A nonsmooth static frictionless contact problem with locking materials

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Abstract

We study a new mathematical model which describes the equilibrium of a locking material in contact with a foundation. The contact is frictionless and is modeled with a nonsmoth multivalued interface law which involves unilateral constraints and subdifferential conditions. We describe the model and derive its weak formulation, which is in the form of an elliptic variational-hemivariatinal inequality for the displacement field. Then, we establish the existence of a unique weak solution to the problem. Next, we introduce a penalty method, for which we state and prove a convergence result. Finally, we consider a particular version of the model for which we prove the continuous dependence of the solution on the bounds which govern the locking and the normal displacement constraints, respectively. We apply this convergence result in the study of an optimization problem associated to the contact model.

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Key words : locking material, frictionless contact, variational-hemivariational inequality, weak solution, penalty method, convergence results, optimization problem.

1 Introduction

This paper deals with the modeling and analysis of a contact problem for locking materials. Locking materials belong to a class of hyperelastic bodies for which the

strain tensor is constrained to belong in a given convex B. Their study started with the pioneering works of Prager [17, 18, 19]. There, the constitutive law of such materials was introduced and various mechanical interpretations have been provided. Reference in the field include [7, 15, 16], for instance. The modeling of torsion of a cylindrical bar made of a locking material was studied in [4].

Processes of contact between deformable bodies abound in industry and everyday life. Their modeling, analysis and numerical simulation made the object of an important number of papers and the literature in the field is extensive. It includes the books [2, 7, 8, 9, 10, 12, 15, 20, 21, 22], as well as the paper [23], for instance. There, various mathematical models of contact have been considered, with both elastic, viscoelastic and viscoplastic materials, and different boundary conditions.

Contact problems with locking materials have been recently considered in [1, 14]. For the problem studied in [14] the contact was described with the Signorini unilateral condition in a form without gap and the friction was modeled with a nonmonotone multivalued subdifferential condition which depends on the slip. As a consequence, the problem was governed by a convex set which describes the constraints of the displacement field and a nonconvex locally Lipschitz potential, which describes the friction. The existence of a unique weak solution to the problem was proved, by using a surjectivity result for pseudomonotone operators combined with the Banach contraction principle. The reference [1] deals with the numerical analysis of the model considered in [14]. There, convergence results and error estimates are provided, together with numerical simulations which represent an evidence of these theoretical results.

This current paper parallels the study initiated in [1, 14]. It deals with the modeling and analysis of a new contact problem which describes the contact of a body made of locking material with a foundation made of a rigid material, covered by a rigid-plastic crust and a deformable layer made of a soft material. The contact is assumed to be frictionless, for simplicity. Nevertheless, we stress that part of the results we present in this paper still remain valid if we complete the model with various friction laws, included the friction law considered in [1, 14]. The statement of the model and its analysis, including various convergence results, represents the traits of novelty of our current work.

The paper is structured as follows. In Section 2 we survey some preliminaries of functional analysis we need in the rest of the paper. In Section 3 we introduce the contact model and describe the boundary conditions. In Section 4 we introduce the function spaces for the displacement and stress field, list the assumptions on the data and derive the variational formulation of the contact problem. Then we state and prove the existence of a unique weak solution of the model. The proof is based, on arguments of variational-hemivariational inequalities which can be found in [13, 22]. In Section 5 we introduce a class of penalized problems, prove their unique solvability and the convergence of the corresponding solutions to the solution of the

original problem, as the penalty parameter converges to zero. Finally, in Section 6 we investigate the continuous dependence of the solution with respect to the bounds which govern the locking and the normal displacement constraints. We apply this result in the study of an optimization problem associated to the contact model.

2 Preliminaries

Everywhere in this section X is assumed to be a reflexive Banach space with norm denoted by $\|\cdot\|_X$ and X^* will represent its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X, by 0_X and 0_{X^*} the zero element of X and X^*, respectively, and by 2^{X^*} set of parts of X^*. All the limits, upper and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly. We start with the definition of some classes of operators.

