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On the quasi-static effective behaviour of poroelastic media containing elastic inclusions

Pascale Royer, Pierre Recho, Claude Verdier

Abstract

The aim of the present study is to derive the effective quasi-static behaviour of a composite medium, made of a poroelastic matrix containing elastic impervious inclusions. For this purpose, the asymptotic homogenisation method is used. On the local scale, the governing equations include Biot's model of poroelasticity in the porous matrix and Navier equations in the inclusions, with elastic properties of the same order of magnitude. Biot's diphasic model of poroelasticity is obtained on the macroscopic scale, but with effective parameters that are strongly impacted by the distribution of inclusions, even at low volume fraction. The impact on fluid flow is strictly geometrical, showing that the inclusions do not play the role of a porous network.

I. Introduction

Composites made of a porous matrix reinforced with solid impervious inclusions occur in several engineering disciplines, involving natural media such as geomaterials Rice et al. (1978), biological tissues Rauch et al. (2018), Loboa et al. (2003), or tumors Xue et al. (2017), as well as man-made structures as, cement-based Lemarchand et al. (2002) or biomimetic materials Rajkhowa et al. (2010). A comprehensive understanding of the overall behaviour of these composites, on the basis of their microstructure, can enhance the knowledge concerning physical scenarios, with respect to key physical properties. In this way, a generalisation of Eshelby's formula is proposed in Berryman (1997), to give the response of a single ellipsoidal elastic inclusion, in a poroelastic whole space, to a uniform strain imposed at infinity. Some other works focus on fluid flow or solute transport, such as the mathematical model developed in Federico and Herzog (2008), of the effect of fibre arrangement on the permeability of a porous fibre-reinforced composite, or the macroscopic models obtained in Baltean et al. (2003) by asymptotic homogenisation, for passive solute transport in a rigid medium made of a porous matrix with impervious inclusions.

The present work is aimed at deriving the effective quasi-static mechanical behaviour of a saturated poroelastic medium containing elastic impervious inclusions and is focused on constituents with elastic properties of the same order of magnitude. For this purpose, the method of asymptotic homogenisation is used and Biot's model of poroelasticity is obtained on the macroscopic scale, but with effective parameters that are strongly affected by the distribution of inclusions. The paper is organised as follows. Section II presents a brief description of the homogenisation methodology. Then, homogenisation of a poroelastic medium with elastic inclusions is detailed in Section III, and the derived macroscopic description is commented in Section IV. Finally, Section V presents a summary of the main theoretical results contained in this work and highlights conclusive remarks.

II. Homogenisation method

i. Medium under consideration

We consider a periodic medium, of characteristic size L, and made of a fluid saturated elastic porous matrix which contains isolated elastic impervious inclusions. We further denote the spacing between two inclusions by l, and and we formulate the condition of separation of scales by $\varepsilon = l/L \ll 1$. Within the periodic cell Ω, we denote by $\Omega_{\rm p}$ the fluid-saturated porous matrix domain, by $\Omega_{\rm c}$ the volume occupied by the inclusion, and by Γ their common interface, as depicted in Fig.1. Using the two characteristic lengths, l and L, and the physical space variable, $\vec{X}$, we define two dimensionless space variables: $\vec{y} = \vec{X}/l$, $\vec{x} = \vec{X}/L$, and $\vec{y}$ and $\vec{x}$ describe variations on the microscopic and the macroscopic scales, respectively. Invoking the differentiation rule of multiple variables, the gradient operator with respect to $\vec{X}$ is written as

$$\vec{\nabla}_X = \frac{1}{l}\vec{\nabla}_y + \frac{1}{L}\vec{\nabla}_x. \tag{1}$$

We further introduce the following cell averages

$$< . >^{\Omega} = < . >^{\Omega_{\rm p}} + < . >^{\Omega_{\rm c}},$$
$$< . >^{\Omega_{\rm ff}} = \frac{1}{\mid \Omega \mid} \int_{\Omega_\alpha} . \, d\Omega \qquad (\alpha = {\rm p}, {\rm c}).$$

ii. Homogenisation procedure

The methodology firstly consists in writing, in dimensionless form, the governing equations which describe the problem on the periodic unit cell. This dimensionless writing of the equations requires the choice of a characteristic length for the dimensionless writing of space derivatives. We arbitrarily choose L as the reference characteristic length. The dimensionless gradient operator is thus $L\vec{\nabla}_X$, which by Eq.(1) is given by

$$\vec{\nabla} = L\vec{\nabla}_X = \varepsilon^{-1}\vec{\nabla}_y + \vec{\nabla}_x. \tag{2}$$

The homogenisation method being used is based upon the fundamental assumption that the unknown fields can be written in the form of asymptotic expansions in powers of ε

$$\psi = \psi^0\left(\vec{y}, \vec{x}\right) + \varepsilon\psi^1\left(\vec{y}, \vec{x}\right) + \varepsilon^2\psi^2\left(\vec{y}, \vec{x}\right) + ..., \tag{3}$$

in which functions ψ^i are Ω-periodic in variable $\vec{y}$. The method consists in incorporating the asymptotic expansions in the dimensionless local description, while taking into account the expression of the dimensionless gradient operator Eq.(2). This leads to approximate governing equations and boundary conditions at the successive orders, which together with the condition of periodicity define well posed boundary value problems within the periodic unit cell, from which functions ψ^i can be determined. Existence of solutions requires that volume averaged equations be satisfied. The latter ones actually describe the macroscopic behaviour at successive orders.

