



# On the multi-symplectic structure of Boussinesq-type systems. I: Derivation and mathematical properties

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ON THE MULTI-SYMPLECTIC STRUCTURE  
OF BOUSSINESQ-TYPE SYSTEMS. I:  
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# ON THE MULTI-SYMPLECTIC STRUCTURE OF BOUSSINESQ-TYPE SYSTEMS. I: DERIVATION AND MATHEMATICAL PROPERTIES

ANGEL DURÁN, DENYS DUTYKH\*, AND DIMITRIOS MITSOTAKIS

**ABSTRACT.** The BOUSSINESQ equations are known since the end of the XIX<sup>st</sup> century. However, the proliferation of various BOUSSINESQ-type systems started only in the second half of the XX<sup>st</sup> century. Today they come under various flavours depending on the goals of the modeller. At the beginning of the XXI<sup>st</sup> century an effort to classify such systems, at least for even bottoms, was undertaken and developed according to both different physical regimes and mathematical properties, with special emphasis, in this last sense, on the existence of symmetry groups and their connection to conserved quantities. Of particular interest are those systems admitting a symplectic structure, with the subsequent preservation of the total energy represented by the HAMILTONIAN. In the present paper a family of BOUSSINESQ-type systems with multi-symplectic structure is introduced. Some properties of the new systems are analyzed: their relation with already known BOUSSINESQ models, the identification of those systems with additional HAMILTONIAN structure as well as other mathematical features like well-posedness and existence of different types of solitary-wave solutions. The consistency of multi-symplectic systems with the full EULER equations is also discussed.

**Key words and phrases:** multi-symplectic structure; long dispersive wave; Boussinesq equations; surface waves

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## 1. Introduction

The first BOUSSINESQ-type equation was proposed by Joseph BOUSSINESQ in 1877 [12], who gave the name to different families of nonlinear wave equations proposed since then. This topic remained a *sleeping beauty*, [44], during almost one hundred years. The modern era was opened by Howell PEREGRINE in his theoretical and numerical investigations of long wave transformations on uniform slopes [58]. A proliferation of various systems started, serving various purposes of the near-shore hydrodynamics. Some reviews of this topic can be found in [16, 28, 55].

At the beginning of the XXI<sup>st</sup> century an effort to classify such systems, at least for even bottoms, was undertaken. In this way, the following four-parameter family

$$\eta_t + [u + \eta u]_x + a u_{xxx} - b \eta_{xxt} = 0, \quad (1.1)$$

$$u_t + [\eta + \frac{1}{2} u^2]_x + c \eta_{xxx} - d u_{xxt} = 0, \quad (1.2)$$

denoted here as the  $(a, b, c, d)$  family, was formulated and analyzed by BONA *et al.*, [6, 7]. In (1.1), (1.2),  $\eta = \eta(x, t)$  and  $u = u(x, t)$  are real-valued functions defined for  $x \in \mathbb{R}$  and  $t \geq 0$ , while the coefficients  $a, b, c, d$  are defined as

$$a \stackrel{\text{def}}{=} \frac{1}{2} (\vartheta^2 - \frac{1}{3}) \nu, \quad b \stackrel{\text{def}}{=} \frac{1}{2} (\vartheta^2 - \frac{1}{3}) \cdot (1 - \nu), \quad (1.3)$$

$$c \stackrel{\text{def}}{=} \frac{1}{2} (1 - \vartheta^2) \mu, \quad d \stackrel{\text{def}}{=} \frac{1}{2} (1 - \vartheta^2) \cdot (1 - \mu), \quad (1.4)$$

where  $\nu, \mu \in \mathbb{R}$  and  $0 \leq \vartheta \leq 1$ . The Systems (1.1), (1.2) are proposed for modelling the two-way propagation of one-dimensional, small amplitude, irrotational long surface waves in a channel of constant depth, see Figure 1. The variables  $x$  and  $t$  represent, respectively, the position along the channel and time, while  $\eta(x, t)$  and  $u(x, t)$  are proportional to the free surface excursion and to the horizontal velocity of the fluid at  $(x, t)$  at a non-dimensional height  $y = -1 + \vartheta(1 + \eta(x, t))$ , respectively. A review of the existing mathematical theory for the Systems (1.1), (1.2) which includes a list of systems of the family of especial interest, well-posedness results of the corresponding Initial – Value Problem (IVP) and periodic IVP as well as existence and stability results of solitary-wave solutions, can be seen in [28]. A symmetric\* variant of the  $(a, b, c, d)$  system of the general form

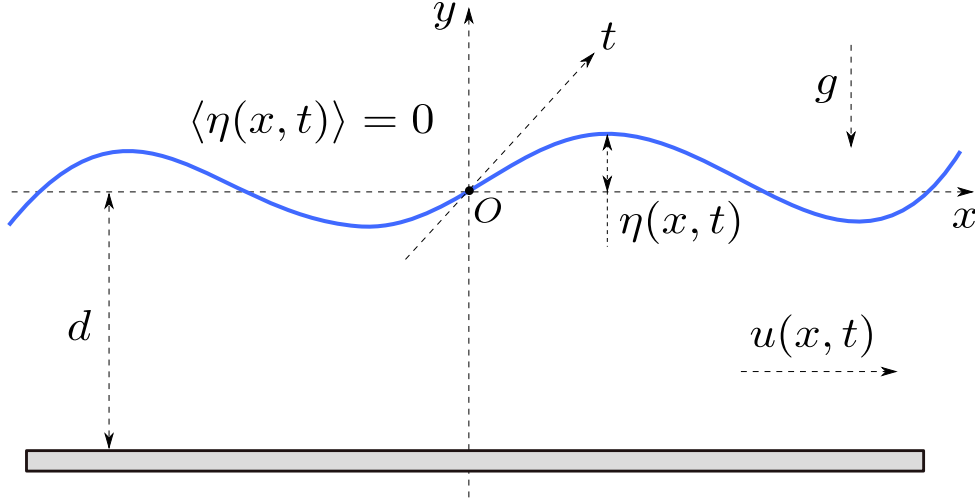
$$\eta_t + [u + \frac{1}{2} \eta u]_x + a u_{xxx} - b \eta_{xxt} = 0, \quad (1.5)$$

$$u_t + [\eta + \frac{1}{4} \eta^2 + \frac{3}{4} u^2]_x + c \eta_{xxx} - d u_{xxt} = 0, \quad (1.6)$$

with  $a, b, c, d$  as in (1.3), (1.4), was proposed by BONA *et al.* in [8]. These systems retain the same order of approximation to the EULER equations as (1.1), (1.2), reduce to a symmetric hyperbolic system when the dispersive third-order derivative terms are omitted,

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\*Here, the symmetry is understood in the sense of FRIEDRICHS–LAX [34].



**Figure 1.** Sketch of the fluid domain. We note that it is always possible to choose dimensionless variables such that  $g = 1$  and  $d = 1$ . That is why these constants are absent in governing Equations (1.5), (1.6) (such a change of dependent and independent variables is given in [28, Equation (3.13)]). The brackets  $\langle \cdot \rangle$  denote the spatial averaging operator, which fix the still water level in our problem.

preserve the  $L^2$  norm

$$\int_{-\infty}^{+\infty} (\eta^2 + u^2) dx, \quad (1.7)$$

and, concerning existence and uniqueness of solutions of the corresponding IVP, are locally well-posed. Higher-order BOUSSINESQ-type systems are proposed in *e.g.* [6] and also by DARIPA in [26]. Although we focus here on surface wave theory, it may be worth mentioning the derivation of BOUSSINESQ systems for internal waves in, *e.g.* [10].

It is well-known that only a sub-family of  $(a, b, c, d)$  systems (1.1), (1.2) with  $b = d$  possesses a co-symplectic HAMILTONIAN structure

$$\begin{pmatrix} \eta_t \\ u_t \end{pmatrix} = \mathbb{J} \cdot \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \eta} \\ \frac{\delta \mathcal{H}}{\delta u} \end{pmatrix}, \quad (1.8)$$

on a suitable functional space for  $(\eta, u)$ , where the co-symplectic structure is given by the nonlocal matrix operator

$$\mathbb{J} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -(1 - b \partial_{xx}^2)^{-1} \circ \partial_x \\ -(1 - b \partial_{xx}^2)^{-1} \circ \partial_x & 0 \end{pmatrix}, \quad (1.9)$$

(the symbol  $\circ$  denotes composition) and  $\mathcal{H}$  is the HAMILTONIAN functional given by

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} \{ \eta^2 + (1 + \eta) u^2 - c \eta_x^2 - a u_x^2 \} dx.$$

with  $\frac{\delta \mathcal{H}}{\delta \eta}$ ,  $\frac{\delta \mathcal{H}}{\delta u}$  standing for the variational (*i.e.* FRÉCHET) derivatives of  $\mathcal{H}$  with respect to  $\eta$  and  $u$ , respectively. (This HAMILTONIAN structure is lost in the symmetric version (1.5), (1.6).) Note that the co-symplectic operator  $\mathbb{J}$  does not depend explicitly neither on time  $t$ , nor on space  $x$ , nor on the solution  $(\eta, u)$ . In this sense the symplectic structure is quite rigid for Partial Differential Equations (PDEs). For each IVP of (1.8), the HAMILTONIAN functional  $\mathcal{H}(t)$  is conserved in time and its value, determined by the corresponding initial conditions, can be considered as a generalized ‘energy’ of the state of the system represented by the solution, [23]. In addition to  $\mathcal{H}$ , the impulse functional

$$\int_{-\infty}^{+\infty} (\eta u + b u_x \eta_x) dx,$$

is preserved by smooth, decaying enough at infinity solutions.

The present paper delves into the properties of BOUSSINESQ-type systems focusing on the multi-symplectic (MS) structure. It is the first part of a study devoted by the authors to this question and deals with the derivation of a MS family of equations of BOUSSINESQ type, the comparison with other existing Boussinesq systems and the description of some mathematical properties, with special emphasis on the existence of traveling-wave solutions. The second part of the study, devoted to the construction of multi-symplectic schemes for the approximation to the systems, will be the subject of a forthcoming paper.

