalman representation of $\mathbf{y}$ *in coordinated form;*

Then (iii) $\iff$ *(iv), in addition, ((i) and (ii))* $\iff$ *(v). If* $\mathbf{y}$ *is coercive, then also ((i) and (ii))* $\iff$ *(v)* $\iff$ *(vi).*

Remark 8 (Alternative formulations of (ii)). If (i) holds, then condition (ii) is equivalent to saying that $\mathbf{y}_i$ does not Granger cause $[\mathbf{y}_j^T,\mathbf{y}_n^T]^T$, $i,j\in\{1,\ldots,n-1\}$, $i\neq j$.

The proof of Theorem 2 can be found in Appendix. The intuition behind this result is the following. For a coordinator to exist, the outputs of the agents should not influence the output of the coordinator, i.e., (i) should hold. Moreover, for $i\neq j$ the output of agent i should not influence the output of agent j, except that information which comes from the output of the coordinator (ii). The condition for minimality (iii) can be explained as follows. It can be shown that a Kalman representation in causal coordinated form is observable, so for minimality, we only have to ensure its reachability. This is equivalent to the components of the state $\mathbf{x}=[\mathbf{x}_1^T,\ldots,\mathbf{x}_n^T]^T$ being linearly independent at each time. The space generated by the components of $\mathbf{x}_n(t)$ is $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}]$ and the space generated by the components of $[\mathbf{x}_i^T,\mathbf{x}_n^T]^T(t)$ is $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_i,\mathbf{y}_n}]$, where $\mathbf{x}_i$ and $\mathbf{x}_n$ are as in (16) and (17). As a result, condition (iii) is an equivalent condition for reachability of a Kalman representation in causal coordinated form.

Using minimal Kalman representations in causal coordinated form is desirable since they are isomorphic to any other minimal Kalman representation of the same process. Hence, any property derived for the minimal Kalman representation in causal coordinated form, if it is invariant under isomorphism, remains valid for any other minimal Kalman representation. Theorem 2 gives a necessary and sufficient condition for existence of Kalman representations in causal coordinated form. From Lemma 1, we know that these Kalman representations behave as minimal ones among all Kalman representations in coordinated form. In particular, they are isomorphic. Existence of a minimal Kalman representation in coordinated form (not causal coordinated form) remains a topic of future research.

C. Computing Kalman representations in coordinated form

Next, we describe a procedure to calculate a Kalman representation of $\mathbf{y}$ in causal coordinated form. Assume that condition (i) in Theorem 2 holds. Consider an LTI-SS representation $(\bar{A},\bar{B},\bar{C},\bar{D},\mathbf{v})$ of $\mathbf{y}$ and the partitions of $\bar{C}$ and $\bar{D}$

$$\bar{C}=\left[\bar{C}_1^T,\ldots,\bar{C}_n^T\right]^T,\ \bar{D}=\left[\bar{D}_1^T,\ldots,\bar{D}_n^T\right]^T\qquad(19)$$

such that $\bar{C}_i$ and $\bar{D}_i$ have $r_i=\dim(\mathbf{y}_i)$ rows for all $i=1,\ldots,n$. Then, $(\bar{A},\bar{B},\left[\bar{C}_i^T\ \bar{C}_n^T\right]^T,\left[\bar{D}_i^T\ \bar{D}_n^T\right]^T,\mathbf{v})$ is an LTI-SS representation of $[\mathbf{y}_i^T,\mathbf{y}_n^T]^T$, $i=1,\ldots,n-1$. Hence, by Corollary 1 the latter can be transformed into a minimal Kalman representation $(\hat{A}_i,\hat{K}_i,\hat{C}_i,I_{r_i+r_n},\mathbf{e}_{i,n}=[\mathbf{e}_i^T,\mathbf{e}_n^T]^T)$ of $[\mathbf{y}_i^T,\mathbf{y}_n^T]^T$ in causal block triangular form, i.e.,

$$\hat{A}_i=\begin{bmatrix}\hat{A}_{ii} & \hat{A}_{in}\\ 0 & \hat{A}_{i,nn}\end{bmatrix},\ \hat{K}_i=\begin{bmatrix}\hat{K}_{ii} & \hat{K}_{in}\\ 0 & \hat{K}_{i,nn}\end{bmatrix},\ \hat{C}_i=\begin{bmatrix}\hat{C}_{ii} & \hat{C}_{in}\\ 0 & \hat{C}_{i,nn}\end{bmatrix},\quad(20)$$

and the process $\mathbf{e}_n(t)=\mathbf{y}_n(t)-E_l[\mathbf{y}_n(t)|\mathcal{H}_{t-}^{\mathbf{y}_n}]$ is the innovation process of $\mathbf{y}_n$ and $\mathbf{e}_i(t)=\mathbf{y}_i(t)-E_l[\mathbf{y}_i(t)|\mathcal{H}_{t-}^{\mathbf{y}_i,\mathbf{y}_n}]$. In addition, $(\hat{A}_{i,nn},\hat{K}_{i,nn},\hat{C}_{i,nn},I_{r_n},\mathbf{e}_n)$ is a minimal Kalman representation of $\mathbf{y}_n$. Since all minimal Kalman representations of $\mathbf{y}_n$ are isomorphic (Proposition 4), there exist non-singular matrices T_i for $i=2,\ldots,n-1$ such that

$$\hat{A}_{1,nn}=T_i\hat{A}_{i,nn}T_i^{-1} \quad \hat{K}_{1,nn}=T_i\hat{K}_{i,nn} \\ \hat{C}_{1,nn}=\hat{C}_{i,nn}T_i^{-1}. \tag{21}$$

Let $\mathbf{e}=[\mathbf{e}_1^T,\mathbf{e}_2^T,\dots,\mathbf{e}_n^T]^T$ and define the matrices A,K and C as in (14) such that for $i=1,\dots,n-1$

$$\begin{aligned} &A_{ii}=\hat{A}_{ii},\ K_{ii}=\hat{K}_{ii},\ C_{ii}=\hat{C}_{ii},\\ &A_{in}=\hat{A}_{in}T_i^{-1},\ K_{in}=\hat{K}_{in},\ C_{in}=\hat{C}_{in}T_i^{-1},\\ &A_{nn}=\hat{A}_{1,nn},\ K_{nn}=\hat{K}_{1,nn},\ C_{nn}=\hat{C}_{1,nn}. \end{aligned} \tag{22}$$

Now, we can state the following result:

Corollary 2. *The following statements hold:*

- *If* $\mathbf{y}$ *satisfies conditions (i) and (ii) in Theorem 2, then* $(A,K,C,I,\mathbf{e})$ *defined by* (20)*,* (21)*, and* (22) *is a Kalman representation of* $\mathbf{y}$ *in causal coordinated form.*
- *If* $\mathbf{y}$ *satisfies (iii), then* $(A,K,C,I,\mathbf{e},\mathbf{y})$ *is also minimal.*

The proof can be found in Appendix. Note that if condition (ii) in Theorem 2 does not hold, then $\{A,K,C,I,\mathbf{e}\}$ can be calculated as above, but the process $\mathbf{e}$ is not necessarily white noise. Hence, if condition (ii) does not hold, then the tuple $\{A,K,C,I,\mathbf{e}\}$ does not necessarily define an LTI-SS representation.

The procedure above is elaborated on in Algorithm 5 and Algorithm 6. Algorithm 5 takes an LTI-SS representation as its input and transforms it into a Kalman representation in causal coordinated form. Algorithm 6 calculates the same representation from covariances of the output. Hence, by using empirical covariances it can be applied to data.

Algorithm 5 Kalman representation in causal coordinated form based on LTI-SS representation

Input $\{\bar{A},\bar{B},\bar{C},\bar{D},\Lambda_0^{\mathbf{v}}\}$: $(\bar{A},\bar{B},\bar{C},\bar{D},\mathbf{v})$ is an LTI-SS representation of $\mathbf{y}$ and $\Lambda_0^{\mathbf{v}}=E[\mathbf{v}(t)\mathbf{v}^T(t)]$

Output $\{A,K,C\}$: system matrices of (14)

for $i=1:n-1$

Step 1 Consider the partition (19) and apply Algorithm 3 with input $\{\bar{A},\bar{B},\bar{C}_i,\bar{D}_i,\Lambda_0^{\mathbf{v}}\}$. Denote its output by $\{\hat{A}_i,\hat{K}_i,\hat{C}_i\}$, where $(\hat{A}_i,\hat{K}_i,\hat{C}_i,I,\mathbf{e}_{i,n},[\mathbf{y}_i^T,\mathbf{y}_n^T]^T)$ is a minimal Kalman representation in block triangular form.

Step 2 If $i>1$, consider the partition (20) of $\hat{A}_i,\hat{K}_i,\hat{C}_i$ and define the non-singular matrix T_i as in (21).

end for

Step 3 Define A,K and C as in (14), such that the submatrices $\{A_{ii},A_{in}C_{ii},C_{in}K_{ii},K_{in}\}_{i=1,\dots,n-1}$ satisfy (22).