III. Quasi-static homogenisation in a poroelastic medium with elastic inclusions

i. Dimensionless governing equations on the local scale

The poroelastic matrix ($\Omega_{\rm p}$) is made of a linear elastic skeleton saturated with a viscous incompressible Newtonian fluid, and its behaviour is described by quasi-static Biot's model Biot (1955)[1]:

$$\begin{cases} \vec{\nabla} \cdot \tilde{\sigma}_{\rm p} = \vec{0}, & (4) \\ \tilde{\sigma}_{\rm p} = \tilde{c}_{\rm p} : \tilde{e}(\vec{u}_{\rm s}) - \tilde{\alpha}_{\rm p}\ p_{\rm f}, & (5) \\ \vec{\nabla} \cdot \vec{v}_{\rm p} = -\tilde{\alpha}_{\rm p} :\ \tilde{e}(\dfrac{\partial \vec{u}_{\rm s}}{\partial t}) - \beta_{\rm p}\dfrac{\partial p_{\rm f}}{\partial t}, & (6) \\ \vec{v}_{\rm p} = \phi_{\rm p}(\vec{v}_{\rm f} - \dfrac{\partial \vec{u}_{\rm s}}{\partial t}) = -\dfrac{\tilde{K}_{\rm p}}{\mu}\vec{\nabla} p_{\rm f}. & (7) \end{cases}$$

The four above equations express the momentum balance, the poroelastic constitutive law, the conservation of fluid mass and Darcy's law, respectively. The distinct quantities involved in the model are the following: $\tilde{\sigma}_{\rm p}$ and

[1] This is possible provided that the matrix pore size, $l_{\rm p}$ be greatly smaller than the spacing between two inclusions, l.

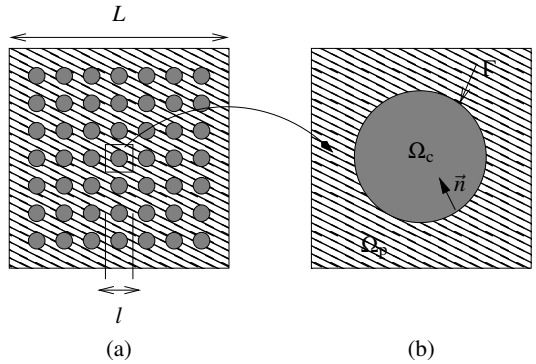

Figure 1: *Porous medium :(a) Macroscopic sample; (b) Periodic unit cell.*

p_f denote the total stress and the interstitial fluid pressure, respectively; $\vec{u}_s$ is the solid displacement of the porous matrix, while $\vec{v}_f$ and $\vec{v}_p$ stand for the mean fluid velocity within the volume of the micropores and the mean fluid relative velocity within the porous matrix; $\tilde{e}(\vec{u}_s) = 1/2(\vec{\nabla}\vec{u}_s + \vec{\nabla}^T\vec{u}_s)$ is the strain tensor, while $\tilde{c}_p$, $\tilde{\alpha}_p$, $\beta_p > 0$, and $\tilde{K}_p$ represent the fourth order elastic tensor of the drained porous matrix, the second order symmetric and positive Biot coupling tensor, Biot's bulk modulus and the second order tensor of permeability of the porous matrix, respectively; ϕ_p and μ denote the porosity of the porous matrix, and the fluid viscosity, respectively.

The inclusion (Ω_c) is linear elastic and satisfies the Navier equations

$$\begin{cases} \vec{\nabla} \cdot \tilde{\sigma}_c = \vec{0}, & (8) \\ \tilde{\sigma}_c = \tilde{c}_c : \tilde{e}(\vec{u}_c), & (9) \end{cases}$$

where $\tilde{\sigma}_c$ and $\vec{u}_c$ stand for the solid stress tensor and displacement, respectively, and where $\tilde{c}_c$ represents the elastic tensor.

The appropriate conditions over the interface Γ between the porous matrix and the inclusion include the continuity of normal stresses and displacements and the normal mean fluid relative velocity within the porous matrix must be set to zero Mikelic and Wheeler (2012):

$$\begin{cases} \tilde{\sigma}_p \cdot \vec{n} = \tilde{\sigma}_c \cdot \vec{n} \quad \text{over } \Gamma, & (10) \\ \vec{u}_s = \vec{u}_c \quad \text{over } \Gamma, & (11) \\ \vec{v}_p \cdot \vec{n} = 0 \quad \text{over } \Gamma, & (12) \end{cases}$$

where $\vec{n}$ denotes the unit vector giving the normal to Γ exterior to Ω_p.

ii. Homogenisation

We consider equations Eqs.(4)-(9) and boundary conditions Eqs.(10)-(12), and we look for solutions in the form of Eq.(3) for $\tilde{\sigma}_p$, $\tilde{\sigma}_c$, $\vec{u}_s$, $\vec{u}_c$, p_f and $\vec{v}_p$. Note that, due to Eq.(2), the expansion of $\vec{v}_p$ starts with a term in ε^{-1}. Furthermore, the strain tensors read

$$\tilde{e}(\vec{u}_\alpha) = \varepsilon^{-1}\tilde{e}_y(\vec{u}_\alpha) + \tilde{e}_x(\vec{u}_\alpha), \qquad (\alpha = \mathrm{p}, \mathrm{c}), \tag{13}$$

and consequently the expansions of both stress tensors also start with a ε^{-1} term. Incorporating the asymptotic expansions and the expressions of the dimensionless gradient operator Eq.(2) and of the strain tensors Eq.(13) into Eqs.(4)-(12), then identifying terms of same power of ε, leads to boundary value problems at the successive orders.