The multi-symplectic theory generalizes the classical HAMILTONIAN formulations, [3], to the case of PDEs such that the space and time variables are treated on the equal footing [13] (see also [49, Chapter 12]). Multi-symplectic formulations are also gaining popularity for both mathematical investigations, [56], along with numerical *structure preserving* modeling, [15, 29].

The history of multi-symplectic formulations can be traced back to V. VOLTERRA (1890) who generalized HAMILTONIAN equations for variational problems involving several variables, [63, 64]. Later these ideas were developed in 1930’s, [27, 50, 66]. Finally, in 1970’s this theory was geometrized by several mathematical physicists, [36, 45–47] similarly to the evolution of symplectic geometry from the ideas of J.-L. LAGRANGE, [48, 59]. In our study we will be inspired by modern works on multi-symplectic PDEs, [13, 56]. Recently this theory has found many applications to the development of structure-preserving integrators with different strategies, [15, 21, 29, 57].

Let us also briefly review the main known equations arising in the modeling of long waves with MS formulation. The KdV equation is among the multi-symplectic veterans, [15, 67], along with the NLS equation, [20]. Among one-way propagation models, the multi-symplectic structure of the BENJAMIN–BONA–MAHONY (BBM) and generalized\* BBM equations was highlighted in [61]. Some interesting numerical results for the BBM

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\*The generalization consisted in taking higher order nonlinearities, *i.e.*  $u u_x \rightsquigarrow u^q u_x$ ,  $q \geq 1$ .



equation were presented in [51]. The ‘good’ scalar BOUSSINESQ equation was studied in [40]. The multi-symplectic structure of the celebrated SERRE–GREEN–NAGHDI (SGN) equations was unveiled in [22]. Finally, the multi-symplectic structure of two-layer SGN system modeling the propagation of long fully nonlinear internal waves was highlighted recently in [25]. We also remind that the full water wave problem is multi-symplectic, [13].

The main highlights of the present paper revolve around the derivation of a family of BOUSSINESQ systems with multi-symplectic structure. The procedure for the derivation starts from a system of  $(a, b, c, d)$  form with a general homogeneous quadratic polynomial as nonlinear term and determines a combination of parameters allowing the multi-symplectic formulation. Additional advantages of the technique implemented are the identification of the MS structure in BOUSSINESQ systems of the family (1.5), (1.6) and the way how to modify the Equations (1.1), (1.2) to admit the MS property.

After the derivation, it is expected to discuss some properties of the new systems. In this sense, the present paper is focused on the identification of those equations of the family with additional symplectic structure, the well-posedness of the corresponding IVP and the existence of different types of solitary-wave solutions. For the last two points, the form of the nonlinearity enables us to make use of the existing literature on them. The discussion of consistency of the EULER equations with the new system is here initiated from the MS structure of Equations (1.5), (1.6). Considered as part of this discussion are also some comparisons of the solitary-wave speed-amplitude relations for the MS systems with systems (1.1), (1.2) and the EULER system. These comparisons suggest the existence of new MS systems, derived in this paper, with good performance in this sense. Nevertheless, a complete analysis of the question deserves a future research. The description of the models is completed with the construction and development of multi-symplectic schemes of approximation, being the subject of a second, forthcoming part.

The paper is organized according to the following structure. After a brief reminder of MS theory for PDEs, Section 2 is devoted to the procedure of derivation of the MS structure. The main steps are first described to obtain the MS formulations of the KdV–BBM equation and the symmetric System (1.5), (1.6). The technique is then extended to construct a family of MS BOUSSINESQ-type systems and to explain how to modify (1.1), (1.2) in such a way that the resulting systems are multi-symplectic. Some properties of the new equations concerning well-posedness and existence of solitary-wave solutions are studied in Section 3. The main conclusions and perspectives are outlined in Section 4.

## 2. Multi-symplectic structure

We recall first some basic facts about multi-symplectic geometry and PDEs. A PDE<sup>\*</sup> (or a system of PDEs) is said to be multi-symplectic in one space dimension  $x \in \mathbb{R}$  if it

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<sup>\*</sup>Note that there is no assumption that the IVP for this PDE is well-posed. To give an example, the multi-symplectic setting may include some elliptic PDEs in space-time which are ill-posed in the sense of HADAMARD.

can be written in the following canonical form:

$$\mathbf{K} \cdot \mathbf{z}_t + \mathbf{M} \cdot \mathbf{z}_x = \nabla_{\mathbf{z}} \mathfrak{S}(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^d, \quad (2.1)$$

for some  $d \geq 3$ , where  $\mathbf{z}(x, t) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}^d$ ,  $\mathbf{K} \in \mathcal{M}_{d \times d}(\mathbb{R})$  and  $\mathbf{M} \in \mathcal{M}_{d \times d}(\mathbb{R})$  are some real, skew-symmetric  $d \times d$  matrices, the dot  $\cdot$  denotes the matrix-vector product in  $\mathbb{R}^d$  and  $\nabla_{\mathbf{z}}$  is the classical gradient operator in  $\mathbb{R}^d$ . The given function  $\mathfrak{S}(\mathbf{z})$  is assumed to be a smooth function of its  $d$  variables  $\mathbf{z} = (z_1, z_2, \dots, z_d)$ .

The formulation (2.1) represents a direct generalization of HAMILTONIAN PDEs where the space and time are treated on equal footing. On the other hand, the main drawback of formulation (2.1) is that it is not intrinsic, *i.e.* not coordinate-free on the base manifold  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ . However, it is sufficient for our purposes, since in wave propagation we would like to keep the distinction between space and time coordinates.

All equations with the multi-symplectic form (2.1), have also a certain number of relevant properties. One is the existence of conservation laws. Note first that Equations (2.1) satisfy the *multi-symplectic conservation law*:

$$\omega_t + \kappa_x = 0, \quad (2.2)$$

with

$$\omega \stackrel{\text{def}}{=} \frac{1}{2} d\mathbf{z} \wedge (\mathbf{K} \cdot d\mathbf{z}), \quad \kappa \stackrel{\text{def}}{=} \frac{1}{2} d\mathbf{z} \wedge (\mathbf{M} \cdot d\mathbf{z}), \quad (2.3)$$

with  $\wedge$  being the standard exterior product of differential forms, [60]. The 2-form  $\omega$  defines a symplectic structure on  $\mathbb{R}^{\text{rank } \mathbf{K}}$  (where  $\text{rank } \mathbf{K}$  denotes the rank of the matrix  $\mathbf{K}$ ), which is associated with the time direction and the 2-form  $\kappa$  defines a symplectic structure  $\mathbb{R}^{\text{rank } \mathbf{M}}$ , which is associated with the space direction, [15]. From definitions of 2-forms (2.3), we can see that the multi-symplectic approach relies on a local concept of symplecticity, since symplectic forms may vary with the solution in space and in time similarly to a LAGRANGIAN tracer in a fluid flow. This observation explains why the class of multi-symplectic PDEs is more general. On the other hand, when designing a geometric integrator, the principal requirement is that the discretization conserves exactly the symplecticity (2.2).

**Remark 1.** In classical field theories sometimes the following 3-form is called a *multi-symplectic form*:

$$\Omega \stackrel{\text{def}}{=} \omega \wedge dx - \kappa \wedge dt - d\mathfrak{S} \wedge dx \wedge dt \in \bigwedge^3 (\mathbb{R}^d \times \mathbb{R}^2).$$

In [15] it was proposed to call  $\Omega$  a meta-symplectic form and we follow this terminology. Throughout our manuscript the term multi-symplectic form refers to the system of PDEs (2.1) or to the couple of differential forms  $\omega, \kappa \in \bigwedge^2 (\mathbb{R}^d)$  depending on the context.

Additionally, other conservation laws can be derived when the function  $\mathfrak{S}(\mathbf{z})$  does not depend explicitly on  $x$  or  $t$ . Defining the generalized energy  $\mathfrak{E}$  and generalized momentum  $\mathfrak{J}$  densities as

$$\mathfrak{E}(\mathbf{z}) \stackrel{\text{def}}{=} \mathfrak{S}(\mathbf{z}) - \frac{1}{2} \langle \mathbf{z}, \mathbf{M} \cdot \mathbf{z}_x \rangle, \quad \mathfrak{J}(\mathbf{z}) \stackrel{\text{def}}{=} \frac{1}{2} \langle \mathbf{z}, \mathbf{K} \cdot \mathbf{z}_x \rangle,$$

and corresponding fluxes

$$\mathfrak{F}(\mathbf{z}) \stackrel{\text{def}}{=} \frac{1}{2} \langle \mathbf{z}, \mathbf{M} \cdot \mathbf{z}_t \rangle, \quad \mathfrak{M}(\mathbf{z}) \stackrel{\text{def}}{=} \mathfrak{S}(\mathbf{z}) - \frac{1}{2} \langle \mathbf{z}, \mathbf{K} \cdot \mathbf{z}_t \rangle,$$

(the symbol  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product in  $\mathbb{R}^d$ ) then the generalized energy  $\mathcal{E} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mathfrak{E}(\mathbf{z}) dx$  and momentum  $\mathcal{J} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mathfrak{J}(\mathbf{z}) dx$  are conserved:

$$\mathfrak{E}_t + \mathfrak{F}_x = 0, \quad \mathfrak{J}_t + \mathfrak{M}_x = 0, \quad (2.4)$$

Note that one of the advantages of the multi-symplectic structure is that a geometric interpretation of the conservation laws can be derived from the application of NOETHER theory. Note also that, in general, it is not possible to conserve energy and momentum (2.4) with *uniform discretizations* additionally to multi-symplecticity (2.2).