Definition 2.1. *An operator $A\colon X \to X^*$ is said to be:*

a) monotone*, if for all u, $v \in X$, we have $\langle Au - Av, u - v\rangle \geq 0$;*

b) bounded*, if A maps bounded sets of X into bounded sets of X^*;*

c) demicontinuous*, if $u_n \to u$ in X implies $Au_n \to Au$ weakly in X^*;*

d) pseudomonotone*, if it is bounded and $u_n \to u$ weakly in X with*

$$\limsup \langle Au_n, u_n - u\rangle \leq 0$$

imply

$$\liminf \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle \quad \text{for all } v \in X.$$

Definition 2.2. *Let K be a subset of X. An operator $P\colon X \to X^*$ is said to be a* penalty operator *of K if P is bounded, demicontinuous, monotone and, moreover, $K = \{x \in X \mid Px = 0_{X^*}\}$.*

Let $K \subset X$ be a nonempty subset. The function ψ_K defined by

$$\psi_K(x) = \begin{cases} 0, & \text{if } \ x \in K, \\ +\infty, & \text{if } \ x \notin K, \end{cases}$$

is called the indicator function of K. Its subdifferential (in the sense of convex analysis) is the multivalued operator $\partial\psi_K : X \to 2^{X^*}$ defined by

$$\partial\psi_K(x) = \begin{cases} \{\, x^* \in X^* \mid \langle x^*, x - v\rangle \geq 0 \ \text{ for all } \ v \in K \,\} & \text{if } x \in K, \\ \emptyset \quad \text{if } x \notin K. \end{cases} \tag{2.1}$$

An element $x^* \in \partial\psi_K(x)$ (if any) is called a subgradient of $\partial\psi_K$ in x.

Definition 2.3. *A function $\varphi\colon K \to \mathbb{R}$ is said to be (strongly)* lower semicontinuous (l.s.c.) *at $u \in K$*

$$\liminf_{n\to\infty} \varphi(u_n) \geq \varphi(u) \tag{2.2}$$

for each sequence $\{u_n\} \subset K$ converging (strongly) to u in X. The function φ is l.s.c. *if it is l.s.c. at every point $u \in K$.*

We now recall the definition of the Clarke subdifferential for a locally Lipschitz function.

Definition 2.4. *A function $h\colon X \to \mathbb{R}$ is said to be* locally Lipschitz*, if for every $x \in X$, there exists a neighborhood of x, denoted U_x, and a constant $L_x > 0$ such that $|h(y) - h(z)| \leq L_x \|y - z\|_X$ for all y, $z \in U_x$.*

We note that a convex continuous function $h\colon X \to \mathbb{R}$ is locally Lipschitz. Moreover, if a function $h\colon X \to \mathbb{R}$ is Lipschitz continuous on bounded sets of X, then it is locally Lipschitz, while the converse does not hold, in general.

Definition 2.5. *Let $h\colon X \to \mathbb{R}$ be a locally Lipschitz function. The* generalized (Clarke) directional derivative *of h at the point $x \in X$ in the direction $v \in X$ is defined by*

$$h^0(x; v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}\,.$$

The generalized gradient (subdifferential) *of h at x is a subset of the dual space X^* given by*

$$\partial h(x) = \{\, \zeta \in X^* \mid h^0(x; v) \geq \langle \zeta, v\rangle \quad \forall\, v \in X \,\}. \tag{2.3}$$

A locally Lipschitz function h is said to be regular *(in the sense of Clarke) at the point $x \in X$ if for all $v \in X$ the one-sided directional derivative $h'(x; v)$ exists and $h^0(x; v) = h'(x; v)$.*

We shall use the following properties of the generalized directional derivative and the generalized gradient.

Proposition 2.6. *Assume that $h\colon X \to \mathbb{R}$ is a locally Lipschitz function. Then the following hold:*

(i) *For every $x \in X$, the function $X \ni v \mapsto h^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $h^0(x; \lambda v) = \lambda h^0(x; v)$ for all $\lambda \geq 0$, $v \in X$ and $h^0(x; v_1 + v_2) \leq h^0(x; v_1) + h^0(x; v_2)$ for all v_1, $v_2 \in X$, respectively.*

(ii) *For every $v \in X$, we have $h^0(x; v) = \max\{\, \langle \xi, v\rangle \mid \xi \in \partial h(x) \,\}$.*

For more details on the definitions and statements above we refer to the monographs [3, 5, 6, 12, 22, 27] as well as to the papers [11, 24, 25, 26].

We now recall some recent results in the study of variational-hemivariaional inequalities. The problem under consideration can be formulated as follows.