ii.1 Boundary value problem for $\tilde{\sigma}_p^{-1}$, $\tilde{\sigma}_c^{-1}$, $\vec{u}_s^0$ and $\vec{u}_c^0$

Considering the leading order of Eqs.(4)-(5) and Eqs.(8)-(11), we deduce the following boundary value problem of unknowns $\vec{u}_s^0$ and

$\vec{u}_{\mathrm{c}}^0$:

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left[c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{u}_{\mathrm{s}}^0)\right] = 0 \text{ within } \Omega_{\mathrm{p}}, & (14) \\ \dfrac{\partial}{\partial y_j}\left[c_{\mathrm{c}_{ijlm}} e_{y_{lm}}(\vec{u}_{\mathrm{s}}^0)\right] = 0 \text{ within } \Omega_{\mathrm{c}}, & (15) \\ \left[c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{u}_{\mathrm{s}}^0)\right] n_j = \\ \left[c_{\mathrm{c}_{ijlm}} e_{y_{lm}}(\vec{u}_{\mathrm{c}}^0)\right] n_j \text{ over } \Gamma, & (16) \\ u_{\mathrm{s}_i}^0 = u_{\mathrm{c}_i}^0 \quad \text{over } \Gamma, & (17) \\ \tilde{\sigma}_{\mathrm{p}}^{-1}, \tilde{\sigma}_{\mathrm{c}}^{-1}, \vec{u}_{\mathrm{s}}^0, \vec{u}_{\mathrm{c}}^0 : \text{periodic in } \vec{y}, & (18) \end{cases}$$

from which it is clear that the displacements $\vec{u}_{\mathrm{s}}^0$ and $\vec{u}_{\mathrm{c}}^0$ are constant over the period

$$\vec{u}_{\mathrm{s}}^0 = \vec{u}_{\mathrm{c}}^0 = \vec{u}^0(x). \tag{19}$$

Since by Eqs.(5) and (9) at $\mathcal{O}(\varepsilon^{-1})$

$$\sigma_{\alpha_{ij}}^{-1} = c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{u}^0) \qquad (\alpha = \mathrm{p}, \mathrm{s}), \tag{20}$$

we consequently get

$$\tilde{\sigma}_{\mathrm{p}}^{-1} = \tilde{\sigma}_{\mathrm{c}}^{-1} = \tilde{0}. \tag{21}$$

ii.2 Boundary value problem for $\vec{v}_p^{-1}$ and p_{f}^0

At the lowest order, Eqs.(6)-(7) lead to

$$\begin{cases} \dfrac{\partial}{\partial y_i}\left[\dfrac{K_{\mathrm{p}_{ij}}}{\mu}\dfrac{\partial p_{\mathrm{f}}^0}{\partial y_j}\right] = 0 \quad \text{within } \Omega_{\mathrm{p}}, & (22) \\ \left[\dfrac{K_{\mathrm{p}_{ij}}}{\mu}\dfrac{\partial p_{\mathrm{f}}^0}{\partial y_j}\right] n_j = 0 \quad \text{over } \Gamma, & (23) \\ p_{\mathrm{f}}^0 : \text{periodic in } \vec{y}. & (24) \end{cases}$$

Consequently, we get:

$$\begin{cases} p_{\mathrm{f}}^0 = p_{\mathrm{f}}^0(\vec{x}), & (25) \\ \vec{v}_{\mathrm{p}}^{-1} = \vec{0}. & (26) \end{cases}$$

ii.3 Boundary value problem for $\tilde{\sigma}_{\mathrm{p}}^0$, $\tilde{\sigma}_{\mathrm{c}}^0$, $\vec{u}_{\mathrm{s}}^1$ and $\vec{u}_{\mathrm{c}}^1$

We now consider the second order of Eqs.(4)-(5) and Eqs.(8)-(11), from which we deduce the following system of unknowns $\vec{u}_{\mathrm{s}}^1$ and $\vec{u}_{\mathrm{c}}^1$:

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left[c_{\mathrm{p}_{ijlm}}[e_{y_{lm}}(\vec{u}_{\mathrm{s}}^1) + e_{x_{lm}}(\vec{u}_{\mathrm{s}}^0)] - \alpha_{\mathrm{p}_{ij}}\, p_f^0\right] = 0 \\ \text{within } \Omega_{\mathrm{p}}, & (27) \\ \dfrac{\partial}{\partial y_j}\left[c_{\mathrm{c}_{ijlm}}[e_{y_{lm}}(\vec{u}_{\mathrm{c}}^1) + e_{x_{lm}}(\vec{u}_{\mathrm{c}}^0)]\right] = 0 \\ \text{within } \Omega_{\mathrm{c}}, & (28) \\ (c_{\mathrm{p}_{ijlm}}[e_{y_{lm}}(\vec{u}_{\mathrm{s}}^1) + e_{x_{lm}}(\vec{u}_{\mathrm{s}}^0)] - \alpha_{\mathrm{p}_{ij}}\, p_f^0)\, n_j = \\ (c_{\mathrm{c}_{ijlm}}(e_{y_{lm}}(\vec{u}_{\mathrm{c}}^1) + e_{x_{lm}}(\vec{u}_{\mathrm{c}}^0)))\, n_j \text{ over } \Gamma, & (29) \\ u_{\mathrm{s}_i}^1 = u_{\mathrm{c}_i}^1 \quad \text{over } \Gamma, & (30) \\ \vec{u}_{\mathrm{s}}^1, \vec{u}_{\mathrm{c}}^1 : \text{periodic in } \vec{y}. & (31) \end{cases}$$