Finally, multi-symplectic PDEs (2.1) automatically possess also the LAGRANGIAN variational structure. Indeed, it is not difficult to check that Equation (2.1) is the EULER-LAGRANGE equation to the following LAGRANGIAN functional:

$$\mathcal{L} = \int_{t_0}^{t_1} \mathcal{L} dt, \quad \mathcal{L} \rightarrow \text{St},$$

with the LAGRANGIAN density defined as

$$\mathcal{L} \stackrel{\text{def}}{=} \int_{x_0}^{x_1} \left\{ \frac{1}{2} \langle \mathbf{K} \cdot \mathbf{z}_t, \mathbf{z} \rangle + \frac{1}{2} \langle \mathbf{M} \cdot \mathbf{z}_x, \mathbf{z} \rangle - \mathfrak{S}(\mathbf{z}) \right\} dx.$$

Here  $t_0, t_1$  denote arbitrary instances of time  $t_1 > t_0 \geq 0$  and, similarly,  $x_0, x_1$  are some locations in space  $x_1 > x_0$ . From the last definition, observe that the function  $\mathfrak{S}(\mathbf{z})$  plays the rôle of the generalized potential energy. The HAMILTON principle states that the physical trajectory corresponds to stationary values of the action functional, *i.e.*  $\mathcal{L} \rightarrow \text{St}$ .

## 2.1. MS structure of the KdV-BBM equation

As a necessary step towards understanding the way how the multi-symplectic structure of the  $(a, b, c, d)$ -type systems will be derived later, we first study the case of the KdV-BBM equation

$$u_t + uu_x + \alpha u_{xxx} - \beta u_{xxt} = 0, \quad (2.5)$$

where  $\alpha, \beta \in \mathbb{R}$  are some real coefficients. The variable  $u(x, t)$  can be related both to the velocity or to the free surface elevation. The KdV-BBM Equation (2.5) arises in the modeling of unidirectional water wave propagation [31] and in the study of non-integrable solitonic gases [32]. It was also used to study the well-posedness of the KdV equation in [11].) The multi-symplectic structure of (2.5) has never been reported before to the best of our knowledge. We provide below an example of such structure in  $\mathbb{R}^5$ .

At the first step, we rewrite Equation (2.5) in a conservative form:

$$u_t + \left[ \frac{1}{2} u^2 + \alpha u_{xx} - \beta u_{xt} \right]_x = 0. \quad (2.6)$$

In order to lower the order of the equation, the following variables are introduced:

$$\phi_x \stackrel{\text{def}}{=} u, \quad v \stackrel{\text{def}}{=} u_x, \quad w \stackrel{\text{def}}{=} u_t.$$

The new variable  $\phi$  is a generalized potential for the field  $u$ . This potential appears, for example, in LAGRANGIAN variational formulations of both KdV and BBM equations. As the final conceptual step, we rewrite Equation (2.6) in the following slightly unusual way:

$$\frac{1}{2} u_t + \underbrace{\left[ \frac{1}{2} u^2 + \frac{1}{2} \phi_t + \alpha v_x - \frac{1}{2} \beta w_x - \frac{1}{2} \beta v_t \right]}_{\equiv p} = 0.$$

The last equation suggests the introduction of an extra variable, which has the meaning of a flux:

$$p \stackrel{\text{def}}{=} \frac{1}{2} u^2 + \frac{1}{2} \phi_t + \alpha v_x - \frac{1}{2} \beta w_x - \frac{1}{2} \beta v_t.$$

Now, we have all elements to present the desired multi-symplectic structure. The vector  $\mathbf{z}$  including the field  $u$  along with ‘conjugate momenta’ is defined as

$$\mathbf{z} \stackrel{\text{def}}{=} (u, \phi, v, w, p) \in \mathbb{R}^5.$$

Note then that (2.5) is equivalent to the system of equations:

$$\frac{1}{2} \phi_t - \frac{1}{2} \beta v_t + \alpha v_x - \beta w_x = p - \frac{1}{2} u^2, \quad (2.7)$$

$$-\frac{1}{2} u_t - p_x = 0, \quad (2.8)$$

$$\frac{1}{2} \beta u_t - \alpha u_x = -\alpha v + \frac{1}{2} \beta w, \quad (2.9)$$

$$\frac{1}{2} \beta u_x = \frac{1}{2} \beta v, \quad (2.10)$$

$$\phi_x = u. \quad (2.11)$$

We now consider the vector field  $\mathfrak{R} : \mathbb{R}^5 \mapsto \mathbb{R}^5$  given by the right hand side of (2.7) – (2.11):

$$\mathfrak{R}(\mathbf{z}) \stackrel{\text{def}}{=} {}^\top \left( p - \frac{1}{2} u^2, 0, -\alpha v + \frac{1}{2} \beta w, \frac{1}{2} \beta v, u \right).$$

Since the JACOBIAN  $\mathfrak{R}'(\mathbf{z})$  of  $\mathfrak{R}$  is symmetric for all  $\mathbf{z}$ , then POINCARÉ lemma implies that  $\mathfrak{R}$  is conservative and thus  $\mathfrak{R} = \nabla \mathfrak{S}$  for some potential  $\mathfrak{S}$ . Therefore, System (2.7) – (2.11) can be written in the canonical matrix-vector form (2.1) with the following skew-symmetric matrices:

$$\mathbf{K} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \beta & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 & 0 & \alpha & -\frac{1}{2} \beta & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -\alpha & 0 & 0 & 0 & 0 \\ \frac{1}{2} \beta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

A potential energy  $\mathfrak{S}(\mathbf{z})$  for the KdV–BBM Equation (2.5) is

$$\mathfrak{S}(\mathbf{z}) = pu - \frac{1}{6} u^3 - \frac{1}{2} \alpha v^2 + \frac{1}{2} \beta vw.$$

This completes the definition of the multi-symplectic structure for (2.5). We conjecture that another multi-symplectic structure for this equation in  $\mathbb{R}^d$ , with  $d < 5$  is not possible.

## 2.2. MS structure of the symmetric $(a, b, c, d)$ Boussinesq system

The previous steps can be adapted to obtain the MS structure of some systems of the family (1.5), (1.6). Similarly to Section 2.1, these can first be written in the conservative form

$$\eta_t + \left[ u + \frac{1}{2} \eta u + a u_{xx} - b \eta_{xt} \right]_x = 0, \quad (2.12)$$

$$u_t + \left[ \eta + \frac{1}{4} \eta^2 + \frac{3}{4} u^2 + c \eta_{xx} - d u_{xt} \right]_x = 0, \quad (2.13)$$

and we introduce the additional variables:

$$\phi_{1x} \stackrel{\text{def}}{=} \eta, \quad v_1 \stackrel{\text{def}}{=} \eta_x, \quad w_1 \stackrel{\text{def}}{=} \eta_t, \quad (2.14)$$

$$\phi_{2x} \stackrel{\text{def}}{=} u, \quad v_2 \stackrel{\text{def}}{=} u_x, \quad w_2 \stackrel{\text{def}}{=} u_t, \quad (2.15)$$

where  $\phi_{1,2}$  are generalized potentials and  $v_{1,2}, w_{1,2}$  are space and time gradients of dynamic variables  $\eta$  and  $u$  correspondingly. Using these variables, we can rewrite Equations (2.12), (2.13) as

$$\begin{aligned} \frac{1}{2} \eta_t + \underbrace{\left[ u + \frac{1}{2} \eta u + \frac{1}{2} \phi_{1t} + a v_{2x} - \frac{1}{2} b v_{1t} - \frac{1}{2} b w_{1x} \right]}_{\equiv p_1} &= 0, \\ \frac{1}{2} u_t + \underbrace{\left[ \eta + \frac{1}{4} \eta^2 + \frac{3}{4} u^2 + \frac{1}{2} \phi_{2t} + c v_{1x} - \frac{1}{2} d v_{2t} - \frac{1}{2} d w_{2x} \right]}_{\equiv p_2} &= 0. \end{aligned}$$

Thus, defining two additional fluxes:

$$p_1 \stackrel{\text{def}}{=} u + \frac{1}{2} \eta u + \frac{1}{2} \phi_{1t} + a v_{2x} - \frac{1}{2} b v_{1t} - \frac{1}{2} b w_{1x}, \quad (2.16)$$

$$p_2 \stackrel{\text{def}}{=} \eta + \frac{1}{4} \eta^2 + \frac{3}{4} u^2 + \frac{1}{2} \phi_{2t} + c v_{1x} - \frac{1}{2} d v_{2t} - \frac{1}{2} d w_{2x}, \quad (2.17)$$

then the vector  $\mathbf{z}$  with auxiliary variables can now be defined:

$$\mathbf{z} \stackrel{\text{def}}{=} (\eta, \phi_1, v_1, w_1, p_1, u, \phi_2, v_2, w_2, p_2) \in \mathbb{R}^{10}. \quad (2.18)$$

$$\begin{aligned}\frac{1}{2} \phi_{1t} - \frac{1}{2} b v_{1t} + a v_{2x} - \frac{1}{2} b w_{1x} &= p_1 - u - \frac{1}{2} \eta u, \\ -\frac{1}{2} \eta_t - p_{1x} &= 0, \\ \frac{1}{2} b \eta_t - a u_x &= \frac{1}{2} b w_1 - a v_2, \\ \frac{1}{2} b \eta_x &= \frac{1}{2} b v_1, \\ \phi_{1x} &= \eta, \\ \frac{1}{2} \phi_{2t} - \frac{1}{2} d v_{2t} + c v_{1x} - \frac{1}{2} d w_{2x} &= p_2 - \eta - \frac{1}{4} \eta^2 - \frac{3}{4} u^2, \\ -\frac{1}{2} u_t - p_{2x} &= 0, \\ \frac{1}{2} d u_t - a \eta_x &= \frac{1}{2} d w_2 - c v_1, \\ \frac{1}{2} d u_x &= \frac{1}{2} d v_2, \\ \phi_{2x} &= u.\end{aligned}$$
$$\mathfrak{R}(z) \stackrel{\text{def}}{=} {}^\top \left( p_1 - u - \frac{1}{2} \eta u, 0, \frac{1}{2} b w_1 - a v_2, \frac{1}{2} b v_1, \eta, \right. \\ \left. p_2 - \eta - \frac{1}{4} \eta^2 - \frac{3}{4} u^2, 0, \frac{1}{2} d w_2 - c v_1, \frac{1}{2} d v_2, u \right),$$
[illegible]

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}b & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 \\ \frac{1}{2}b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad (2.20)$$

and where a potential energy for the symmetric  $(a, b, a, d)$  sub-family reads:

$$\mathfrak{S}(z) \stackrel{\text{def}}{=} p_1 \eta - \eta u - \frac{1}{4} \eta^2 u + \frac{1}{2} b v_1 w_1 - \frac{1}{4} u^3 + \frac{1}{2} d v_2 w_2 - a v_1 v_2 + p_2 u.$$

This completes the presentation of the multi-symplectic structure of Equations (2.12), (2.13). It is noted that only a sub-class of symmetric Systems (1.5), (1.6) (with  $a = c$ ) possesses a multi-symplectic structure.