Remark 9 (Correctness of Algorithms 5–6). Consider a ZM-SIR process $\mathbf{y}=[\mathbf{y}_1^T,\dots,\mathbf{y}_n^T]^T$ with covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$ and an LTI-SS representation $(\bar{A},\bar{B},\bar{C},\bar{D},\mathbf{v})$ of $\mathbf{y}$. Let $\mathbf{e}$ be the innovation process of $\mathbf{y}$ and N be any number larger than or equal to the dimension of a minimal LTI-SS representation of $\mathbf{y}$. Assume that $\mathbf{y}$ satisfies conditions (i) and (ii) in Theorem 2 and note that Algorithms 3–4 calculate a minimal Kalman representation in causal block triangular form (Remark 5). Then it follows form Corollary 2 that if $\{A,K,C\}$ is the output of Algorithm 5 with input $\{\bar{A},\bar{B},\bar{C},\bar{D},\Lambda_0^{\mathbf{v}}=E[\mathbf{v}(t)\mathbf{v}^T(t)]\}$, then $(A,K,C,I,\mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in causal coordinated form. In addition, $(A,K,C,I,\mathbf{e})$ is minimal if and only if (18) holds. Similarly, it follows from Corollary 2 that if $\{A,K,C\}$ is the output of Algorithm 6 with input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$, then $(A,K,C,I,\mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in causal coordinated form and is minimal if and only if (18) holds.

Algorithm 6 Kalman representation in causal coordinated form based on output covariances

Input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$: Markov parameters of $\mathbf{y}=\left[\mathbf{y}_1^T,..,\mathbf{y}_n^T\right]^T$

Output $\{A,K,C\}$: system matrices of (14)

for $i=1:n-1$

Step 1 Denote the kth Markov parameters for $\mathbf{y}_{i,n}=[\mathbf{y}_i^T,\mathbf{y}_n^T]^T$ by $\Lambda_k^{\mathbf{y}_{i,n}}=E[\mathbf{y}_{i,n}\mathbf{y}_{i,n}^T]$.

Step 2 Calculate the rank of the Hankel matrix formed by $\{\Lambda_k^{\mathbf{y}_{i,n}}\}_{k=0}^{2N-1}$ and denote it by N_i. Call Algorithm 4 for $\{\Lambda_k^{\mathbf{y}_{i,n}}\}_{k=1}^{2N_i}$ and denote its output by $\{\hat{A}_i,\hat{K}_i,\hat{C}_i\}$.

Step 3 Step 2 of Algorithm 5.

end for

Step 4 Step 3 of Algorithm 5.

Remark 10. Using Remark 2 and Remark 6, the computational complexity of Algorithm 6 and 5 is polynomial. Algorithm 5 is polynomial in the dimension of the state, output and noise processes of the LTI-SS representation $(\bar{A},\bar{B},\bar{C},\bar{D},\mathbf{v})$. Algorithm 6 is polynomial in the number and size of the output covariances.

Remark 11 (Checking (i)–(ii)). In contrast to Remark 7 in Section III, the algorithms of this section cannot be directly used to check conditions (i) and (ii) in Theorem 2. However, (i) is just a Granger non-causality condition, and by Remark 8, (ii) can also be reformulated as (non-conditional) Granger non-causality. Therefore, by Remark 7, Algorithms 3 and 4 can be used to check these conditions.

Note that Algorithm 6 operates in a distributed manner; it combines subsystems belonging to an agent and the coordinator for which it does not need the observation of any other agent. Furthermore, it only uses the covariances of the observed process, thus using empirical covariances, it is suitable to estimate Kalman representations in causal coordinated form based on data. Due to the distributed conception, Algorithm 6 is possibly advantageous in terms of estimation error compared to non-distributed procedures. In the next section we illustrate Algorithm 6 with an example.

V. Example for coordinated representation

In this section we adopt a case study from [16, Section 8.1] to illustrate the results of the paper. The focus of this study is the dynamics of three underwater vehicles that track a reference path in a fixed formation. Among the vehicles there is one acting as a coordinator that tracks a reference path and two others acting as agents that track the coordinator.

In comparison with [16, Section 8.1] we made the following changes: (1) to ensure stationarity, the coordinator follows the zero position; (2) for convenience, we consider the movements of the vehicles along the first coordinate; (3) besides the position disturbance we include measurement noise.

We will show that the relative positions (concerning the formation) of the vehicles are ZMSIR processes that can be modeled by a minimal Kalman representation in causal coordinated form. In fact, we reverse engineer the coordinated network topology from the observed process in the following way: We verify that conditions (i) and (ii) in Theorem 2 hold by calculating Granger non-causal relations based on Remark 7. Then, we calculate a minimal Kalman representation in causal coordinated form using Algorithm 6.

Model description: Assume that we have three underwater vehicles V_1, V_2 and V_c where V_1,V_2 act as agents and V_c as the coordinator. For $j \in \{1, 2, c\}$ denote the first coordinate at time $t \in \mathbb{Z}$ of the position, velocity, acceleration, position disturbance and measurement noise of V_j by $p_j(t)$, $s_j(t)$, $a_j(t)$, $w_j(t)$ and $\tilde{w}_j(t)$, respectively. Also, denote the first coordinate of the reference position and velocity of V_j by $p_j^R(t)$ and $s_j^R(t)$, respectively. Let $p_c^R(t) = -(p_c(t) + \tilde{w}_c(t))$ and $p_j^R(t) = (p_c(t) + \tilde{w}_c(t)) + \Delta_j$ for $j = 1, 2$. That is, V_c follows the zero position based on its own measured position and for $j=1, 2$, V_j follows V_c in a distance Δ_j based on the same information. To shorten the expressions, for a process $l(t)$ we write l and we use σ to denote the forward time shift operator defined as follows: $\sigma l(t) = l(t + 1)$.

The dynamics of $[p_j, s_j]^T$, $j \in \{1, 2, c\}$ is given by

$$\sigma \begin{bmatrix} p_j \\ s_j \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\tau-1}{\tau} \end{bmatrix} \begin{bmatrix} p_j \\ s_j \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} a_j + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_j \tag{23}$$

where a_j is the control input and τ is a time constant. The reference signals $[p_j^R, s_j^R]^T$ are estimated by the observer

$$\sigma \begin{bmatrix} \hat{p}_j^R \\ \hat{s}_j^R \end{bmatrix} = \begin{bmatrix} 1 - G_j^p & 1 \\ -G_j^s & \frac{\tau-1}{\tau} \end{bmatrix} \begin{bmatrix} \hat{p}_j^R \\ \hat{s}_j^R \end{bmatrix} + \begin{bmatrix} G_j^p \\ G_j^s \end{bmatrix} p_j^R \tag{24}$$

where G_j^p, G_j^s are constant gains. The linear feedback control is $a_j = [F_j^p F_j^s] \begin{bmatrix} p_j - \hat{p}_j^R \\ s_j - \hat{s}_j^R \end{bmatrix}$. Combining (23) and (24) and assuming that $\mathbf{x}_j := [p_j - \Delta_j, s_j, \hat{p}_j^R - \Delta_j, \hat{s}_j^R]^T$, $j = 1, 2$, and $\mathbf{v} := [w_1, w_2, w_c, \tilde{w}_1, \tilde{w}_2, \tilde{w}_c]^T$ is a white noise process we can define the following LTI-SS representation of the process $\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_c]^T := [p_1 - \Delta_1, p_2 - \Delta_2, p_c]^T$:

$$\sigma \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_c\text{-}B_cF & \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_c \end{bmatrix} + \underbrace{\begin{bmatrix} E & 0 & 0 & 0 & 0 & B_1 \\ 0 & E & 0 & 0 & 0 & B_2 \\ 0 & 0 & E & 0 & 0 & \text{-}B_c \end{bmatrix}}_{B} \mathbf{v},$$

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_c \end{bmatrix} = \underbrace{\begin{bmatrix} E^T & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & E^T \end{bmatrix}}_{C} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_c \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D} \mathbf{v}.$$ where

for $j = 1, 2$, $A_j = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \frac{1}{\tau}F_j^p & \frac{\tau-1}{\tau} + \frac{1}{\tau}F_j^s & -\frac{1}{\tau}F_j^p & -\frac{1}{\tau}F_j^s \\ 0 & 0 & 1 - G_j^p & 1 \\ 0 & 0 & -G_j^s & \frac{\tau-1}{\tau} \end{bmatrix}$, and $B_j^T = \begin{bmatrix} 0 & 0 & G_j^p & G_j^s \end{bmatrix}^T$.

Parameter settings: Following the approach of [16], we take $F_1^p = F_2^p = F_c^p$ and $F_1^s = F_2^s = F_c^s$ as the solution of the linear quadratic problem $\min_{a_c}\{||\mathbf{z}_1^2||^2 + \alpha||a_c^2||^2\}$ with respect to the dynamics $\sigma \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\tau-1}{\tau} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} a_c$. Accordingly, for $\tau = 2$ and $\alpha = 10$ the optimal solution is $F_1^p = F_2^p = F_c^p = -0.3$ and $F_1^s = F_2^s = F_c^s = -0.5$. The gain constants were chosen to be $G_1^p = 1.5$, $G_1^s = 0.3$, $G_2^p = 1.2$, $G_2^s = 0.1$, $G_{R_c}^p = 0.9$, and $G_{R_c}^s = 0.5$ for which the matrix A is stable. Finally, the joint noise process $\mathbf{v}$ is chosen to be a normalized Gaussian white noise process.