The above set of equations constitutes a well-posed problem for $\vec{u}_{\mathrm{s}}^1$ and $\vec{u}_{\mathrm{c}}^1$, and by virtue of linearity, the solutions read (*Cf.* A):

$$\begin{cases} u_{\mathrm{s}_i}^1 = \omega_{\mathrm{p}_i}^{kh} e_{x_{kh}}(\vec{u}^0) - \pi_{\mathrm{p}_i}\, p_{\mathrm{f}}^0 + \bar{u}_{\mathrm{s}_i}^1(\vec{x}), & (32) \\ u_{\mathrm{c}_i}^1 = \omega_{\mathrm{c}_i}^{kh} e_{x_{kh}}(\vec{u}^0) + \bar{u}_{\mathrm{c}_i}^1(\vec{x}), & (33) \end{cases}$$

where $\bar{u}_{\mathrm{s}_i}^1(\vec{x})$ and $\bar{u}_{\mathrm{c}_i}^1(\vec{x})$ are arbitrary functions. Note that, to render the solution unique, we impose that $\tilde{\omega}_{\mathrm{p}}$, $\vec{\pi}_{\mathrm{p}}$ and $\tilde{\omega}_{\mathrm{c}}$ are with zero average Bensoussan et al. (1978), Sanchez-Palencia (1980):

$$< \omega_{\mathrm{p}_i}^{kh} >^{\Omega_{\mathrm{p}}} = 0, \; < \omega_{\mathrm{c}_i}^{kh} >^{\Omega_{\mathrm{c}}} = 0, \; < \pi_{\mathrm{p}_i} >^{\Omega_{\mathrm{p}}} = 0.$$

By Eqs.(5) and (9) at $\mathcal{O}(\varepsilon^0)$, we obtain

$$\begin{cases} \sigma_{\mathrm{p}_{ij}}^0 = c_{\mathrm{p}_{ijlm}}[e_{y_{lm}}(\vec{u}_{\mathrm{s}}^1) + e_{x_{lm}}(\vec{u}_{\mathrm{s}}^0)] \\ -\alpha_{\mathrm{p}_{ij}}\, p_f^0, & (34) \\ \sigma_{\mathrm{c}_{ij}}^0 = c_{\mathrm{c}_{ijlm}}[e_{y_{lm}}(\vec{u}_{\mathrm{c}}^1) + e_{x_{lm}}(\vec{u}_{\mathrm{c}}^0)], & (35) \end{cases}$$

and then employing Eqs.(32)-(33), we deduce

$$\begin{cases} \sigma_{\mathrm{p}_{ij}}^0 = (c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{\omega}^{kh}) + c_{\mathrm{p}_{ijkh}}) e_{x_{kh}}(\vec{u}^0), \\ -(c_{\mathrm{p}_{ijkh}} e_{y_{kh}}(\vec{\pi}) + \alpha_{\mathrm{p}_{ij}}) p_{\mathrm{f}}^0, & (36) \\ \sigma_{\mathrm{c}_{ij}}^0 = (c_{\mathrm{c}_{ijlm}} e_{y_{lm}}(\vec{\omega}^{kh}) + c_{\mathrm{c}_{ijkh}}) e_{x_{kh}}(\vec{u}^0), & (37) \end{cases}$$

ii.4 Boundary value problem for $\vec{v}_{\mathrm{p}}^0$ and p_{f}^1

We now consider Eq.(6) at $\mathcal{O}(\varepsilon^{-1})$, Eq.(7) at $\mathcal{O}(\varepsilon^{O})$, and boundary condition Eq.(12) at

$\mathcal{O}(\varepsilon^0)$, from which we deduce the differential system

$$\begin{cases} \dfrac{\partial}{\partial y_i}\left[\dfrac{K_{\mathrm{p}_{ij}}}{\mu}(\dfrac{\partial p^1_{\mathrm{f}}}{\partial y_j}+\dfrac{\partial p^0_{\mathrm{f}}}{\partial x_j})\right]=0 \text{ in } \Omega_{\mathrm{p}}, & (38)\\ \left[\dfrac{K_{\mathrm{p}_{ij}}}{\mu}(\dfrac{\partial p^1_{\mathrm{f}}}{\partial y_j}+\dfrac{\partial p^0_{\mathrm{f}}}{\partial x_j})\right]n_i=0 \text{ over } \Gamma, & (39)\\ p^1_{\mathrm{f}} : \text{periodic in } \vec{y}. & (40)\end{cases}$$

The above set of equations is a well-posed boundary value problem of unknown p^1_{f}, from which it appears that p^1_{f} is a linear function of $\vec{\nabla}_x p^0_f$:

$$p^1_{\mathrm{f}} = \chi_{\mathrm{p}_i}\,\frac{\partial p^0_{\mathrm{f}}}{\partial x_i} + \bar{p}^1_{\mathrm{f}}(\vec{x},t), \qquad (41)$$

where $\bar{p}^1_{\mathrm{f}}(\vec{x},t)$ is an arbitrary function and where

$$<\vec{\chi}_{\mathrm{p}}>^{\Omega_{\mathrm{p}}}_{\Omega}=\vec{0}. \qquad (42)$$