### 2.3. A MS family of Boussinesq-type systems

The construction presented in the previous sections can be conformed to derive a new family of BOUSSINESQ-type systems with MS structure. We start from a system of equations of the general form

$$\eta_t + [u + \mathfrak{A}(\eta, u) + a u_{xx} - b \eta_{xt}]_x = 0, \quad (2.21)$$

$$u_t + [\eta + \mathfrak{B}(\eta, u) + c \eta_{xx} - d u_{xt}]_x = 0, \quad (2.22)$$

of BOUSSINESQ type. The parameters  $(a, b, c, d)$  are as in (1.3), (1.4) and the nonlinearities  $\mathfrak{A}$  and  $\mathfrak{B}$  are homogeneous, quadratic polynomials

$$\mathfrak{A}(\eta, u) \stackrel{\text{def}}{=} \alpha_{11} \eta^2 + \alpha_{12} \eta u + \alpha_{22} u^2,$$

$$\mathfrak{B}(\eta, u) \stackrel{\text{def}}{=} \beta_{11} \eta^2 + \beta_{12} \eta u + \beta_{22} u^2,$$

with real coefficients  $\alpha_{ij}, \beta_{ij}$  which can be chosen on modeling or geometric structure bases. System of the form (2.21), (2.22) have been used for modelling nonlinear waves in different situations. Several examples are given below:

- (1) The  $(a, b, c, d)$  System (1.1), (1.2) trivially corresponds to the choice

$$\alpha_{11} = 0, \quad \alpha_{12} = 1, \quad \alpha_{22} = 0,$$

$$\beta_{11} = 0, \quad \beta_{12} = 0, \quad \beta_{22} = \frac{1}{2}.$$

(2) The symmetric version (1.5), (1.6) is obtained from (2.21), (2.22) with

$$\begin{aligned}\alpha_{11} &= 0, & \alpha_{12} &= \frac{1}{2}, & \alpha_{22} &= 0, \\ \beta_{11} &= \frac{1}{4}, & \beta_{12} &= 0, & \beta_{22} &= \frac{3}{4}.\end{aligned}$$

(3) Some examples of (2.21), (2.22) also appear by choosing specific values of  $(a, b, c, d)$  instead of the parameters of the nonlinearities. This is the case of the system of KdV type considered by BONA *et al.* in [5]:

$$\eta_t + [\mathfrak{A}(\eta, u) + u_{xx}]_x = 0, \quad (2.23)$$

$$u_t + [\mathfrak{B}(\eta, u) + \eta_{xx}]_x = 0, \quad (2.24)$$

with  $\mathfrak{A}$  and  $\mathfrak{B}$  as above, which corresponds to taking  $a = c = 1, b = d = 0$  in (2.21), (2.22) and where the terms  $u_x$  and  $\eta_x$  can be omitted by a suitable change of variables.

(4) Other examples can be found in the GEAR–GRIMSHAW system, [35], for internal wave propagation or in the coupled systems of BBM-type considered in [39].

In order to obtain a MS structure in the Equations (2.21), (2.22), we consider (2.14), (2.15) and generalize (2.16), (2.17) by introducing the auxiliary variables

$$p_1 \stackrel{\text{def}}{:=} u + \mathfrak{A}(\eta, u) + \frac{1}{2} \phi_{1t} + a v_{2x} - \frac{1}{2} b v_{1t} - \frac{1}{2} b w_{1x}, \quad (2.25)$$

$$p_2 \stackrel{\text{def}}{:=} \eta + \mathfrak{B}(\eta, u) + \frac{1}{2} \phi_{2t} + c v_{1x} - \frac{1}{2} d v_{2t} - \frac{1}{2} d w_{2x}. \quad (2.26)$$

Then, Equations (2.21), (2.22) can be written in an equivalent form as the following system of first-order differential relations:

$$\begin{aligned}\frac{1}{2} \phi_{1t} - \frac{1}{2} b v_{1t} + a v_{2x} - \frac{1}{2} b w_{1x} &= p_1 - u - \mathfrak{A}(\eta, u), \\ -\frac{1}{2} \eta_t - p_{1x} &= 0, \\ \frac{1}{2} b \eta_t - a u_x &= \frac{1}{2} b w_1 - a v_2, \\ \frac{1}{2} b \eta_x &= \frac{1}{2} b v_1, \\ \phi_{1x} &= \eta, \\ \frac{1}{2} \phi_{2t} - \frac{1}{2} d v_{2t} + c v_{1x} - \frac{1}{2} d w_{2x} &= p_2 - \eta - \mathfrak{B}(\eta, u), \\ -\frac{1}{2} u_t - p_{2x} &= 0, \\ \frac{1}{2} d u_t - a \eta_x &= \frac{1}{2} d w_2 - c v_1, \\ \frac{1}{2} d u_x &= \frac{1}{2} d v_2, \\ \phi_{2x} &= u.\end{aligned}$$



Now the vector field  $\mathfrak{R}(\mathbf{z})$  (with vector  $\mathbf{z}$  defined as in (2.18)) on the right hand side is of the form

$$\mathfrak{R}(\mathbf{z}) \stackrel{\text{def}}{=} {}^\top \left( p_1 - u - \mathfrak{A}(\eta, u), 0, \frac{1}{2} b w_1 - a v_2, \frac{1}{2} b v_1, \eta, \right. \\ \left. p_2 - \eta - \mathfrak{B}(\eta, u), 0, \frac{1}{2} d w_2 - a v_1, \frac{1}{2} d v_2, u \right).$$

Imposing the symmetry of the JACOBIAN leads to

$$a = c, \quad \frac{\partial \mathfrak{A}}{\partial u} = \frac{\partial \mathfrak{B}}{\partial \eta}.$$

The last condition holds for any values of  $\eta$  and  $u$  if and only if

$$\alpha_{12} = 2\beta_{11}, \quad \beta_{12} = 2\alpha_{22}. \quad (2.27)$$

Henceforth, by POINCARÉ lemma,  $\mathfrak{R}(\mathbf{z})$  is conservative, *i.e.*  $\mathfrak{R}(\mathbf{z}) = \nabla_{\mathbf{z}} \mathfrak{S}(\mathbf{z})$ , when  $a = c$  and (2.27) hold. All generalized Systems (2.21), (2.22) with such coefficients are multi-symplectic with a potential functional given *e.g.* by

$$\mathfrak{S}(\mathbf{z}) \stackrel{\text{def}}{=} p_1 \eta - \eta u - \frac{1}{3} \alpha_{11} \eta^3 - \beta_{11} \eta^2 u - \frac{1}{2} \beta_{12} \eta u^2 + \frac{1}{2} b v_1 w_1 \\ - \frac{1}{3} \beta_{22} u^3 + \frac{1}{2} d v_2 w_2 - a v_1 v_2 + p_2 u.$$

The skew-symmetric matrices  $\mathbf{K}$  and  $\mathbf{M}$  are defined as in (2.19), (2.20).

**Remark 2.** *It may be worth describing the application of this result to the above mentioned particular cases:*

- (1) *From conditions (2.27) it can be readily seen that the asymptotically consistent  $(a, b, a, d)$  family (1.1), (1.2) is not multi-symplectic. Indeed,*

$$1 = a_{12} \neq 2\beta_{11} = 0, \\ 0 = \beta_{12} = 2a_{22} = 0.$$

*From these observations it follows that the  $(a, b, a, d)$  family (1.1), (1.2) can be made naturally multi-symplectic with minimal modifications if we take  $\beta_{11} = \frac{1}{2}$ . All other coefficients are kept unchanged. The resulting multi-symplectic BOUSSINESQ-type system reads:*

$$\eta_t + [u + \eta u]_x + a u_{xxx} - b \eta_{xxt} = 0, \quad (2.28)$$

$$u_t + \left[ \eta + \underbrace{\frac{1}{2} \eta^2}_{(*)} + \frac{1}{2} u^2 \right]_x + c \eta_{xxx} - d u_{xxt} = 0. \quad (2.29)$$

*The new term  $(*)$  is asymptotically small, since its magnitude, in non-dimensional, scaled variables, is  $\mathcal{O}(\varepsilon^2)$ , where  $\varepsilon$  is the nonlinearity order parameter.*

- (2) *It is clear that the MS structure of the symmetric family of (1.5), (1.6), shown in Section 2.2, can be alternatively obtained from the verification of the conditions derived here.*

(3) In the Systems (2.23), (2.24), since  $a = c = 1$ , the MS structure holds for those equations for which the coupled nonlinear terms satisfy (2.27). Furthermore, it can be seen that the conditions (2.27) are related to the reduces system in [5] to simplify the study of existence and stability of the solitary wave solutions.

**Remark 3.** Conditions  $a = c$ , and (2.27) for a multi-symplectic structure of Systems (2.21), (2.22) were obtained by using the set of variables (2.14), (2.15) and (2.25), (2.26). The derivation of other conditions with different variables or with extensions of the definition in (2.1) which allows the matrices  $\mathbf{K}$  and  $\mathbf{M}$  be dependent of  $\mathbf{z}$ , cf. [13], is not discarded. Thus, the BOUSSINESQ System (1.3), (1.4) with  $a = 0$ ,  $b = d = 1/3$  and  $c = -1/3$  is multi-symplectified in [14].