Reverse engineering of the coordinated network graph: Assume that the output process $\mathbf{y}$ of $\mathcal{S} := (A, B, C, D, \mathbf{v}, \mathbf{y})$ is observed. By the result of this paper, we will calculate a minimal Kalman representation of $\mathbf{y}$ in causal coordinated form. Note that we do not use prior knowledge of the coordinated structure of the network graph. In fact, this representation reconstructs the coordinated network graph of $\mathcal{S}$.

First, we check the Granger non-causal relations among the components of $\mathbf{y}$ using the covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$ of $\mathbf{y}$ where N is larger than or equal to the dimension of a minimal LTI-SS representation of $\mathbf{y}$. For this, we calculate a Kalman representation $(A, K, C, I, \mathbf{e})$ of $\mathbf{y}$ and verify that $\mathbf{y}$ is coercive by checking that $A - KC$ is invertible (see the last paragraph of Section II-B). In view of Corollary 1, a Granger non-causal relation can be verified by observing the output matrix K of Algorithm 4. More specifically, if the left lower block of the matrix K is zero, then an appropriate Granger non-causal relation holds (see Remark 7). Following this method, we apply Algorithm 4 choosing the coordinator to be $\mathbf{y}_1$,$\mathbf{y}_2$,$\mathbf{y}_c$, $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$, $[\mathbf{y}_1^T, \mathbf{y}_c^T]^T$ and $[\mathbf{y}_2^T, \mathbf{y}_c^T]^T$, thus trying all the possibilities. We obtain that $[\mathbf{y}_1, \mathbf{y}_2]^T$ does not Granger cause $\mathbf{y}_c$ and $\mathbf{y}_j$ does not Granger cause $[\mathbf{y}_i, \mathbf{y}_c]$, $i, j = 1, 2$, $i \neq j$ thus conditions (i) and (ii) in Theorem 2 hold for the partition $\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_c]^T$.

Second, in order to calculate a Kalman representation of $\mathbf{y}$ in coordinated form, we apply Algorithm 6 with the covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$ as its input. Accordingly, first the minimal Kalman representations $(A_{k,1}, K_1, C_{k,1}, I, \mathbf{e}_{1,c}, [\mathbf{y}_1, \mathbf{y}_c]^T)$ and $(A_{k,2}, K_2, C_{k,2}, I, \mathbf{e}_{2,c}, [\mathbf{y}_2, \mathbf{y}_c]^T)$ are calculated in causal block triangular form using Algorithm 3 with the covariances of $[\mathbf{y}_1, \mathbf{y}_c]^T$ and $[\mathbf{y}_2, \mathbf{y}_c]^T$ as its input. With our parameter settings, these matrices are in the form

$$A_{k,1} = \left[\begin{array}{cccc|cccc} 0.4 & -0.3 & -0.1 & 0.1 & 0.0 & 0.1 & 0.2 & -0.1 \\ 0.2 & 0.4 & 0.6 & 0.2 & 0.1 & 0.0 & 0.2 & 0.1 \\ 0.0 & -0.3 & 0.4 & -0.1 & 0.0 & 0.0 & 0.0 & -0.4 \\ -0.2 & 0 & -0.2 & 0.7 & 0.0 & -0.1 & 0.0 & 0.0 \\ \hline 0 & 0 & 0 & 0 & 0.2 & -0.9 & 0.1 & 0.0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.1 & 0.0 & -0.5 & 0.3 \end{array}\right]$$

$$K_1 = \left[\begin{array}{c|c} 0.1 & 0.2 \\ -0.1 & -0.1 \\ 0.0 & 0.0 \\ -0.3 & -0.1 \\ \hline 0 & 0.1 \\ 0 & 0.2 \\ 0 & 0.2 \\ 0 & -0.1 \end{array}\right], \quad C_{k,1}^T = \left[\begin{array}{c|c} -0.4 & 0 \\ -0.3 & 0 \\ 0.4 & 0 \\ -1.8 & 0 \\ \hline 0.2 & 0.5 \\ -0.2 & 0.0 \\ 0.0 & -0.2 \\ 0.2 & 0.0 \end{array}\right]$$

$$A_{k,2} = \left[\begin{array}{cccc|cccc} 0.5 & 0.2 & 0 & -0.1 & 0 & -0.1 & -0.2 & 0.2 \\ -0.2 & 0.4 & 0.7 & 0.0 & 0.1 & 0.0 & 0.1 & 0.1 \\ 0.0 & -0.1 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 \\ 0.1 & 0.0 & 0.0 & 0.8 & 0.0 & -0.1 & 0.0 & 0.0 \\ \hline 0 & 0 & 0 & 0 & 0.2 & -0.8 & 0.1 & 0.0 \\ 0 & 0 & 0 & 0 & 0.7 & 0.4 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.1 & 0.1 & -0.3 & 0.2 \end{array}\right]$$

$$K_2 = \left[\begin{array}{c|c} -0.1 & -0.2 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \\ -0.3 & -0.1 \\ \hline 0 & 0.1 \\ 0 & 0.2 \\ 0 & 0.2 \\ 0 & -0.1 \end{array}\right], \quad C_{k,2}^T = \left[\begin{array}{c|c} 0.5 & 0 \\ -0.1 & 0 \\ -0.1 & 0 \\ -1.9 & 0 \\ \hline 0.2 & 0.5 \\ -0.1 & -0.1 \\ 0.0 & -0.2 \\ 0.1 & 0.0 \end{array}\right].$$

Following this, we define a transformation matrix $T := \left(\left[C_{k,23}^T \; A_{k,23}C_{k,23}^T\right]^T\right)^{-1}\left[C_{k,13}^T \; A_{k,13}C_{k,13}^T\right]^T$ with which, the output matrices $\{A_k, K, C_k\}$ of Algorithm 6 are calculated as below. $A_k = \left[\begin{array}{c|c|c} A_{k,11} & 0 & A_{k,1c} \\ \hline 0 & A_{k,22} & T\hat{A}_{k,2c}T^{-1} \\ \hline 0 & 0 & A_{k,cc} \end{array}\right]$ $K = \left[\begin{array}{c|c|c} K_{11} & 0 & A_{1c} \\ \hline 0 & K_{22} & T\hat{A}_{2c} \\ \hline 0 & 0 & K_{cc} \end{array}\right]$, $C_k = \left[\begin{array}{c|c|c} C_{k,11} & 0 & C_{k,1c} \\ \hline 0 & C_{k,22} & \hat{C}_{k,2c}T^{-1} \\ \hline 0 & 0 & C_{k,cc} \end{array}\right]$. In view of Remark 9, the representation $\mathcal{S}_k := (A_k, K, C_k, I, \mathbf{e}, \mathbf{y})$ is a Kalman representation in causal coordinated form. Furthermore, it is easy to check that $\mathcal{S}_k$ is minimal, which implies that condition (iii) in Theorem 2 holds. The calculation of $\mathcal{S}_k$ only requires the second order statistics of the output process and does not use prior knowledge of the network topology. Therefore, the coordinated network graph of $\mathcal{S}$ is reverse engineered as $\mathcal{S}_k$. Moreover, by Theorem 2, the reconstructed representation $\mathcal{S}_k$ not only shows the coordinated structure but also characterizes the causal relations that describe the coordinated relationship in the observed process. The procedure can be repeated based on data, using empirical covariances which provide an estimation of $\mathcal{S}_k$. Note that $\mathcal{S}_k$ is calculated in a distributed way which possibly reduces estimation error.

Conclusions

In this paper we studied the relationship between coordinated state-space representations and (conditional) Granger non-causality. Our results show that certain (conditional) Granger non-causalities among the components of a process are equivalent to the existence of an LTI-SS representation with the star graph as its network graph, called Kalman representation in coordinated form. We provided algorithms for calculating this structured representation, in particular, calculating it from the covariance sequence of the observed output process. The covariances can be estimated from data. Hence, our results open up the possibility of calculating this representation from output data.

In systems biology and neuroscience the interactions between subsystems of a dynamical system are often hard to detect. By contrast, causality of the observed process is easy to estimate. If a star-like causality structure is detected among the components of a process ((i) and (ii) in Theorem 2 hold), then, by the results of this paper, we can calculate a Kalman representation of this process in causal coordinated form. This representation has the star graph as its network graph, which characterizes the detected causalities. By this, we can reverse engineer the network graph of dynamical systems without prior knowledge of the internal interconnection structure.

Kalman representation in coordinated form allows distributed estimation. In particular, the algorithms of this paper estimate the coordinator and each agent separately and thus are also suitable for distributed parameter estimation.

The results can be of interest to structure preserving model reduction. Assume that a process has a Kalman representation in coordinated form. Then, one can reduce the model order of a subsystem belonging to an agent without modifying the subsystems belonging to other agents or the coordinator and preserving the coordinated interconnection structure.

There are three main restrictions of our work in real life applications: 1) the coordinated interconnection structure, i.e., the restriction of the network graph to a star graph 2) the autonomy of the system and 3) the linearity of the system. The extension of the results concerning these aspects remains a topic of future research.