Vector χ_{p_k} is the specific solution for p^1_{f}, corresponding to $\partial p^0_{\mathrm{f}}/\partial x_j=\delta_{jk}$. Then, by Eq.(7) at $\mathcal{O}(\varepsilon^{-1})$

$$v^0_{\mathrm{p}_i} = -\frac{K_{\mathrm{p}_{ij}}}{\mu}(\frac{\partial p^1_{\mathrm{f}}}{\partial y_j}+\frac{\partial p^0_{\mathrm{f}}}{\partial x_j}), \qquad (43)$$

and by Eq.(41), we deduce the following expression for $\vec{v}^0_{\mathrm{p}}$:

$$v^0_{\mathrm{p}_i} = -\frac{K_{p_{ij}}}{\mu}(\frac{\partial \chi_k}{\partial y_j}+\delta_{jk})\frac{\partial p^0_{\mathrm{f}}}{\partial x_k}, \qquad (44)$$

where $\tilde{\delta}$ denotes Kronecker's symbol.

ii.5 Macroscopic momentum balance

Let now consider Eqs.(4), (8), (10) at the third order:

$$\begin{cases} \dfrac{\partial \sigma^1_{\mathrm{p}_{ij}}}{\partial y_j}+\dfrac{\partial \sigma^0_{\mathrm{p}_{ij}}}{\partial x_j}=0 & \text{within } \Omega_{\mathrm{p}}, \quad (45)\\ \dfrac{\partial \sigma^1_{\mathrm{c}_{ij}}}{\partial y_j}+\dfrac{\partial \sigma^0_{\mathrm{c}_{ij}}}{\partial x_j}=0 & \text{within } \Omega_{\mathrm{c}}, \quad (46)\\ \sigma^1_{\mathrm{p}_{ij}}\,n_j=\sigma^1_{\mathrm{c}_{ij}}\,n_j & \text{over } \Gamma. \quad (47)\end{cases}$$

The homogenisation procedure consists now in integrating Eqs.(45) and (46) over Ω_{p} and Ω_{c}, respectively. This leads to a compatibility condition, *i.e.* a necessary and sufficient condition for the existence of solutions for $\vec{u}^2_{\mathrm{s}}$ and $\vec{u}^2_{\mathrm{c}}$, which further represents the first order macroscopic momentum balance. Invoking Gauss' theorem, integration of Eq.(45) over Ω_{p} yields

$$\frac{1}{\mid\Omega\mid}\int_{\Gamma}\sigma^1_{\mathrm{p}_{ij}}\,n_j\,dS+\frac{1}{\mid\Omega\mid}\int_{\Omega_{\mathrm{p}}}\frac{\partial\sigma^0_{\mathrm{p}_{ij}}}{\partial x_j}\,d\Omega=0, \qquad (48)$$

where the contribution over the cell boundaries, $\delta\Omega\cap\delta\Omega_{\mathrm{p}}$, cancel due to $\vec{y}$-periodicity. Then, employing Eq.(47) and Gauss' theorem, we get

$$\frac{1}{\mid\Omega\mid}\int_{\Gamma}\sigma^1_{\mathrm{p}_{ij}}n_j d\Omega=\frac{1}{\mid\Omega\mid}\int_{\Omega_{\mathrm{c}}}\frac{\partial\sigma^0_{\mathrm{c}_{ij}}}{\partial x_j}d\Omega. \qquad (49)$$

Finally, Eq.(48) becomes

$$\frac{\partial<\sigma^0_{\mathrm{p}_{ij}}>^{\Omega_{\mathrm{p}}}}{\partial x_i}+\frac{\partial<\sigma^0_{\mathrm{c}_{ij}}>^{\Omega_{\mathrm{c}}}}{\partial x_i}=0. \qquad (50)$$

Let us define the total stress $\tilde{\sigma}_{\mathrm{T}}$ as

$$\tilde{\sigma}_{\mathrm{T}}=\begin{cases}\tilde{\sigma}_{\mathrm{p}} & \text{in } \Omega_{\mathrm{p}},\\ \tilde{\sigma}_{\mathrm{c}} & \text{in } \Omega_{\mathrm{c}}.\end{cases}$$

Thus, Eq.(50) is rewritten as

$$\begin{cases}\dfrac{\partial<\sigma^0_{\mathrm{T}_{ij}}>^{\Omega}}{\partial x_i}=0, & (51)\\ <\sigma^0_{\mathrm{T}_{ij}}>^{\Omega}=<\sigma^0_{\mathrm{p}_{ij}}>^{\Omega_{\mathrm{p}}}+<\sigma^0_{\mathrm{c}_{ij}}>^{\Omega_{\mathrm{c}}} & (52)\end{cases}$$

Finally, using Eqs.(36)-(37), we get

$$<\sigma^0_{\mathrm{T}_{ij}}>^{\Omega}=C^{\mathrm{eff}}_{ijkh}\,e_{x_{kh}}(\vec{u}^0)-A^{\mathrm{eff}}_{ij}\,p^0_{\mathrm{f}}, \qquad (53)$$

with

$$\begin{cases} C^{\mathrm{eff}}_{ijkh}=<c_{\mathrm{p}_{ijlm}}e_{y_{lm}}(\vec{\omega}^{kh}_{\mathrm{p}})+c_{\mathrm{p}_{ijkh}}>^{\Omega_{\mathrm{p}}}_{\Omega} & \\ +<c_{\mathrm{c}_{ijlm}}e_{y_{lm}}(\vec{\omega}^{kh}_{\mathrm{c}})+c_{\mathrm{c}_{ijkh}}>^{\Omega_{\mathrm{c}}}_{\Omega}, & (54)\\ A^{\mathrm{eff}}_{ij}=<c_{p_{ijkh}}e_{y_{kh}}(\vec{\pi}_{\mathrm{p}})+\alpha_{p_{ij}}>^{\Omega_{\mathrm{p}}}_{\Omega}. & (55)\end{cases}$$