### 2.3.1. Systems with symplectic and multi-symplectic structure

We note that if  $b = d$ , the Equations (2.21), (2.22) have a HAMILTONIAN structure (1.8), with the structure operator  $\mathbb{J}$  as in (1.9), when the coefficients of the nonlinear terms  $\mathfrak{A}(\eta, u)$  and  $\mathfrak{B}(\eta, u)$  satisfy

$$\beta_{12} = 2\alpha_{11}, \quad \alpha_{12} = 2\beta_{22}. \quad (2.30)$$

The HAMILTONIAN is now given by

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} \{ \eta^2 + u^2 - c\eta_x^2 - au_x^2 + 2\mathfrak{G}(\eta, u) \} dx,$$

where

$$\mathfrak{G}(\eta, u) \stackrel{\text{def}}{=} \frac{\beta_{11}}{3} \eta^3 + \frac{\beta_{12}}{2} \eta^2 u + \beta_{22} \eta u^2 + \frac{\alpha_{22}}{3} u^3.$$

This allows for studying whether the intersection of symplectic and multi-symplectic BOUSSINESQ-type Systems in (2.21), (2.22) is non-empty. It is not hard to see that the compatibility of the conditions (2.27) and (2.30) holds when

$$\beta_{12} = 2\alpha_{11} = 2\alpha_{22}, \quad \alpha_{12} = 2\beta_{11} = 2\beta_{22}, \quad (2.31)$$

and then the corresponding  $(a, b, a, b)$  System (2.21), (2.22) is multi-symplectic and symplectic. The family is determined by two dispersive parameters (say  $(a, b)$ ) and two nonlinear parameters (say  $(\beta_{11}, \beta_{12})$ ). The corresponding HAMILTONIAN is given by:

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} \{ \eta^2 + u^2 - a(\eta_x^2 + u_x^2) + 2\mathfrak{G}(\eta, u) \} dx,$$

where

$$\mathfrak{G}(\eta, u) \stackrel{\text{def}}{=} \frac{\beta_{11}}{3} \eta^3 + \frac{\beta_{12}}{2} \eta^2 u + \beta_{11} \eta u^2 + \frac{\beta_{12}}{6} u^3.$$

Note that the family (1.5), (1.6) does not contain any system with both structures. (Actually, as mentioned before, any of them does not have a HAMILTONIAN formulation.) On the other hand, systems of KdV type (2.23), (2.24) (or of BBM type, [39]) with homogeneous quadratic nonlinearities, are symplectic and multi-symplectic when (2.31) holds. Finally,

it can be seen that System (2.28), (2.29) with  $b = d$  and  $a = c$  is symplectic and multi-symplectic (for  $\beta_{11} = \frac{1}{2}, \beta_{12} = 0$ ).

### 3. Some properties of the MS Boussinesq-type systems

In this Section some additional properties of the MS BOUSSINESQ Systems (2.21), (2.22) satisfying  $a = c$  and (2.27) will be discussed. They will be focused on the well-posedness of the corresponding IVP and the existence of solitary wave solutions. The results are mainly based on the literature about these questions for  $(a, b, c, d)$  systems.

#### 3.1. Linear well-posedness

Linear well-posedness can be studied using the arguments considered in [6] for the  $(a, b, c, d)$  System (1.1), (1.2), since the linear part coincides with that of Equations (2.21), (2.22). According to this, and if

$$\omega_1(k) \stackrel{\text{def}}{=} \frac{1 - ak^2}{1 + bk^2}, \quad \omega_2(k) \stackrel{\text{def}}{=} \frac{1 - ck^2}{1 + dk^2}, \quad k \in \mathbb{R},$$

then the linearized problem is well-posed when the rational function  $\omega_1(k)/\omega_2(k)$  has neither zeros or poles on the real axis. This limits the range of the parameters  $(a, b, c, d)$  to the three ‘admissible’ cases derived in [6, Proposition 3.1]. The MS structure, which requires the condition  $a = c$ , reduces them to

$$(L_1) : \quad b \geq 0, \quad d \geq 0, \quad a = c,$$

$$(L_2) : \quad b = d < 0, \quad a = c > 0.$$

Furthermore, for  $(a, b, c, d)$  satisfying one of the conditions  $(L_1), (L_2)$ , if  $\ell$  denotes the order of the symbol

$$g(k) \stackrel{\text{def}}{=} \sqrt{\frac{\omega_1(k)}{\omega_2(k)}}, \quad k \in \mathbb{R},$$

and  $m_1 \stackrel{\text{def}}{=} \max\{0, -\ell\}$ ,  $m_2 \stackrel{\text{def}}{=} \max\{0, \ell\}$ , then [6, Theorem 3.2] applies to have linear well-posedness of the IVP in the  $L^2$  based SOBOLEV spaces  $H^{s+m_1} \times H^{s+m_2}$  for any  $s \geq 0$ .

#### 3.2. Nonlinear well-posedness

The study of local well-posedness of the full nonlinear MS System (2.21), (2.22) may follow the lines established in [7] for the IVP of the  $(a, b, c, d)$  System (1.1), (1.2). This

is due to the fact that, although the nonlinear terms may be different, they are always homogeneous quadratic polynomials and, therefore, arguments and estimates used in [7] can be taken in the new systems. This leads to the following cases of local well-posedness in suitable SOBOLEV spaces:

$$(N_1) : \quad a = c, \quad b > 0, \quad d > 0.$$

$$(N_2) : \quad a = c > 0, \quad b = d = 0 \quad (\text{cf. [5]}),$$

$$(N_3) : \quad a = c, \quad b = 0, \quad d > 0.$$

$$(N_4) : \quad a = c \geq 0, \quad b > 0, \quad d = 0.$$

We also note that global well-posedness results can be obtained for those multi-symplectic and symplectic Systems (2.21), (2.22) with  $a = c$ ,  $b = d$  and satisfying (2.31).

### 3.3. Existence and classification of solitary waves

A second property to discuss here is the existence of solitary wave solutions

$$\zeta(x, t) \stackrel{\text{def}}{=} \zeta_s(x - c_s t), \quad u(x, t) \stackrel{\text{def}}{=} u_s(x - c_s t), \quad (3.1)$$

where  $c_s$  is the speed of propagation of the waves and the profiles  $\eta_s = \eta_s(X)$ ,  $u_s = u_s(X)$ ,  $X = x - c_s t$ , are smooth, positive and even functions. Substituting (3.1) into (2.21) and (2.22) (with  $a = c$ ), integrating once and setting the integration constant equals zero, we are interested in the profiles solutions of the nonlinear ordinary differential system

$$-c_s \zeta_s + u_s + \mathfrak{A}(\zeta_s, u_s) + a u_s'' + b c_s \zeta_s'' = 0, \quad (3.2)$$

$$-c_s u_s + \zeta_s + \mathfrak{B}(\zeta_s, u_s) + a \zeta_s'' + d c_s u_s'' = 0, \quad (3.3)$$

with  $\mathfrak{A}$ ,  $\mathfrak{B}$  satisfying the constraints (2.27). As in the previous section, our study will be based on the existing results on this subject in the literature for other BOUSSINESQ systems, especially for (1.1), (1.2). In order to derive the corresponding existence results for (3.2), (3.3), we will focus on the application of the *Normal Form Theory*, (see *e.g.* [18] and references therein). We refer to [28] for more details. The application of other existing theories, such as the *Positive Operator Theory*, [4], the *Concentration-Compactness Theory*, [52, 53] or TOLAND's Theory, [62], will be discussed in Section 4.

Normal Form Theory is useful to obtain the existence of solutions of (3.2), (3.3) for speeds  $c_s$  greater than but close to one and allows for distinguishing different kinds of solutions depending on the parameters  $a$  ( $= c$ ),  $b$  and  $d$ . The arguments exposed in, *e.g.* [28] for the Systems (1.1), (1.2) can be applied here from rewriting (3.2), (3.3) as a

first-order differential system for  $U = {}^\top(u_1, u_2, u_3, u_4) = {}^\top(\zeta_s, \zeta'_s, u_s, u'_s)$  as

$$U' = \mathcal{T}(U, c_s) = \mathfrak{L}(c_s)U + \mathcal{R}(U, c_s), \quad (3.4)$$

$$\mathfrak{L}(c_s) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{dc_s^2 + a}{\mathfrak{D}} & 0 & -\frac{c_s}{\mathfrak{D}}(a + d) & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{c_s(a + b)}{\mathfrak{D}} & 0 & \frac{1}{\mathfrak{D}}(bc_s^2 + a) & 0 \end{pmatrix}, \quad (3.5)$$

$$\mathcal{R}(U, c_s) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \frac{1}{\mathfrak{D}}(-dc_s\mathfrak{A}(u_1, u_3) + a\mathfrak{B}(u_1, u_3)) \\ 0 \\ \frac{1}{\mathfrak{D}}(-bc_s\mathfrak{B}(u_1, u_3) + a\mathfrak{A}(u_1, u_3)) \end{pmatrix}, \quad (3.6)$$

where  $\mathfrak{D} \stackrel{\text{def}}{=} bdc_s^2 - a^2$  is assumed to be nonzero. As in [28], the form of the nonlinear term (3.6) makes possible to study the existence of solutions of (3.4) for positive and small  $c_s - 1$  by using the linear part (3.5) and the *Normal Form Theory*, see e.g. [18] and references therein for details. This leads to the classification displayed in Table 1, where, depending on the choice of the parameters ( $a$ ,  $b$  and  $d$ ), two types of solitary wave solutions are distinguished: classical solitary waves (**Class**), that is, smooth and monotonically decaying at infinity travelling wave solutions (or, equivalently, orbits homoclinic to zero at infinity) and generalized solitary waves (**Gen**) or travelling waves which are homoclinic to periodic solutions at infinity, [54].