Problem 1. *Find an element $u \in K$ such that*

$$\langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) \geq \langle f, v-u\rangle \ \text{ for all } v \in K.$$

In the study of this problem we consider the following hypotheses on the data.

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.4}$$

$$\begin{cases} A\colon X \to X^* \text{ is such that} \\ \quad \text{(a) it is pseudomonotone,} \\ \quad \text{(b) there exist } \alpha_A > 0, \beta, \gamma \in \mathbb{R} \text{ and } u_0 \in K \text{ such that} \\ \qquad \langle Av, v-u_0\rangle \geq \alpha_A \|v\|_X^2 - \beta \|v\|_X - \gamma \quad \text{for all } v \in X, \\ \quad \text{(c) strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \qquad \langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \text{for all } v_1, v_2 \in X. \end{cases} \tag{2.5}$$

$$\varphi\colon K \to \mathbb{R} \text{ is convex and lower semicontinuous.} \tag{2.6}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j \text{ is locally Lipschitz,} \\ \quad \text{(b) } \|\partial j(v)\|_{X^*} \leq c_0 + c_1 \|v\|_X \text{ for all } v \in X \text{ with } c_0, c_1 \geq 0, \\ \quad \text{(c) there exists } \alpha_j > 0 \text{ such that} \\ \qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \\ \qquad \text{for all } v_1, v_2 \in X. \end{cases} \tag{2.7}$$

$$f \in X^*. \tag{2.8}$$

Note that in the statement of Problem 1, the function φ is assumed to be convex and the function j is locally Lipschitz and, in general, nonconvex. For this reason, following the terminology introduced in [16], the inequality in Problem 1 is a variational-hemivariational inequality. Its unique solvability is provided by the following existence and uniqueness result.

Theorem 2.7. *Assume that* (2.4)–(2.8) *hold and, in addition, assume that*

$$\alpha_j < m_A, \tag{2.9}$$

Then, Problem 1 *has a unique solution $u \in K$.*

Theorem 2.7 represents a particular case of a result proved in [13, 22]. There, the more general case in which the function φ depends on solution was considered. Its proof was carried out in several steps, based on a surjectivity result for the sum of a multivalued pseudomonotone operator and a maximal monotone one, combined with the Banach fixed point argument.

Assume now that $P : X \to X^*$. Then, for every $\lambda > 0$, we consider the following problem.

Problem 2. *Find an element $u_\lambda \in X$ such that*

$$\langle Au_\lambda, v - u_\lambda\rangle + \frac{1}{\lambda}\langle Pu_\lambda, v - u_\lambda\rangle + \varphi(v) - \varphi(u_\lambda) + j^0(u_\lambda; v - u_\lambda) \geq \langle f, v - u_\lambda\rangle \quad \text{for all } v \in X.$$

In the study of Problem 2 we assume the following additional hypotheses.

$$P\colon X \to X^* \text{ is a penalty operator of } K. \tag{2.10}$$

$$\begin{cases} \limsup j^0(u_n; v - u_n) \leq j^0(u; v - u) \\ \text{for all } u, v \in X \text{ and } u_n \to u \text{ weakly in } X. \end{cases} \tag{2.11}$$

We have the following result, proved in [13].

Theorem 2.8. *Assume that* (2.4)–(2.11) *hold. Then:*

(i) *For each $\lambda > 0$, there exists a unique solution $u_\lambda \in X$ to Problem* 2.

(ii) *$u_\lambda \to u$ in X, as $\lambda \to 0$, where $u \in K$ is the unique solution to Problem* 1.

Note that Problem 2 represents a penalized version of Problem 1. It is formulated on whole space, since the constraint on the solution was the removed. Theorem 2.8 states that the penalized problem has a unique solution which converges to the solution of the original problem, as the penalty parameter converges to zero. Its proof is obtained in several steps, by using well-known arguments of monotonicity, compactness and lower semicontinuity. In this first step it is proved that the sequence $\{u_\lambda\}$ is bounded and, therefore, it converges weakly to an element $\widetilde{u} \in X$, as $\lambda \to 0$. In the second step it is proved that $\widetilde{u} = u$, which guarantees that the whole sequence $\{u_\lambda\}$ converges weakly to u in X, as $\lambda \to 0$. Finally, in the last step the strong convergence is proved, by using the strong monotonicity of the operator A.