The first order momentum balance is thus described by Eqs.(51), (53).

ii.6 Macroscopic mass balance

At the second order, Eqs.(6) and (12) yield

$$\begin{cases} \dfrac{\partial v^1_{\mathrm{p}_i}}{\partial y_i} + \dfrac{\partial v^0_{\mathrm{p}_i}}{\partial x_i} = -\alpha_{\mathrm{p}_{ij}} \left[e_{y_{ij}}(\dfrac{\partial \vec{u}^1_{\mathrm{s}}}{\partial t}) + e_{x_{ij}}(\dfrac{\partial \vec{u}^0}{\partial t}) \right] \\ -\beta_{\mathrm{p}} \dfrac{\partial p^0_{\mathrm{f}}}{\partial t} \text{ within } \Omega_{\mathrm{p}}, & (56) \\ v^1_{\mathrm{p}_i}\, n_i = 0 \quad \text{ over } \Gamma. & (57) \end{cases}$$

Integrating Eq.(56) and invoking Gauss' theorem, while taking boundary condition Eq.(57) into account, together with the condition of periodicity, yields

$$\begin{aligned} &\frac{\partial < v^0_{\mathrm{p}_i} >^{\Omega_{\mathrm{p}}}}{\partial x_i} = \\ &- < \alpha_{\mathrm{p}_{ij}} (e_{y_{ij}}(\frac{\partial \vec{u}^1_{\mathrm{s}}}{\partial t}) + e_{x_{ij}}(\frac{\partial \vec{u}^0}{\partial t}) >^{\Omega_{\mathrm{p}}}_{\Omega} \qquad (58) \\ &- < \beta_{\mathrm{p}} >^{\Omega_{\mathrm{p}}}_{\Omega} \frac{\partial p^0_{\mathrm{f}}}{\partial t}. \end{aligned}$$

Using Eq.(32), the above equation can be written as

$$\frac{\partial < v^0_{\mathrm{p}_i} >^{\Omega_{\mathrm{p}}}}{\partial x_i} = -G^{\mathrm{eff}}_{lm}\, e_{x_{lm}}(\frac{\partial \vec{u}^0}{\partial t}) - B^{\mathrm{eff}} \frac{\partial p^0_{\mathrm{f}}}{\partial t}, \qquad (59)$$

where

$$\begin{cases} G^{\mathrm{eff}}_{lm} =< \alpha_{p_{ij}} e_{y_{ij}}(\vec{\omega}^{lm}_{\mathrm{p}}) + \alpha_{p_{lm}} >^{\Omega_{\mathrm{p}}}, & (60) \\ B^{\mathrm{eff}} =< \beta_p - \alpha_{p_{ij}} e_{y_{ij}}(\vec{\pi}_{\mathrm{p}}) >^{\Omega_{\mathrm{p}}}. & (61) \end{cases}$$

Now, by Eq.(44), we get

$$\begin{cases} < v^0_{\mathrm{p}_i} >^{\Omega_{\mathrm{p}}} = -\dfrac{K^{\mathrm{eff}}_{ij}}{\mu} \dfrac{\partial p^0_{\mathrm{f}}}{\partial x_j}, & (62) \\ K^{\mathrm{eff}}_{ik} =< K_{\mathrm{p}_{ij}} (\dfrac{\partial \chi_k}{\partial y_j} + \delta_{jk}) >^{\Omega_{\mathrm{p}}}. & (63) \end{cases}$$

The first order macroscopic mass balance is thus given by Eqs.(59) and (62).

IV. Macroscopic description

The first order macroscopic description thus consists of by Eqs.(51), (53), (59) and (62), with the effective properties defined by Eqs.(54), (55), (60), (61) and (63). From its definition, it is clear that tensor C^{eff}_{ijkh} is the effective elastic tensor of the whole empty medium, made of the empty porous matrix and the inclusions. It therefore coincides with the effective elasticity tensor that would be obtained for a two-phase elastic composite and thus possesses all the required symmetry properties that characterise an elastic tensor (*e.g.* Penta and Gerish (2017)). Now, from the variational formulation associated with the definitions of $\vec{u}^1_{\mathrm{s}}$ and $\vec{u}^1_{\mathrm{c}}$, we show that the coupling tensors $\tilde{A}^{\mathrm{eff}}$ and $\tilde{G}^{\mathrm{eff}}$ are equal (*Cf.* B), and that the bulk modulus B^{eff} is positive (*Cf.* C). Furthermore, from the symmetries of $\tilde{c}_{\mathrm{p}}$ and of $\tilde{\alpha}_{\mathrm{p}}$, it follows that $\tilde{A}^{\mathrm{eff}}$ is symmetric. With the above mentioned properties, the macroscopic description, Eqs.(51), (53), (59) and (62), is a Biot diphasic model of poroelasticity, but in which the effective properties are strongly impacted by the local distribution of inclusions. An illustration of this appears when considering homogeneous materials. The definitions of the effective Biot parameters and permeability then reduce to