In order to illustrate the way to identify the structure of the homoclinic solutions to (3.4), two cases (one for each class of solitary waves) will be described in more detail. The first one takes  $a = c = 0$ ,  $b = d > 0$ . Note first that the system is reversible in the sense that if  $\mathcal{S}$  is the  $4 \times 4$  diagonal matrix with diagonal entries given by the vector  ${}^\top(1, -1, 1, -1)$ , then

$$\mathcal{S}\mathcal{T}(U, c_s) = -\mathcal{T}(\mathcal{S}U, c_s). \quad (3.7)$$

On the other hand, when  $c_s > 1$ , the spectrum of the linearization at the origin  $U = 0$  of (3.4) consists of four different, real eigenvalues  $\{\pm\lambda_-, \pm\lambda_+\}$  with

$$\lambda_{\pm} = \sqrt{\frac{c_s \pm 1}{c_s b}},$$

satisfying  $\lambda_- < 0 < \lambda_+$ . When  $c_s = 1$ ,  $\mathfrak{L}(1)$  has two simple eigenvalues  $\pm\sqrt{2/b}$  and the zero eigenvalue with geometric multiplicity equals one and algebraic multiplicity equals two. Let  $\{w_1, w_2, w_3, w_4\}$  a basis of generalized eigenvectors, with  $w_3, w_4$  eigenvectors of  $\sqrt{2/b}$  and  $-\sqrt{2/b}$ , respectively,  $w_1$  eigenvector associated to the zero eigenvalue and

$w_2$  such that  $\mathfrak{L}(1) w_2 = w_1$ . Explicitly, we take

$$\begin{aligned} w_1 &= {}^\top (1, 0, 1, 0), & w_2 &= {}^\top (0, -1, 0, -1), \\ w_3 &= {}^\top \left( \sqrt{\frac{b}{2}}, 1, -\sqrt{\frac{b}{2}}, -1 \right), & w_4 &= {}^\top \left( \sqrt{\frac{b}{2}}, 1, -\sqrt{\frac{b}{2}}, -1 \right). \end{aligned}$$

Note that  $w_1, w_2$  additionally satisfy  $\mathcal{S} w_1 = w_1, \mathcal{S} w_2 = -w_2$ . If  $\mathcal{P}$  is the  $4 \times 4$  matrix with columns given by the  $w_j$ 's, then we consider the new variables  $V = {}^\top (v_1, v_2, v_3, v_4)$  such that  $U = \mathcal{P} V$ . If  $\mu \stackrel{\text{def}}{=} c_s - 1 > 0$ , the System (3.4) in the new variables reads

$$v'_1 = v_2, \quad (3.8)$$

$$v'_2 = -\frac{\mu v_1}{b} + \frac{v_1^2}{b} \sum_{i,j=1}^2 (\alpha_{ij} + \beta_{ij}) + \mathcal{O}(\mu |V| + |V|^2), \quad (3.9)$$

$$v'_3 = \sqrt{\frac{2}{b}} v_3 + \mathcal{O}(\mu |V| + |V|^2), \quad (3.10)$$

$$v'_4 = -\sqrt{\frac{2}{b}} v_4 + \mathcal{O}(\mu |V| + |V|^2), \quad (3.11)$$

as  $\mu, |V| \rightarrow 0$  ( $|V|$  stands for the EUCLIDEAN norm of  $V$  in  $\mathbb{R}^4$ ) and where we assume that

$$\sum_{i,j=1}^2 (\alpha_{ij} + \beta_{ij}) > 0. \quad (3.12)$$

Then, for  $\mu$  positive and sufficiently small, the center-manifold reduction theorem, [41], can be applied to see that bounded solutions of the System (3.8) – (3.11) are on a locally invariant, center manifold which determines a dependence  $(v_3, v_4) = h(\mu, v_1, v_2)$  for some smooth  $h(\mu, v_1, v_2) = \mathcal{O}(\mu |{}^\top(v_1, v_2)| + |{}^\top(v_1, v_2)|^2)$  as  $\mu, |{}^\top(v_1, v_2)| \rightarrow 0$ , see [42, Theorem 3.2]. Furthermore, every solution  $v_1, v_2$  of the reduced system obtained from (3.8), (3.9) with  $(v_3, v_4) = h(\mu, v_1, v_2)$  leads to a solution of Equations (3.8) – (3.11) through this dependence. The normal form system can be written as

$$v'_1 = v_2, \quad v'_2 = c_1(\mu) v_1 + c_2(\mu) v_2, \quad (3.13)$$

for some coefficients  $c_1(\mu), c_2(\mu)$ . A suitable  $\mu$ -scaling of the variables transforms (3.13) into

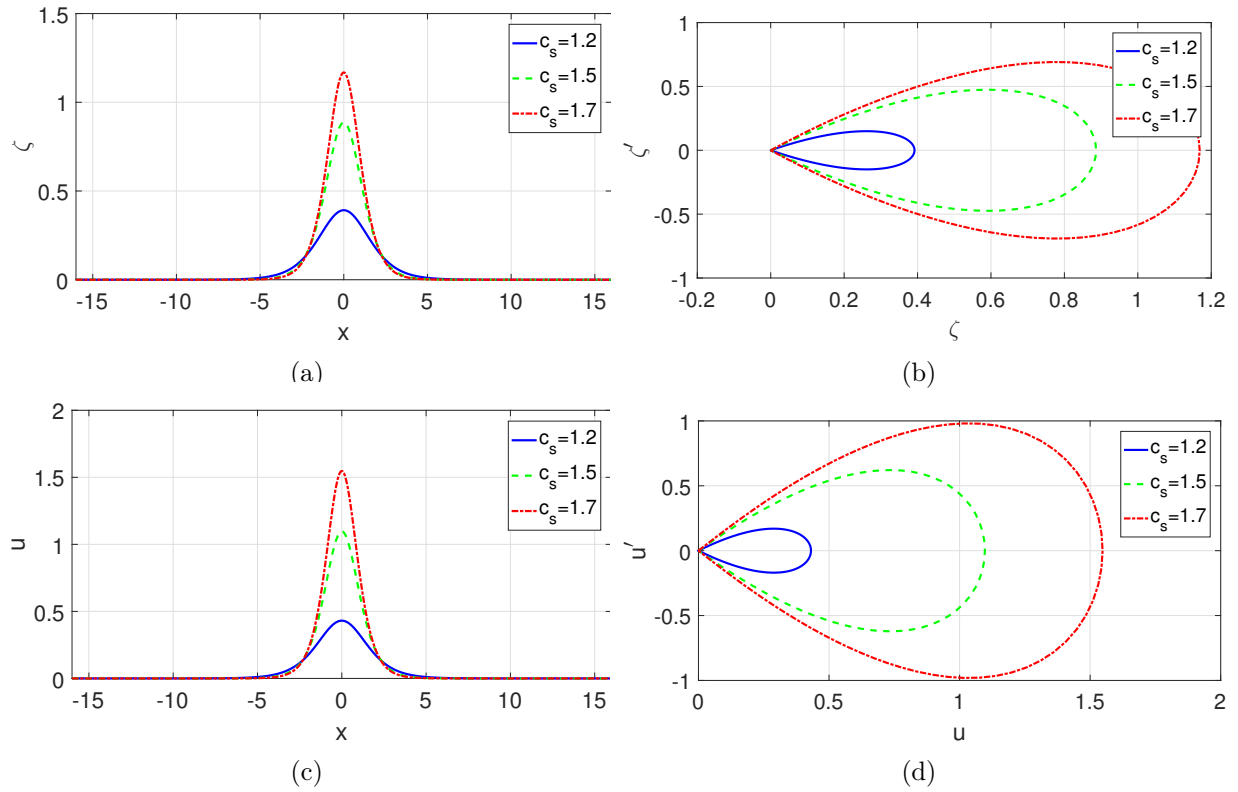
$$v'_1 = v_2, \quad v'_2 = \text{sign}(\mu) v_1 - \frac{3}{2} v_1^2 + \mathcal{O}(\mu), \quad (3.14)$$

For  $\mu > 0$ , (3.14) admits a solution of the form  $v_1(x) = \text{sech}^2(x/2) + \mathcal{O}(\mu)$ ,  $v_2 = v'_1$ . The persistence of this homoclinic orbit from the perturbation which connects to the original System (3.4) can be proved by using similar arguments to those of [42] (see also [18]).

The form of the waves is illustrated in Figures 2, which displays  $\zeta$  and  $u$  profiles (with the corresponding phase portraits) solutions of (3.2), (3.3), with the coefficients specified in the caption, and for several values of the speed  $c_s$ .

**Table 1.** Classification of solitary wave solutions using normal form theory for small and positive values of  $c_s - 1$ .

| Parameters                              | Solitary wave type |
|-----------------------------------------|--------------------|
| $a = c > 0, b = d = 0$                  | <b>Gen</b>         |
| $a = c < 0, b = 0, d > 0$               | <b>Gen</b>         |
| $a = c > 0, d = 0, b > 0$               | <b>Gen</b>         |
| $a = c < 0, b > 0, d > 0, bd - a^2 > 0$ | <b>Class</b>       |
| $a = c < 0, b > 0, d > 0, bd - a^2 < 0$ | <b>Gen</b>         |
| $a = c > 0, b > 0, d > 0, bd - a^2 > 0$ | <b>Class</b>       |
| $a = c > 0, b > 0, d > 0, bd - a^2 < 0$ | <b>Gen</b>         |
| $a = c > 0, b = d < 0$                  | <b>Gen</b>         |
| $a = c = 0, b > 0, d > 0$               | <b>Class</b>       |



**Figure 2.** Approximate classical solitary wave profile solutions of Equations (3.2) with  $a_{11} = 0$ ,  $a_{12} = 0.46$ ,  $a_{22} = 0$ ,  $\beta_{11} = 0.23$ ,  $\beta_{12} = 0$ ,  $\beta_{22} = 0.73$  and  $a = c = 0$ ,  $b = d = 1/6$ . (a)  $\zeta$  profiles; (b) phase portrait of (a); (c)  $u$  profiles; (d) phase portrait of (c).