Appendix - Technical proofs

We will use the following notation: the sum of two subspaces $U, V \subseteq \mathcal{H}$ is written by $U+V := \{u+v | u \in U, v \in V\}$ and the orthogonal complement of U in V (with respect to $\mathcal{H}$) by $V \ominus U$; if $U \cap V = \{0\}$ then the direct sum of them is denoted by $U \dot{+} V$; if U and V are orthogonal then we write the orthogonal direct sum as $U \oplus V$. Also, we denote the orthogonality of a process $\mathbf{y}$ to a closed subspace $U \subseteq \mathcal{H}$ by $\mathbf{y} \perp U$. Furthermore, an LTI-SS representation $(A, B, C, D, \mathbf{v})$ of a white noise process $\mathbf{y}$ has zero dimension, thus A, B, C are absent, and it is the trivial equation $\mathbf{y} = D\mathbf{v}$. A zero dimensional LTI-SS representation is minimal, observable and controllable by convention.

Proof of Theorem 1. First, we discuss the trivial implications: since any minimal Kalman representation in causal block triangular form is a Kalman representation in a causal block triangular form and any Kalman representation in a causal block triangular form is a Kalman representation in block triangular form (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) follow. In addition, the implication (ii) $\Longrightarrow$ (i) is easy to see; if $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$ in causal block triangular form (7), then $(A_{22}, K_{22}, C_{22}, I_{r_2}, \mathbf{e}_2)$ is a minimal Kalman representation of $\mathbf{y}_2$, and hence $\mathbf{e}_2(t) = \mathbf{y}_2(t) - E_l[\mathbf{y}_2(t) \mid \mathcal{H}_{t-}^{\mathbf{y}}]$ equals the innovation process of $\mathbf{y}_2$. By [23, Proposition 2.3], the latter implies that $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$.

Proof of (iv) $\Longrightarrow$ (i). If $\mathbf{y}$ is coercive, then Granger non-causality is equivalent to the transfer matrix of a Kalman representation of $\mathbf{y}$ having a block triangular structure, see [25, Theorem 2.2.]. Since the transfer function of a Kalman representation in block triangular form has a triangular structure described in [24], [25], the implication (iv) $\Longrightarrow$ (i) follows.

Proof of (i) $\Longrightarrow$ (ii). To begin with, from Proposition 2 we know that the ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ has a minimal Kalman representation $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e})$. Assuming that (i) in Theorem 1 holds, we transform this representation into causal block triangular form. Consider the partition $\hat{C} = [\hat{C}_1^T, \hat{C}_2^T]^T$ such that $C_i \in \mathbb{R}^{r_i \times p}$ where $r_i = \dim(\mathbf{y}_i)$ is the dimension of $\mathbf{y}_i$, $i = 1, 2$ and p is the dimension of $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e}, \mathbf{y})$. We assume that $p > 0$; if $p = 0$, then $\mathbf{y} = \mathbf{e}$ defines a minimal Kalman representation in causal block triangular form. Take the non-singular matrix T which brings $(\hat{A}, \hat{C}_2)$ into observability staircase form, i.e., T is such that

$$T\hat{A}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \ \hat{C}_2 T^{-1} = \begin{bmatrix} 0 & C_{22} \end{bmatrix}, \qquad (25)$$

where (A_{22}, C_{22}) is observable and $A_{11} \in \mathbb{R}^{p_1 \times p_1}$, $A_{22} \in \mathbb{R}^{p_2 \times p_2}$ such that p_2 is the rank of the observability matrix of the pair $(\hat{A}, \hat{C}_2)$. Define $A := T\hat{A}T^{-1}, K := T\hat{K}, C := \hat{C}T^{-1}$ and notice that $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is a minimal Kalman representation since it is isomorphic to $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e}, \mathbf{y})$. Note that if $(\hat{A}, \hat{C}_2)$ is observable, then $p_1 = 0$ and A_{11}, A_{12} are absent in (25). If the observability matrix of $(\hat{A}, \hat{C}_2)$ has zero rank, then $p_2 = 0$ and A_{12}, A_{22}, C_{22} are absent. Moreover, if $p_2 = 0$, then $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is already in causal block triangular form (see Remark 3). Hence, we can assume that $p_2 > 0$. Next, we show that $K_{21} = 0$ where $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ and $K_{ij} \in \mathbb{R}^{p_i \times r_j}$ for $i = 1, 2$. Denote the state of $(A, K, C, I, \mathbf{e}, \mathbf{y})$ by $\mathbf{x}$. Take the partition $\mathbf{e} = [\mathbf{e}_1^T, \mathbf{e}_2^T]^T$ and $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$ where $\mathbf{e}_i \in \mathbb{R}^{r_i}$ and $\mathbf{x}_i \in \mathbb{R}^{p_i}$, $i = 1, 2$. Notice that $C = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}$ and $A^k = \begin{bmatrix} A_{11}^k & (A^k)_{12} \\ 0 & A_{22}^k \end{bmatrix}$, where $C_{1i} \in \mathbb{R}^{r_1 \times p_i}$, $i = 1, 2$ and $(A^k)_{12} \in \mathbb{R}^{p_1 \times p_2}$ denotes the right upper block of A^k. It then follows that $C_2 A^k \mathbf{x}(t) = C_{22} A_{22}^k \mathbf{x}_2(t)$ and for $k > 0$

$$\mathbf{y}_2(t+k) = C_{22} A_{22}^k \mathbf{x}_2(t) + \sum_{l=0}^{k-1} M_l \mathbf{e}(t+k-l) \qquad (26)$$

for some matrices $M_0, \ldots, M_{k-1}$. Since $\mathbf{e}$ is the innovation process of $\mathbf{y}$, it implies that $\mathbf{e}(t+k-l) \perp \mathcal{H}_{t-}^{\mathbf{y}}$ and $\mathcal{H}_{t-}^{\mathbf{e}} = \mathcal{H}_{t-}^{\mathbf{y}}$, for $k - l \geq 0$, $t \in \mathbb{Z}$. Furthermore, from $\mathbf{x}(t) = \sum_{k=1}^{\infty} CA^{k-1}K\mathbf{e}(t-k)$, the components of $\mathbf{x}(t)$ belong to $\mathcal{H}_{t-}^{\mathbf{e}} = \mathcal{H}_{t-}^{\mathbf{y}}$. Using (26), it then follows that $E_l[\mathbf{y}_2(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}] = C_{22} A_{22}^k \mathbf{x}_2(t)$. As $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$ we know that $E_l[\mathbf{y}_2(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_2}] = E_l[\mathbf{y}_2(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}]$ for all $k \geq 0$, and thus $E_l[\mathbf{y}_2(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_2}] = C_{22} A_{22}^k \mathbf{x}_2(t) \in \mathcal{H}_{t-}^{\mathbf{y}_2}$. Let O_2 be the observability matrix of (A_{22}, C_{22}) and denote its left inverse by O_2^+. Then $\mathbf{x}_2(t) = O_2^+ E_l[\begin{bmatrix} \mathbf{y}_2(t) \\ \vdots \\ \mathbf{y}_2(t+n-1) \end{bmatrix} | \mathcal{H}_{t-}^{\mathbf{y}_2}]$ and thus the elements of $\mathbf{x}_2(t)$ belong to $\mathcal{H}_{t-}^{\mathbf{y}_2}$. Note that since $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$, by [23, Proposition 2.3] $\mathbf{e}_2$ is the innovation process of $\mathbf{y}_2$, and hence $\mathcal{H}_{(t+1)-}^{\mathbf{y}_2} = \mathcal{H}_{t-}^{\mathbf{y}_2} \oplus \mathcal{H}_t^{\mathbf{e}_2}$. Therefore, $\mathbf{x}_2(t+1) = E_l[\mathbf{x}_2(t+1)|\mathcal{H}_{(t+1)-}^{\mathbf{y}_2}] = E_l[\mathbf{x}_2(t+1)|\mathcal{H}_{t-}^{\mathbf{y}_2}] + E_l[\mathbf{x}_2(t+1)|\mathcal{H}_t^{\mathbf{e}_2}]$. From $\mathbf{e}(t) \perp \mathcal{H}_{t-}^{\mathbf{y}} \supseteq E_l[\mathbf{x}_2(t+1)|\mathcal{H}_{t-}^{\mathbf{y}_2}]$ and $E_l[\mathbf{x}_2(t+1)|\mathcal{H}_t^{\mathbf{e}_2}] \subseteq \mathcal{H}_t^{\mathbf{e}}$ we have that $E_l[\mathbf{x}_2(t+1)|\mathcal{H}_t^{\mathbf{e}}] = E_l[\mathbf{x}_2(t+1)|\mathcal{H}_t^{\mathbf{e}_2}] = \hat{R}\mathbf{e}_2(t)$ for a suitable $\hat{R}$ matrix. Then $\mathbf{x}_2(t+1) = A_{22}\mathbf{x}_2(t) + \begin{bmatrix} K_{21} & K_{22} \end{bmatrix} \mathbf{e}(t)$ and $\mathbf{e}(t) \perp \mathcal{H}_t^{\mathbf{x}_2}$ implies that $E_l[\mathbf{x}_2(t+1)|\mathcal{H}_t^{\mathbf{e}}] = \begin{bmatrix} K_{21} & K_{22} \end{bmatrix} \mathbf{e}(t) = \hat{R}\mathbf{e}_2(t)$. Using that $\mathbf{y}$ is full rank, $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent, and hence $K_{21} = 0$, $K_{22} = \hat{R}$. That is, $(A, K, C, I, \mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in block triangular form. In order to see that $(A, K, C, I, \mathbf{e})$ is in causal block triangular form, we need to show that $(A_{22}, K_{22}, C_{22}, I_{r_2}, \mathbf{e}_2)$ is a minimal Kalman representation of $\mathbf{y}_2$. From Granger non-causality, $\mathbf{e}_2$ is the innovation process of $\mathbf{y}_2$, hence we only need to prove minimality. Note that if $p_1 = 0$, then $A = A_{22}, K = K_{22}, C = C_{22}$ thus $(A_{22}, K_{22}, C_{22}, I_{r_2}, \mathbf{e}_2)$ is minimal. In view of Proposition 3, it is sufficient to show that (A_{22}, C_{22}) is observable and (A_{22}, K_{22}) is controllable. The former follows from the construction. Assume now indirectly that (A_{22}, K_{22}) is uncontrollable, i.e., that for some vector $\eta \neq 0$, $\eta^T A_{22}^k K_{22} = 0$ for all $k \geq 0$. However, $A^k K = \begin{bmatrix} A_{11}^k K_{11} & (A^k K)_{12} \\ 0 & A_{22}^k K_{22} \end{bmatrix}$, where $(A^k K)_{12}$ denotes the right upper block of $A^k K$ with suitable dimensions. It follows that $\begin{bmatrix} 0 & \eta^T \end{bmatrix} A^k K = 0$ for all $k \geq 0$, which implies that (A, K) is not controllable. Since $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$, by Proposition 3 (A, K) is controllable, which is a contradiction. □