$$\begin{cases} A^{\mathrm{eff}}_{ij} = c_{p_{ijkh}} < e_{y_{kh}}(\vec{\pi}_{\mathrm{p}}) >^{\Omega_{\mathrm{p}}} + (1 - n_{\mathrm{c}}) \alpha_{p_{ij}}, \\ B^{\mathrm{eff}} = (1 - n_{\mathrm{c}}) \beta_{\mathrm{p}} - \alpha_{p_{ij}} < e_{y_{ij}}(\vec{\pi}_{\mathrm{p}}) >^{\Omega_{\mathrm{p}}}, \\ K^{\mathrm{eff}}_{ij} = K_{\mathrm{p}_{ik}} < \dfrac{\partial \chi_j}{\partial y_k} + \delta_{kj} >^{\Omega_{\mathrm{p}}}, \end{cases}$$

where $n_{\mathrm{c}} = \mid \Omega_{\mathrm{c}} \mid / \mid \Omega \mid$ denotes the inclusion volume fraction. We firstly note that at low inclusion concentration, *i.e.* when $n_{\mathrm{c}} \approx 0$, the inclusions still have an impact since $\tilde{A}^{\mathrm{eff}} \neq \tilde{\alpha}_{\mathrm{p}}$ and $B^{\mathrm{eff}} \neq \beta_{\mathrm{p}}$. Furthermore, when the porous matrix is incompressible, *i.e* when $\tilde{\alpha}^{\mathrm{eff}}_{\mathrm{p}} = \tilde{I}$ and $\beta^{\mathrm{eff}}_{\mathrm{p}} = 0$, the whole poroelastic composite remains compressible since $\tilde{A}^{\mathrm{eff}} \neq \tilde{I}$ and $B^{\mathrm{eff}} \neq 0$, even at low inclusion volume fraction. Finally, we see that the permeability is such that $K^{\mathrm{eff}}_{ij} = K_{\mathrm{p}_{ik}} T_{\mathrm{p}_{kj}}$, where

$$T_{\mathrm{p}_{kj}} =< \frac{\partial \chi_j}{\partial y_k} + \delta_{kj} >^{\Omega_{\mathrm{p}}}.$$

Tensor $\tilde{T}_{\mathrm{p}}$ is a purely geometrical parameter, and we note that $\phi^{-1}\tilde{T}_{\mathrm{p}}$ actually represents the tortuosity (*e.g.* see Royer and Cherblanc (2010)) associated with the distribution of inclusions.

V. Conclusion

We have thus shown that the first order macroscopic behaviour, *i.e.* with precision in the order of $\mathcal{O}(\varepsilon)$, of a poroelastic matrix containing elastic inclusions is described by Biot's diphasic model of poroelasticity

$$\begin{cases} \dfrac{\partial < \sigma_{\mathrm{T}_{ij}} >^{\Omega}}{\partial x_i} = 0, \\ < \sigma_{\mathrm{T}_{ij}} >^{\Omega} = C^{\mathrm{eff}}_{ijkh}\ e_{x_{kh}}(\vec{u}) - A^{\mathrm{eff}}_{ij}\ p_{\mathrm{f}}, \\ \dfrac{\partial < v_{\mathrm{p}_i} >^{\Omega_{\mathrm{p}}}}{\partial x_i} = -A^{\mathrm{eff}}_{lm}\ e_{x_{lm}}(\dfrac{\partial \vec{u}}{\partial t}) - B^{\mathrm{eff}}\ \dfrac{\partial p_{\mathrm{f}}}{\partial t}, \\ < v_{\mathrm{p}_i} >^{\Omega_{\mathrm{p}}} = -\dfrac{K^{\mathrm{eff}}_{ij}}{\mu}\dfrac{\partial p_{\mathrm{f}}}{\partial x_j}. \end{cases}$$

The effective parameters are strongly affected by the distribution of inclusions, even at low volume fraction. In the above developments, it is implicitely assumed that the inclusion size, l_c, is of same order of magnitude as the distance between two inclusions. Considering the case of low inclusion concentration, $l_c \ll l$, would not modify the macroscopic behaviour. But, since this introduces the additional small parameter $l_c/l \ll 1$, simplified formulas can be obtained for the effective parameters. While sometimes qualified as a multiporous medium and although three distinct scales are actually involved, this composite is distinct from a double porosity microstructure (*e.g.* Auriault and Royer (1993), Royer and Boutin (2012), Boutin and Royer (2015)), as the distribution of the inclusions does not play the role of a porous network. Indeed, the impact of inclusions on fluid flow transfer is characterised by a purely geometrical parameter, with no reference to their fluid conductivity. Finally, we shall underline that all the above results are valid for elastic properties of both constituents in the same order of magnitude and for perfect interface bonding. A particular case of the model is derived and solved in Chen et al. (2019), in which an incompressible istropic porous matrix reinforced with isotropic fibers is considered to model a construct for tissue engineering. Further work should include numerical simulations on specific geometries, so as to analyse the sensitivity of inclusion concentration.