**Remark 4.** As mentioned in the Introduction (see also Section 4), the question of the consistency of the EULER equations with the  $(a, b, a, b)$  MS Systems (2.21), (2.22) will be a subject of future research. Some related preliminary remarks can be made here. Recall first that the consistency of the EULER equations with the Systems (1.5), (1.6) was established in [8]. (For the case of (1.1), (1.2), see [7].) In particular, consistency holds for the MS family  $(a = c)$ . On the other hand, Figure 3 compares the solitary-wave speed-amplitude relations, for the case  $a = c = 0$ ,  $b = d > 0$ , of the Systems (1.1), (1.2), the symmetric version (1.5), (1.6) and the MS System (2.21), (2.22) used in Figure 2 with those of the EULER system, computed as in [24, 30]. These results (and those of other experiments not shown here) suggest the existence of MS Systems (2.21), (2.22), close to the symmetric (1.5), (1.6), with a comparable speed-amplitude behaviour.

In order to illustrate the arguments leading to the existence of generalized solitary waves of Table 1, we will consider the case  $a = c > 0$ ,  $b = d = 0$  (see [9] for the KdV–KdV System (1.1), (1.2)). Note first that (3.4) is still reversible as it satisfies (3.7). Now, as for the spectrum of the matrix  $\mathfrak{L}$  in (3.5) at  $c_s = 1$ , the difference with the case of classical solitary waves, considered above, is the presence of two simple, pure imaginary eigenvalues  $\pm i\sqrt{2/a}$ , while zero is also an eigenvalue with geometric multiplicity one and algebraic multiplicity two. The basis  $\{w_1, w_2, w_3, w_4\}$ , in  $\mathbb{C}^4$ , with  $w_1, w_2$  as above, contains the eigenvectors

$$\begin{aligned} w_3 &= {}^\top \left( i\sqrt{\frac{a}{2}}, 1, -i\sqrt{\frac{a}{2}}, -1 \right), \\ w_4 &= {}^\top \left( i\sqrt{\frac{a}{2}}, 1, -i\sqrt{\frac{a}{2}}, -1 \right), \end{aligned}$$

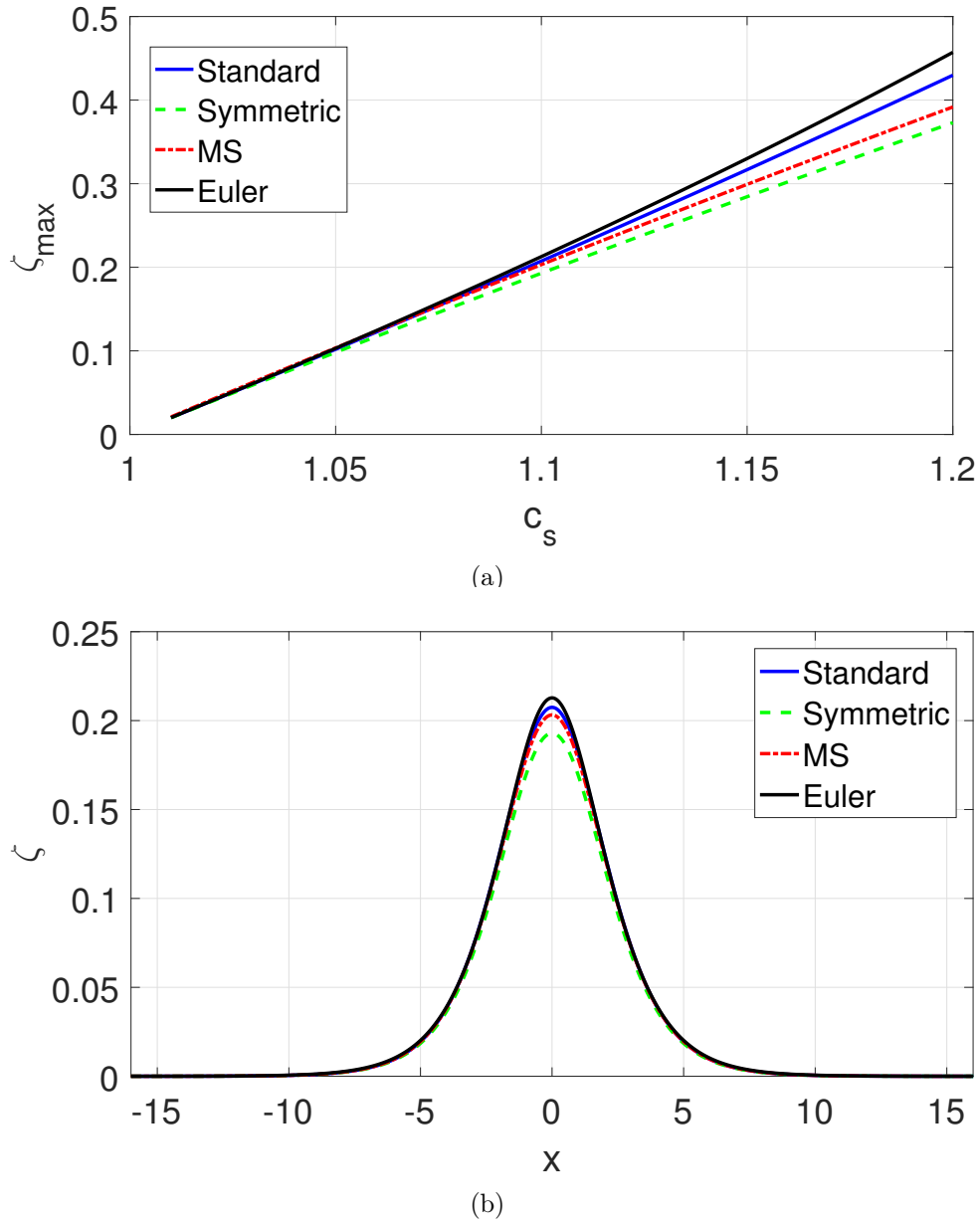
associated to  $i\sqrt{2/a}$  and  $-i\sqrt{2/a}$ , respectively. The application of the *Normal Form Theory* in this case may follow the direct approach of [42, Section 32] or, alternatively, that of LOMBARDI in [54]. In this last case, let  $\{w_1^*, w_2^*, w_3^*, w_4^*\}$  be the corresponding dual basis (with, in particular,  $w_1^* = {}^\top (0, 1/2, 0, 1/2)$ ). If  $D_{\mu, U}^2 \mathcal{T}(U, \mu)$  denotes the derivative, with respect to  $\mu$ , of the JACOBIAN matrix of  $\mathcal{T}$  in (3.4) and  $D_{U, U} \mathcal{T}(U, \mu)^2$  the HESSIAN (matrix) of  $\mathcal{T}$ , then

$$\begin{aligned} c_{10} &\stackrel{\text{def}}{=} \langle w_1^*, D_{\mu, U}^2 \mathcal{T}(0, 0) w_0 \rangle = \frac{1}{a} > 0, \\ c_{20} &\stackrel{\text{def}}{=} \frac{1}{2} \langle w_1^*, D_{U, U}^2 \mathcal{T}(0, 0) [w_0, w_0] \rangle = -\frac{1}{2a} \sum_{i,j=1}^2 (\alpha_{ij} + \beta_{ij}) \neq 0, \end{aligned}$$

where (3.12) is assumed. Therefore [54, Theorem 7.1.1] applies to prove that, for  $\mu$  small enough,  $\mathcal{T}(U, \mu)$  admits, near the fixed point  $U = 0$ , a one-parameter family of periodic orbits of arbitrarily small amplitude (depending on  $\mu$ ) and a pair of reversible, homoclinic connections to them. A direct application of *Normal Form Theory* to the reduced system, in a similar way to the case of classical solitary waves, can be seen in [42].

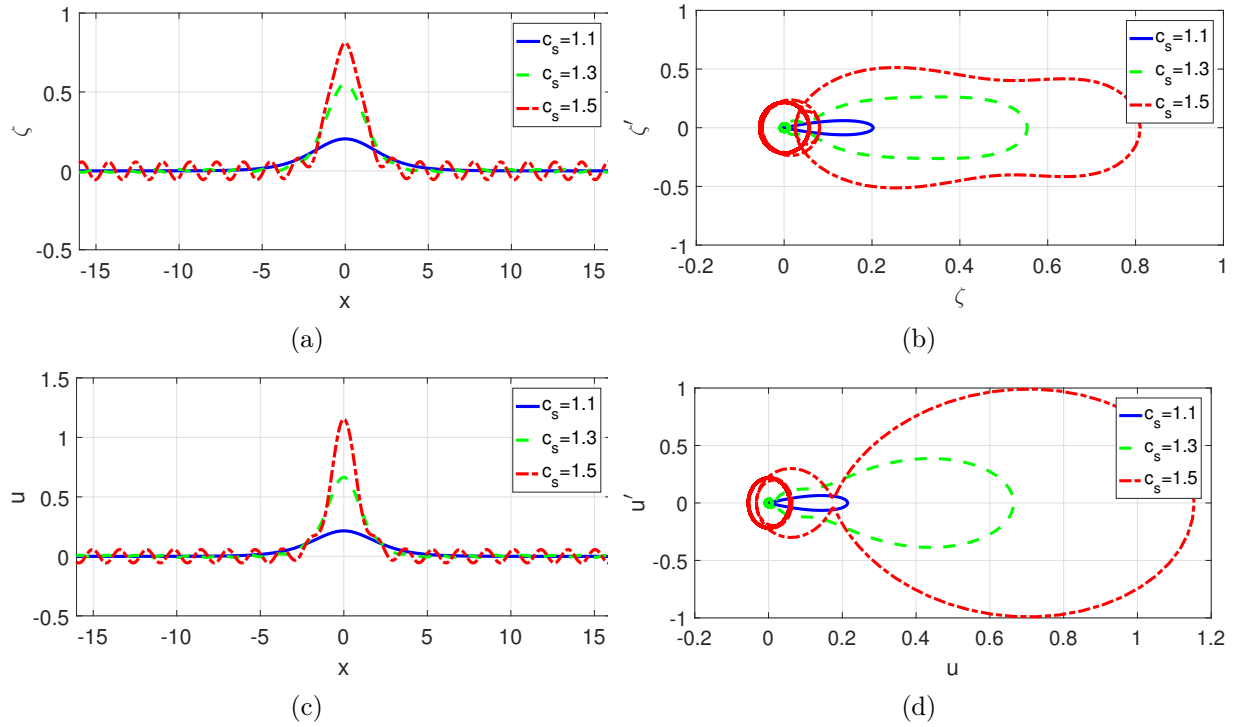
The form of the waves is illustrated in Figure 4 for different speeds. The coefficients of the nonlinear part of the MS system are the same as those of Figure 2 while  $a = c = 1/6$ ,