Proof of Corollary 1. The construction of the Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$ coincides with the one described in the proof of the implication (i) $\Longrightarrow$ (ii) in Theorem 1. Hence, if $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$, then the above-mentioned proof implies that either K_{21} is absent or $K_{21} = 0$ and that $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$ in causal block triangular form. Conversely, if K_{21} is absent or $K_{21} = 0$, and $\mathbf{y}$ is coercive, then $(A, K, C, I, \mathbf{e})$ is a minimal Kalman representation of $\mathbf{y}$ in block triangular form. Hence, by the implication (iii) $\Longrightarrow$ (i) of Theorem 1, $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$. □

Proof of Lemma 1. Consider a process $\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T]^T$ where $\mathbf{y}_i \in \mathbb{R}^{r_i}$, for $r_i > 0$, $i = 1, \ldots, n$. Let $(A, K, C, I, \mathbf{e})$ and $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e})$ be two Kalman representations of $\mathbf{y}$ in causal coordinated form (14) with blocks $A_{ij} \in \mathbb{R}^{p_i \times p_j}, K_{ij} \in \mathbb{R}^{p_i \times r_j}, C_{ij} \in \mathbb{R}^{r_i \times p_j}$ and $\hat{A}_{ij} \in \mathbb{R}^{p_i \times p_j}, \hat{K}_{ij} \in \mathbb{R}^{p_i \times r_j}, \hat{C}_{ij} \in \mathbb{R}^{r_i \times p_j}$ for $i, j = 1, \ldots, n$. Let $\mathcal{S}_i$ be the Kalman representation (15), and let $\hat{\mathcal{S}}_i$ be the counterpart of (15), obtained by replacing A_{ij}, K_{ij}, C_{ij} by the matrices $\hat{A}_{ij}, \hat{K}_{ij}, \hat{C}_{ij}$ for $j = i, n$, $i = 1, \ldots, n$. From Definition 6 it follows that $\mathcal{S}_i$ and $\hat{\mathcal{S}}_i$ are minimal Kalman representations of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in block triangular form, thus there exists an isomorphism T_i from $\hat{\mathcal{S}}_i$ to $\mathcal{S}_i$, $i = 1, \ldots, n-1$. We will show that T_i is of the form $T_i = \begin{bmatrix} T_{ii} & T_{in} \\ 0 & T_{nn} \end{bmatrix}$, where $T_{ij} \in \mathbb{R}^{p_i \times p_j}$, $j = i, n$. This then implies that $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e})$ and $(A, K, C, I, \mathbf{e})$ are isomorphic such as $A = T\hat{A}T^{-1}$, $K = T\hat{K}$ and $C = \hat{C}T^{-1}$ with the matrix T defined by $T = \left[\begin{array}{cccc|c} T_{11} & 0 & \ldots & 0 & T_{1n} \\ 0 & T_{22} & \ldots & 0 & T_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & T_{(n-1)(n-1)} & T_{(n-1)n} \\ \hline 0 & 0 & \ldots & 0 & T_{nn} \end{array}\right]$. Consider the

partition $T_i = \begin{bmatrix} T_{ii} & T_{in} \\ T_{in} & T_{nn} \end{bmatrix}$, where $T_{kl} \in \mathbb{R}^{p_k \times p_l}$ for $k, l = i, n$. By reordering the rows of the observability matrices $\mathscr{O}_i$ and $\hat{\mathscr{O}}_i$ of $\mathcal{S}_i$ and $\hat{\mathcal{S}}_i$ we obtain that $\mathscr{O}_i = \begin{bmatrix} O_i & O_{in} \\ 0 & O_n \end{bmatrix}$ and $\hat{\mathscr{O}}_i = \begin{bmatrix} \hat{O}_i & \hat{O}_{in} \\ 0 & \hat{O}_n \end{bmatrix}$, where O_n and $\hat{O}_n$ are the observability matrices of (A_{nn}, C_{nn}) and $(\hat{A}_{nn}, \hat{C}_{nn})$, respectively. Since $\mathscr{O}_i T_i = \hat{\mathscr{O}}_i$, it follows that $O_n T_{in} = 0$. Since $\mathcal{S}_i$ is a Kalman representation in causal block triangular form, (A_{nn}, C_{nn}) is observable and therefore, O_n is full row rank. Then $O_n T_{in} = 0$ implies $T_{in} = 0$. □

To prove Theorem 2, we need an auxiliary result. For the sake of simplicity, a ZMSIR process $[\mathbf{y}_{j_1}^T, \ldots, \mathbf{y}_{j_k}^T]^T$ is shortened by $\mathbf{y}_J$ where $J = \{j_1, \ldots, j_k\}$ and the Hilbert spaces generated by the present, past, and future values of $\mathbf{y}_J$ are written by $\mathcal{H}_t^{\mathbf{y}_J}$, $\mathcal{H}_{t-}^{\mathbf{y}_J}$ and $\mathcal{H}_{t+}^{\mathbf{y}_J}$, respectively.

Lemma 2. *Consider a ZMSIR process* $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T, \mathbf{y}_4^T]^T$. *Then* $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *conditionally do not Granger cause* $\mathbf{y}_3$ *with respect to* $\mathbf{y}_4$ *if and only if* $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ *conditionally does not Granger cause* $\mathbf{y}_3$ *with respect to* $\mathbf{y}_4$.

Proof. if: By definition, the joint process $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ conditionally does not Granger cause $\mathbf{y}_3$ with respect to $\mathbf{y}_4$ if for all $t, k \in \mathbb{Z}, k \geq 0$ $E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}] = E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}]$. By projecting both sides onto $\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}}$ and to $\mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}$ we have that

$$\begin{aligned} E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}] &= E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}}] \\ E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}] &= E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}], \end{aligned} \tag{27}$$

which implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ conditionally does not Granger cause $\mathbf{y}_3$ with respect to $\mathbf{y}_4$.

only if: For $t, k \in \mathbb{Z}$, $k \geq 0$ define the process $\alpha_{t+k} := \mathbf{y}_3(t+k) - E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}]$. Then, $\alpha_{t+k} \perp \mathcal{H}_{t-}^{\mathbf{y}_{3,4}}$ and from the Granger non-causality conditions we obtain that also $\alpha_{t+k} \perp \mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}}$ and $\alpha_{t+k} \perp \mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}$ hold. Therefore, α_{t+k} is orthogonal to $\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}} + \mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}$, thus also to $\mathcal{H}_{t-}^{\mathbf{y}}$, the Hilbert space generated by $\mathbf{y}$. By projecting α_{t+k} onto $\mathcal{H}_{t-}^{\mathbf{y}}$ we obtain that $E_l[\alpha_{t+k}|\mathcal{H}_{t-}^{\mathbf{y}}] = 0$ thus $E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}]$, which by definition is that $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ conditionally does not Granger cause $\mathbf{y}_3$ with respect to $\mathbf{y}_4$. □

Proof of Theorem 2. To start with, any Kalman representation in causal coordinated form is a Kalman representation in coordinated form, hence (v) $\Longrightarrow$ (vi) follows. We assume now that $\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T]^T$ is a ZMSIR process where $\mathbf{y}_i \in \mathbb{R}^{r_i}$, $r_i > 0$, $i = 1, \ldots, n$ and we continue with the proof of the remaining implications.