A. Boundary value problem for $\vec{u}^1_{\mathrm{s}}$ and $\vec{u}^1_{\mathrm{c}}$

Let us multiply the system Eqs.(27)-(31) by a vectorial test function $\vec{\gamma}$, and then let us integrate over Ω. We obtain the following variational formulation:

$$\begin{aligned} &\int_{\Omega_{\mathrm{p}}} c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{u}^1_{\mathrm{s}}) e_{y_{ij}}(\vec{\gamma})\ d\Omega \\ &+ \int_{\Omega_{\mathrm{c}}} c_{\mathrm{c}_{ijlm}} e_{y_{lm}}(\vec{u}^1_{\mathrm{c}}) e_{y_{ij}}(\vec{\gamma})\ d\Omega = \\ &- \int_{\Omega_{\mathrm{p}}} c_{\mathrm{p}_{ijlm}} e_{y_{ij}}(\vec{\gamma})\ d\Omega\ e_{x_{lm}}(\vec{u}^0) \\ &- \int_{\Omega_{\mathrm{c}}} c_{\mathrm{c}_{ijlm}} e_{y_{ij}}(\vec{\gamma})\ d\Omega\ e_{x_{lm}}(\vec{u}^0) \\ &+ \int_{\Omega_{\mathrm{p}}} \alpha_{p_{ij}} e_{y_{ij}}(\vec{\gamma})\ d\Omega\ p^0_{\mathrm{f}}, \end{aligned} \tag{64}$$

from which it appears that $\vec{u}^1_{\mathrm{s}}$ is a linear vectorial function of $\tilde{e}_x(\vec{u}^0)$ and p^0_f, Eq.(32), and that $\vec{u}^1_{\mathrm{c}}$ is as a linear vectorial function of $\tilde{e}_x(\vec{u}^0)$, Eq.(33). Third-order tensors $\omega^{kh}_{\mathrm{p}_i}$ and $\omega^{kh}_{\mathrm{c}_i}$ are the specific solutions, $\vec{u}^1_{\mathrm{s}} = \vec{\omega}^{kh}_{\mathrm{p}}$, $\vec{u}^1_{\mathrm{c}} = \vec{\omega}^{kh}_{\mathrm{c}}$, to system Eqs.(27)-(31), corresponding to

$$\begin{cases} e_{x_{lm}}(\vec{u}^0) = \dfrac{1}{2}\left(\delta_{lk}\delta_{mh} + \delta_{mk}\delta_{lh}\right), \\ p^0_{\mathrm{f}} = 0. \end{cases}$$

As for vector $\vec{\pi}_{\mathrm{p}}$, it is the specific solution for $\vec{u}^1_{\mathrm{s}}$ when $p^0_{\mathrm{f}} = -1$ and $e_{x_{lm}}(\vec{u}^0_s) = 0$.

B. Equality of coupling tensors $\tilde{A}^{\mathrm{eff}}$ and $\tilde{G}^{\mathrm{eff}}$

By taking $\vec{u}^1_{\mathrm{s}} = \vec{\omega}^{kh}_{\mathrm{p}}$, $\vec{u}^1_{\mathrm{c}} = \vec{\omega}^{kh}_{\mathrm{c}}$, $\vec{\gamma} = \vec{\pi}_{\mathrm{p}}$ in Ω_{p}, and $\vec{\gamma} = \vec{0}$ in Ω_{c}, in the variational formulation Eq.(64), we get

$$\begin{aligned} &\int_{\Omega_{\mathrm{p}}} c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{\omega}^{kh}_{\mathrm{p}}) e_{y_{ij}}(\vec{\pi}_{\mathrm{p}})\ d\Omega = \\ &- \int_{\Omega_{\mathrm{p}}} c_{\mathrm{p}_{ijkh}} e_{y_{ij}}(\vec{\pi}_{\mathrm{p}})\ d\Omega. \end{aligned} \tag{65}$$

Next, we take $\vec{u}_{\mathrm{s}}^1 = \vec{\pi}_{\mathrm{p}}$ and $\vec{\gamma} = \vec{\omega}_{\mathrm{p}}^{kh}$ in Eq.(64):

$$\int_{\Omega_{\mathrm{p}}} c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{\pi}_{\mathrm{p}}) e_{y_{ij}}(\vec{\omega}_{\mathrm{p}}^{kh})\, d\Omega = - \int_{\Omega_{\mathrm{p}}} \alpha_{p_{ij}} e_{y_{ij}}(\vec{\omega}_{\mathrm{p}}^{kh})\, d\Omega. \tag{66}$$

The left hand sides of Eqs.(65) and (66) are equal. Then, from the equality of both right hand sides, it appears by Eqs.(55), (60), that $\tilde{A}^{\mathrm{eff}} = \tilde{G}^{\mathrm{eff}}$.

C. Positiveness of the effective Biot bulk modulus B^{eff}

Considering $\vec{u}_{\mathrm{s}}^1 = \vec{\pi}_{\mathrm{p}}$, $\vec{\gamma} = \vec{\pi}_{\mathrm{p}}$, in Eq.(64), we get:

$$\int_{\Omega_{\mathrm{p}}} c_{\mathrm{p}_{ijlm}} e_{y_{lm}}(\vec{\pi}_{\mathrm{p}}) e_{y_{ij}}(\vec{\pi}_{\mathrm{p}}) d\Omega = - \int_{\Omega_{\mathrm{p}}} \alpha_{p_{ij}} e_{y_{ij}}(\vec{\pi}_{\mathrm{p}}) d\Omega.$$

The left hand side of the above equation is positive, due to the positiveness of the local strain energy. Then, from the positiveness of the right hand side, it follows by Eq.(61), that $B^{\mathrm{eff}} > 0$.

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