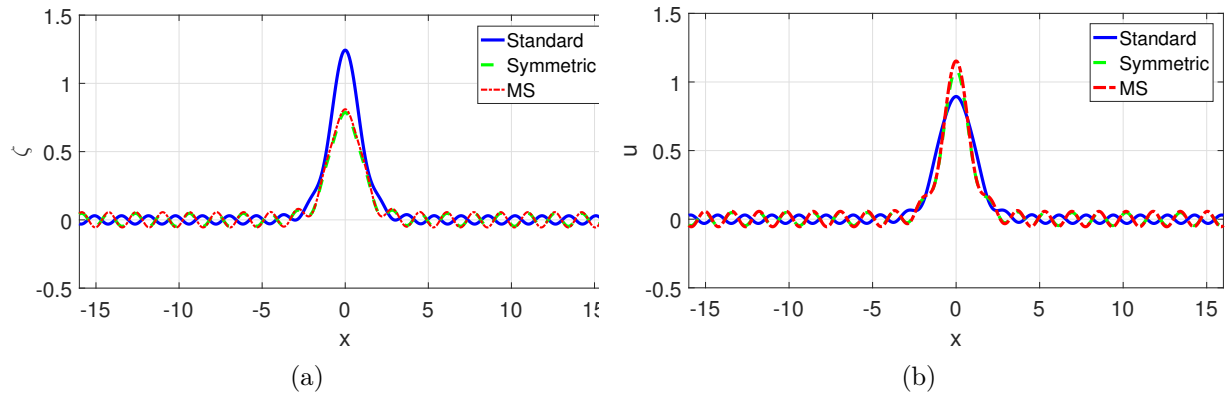


**Figure 3.** (a) Speed-amplitude relation; (b) Comparison of  $\zeta$  profiles for  $c_s = 1.1$ .

$b = d = 0$ . Finally, Figure 5 compares the approximate generalized profile corresponding to  $c_s = 1.5$  with those of the KdV–KdV System (1.1), (1.2) and of the symmetric version (1.5), (1.6).



**Figure 4.** Approximate generalized solitary wave profile solutions of Equations (3.2) with  $a_{11} = 0$ ,  $a_{12} = 0.46$ ,  $a_{22} = 0$ ,  $\beta_{11} = 0.23$ ,  $\beta_{12} = 0$ ,  $\beta_{22} = 0.73$  and  $a = c = 1/6$ ,  $b = d = 0$ . (a)  $\zeta$  profiles; (b) phase portraits of (a); (c)  $u$  profiles; (d) phase portraits of (c).



**Figure 5.** Comparison with KdV-KdV System (1.1), (1.2) the symmetric version (1.5), (1.6) and the MS System (2.21), (2.22) of Figure 4 for  $c_s = 1.5$ . (a)  $\zeta$  profiles; (b)  $u$  profiles.

## 4. Discussion

Below we discuss the main conclusions and perspectives of our study.

### 4.1. Conclusions

In the present manuscript we extended the class of known multi-symplectic systems arising in the modeling nonlinear shallow water waves, with especial emphasis on BOUSSINESQ-type equations for surface wave propagation. Having in mind that the full EULER equations are multi-symplectic, [13], it seems reasonable to expect from an approximate model to preserve this property. This is the case of, for example, the fully nonlinear SGN equations, [22, 25], or the simpler, but fundamental, NLS, KdV and BBM equations, [15, 20, 61, 67]. Because of the last two, it is expected for the KdV–BBM equation to be the next example. The MS structure of the KdV–BBM equation was derived in this paper and with a procedure that can be extended to other equations. Focused on BOUSSINESQ systems, our first result was the proof of the MS structure of the symmetric, four-parameter  $(a, b, a, d)$  family proposed in [8]. This is, however, a particular case of a second, more general approach. The main steps of the procedure helped in constructing a family of MS BOUSSINESQ-type systems, from a four-parameter  $(a, b, c, d)$  structure in the dispersive part and general homogeneous quadratic polynomials as nonlinear terms, by identifying those combinations of the parameters leading to the MS structure. Two main consequences of the method can be emphasized. The first one is that no one of the  $(a, b, a, d)$  sub-family of BOUSSINESQ systems, introduced and analyzed in [6, 7], is multi-symplectic; however, a modification of the nonlinear part (which, in scaled variables, belongs to a higher order of the asymptotic derivation and therefore is asymptotically negligible compared to the first-order terms) enables us to obtain a related sub-family of MS BOUSSINESQ equations. The second consequence is the derivation of a four-parameter, symplectic and multi-symplectic class of equations, with two parameters in the dispersive part and two from the nonlinearity. In addition to this, some mathematical properties of the new systems were studied. The discussion was focused on well-posedness of the corresponding IVP and the existence of solitary wave solutions. The derivation of results makes use of the pertinent literature for other BOUSSINESQ systems.

### 4.2. Perspectives

The future research may follow several directions. The first one concerns the consistency of the EULER system with the new family of MS BOUSSINESQ-type equations. The study on this matter developed in [8] can be considered as starting point of our future analysis, as well as some comparison of the speed-amplitude relation for classical solitary waves computed in Section 3. In the first case, the results concern the MS family within the symmetric Systems (1.5), (1.6). The second point is illustrated here by comparing the

speed-amplitude relation for solitary waves of the EULER equations with the Systems (1.1), (1.2) and MS Systems (2.21), (2.22). The results suggest that close to the MS family in (1.5), (1.6) there are systems improving this relation and the corresponding approximation to the solitary waves of the EULER system. This may be understood as an indication of consistency and deserves to be explored in more detail in a future research.

On the other hand, the results on the existence of solitary wave solutions derived in this paper made use of the Normal Form Theory, following previous applications, [18, 28]. This has the limitation of the local character of this theory, leading to an existence of solutions which is ensured for speeds  $c_s$  greater than but close enough to one. For the case of classical solitary waves, some other strategies for the existence are available. Of particular interest for us are two:

- The Positive Operator Theory, [4], which is based on writing the equations for the traveling waves in the form of a fixed-point system and analyzing the existence of solution from the properties of the fixed-point operator on cones in a suitable FRÉCHET space. In this sense, we think that the approach carried out by BONA and CHEN in [19] to derive existence results of classical solitary waves of BOUSSINESQ-type systems can be applied to the new MS equations. The corresponding results would not be limited by the magnitude of  $c_s - 1$ .
- A second, interesting for us, approach is TOLAND's Theory, [62]. This is based on the qualitative theory for second-order differential systems and, under suitable conditions, determines a curve  $f = 0$  satisfying that the corresponding solution with initial data on it leads to an orbit which is homoclinic to zero at infinity, being therefore identified as a classical solitary wave profile. In order to adapt this theory to the new systems, two first questions should be overcome (*cf.* [28] and references therein). The first one concerns the symmetric, conservative form in which the system for the profiles must be written. The second one is the potential energy associated to the nonlinear part of the resulting equations. Due to the general quadratic form of the nonlinearity of the new systems, this potential energy is different and leads to a curve of initial data of cubic degree, unlike other cases like some  $(a, b, c, d)$  systems of [6, 7], where  $f$  is quadratic. Finally, the indefinite character of the kinetic energy of the steady system (3.2), (3.3) (leading to the condition  $a \cdot c_s \cdot (b - d) < 0$ ) must be considered.

Concerning the stability of the solitary waves, the HAMILTONIAN structure is known to be used in some cases to analyze the orbital stability, *e.g.* [37, 38, 65], using the characterization of the solitary waves as equilibria of the HAMILTONIAN relative to a fixed value of the momentum. In addition to this, BRIDGES and DERKS, [14], present a characterization of the EVANS functions in HAMILTONIAN PDEs and use it to analyze the linear stability of three examples: the classical semilinear wave equation, the generalized KdV equation and a BOUSSINESQ system. These results suggest as future research the use of the multi-symplectic structure to investigate some aspects of the stability of the solitary waves of the family of MS BOUSSINESQ-type equations.

Finally, our study opens new perspectives in the construction of structure-preserving integrators for BOUSSINESQ-type equations, since only a sub-family  $(a, b, c, b)$  possesses the HAMILTONIAN formulation to the best of our knowledge. Fortunately, all of them are multi-symplectic. Thus, with the multi-symplectic structure unveiled in this paper, the so-called multi-symplectic integrators can be readily applied to BOUSSINESQ-type systems as well. We remind that these discretizations can be of finite difference [15] or of pseudo-spectral [43] types to satisfy the most stringent accuracy requirements. Moreover, the major advantage of these methods is that they preserve *exactly* (by construction) the multi-symplectic form conservation law (2.2) on the discrete level. The particular application to linear wave equations gave interesting results in [33]. Furthermore, a numerical comparison of symplectic and multi-symplectic schemes applied to the KdV equation and its solitonic gas [17, 32] was undertaken in [1, 2, 29]. A comparative study in similar terms for the MS BOUSSINESQ systems will be performed in a second, forthcoming paper.

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