Proof of ((i) and (ii)) $\Longrightarrow$ (v). Condition (i) and Theorem 1 imply the existence of minimal Kalman representations $(\hat{A}_i, \hat{K}_i, \hat{C}_i, I, \mathbf{e}_{i,n})$ of $\mathbf{y}_{i,n} = [\mathbf{y}_i^T, \mathbf{y}_n^T]^T$, $i = 1, \ldots n-1$ in causal block triangular form. Note that $\mathbf{e}_{i,n} = [\mathbf{e}_i^T, \mathbf{e}_n^T]^T$ is the innovation process of $\mathbf{y}_{i,n}$ and $\mathbf{e}_n$ is the innovation process of $\mathbf{y}_n$. By using (i), (ii) and Lemma 2, we get $\mathbf{e}_i(t) = \mathbf{y}_i(t) - E_l[\mathbf{y}_i(t)|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] = \mathbf{y}_i(t) - E_l[\mathbf{y}_i(t)|\mathcal{H}_{t-}^{\mathbf{y}}]$, $\mathbf{e}_n(t) = \mathbf{y}_n(t) - E_l[\mathbf{y}_n(t)|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathbf{y}_n(t) - E_l[\mathbf{y}_n(t)|\mathcal{H}_{t-}^{\mathbf{y}}]$, thus $\mathbf{e} = [\mathbf{e}_1^T, \ldots, \mathbf{e}_n^T]^T$ is the innovation process of $\mathbf{y}$. Consider the partition $\hat{A}_i = \begin{bmatrix} \hat{A}_{ii} & \hat{A}_{in} \\ 0 & \hat{A}_{i,nn} \end{bmatrix}$, $\hat{K}_i = \begin{bmatrix} \hat{K}_{ii} & \hat{K}_{in} \\ 0 & \hat{K}_{i,nn} \end{bmatrix}$, $\hat{C}_i = \begin{bmatrix} \hat{C}_{ii} & \hat{C}_{in} \\ 0 & \hat{C}_{i,nn} \end{bmatrix}$ as in (20). Let T_i be the matrix as in (21) that transforms $(\hat{A}_{i,nn}, \hat{K}_{i,nn}, \hat{C}_{i,nn}, I, \mathbf{e}_n, \mathbf{y}_n)$ into $(\hat{A}_{1,nn}, \hat{K}_{1,nn}, \hat{C}_{1,nn}, I, \mathbf{e}_n, \mathbf{y}_n)$. Then define A_{in}, C_{in}, K_{in} as in (22) and A, K, C as in (14). Note that the stability of $\hat{A}_i$, $i = 1, \ldots, n-1$ implies the stability of A. Then, $(A, K, C, I, \mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in coordinated form. Finally, since $(\hat{A}_i, \hat{K}_i, \hat{C}_i, I, \mathbf{e}_{i,n})$ is isomorphic with

$$\left(\begin{bmatrix} A_{ii} & A_{in} \\ 0 & A_{nn} \end{bmatrix}, \begin{bmatrix} K_{ii} & K_{in} \\ 0 & K_{nn} \end{bmatrix}, \begin{bmatrix} C_{ii} & C_{in} \\ 0 & C_{nn} \end{bmatrix}, I, \mathbf{e}_{i,n} \right) \tag{28}$$

by the isomorphism defined by the transformation matrix $\begin{bmatrix} I & 0 \\ 0 & T_i \end{bmatrix}$, it follows that the LTI-SS representation (28) is also a Kalman representation of $\mathbf{y}_{i,n}$ in causal block triangular form for all $i = 1, \ldots, n-1$. As a result, $(A, K, C, I, \mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in causal coordinated form.

Proof of (iii) $\Longrightarrow$ (iv). Consider the Kalman representation $(A, K, C, I, \mathbf{e})$ of $\mathbf{y}$ in causal coordinated form which was constructed in the proof of ((i) and (ii)) $\Longrightarrow$ (v). First, we show that (A, C) is an observable pair thus $\begin{bmatrix} C \\ (A - \lambda I) \end{bmatrix}$ is full column rank for all $\lambda \in \mathbb{C}$. Minimality of (28) implies that (A_{ii}, C_{ii}) are observable pairs for $i = 1, \ldots, n$ so that the matrices $\begin{bmatrix} C_{ii} \\ (A_{ii} - \lambda I) \end{bmatrix}$ are full column rank for all $\lambda \in \mathbb{C}$. Notice that $\begin{bmatrix} C \\ (A - \lambda I) \end{bmatrix}$ can be transformed into an upper block triangular form such that the diagonal blocks are $\begin{bmatrix} C_{ii} \\ (A_{ii} - \lambda I) \end{bmatrix}$ for $i = 1, \ldots, n$. Hence, $\begin{bmatrix} C \\ (A - \lambda I) \end{bmatrix}$ is full column rank for all $\lambda \in \mathbb{C}$, which implies that (A, C) is observable.

Next, we prove that if condition (iii) holds, then the components of $\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_n^T]^T$ which are consistent with (16) and (17) are linearly independent. To this end, we first show that $\dim(\mathcal{H}_t^{\mathbf{x}}) = \sum_{i=1}^n \dim(\mathcal{H}_t^{\mathbf{x}_i})$, where $\dim(\mathcal{H}_t^{\mathbf{z}})$ denotes the number of scalar components of a basis in $\mathcal{H}_t^{\mathbf{z}}$ for a process $\mathbf{z}$. Notice that the representations (28) are minimal Kalman representations. Hence, the components of $\mathbf{x}_{i,n} = [\mathbf{x}_i^T, \mathbf{x}_n^T]^T$ are linearly independent and $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$, $t \in \mathbb{Z}$ for $i = 1, \ldots, n-1$. Recall that the orthogonal complement of $B \subseteq \mathcal{H}_{t-}^{\mathbf{x}} + \mathcal{H}_{t+}^{\mathbf{x}}$ in $A \subseteq \mathcal{H}_{t-}^{\mathbf{x}} + \mathcal{H}_{t+}^{\mathbf{x}}$ is denoted by $A \ominus B$. It then follows that $\mathcal{H}_t^{\mathbf{x}_{i,n}} = (\mathcal{H}_t^{\mathbf{x}_i} \ominus \mathcal{H}_t^{\mathbf{x}_n}) \oplus \mathcal{H}_t^{\mathbf{x}_n}$ and because of $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$, also $\dim(\mathcal{H}_t^{\mathbf{x}_i} \ominus \mathcal{H}_t^{\mathbf{x}_n}) = \dim(\mathcal{H}_t^{\mathbf{x}_i})$. Below, we will show that using (18) in Theorem 2, $\mathcal{H}_t^{\mathbf{x}}$ can be decomposed as

$$\mathcal{H}_t^{\mathbf{x}} = \mathcal{H}_t^{\mathbf{x}_n} \oplus \left((\mathcal{H}_t^{\mathbf{x}_1} \ominus \mathcal{H}_t^{\mathbf{x}_n}) \dot{+} \ldots \dot{+} (\mathcal{H}_t^{\mathbf{x}_{n-1}} \ominus \mathcal{H}_t^{\mathbf{x}_n}) \right), \tag{29}$$

from which it follows that $\dim(\mathcal{H}_t^{\mathbf{x}}) = \sum_{i=1}^n \dim(\mathcal{H}_t^{\mathbf{x}_i})$, i.e., the components of $\mathbf{x}(t)$ are linearly independent. Take the observability matrix $O_i^N := \left[C_{ii}^T \ (C_{ii}A_{ii})^T \ \ldots \ (C_{ii}A_{ii}^{N-1})^T \right]^T$ of (A_{ii}, C_{ii}), $i = 1, \ldots, n$ where $N \geq \dim(\mathbf{x})$. By defining $Y_i(t) := [\mathbf{y}_i^T(t), \ldots, \mathbf{y}_i^T(t+N-1)]^T$, we have that $Y_i(t)$ spans $\mathcal{H}_{t+}^{\mathbf{y}_n}$ for $i = 1, \ldots, n$. Then the following equations hold with an appropriate M matrix: $E_l[Y_n(t)|\mathcal{H}_{t-}^{\mathbf{y}_n}] = O_n^N \mathbf{x}_n(t)$, $E_l[Y_j(t)|\mathcal{H}_{t-}^{\mathbf{y}_j, \mathbf{y}_n}] = O_j^N \mathbf{x}_j(t) + M\mathbf{x}_n(t)$. It implies that $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] \subseteq \mathcal{H}_t^{\mathbf{x}_{i,n}}$. Since

(A_{ii}, C_{ii}) is an observable pair, O_i^N has left inverse and we also have that $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] \supseteq \mathcal{H}_t^{\mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] + \mathcal{H}_t^{\mathbf{x}_n} \supseteq \mathcal{H}_t^{\mathbf{x}_i}$, $i=1,\dots,n$. Hence,

$$\begin{aligned} &E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathcal{H}_t^{\mathbf{x}_n} \\ &E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] + E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathcal{H}_t^{\mathbf{x}_{i,n}}. \end{aligned} \tag{30}$$

Notice that $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] \cap E_l[\mathcal{H}_{t+}^{\mathbf{y}_j}|\mathcal{H}_{t-}^{\mathbf{y}_{j,n}}]|E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \{0\}$ if and only if $\left(E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}]+E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}]\right) \cap \left(E_l[\mathcal{H}_{t+}^{\mathbf{y}_j}|\mathcal{H}_{t-}^{\mathbf{y}_{j,n}}]+E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}]\right)|E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \{0\}$. and similarly, that $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_j}|\mathcal{H}_t^{\mathbf{x}_n} = \{0\} \iff \mathcal{H}_t^{\mathbf{x}_{i,n}} \cap \mathcal{H}_t^{\mathbf{x}_{j,n}}|\mathcal{H}_t^{\mathbf{x}_n} = \{0\}$. By using the equations (30), we obtain that the condition (18) in Theorem 2 is equivalent to $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_j}|\mathcal{H}_t^{\mathbf{x}_n} = \{0\}$ which implies that $\mathcal{H}_t^{\mathbf{x}}$ can be decomposed as in (29). Hence, the components of $\mathbf{x}$ are linearly independent and thus (A, K) is controllable. By Proposition 3, the observability of (A, C) and the controllability of (A, K) implies the minimality of the Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$.