3 The contact model

The physical setting, already considered in many papers and surveys, can be resumed as follows. A deformable body occupies, in its reference configuration, a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$, with a Lipschitz continuous boundary Γ. The boundary is divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3, such that the $d - 1$ Lebesgue measure of Γ_1, denoted by $meas\,(\Gamma_1)$, is positive. The body is fixed on Γ_1, is acted upon by given surface tractions on Γ_2, and is in contact with an obstacle on Γ_3. The equilibrium of the body in this physical setting can be described by

various mathematical models, obtained by using different mechanical assumptions. The mathematical model we consider in this paper is based on specific constitutive law and interface boundary conditions which will be described below in this section. Its statement is as follows.

Problem 3. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *and two interface functions* $\eta_\nu : \Gamma_3 \to \mathbb{R}$*,* $\xi_\nu : \Gamma_3 \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} \in \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \partial\psi_B\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$\left.\begin{aligned} &u_\nu \le g, \\ &\sigma_\nu + \eta_\nu + \xi_\nu \le 0, \\ &(u_\nu - g)(\sigma_\nu + \eta_\nu + \xi_\nu) = 0, \\ &0 \le \eta_\nu \le F, \\ &\eta_\nu = \begin{cases} 0 & \text{if} \quad u_\nu < 0, \\ F & \text{if} \quad u_\nu > 0, \end{cases} \\ &\xi_\nu \in \partial j_\nu(u_\nu) \end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_3. \tag{3.6}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Moreover, $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\boldsymbol{0}$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d, \\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \end{aligned}$$

where the indices i, j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. In addition, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$, i.e., $u_{i,j} = \partial u_i/\partial x_j$. Also, $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad \operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$$

and, therefore, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearized strain tensor. Moreover, $\boldsymbol{\nu} = (\nu_i)$ denotes the outward unit normal at Γ and u_ν, $\boldsymbol{u}_\tau$ represent the normal and tangential components of $\boldsymbol{u}$ on Γ given by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. Finally, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the normal and tangential stress on Γ, that is $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

We now provide a short description of the equations and boundary conditions in Problem 3.

First, equation (3.1) represents the constitutive law of the locking material in which $\mathcal{A}$ is the elasticity operator, assumed to be nonlinear, ψ_B is the indicator function of the set $B \subset \mathbb{S}^d$ and $\partial\psi_B$ represents its subdifferential, see (2.1). Example of operators $\mathcal{A}$ which satisfy the conditions presented below in this paper can be found in our books [12, 21, 22]. For the set B, which describes the locking constraints of the material, various examples can be found in the literature, as explained in [7]. A typical example is given by

$$B = \{\, \boldsymbol{\tau} \in \mathbb{S}^d \ : \ \mathcal{F}(\boldsymbol{\tau}) \leq k \,\}, \tag{3.7}$$

where $\mathcal{F} : \mathbb{S}^d \to \mathbb{R}$ is a convex continuous function such that $\mathcal{F}(\mathbf{0}) = 0$ and k is a positive constant. It is easy to see that in this case the set B is a nonempty convex closed subset of $\mathbb{S}^d$. Using (3.7) with the choice

$$\mathcal{F}(\boldsymbol{\tau}) = \frac{1}{2} \|\boldsymbol{\tau}^D\|\,, \tag{3.8}$$

where $\boldsymbol{\tau}^D$ denotes the deviator of the tensor $\boldsymbol{\tau} \in \mathbb{S}^d$, leads to the Von Mises convex. This convex set was considered in [17, 18] to model the ideal-locking effect. The choice

$$\mathcal{F}(\boldsymbol{\tau}) = \mathrm{tr}(\boldsymbol{\tau}) \tag{3.9}$$

where $\mathrm{tr}(\boldsymbol{\tau})$ denotes the trace of the tensor $\boldsymbol{\tau} \in \mathbb{S}^d$ leads to the class of materials with limited compressibility considered in [18].

Equation (3.2) is the equation of equilibrium that we use here since the process is assumed to be static. Conditions (3.3), (3.4) represent the displacement and the traction boundary conditions, respectively. Finally, condition (3.6) represents the frictionless contact condition.

We now turn on the contact condition (3.5) in which our main interest is. Here $g > 0$, F and j_ν are given functions which will be described below and ∂j_ν denotes the Clarke subdifferential of j_ν. This condition models the contact with a foundation made of a rigid body covered by a deformable layer of thickness g and a crust. It can be derived in the following way.

First, the rigid body does not allow penetration and, therefore, the normal displacement is limited by the bound g, i.e.,

$$u_\nu \leq g \quad \text{on } \Gamma_3. \tag{3.10}$$

Next, we assume that the normal stress has an additive decomposition of the form

$$\sigma_\nu = \sigma_\nu^C + \sigma_\nu^D + \sigma_\nu^R \quad \text{on } \Gamma_3, \tag{3.11}$$

in which the quantities σ_ν^C, σ_ν^D and σ_ν^R describe the reaction of the crust, the deformable layer and the rigid body, respectively. The part σ_ν^C of the normal stress is such that

$$-F \le \sigma_\nu^C \le 0, \quad -\sigma_\nu^C = \begin{cases} 0 & \text{if } u_\nu < 0, \\ F & \text{if } u_\nu > 0 \end{cases} \quad \text{on } \Gamma_3, \tag{3.12}$$

where F is a given nonnegative function. Assume $F > 0$. Then, using (3.12) we have

$$\begin{aligned} -F < \sigma_\nu^C \le 0 \quad &\Longrightarrow \quad u_\nu \le 0, \\ \sigma_\nu^C = -F \quad &\Longrightarrow \quad u_\nu \ge 0. \end{aligned}$$

This shows that the crust does not allow penetration (and, therefore, it behaves like a rigid body) as far as the inequality $-F < \sigma_\nu^C \le 0$ holds. It could allow penetration only when $\sigma_\nu^C = -F$ and, in this case, it offers no additional resistance. We conclude from here that the crust has a rigid-plastic behaviour and the function F could be interpreted as its yield limit.

We also assume that that σ_ν^D satisfies a multivalued normal compliance contact condition, that is

$$-\sigma_\nu^D \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3. \tag{3.13}$$

Details on such condition, including examples and various mechanical interpretations, can be found in [12]. Here we restrict ourselves to mention that it models the contact with a deformable foundation. Finally, the part $\sigma_\nu^R(t)$ of the normal stress satisfies the well known Signorini condition in the form with the gap g, i.e.,

$$\sigma_\nu^R \le 0, \quad \sigma_\nu^R(u_\nu - g) = 0 \quad \text{on } \Gamma_3. \tag{3.14}$$

We denote $-\sigma_\nu^C = \eta_\nu$, $-\sigma_\nu^D = \xi_\nu$ and use (3.11) to see that

$$\sigma_\nu^R = \sigma_\nu + \eta_\nu + \xi_\nu \quad \text{on } \Gamma_3. \tag{3.15}$$

Then we substitute equality (3.15) in (3.14) and use (3.10), (3.12), (3.13) to obtain the contact condition (3.5).

4 Existence and uniqueness

In the study of Problem 3 we need to introduce further notation and preliminary material. Everywhere in this paper we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$, $L^2(\Gamma_3)^d$ and $H^1(\Omega)^d$, endowed with their canonical inner products

and associated norms. Moreover, we recall that for an element $\boldsymbol{v} \in H^1(\Omega)^d$ we usually write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ of $\boldsymbol{v}$ to Γ. In addition, we consider the following spaces:

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\},$$
$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \,\}.$$

The spaces V and Q are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx. \tag{4.1}$$

The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Recall that the completeness of the space V follows from the assumption $meas\,(\Gamma_1) > 0$ which allows the use of Korn's inequality. We denote by V^* and $\langle\cdot,\cdot\rangle$ the topological dual of V and the duality pairing between V^* and V, respectively. We also denote by $\boldsymbol{0}_V$ the zero element of V and, for any element $\boldsymbol{v} \in V$, we denote by v_ν and $\boldsymbol{v}_\tau$ its normal and tangential components on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, respectively. Recall that, for a regular stress function $\boldsymbol{\sigma}$, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^d. \tag{4.2}$$

Finally, we recall that the Sobolev trace theorem yields

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le \|\gamma\|\, \|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V, \tag{4.3}$$

$\|\gamma\|$ being the norm of the trace operator $\gamma\colon V \to L^2(\Gamma_3)^d$.