Proof of (v) $\Longrightarrow$ ((i) and (ii)). Assume that $(A, K, C, I, \mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in causal coordinated form where the matrices A, K, C are as in (14). Notice that $(A, K, C, I, \mathbf{e})$ is a Kalman representation of $[\mathbf{y}_{1,\dots,n-1}^T, \mathbf{y}_n^T]^T$ in block triangular form and hence by Theorem 1, $\mathbf{y}_{1,\dots,n-1}$ does not Granger cause $\mathbf{y}_n$. Apply Lemma 2 for $[\mathbf{y}_i^T, \mathbf{y}_{1,\dots,i-1,i+1,\dots,n-1}^T, \mathbf{y}_n^T]^T$, $i = 1,\dots,n-1$. [4] Then Lemma 2 implies condition (i). By Remark 8, (ii) is equivalent to saying that $\mathbf{y}_j$ does not Granger cause $\mathbf{y}_{i,n}$ for all $i, j = 1,\dots,n-1$, $i \neq j$. By Lemma 2, this is equivalent to saying that $\mathbf{y}_{1,2,\dots,i-1,i+1,\dots,n-1}$ does not Granger cause $\mathbf{y}_{i,n}$, which is further equivalent to $\mathbf{e}_{i,n}$ being the innovation process of $\mathbf{y}_{i,n}$. Since (15) is a minimal Kalman representation of $\mathbf{y}_{i,n}$, by Definition 6, $\mathbf{e}_{i,n}$ is indeed the innovation process of $\mathbf{y}_{i,n}$.

Proof of (iv) $\Longrightarrow$ (iii). Let $(A, K, C, I, \mathbf{e})$ be a minimal Kalman representation of $\mathbf{y}$ in causal coordinated form, and assume that A, K, C satisfy (14). It is easy to see that

$$\begin{aligned} &E_l[\mathbf{y}_n(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_n}] = C_{nn}A_{nn}^k\mathbf{x}_n(t) \\ &E_l[\mathbf{y}_i(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] = C_{ii}A_{ii}^k\mathbf{x}_i(t) + C_{in}A_{nn}^k\mathbf{x}_n(t) \end{aligned} \tag{31}$$

for $k \geq 0$, from which $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] \subseteq \mathcal{H}_t^{\mathbf{x}_{i,n}}$. Since $(A_{nn}, K_{nn}, C_{nn}, I, \mathbf{e}_n)$ is a minimal Kalman representation of $\mathbf{y}_n$, the pair (A_{nn}, C_{nn}) is observable. Define the observability matrix of (A_{nn}, C_{nn}) as $O_n^N := \left[C_{nn}^T \ (C_{nn}A_{nn})^T \ \dots \ (C_{nn}A_{nn}^{N-1})^T\right]^T$ where $N \geq \dim(\mathbf{x}_n)$. Then from (31) we obtain that $E_l[\begin{bmatrix} \mathbf{y}_n(t) \\ \mathbf{y}_n(t+1) \\ \vdots \\ \mathbf{y}_n(t+N-1) \end{bmatrix}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = O_n^N\mathbf{x}_n(t)$. From the observability of (A_{nn}, C_{nn}) we know that O_n^N has left inverse and thus $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathcal{H}_t^{\mathbf{x}_n}$. Since $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is minimal, the components of $\mathbf{x}(t)$ are linearly independent for each $t \in \mathbb{Z}$. In particular, this means that $\mathcal{H}_t^{\mathbf{x}_{i,j}} \cap \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$ and $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_j} = \{0\}$ for $i, j = 1,\dots,n-1$, $i \neq j$. In turn, this implies that $\mathcal{H}_t^{\mathbf{x}_{i,n}} \cap \mathcal{H}_t^{\mathbf{x}_{j,n}}|\mathcal{H}_t^{\mathbf{x}_n} = \{0\}$. By combining it with $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i}|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] \subseteq \mathcal{H}_t^{\mathbf{x}_{i,n}}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n}|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathcal{H}_t^{\mathbf{x}_n}$ we can conclude that condition (18) in Theorem 2 holds.

[4] It is a special case of Lemma 2 when the component $\mathbf{y}_4$ of $\mathbf{y}$ is absent.

Proof of ((i) and (ii)) $\iff$ (vi) if $\mathbf{y}$ is coercive. We have shown that ((i) and (ii)) $\Longrightarrow$ (v) and that the implication (v) $\Longrightarrow$ (vi) is trivial. It is thus left to show that (vi) $\Longrightarrow$ ((i) and (ii)). Assume that $(A, K, C, I, \mathbf{e})$ is a Kalman representation of $\mathbf{y}$ in coordinated form satisfying (14). Since $\mathbf{y}$ is coercive, $\mathbf{e}(t) = \mathbf{y}(t) + \sum_{k=1}^{\infty} C(A-KC)^{k-1}K\mathbf{y}(t-k)$. From (14) it is easy to see that for any $k \geq 1$,

$$C(A-KC)^{k-1}K = \begin{bmatrix} M_{k,11} & 0 & 0 & \cdots & 0 & M_{k,1n} \\ 0 & M_{k,22} & 0 & \cdots & 0 & M_{k,2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & M_{k,nn} \end{bmatrix},$$

where $M_{k,ii} = C_{ii}(A_{ii} - K_{ii}C_{ii})^{k-1}K_{ii}$, $i = 1,\dots,n$ and $M_{k,in}$ are suitable matrices for $i = 1,\dots,n-1$. Hence, $\mathbf{e}_n(t) = \mathbf{y}_n(t) + \sum_{k=1}^{\infty} M_{k,nn}\mathbf{y}_n(t-k)$, $\mathbf{e}_i(t) = \mathbf{y}_i(t) + \sum_{k=1}^{\infty} M_{k,ii}\mathbf{y}_i(t-k) + M_{k,in}\mathbf{y}_n(t-k)$, where $\mathbf{e}_i(t) = \mathbf{y}_i(t) - E_l[\mathbf{y}_i(t) \mid \mathcal{H}_{t-}^{\mathbf{y}}]$ for $i = 1,\dots,n$. This implies that $E_l[\mathbf{y}_n(t) \mid \mathcal{H}_{t-}^{\mathbf{y}}] \subseteq \mathcal{H}_{t-}^{\mathbf{y}_n}$ and $E_l[\mathbf{y}_i(t) \mid \mathcal{H}_{t-}^{\mathbf{y}}] \subseteq \mathcal{H}_{t-}^{\mathbf{y}_{i,n}}$. Therefore, $E_l[\mathbf{y}_n(t) \mid \mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_n(t) \mid \mathcal{H}_{t-}^{\mathbf{y}_n}]$ and $E_l[\mathbf{y}_i(t) \mid \mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_i(t) \mid \mathcal{H}_{t-}^{\mathbf{y}_{i,n}}]$. It follows from [23, Proposition 2.3] that $\mathbf{y}_i$ does not Granger cause $\mathbf{y}_n$, and $\mathbf{y}_j$ does not Granger cause $\mathbf{y}_{i,n}$ for all $i, j \in \{1,\dots,n-1\}$, $i \neq j$. In view of Remark 8 this is equivalent to (i) and (ii). □

Proof of Corollary 2. Consider the LTI-SS representation $(A, K, C, I, \mathbf{e})$ defined by (20), (21), and (22) before Corollary 2. Then $(A, K, C, I, \mathbf{e})$ coincides with the Kalman representation defined in the proof of Theorem 2. Hence, the first statement of Corollary 2 is a consequence of the implication ((i) and (ii)) $\Longrightarrow$ (v) of Theorem 2. Similarly, the second statement of Corollary 2 is a direct consequence of the implication (iii) $\Longrightarrow$ (iv) of Theorem 2. □

References

[1] C. W. J. Granger, "Economic processes involving feedback," *Information and Control*, vol. 6, no. 1, pp. 28–48, 1963.

[2] Y. Yuan, K. Glover, and J. Gonçalves, "On minimal realisations of dynamical structure functions," *Automatica*, vol. 55, pp. 159–164, 2015.

[3] Y. Yuan, G.-B. Stan, S. Warnick, and J. Gonçalves, "Robust dynamical network structure reconstruction," *Automatica*, vol. 47, no. 6, pp. 1230–1235, 2011.

[4] T. E. M. Nordling and E. W. Jacobsen, "On Sparsity as a Criterion in Reconstructing Biochemical Networks," in *18th IFAC World Congress*, 2011.

[5] A. A. Julius, M. Zavlanos, S. Boyd, and G. J. Pappas, "Genetic network identification using convex programming," *Systems Biology, IET*, vol. 3, pp. 155–166, 2009.

[6] T. Kang, R. Moore, Y. Li, E. D. Sontag, and L. Bleris, "Discriminating direct and indirect connectivities in biological networks," *National Academy of Sciences USA*, vol. 112, pp. 12 893–12 898, 2015.

[7] A. Roebroeck, A. K. Seth, and P. A. Valdes-Sosa, "Causal time series analysis of functional magnetic resonance imaging data," *Journal of Machine Learning Research, Proceedings Track*, vol. 12, pp. 65–94, 2011.

[8] P. A. Valdes-Sosa, A. Roebroeck, J. Daunizeau, and K. J. Friston, "Effective connectivity: Influence, causality and biophysical modeling," *Neuroimage*, vol. 58, no. 2, pp. 339–361, 2011.

[9] S. Bolognani, N. Bof, D. Michelotti, R. Muraro, and L. Schenato, "Identification of power distribution network topology via voltage correlation analysis," in *52nd IEEE Conference on Decision and Control*, 2013, pp. 1659–1664.

[10] W. Zhang, W. Liu, C. Zang, and L. Liu, "Multi-agent system based integrated solution for topology identification and state estimation," *Industrial Informatics, IEEE Transactions on*, vol. 13, no. 2, pp. 714–724, 2017.

[11] K. J. Friston, L. Harrison, and W. Penny, "Dynamic causal modeling," *Neuroimage*, vol. 19, no. 4, pp. 1273 – 1302, 2003.

[12] R. Goebel, A. Roebroeck, D.-S. Kim, and E. Formisano, "Investigating directed cortical interactions in time-resolved fMRI data using vector autoregressive modeling and Granger causality mapping," *Magnetic Resonance Imaging*, vol. 21, pp. 1251–1261, 2003.

[13] O. David, "fMRI connectivity, meaning and empiricism: Comments on: Roebroeck et al. the identification of interacting networks in the brain using fMRI: Model selection, causality and deconvolution," *NeuroImage*, vol. 58, no. 2, pp. 306 – 309, 2011.

[14] A. Roebroeck, E. Formisano, and R. Goebel, "Reply to friston and david: After comments on: The identification of interacting networks in the brain using fmri: Model selection, causality and deconvolution," *NeuroImage*, vol. 58, no. 2, pp. 310 – 311, 2011.

[15] P. L. Kempker, A. C. M. Ran, and J. H. van Schuppen, "Lq control for coordinated linear systems," *Automatic Control, IEEE Transactions on*, vol. 59, no. 4, pp. 851–862, 2014.

[16] P. L. Kempker, "Coordination control of linear systems," Ph.D. dissertation, Amsterdam: Vrije Universiteit, 2012.

[17] A. J. van der Schaft, "Physical network systems and model reduction," in *Mathematical Control Theory II: Behavioral Systems and Robust Control*, M. N. Belur, M. K. Camlibel, P. Rapisarda, and J. M. Scherpen, Eds. Springer, Cham, 2015, pp. 199–219.

[18] H. Sandberg and R. M. Murray, "Model reduction of interconnected linear systems," *Optimal Control Applications and Methods*, vol. 30, no. 3, pp. 225–245, 2009.

[19] N. Monshizadeh, H. L. Trentelman, and M. K. Camlibel, "Projection based model reduction of multi-agent systems using graph partitions," *Control of Network Systems, IEEE Transactions on*, vol. 1, no. 2, pp. 145–154, 2014.

[20] Z. Yue, J. Thunberg, Y. Yuan, and J. Gonalves, "Dynamical structure function and Granger causality: Similarities and differences," in *54th IEEE Conference on Decision and Control*, 2015, pp. 889–894.

[21] J. Gonçalves and S. Warnick, "Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks," *Automatic Control, IEEE Transactions on*, vol. 53, no. 7, pp. 1670–1674, 2008.

[22] M. Eichler, "Graphical modelling of multivariate time series," *Probability Theory and Related Fields*, vol. 153, no. 1, pp. 233–268, 2012.

[23] J.-M. Dufour and E. Renault, "Short run and long run causality in time series: Theory," *Econometrica*, vol. 66, no. 5, pp. 1099–1125, 1998.

[24] C. Hsiao, "Autoregressive modelling and causal ordering of econometric variables," *Journal of Economic Dynamics and Control*, vol. 4, pp. 243–259, 1982.

[25] P. E. Caines, "Weak and strong feedback free processes," *Automatic Control, IEEE Transactions on*, vol. 21, no. 5, pp. 737–739, 1976.

[26] M. R. Gevers and B. Anderson, "On jointly stationary feedback-free stochastic processes," *Automatic Control, IEEE Transactions on*, vol. 27, no. 2, pp. 431–436.

[27] L. Barnett and A. K. Seth, "Granger causality for state space models," *Physical Review E*, vol. 91, no. 4, pp. 737–739, 2015.

[28] P. E. Caines, R. Deardon, and H. P. Wynn, "Conditional orthogonality and conditional stochastic realization," in *Directions in Mathematical Systems Theory and Optimization*, A. Rantzer and C. I. Byrnes, Eds. Springer Berlin Heidelberg, vol. 286, pp. 71–84.

[29] P. E. Caines and H. P. Wynn, "An algebraic framework for bayes nets of time series," in *Modeling, Estimation and Control*, ser. Lecture Notes in Control and Information Sciences, A. Chiuso, S. Pinzoni, and A. Ferrante, Eds. Springer Berlin, 2007, vol. 364, pp. 45–57.

[30] P. E. Caines, R. Deardon, and H. P. Wynn, "Bayes nets of time series: Stochastic realizations and projections," in *Optimal Design and Related Areas in Optimization and Statistics*, ser. Springer Optimization and Its Applications, L. Pronzato and A. Zhigljavsky, Eds. Springer New York, 2009, vol. 28, pp. 155–166.

[31] P. L. Kempker, A. C. M. Ran, and J. H. van Schuppen, "Construction and minimality of coordinated linear systems," *Linear Algebra and its Applications*, vol. 452, pp. 202–236, 2014.

[32] A. C. M. Ran and J. H. van Schuppen, "Coordinated linear systems," in *Coordination Control of Distributed Systems*, ser. Lecture Notes in Control and Information Sciences, vol. 456, 2014, pp. 113–121.

[33] N. Pambakian, "LQG coordination control," Master's thesis, Delft University of Technology, 2011.

[34] M. Jozsa, M. Petreczky, and M. K. Camlibel, "Towards realization theory of interconnected linear stochastic systems," in *22nd International Symposium on Mathematical Theory of Networks and Systems*, 2016.

[35] ——, "Relationship between causality of stochastic processes and zero blocks of their joint innovation transfer matrices," in *20th IFAC World Congress*, 2017.

[36] ——, "Causality based graph structure of stochastic linear state-space representations," in *56th IEEE Conference on Decision and Control*, 2017.

[37] A. Lindquist and G. Picci, *Linear Stochastic Systems*, ser. Series in Contemporary Mathematics. Springer Berlin, 2015, vol. 1.

[38] T. Katayama, *Subspace Methods for System Identification*, ser. Communications and Control Engineering. Springer London, 2005.

[39] H. G. Golub and C. F. Van Loan, *Matrix computations*. The Johns Hopkins University Press, 2013.

[40] D. Bini, B. Iannazzo, and B. Meini, *Numerical Solution of Algebraic Riccati Equations*. Society for Industrial and Applied Mathematics, 2011.

[41] H. H. Rosenbrock, *State-Space and Multivariable Theory*. John Wiley, 1970.

Mónika Józsa received the M.S. degree in applied mathematics from Eötvös Loránd University in Budapest, Hungary in 2013. From 2013 she worked for Research Institute of Agricultural Economics in Budapest and from 2014 she started her Ph.D in systems and control at the University of Groningen, Netherlands. From 2016 she continued her Ph.D. at IMT Lille-Douai, France. Her research is on stochastic processes and realization/identification theory of stochastic systems.

Mihály Petreczky received the Ph.D. degree from Vrije Universiteit in Amsterdam, The Netherlands in 2006. In the past, he held appointments at Johns Hopkins University, USA, Eindhoven University of Technology, and Maastricht University, The Netherlands and at Ecole des Mines de Douai, France. He is currently a CNRS researcher at Centre de Recherche en Informatique, Signal et Automatique de Lille (CRIStAL), UMR CNRS 9189. His research interests include control and systems theory of hybrid and LPV systems.

M. Kanat Camlibel (M'05) received the Ph.D. degree in mathematical theory of systems and control from Tilburg University in 2001. Currently he is an Associate Professor with Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, where he served as an Assistant Professor between 2007 and 2013. From 2001 to 2007, he held Post- Doctoral/Assistant Professor positions with the University of Groningen, Tilburg University, and Eindhoven Technical University. His research interests include differential variational inequalities, complementarity problems, optimization, piecewise affine dynamical systems, switched linear systems, constrained linear systems, multiagent systems, model reduction, and geometric theory of linear systems.