In the study of the mechanical problem (3.1)–(3.6) we assume that the elasticity operator $\mathcal{A}$ satisfies the following conditions.

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{4.4}$$

We also assume that the set of locking constraints, the densities of body forces and tractions, the yield limit of the crust and the bound of the normal displacement are such that

$$B \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0} \in B. \tag{4.5}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d. \tag{4.6}$$

$$\boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{4.7}$$

$$F \in L^2(\Gamma_3), \qquad F(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{4.8}$$

$$g > 0. \tag{4.9}$$

Finally, the normal compliance function j_ν satisfies the follwing condition.

$$\left\{\begin{array}{l} j_\nu \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3), \\ \quad \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \quad \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \leq \bar{c}_0 + \bar{c}_1 \, |r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \qquad \text{for all } r \in \mathbb{R} \text{ with } \bar{c}_0,\, \bar{c}_1 \geq 0, \\ \quad \text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \leq \alpha_{j_\nu} \, |r_1 - r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } r_1,\, r_2 \in \mathbb{R} \text{ with } \alpha_{j_\nu} \geq 0. \end{array}\right. \tag{4.10}$$

Here and below we denote by $\partial j_\nu(\boldsymbol{x}, \cdot)$ and $j^0(\boldsymbol{x}, \cdot; \cdot)$ the generalized gradient and the generalized directional derivative of j_ν with respect to the second variable, for a.e. $\boldsymbol{x} \in \Gamma_3$.

We now turn to the variational formulation of Problem 3 and, to this end, we introduce the sets U, W and K defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \leq g \ \text{ a.e. on } \Gamma_3 \,\}, \tag{4.11}$$

$$W = \{\, \boldsymbol{v} \in V : \boldsymbol{\varepsilon}(\boldsymbol{v}) \in B \ \text{ a.e. in } \Omega \,\}, \tag{4.12}$$

$$K = U \cap W. \tag{4.13}$$

Assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.1)–(3.6). Then, using (3.5) it follows that $\boldsymbol{u} \in U$ and, using (3.1) and (2.1) we deduce that $\boldsymbol{u} \in W$. Thus, definition (4.13) implies that

$$\boldsymbol{u} \in K. \tag{4.14}$$

Let $\boldsymbol{v} \in K$. We use Green's formula (4.2) and equalities (3.2)–(3.4), to see that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})) \, dx \\ &\quad = \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u}) \, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u}) \, da + \int_{\Gamma_3} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, da. \end{aligned}$$

Moreover, since

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, da = \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, da + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da,$$

the frictionless condition (3.6) yields

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx \\ &\quad = \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, da + \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, da. \end{aligned} \tag{4.15}$$

We now use the constitutive law (3.1) and (2.1) to deduce that

$$\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx \geq \int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx. \tag{4.16}$$

On the other hand, using the contact conditions (3.5), assumption (4.8) and equality (2.3) it is easy to see that

$$\begin{aligned} &(\sigma_\nu + \eta_\nu + \xi_\nu)(v_\nu - g) \geq 0, \qquad (\sigma_\nu + \eta_\nu + \xi_\nu)(g - u_\nu) = 0 \\ &-\eta_\nu (v_\nu - u_\nu) \geq F(u_\nu^+ - v_\nu^+), \qquad -\xi_\nu (v_\nu - u_\nu) \geq -j_\nu^0(u_\nu; v_\nu - u_\nu) \end{aligned}$$

a.e. on Γ_3 where, here and below, r^+ represent the positive part of r. Therefore, writing

$$\begin{aligned} \sigma_\nu (v_\nu - u_\nu) &= (\sigma_\nu + \eta_\nu + \xi_\nu)(v_\nu - g) + (\sigma_\nu + \eta_\nu + \xi_\nu)(g - u_\nu) \\ &\quad -\eta_\nu (v_\nu - u_\nu) - \xi (v_\nu - u_\nu) \end{aligned}$$

we deduce that

$$\sigma_\nu (v_\nu - u_\nu) \geq F(u_\nu^+ - v_\nu^+) - j_\nu^0(u_\nu; v_\nu - u_\nu) \quad \text{a.e. on } \Gamma_3,$$

which implies that

$$\int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, da \geq \int_{\Gamma_3} F(u_\nu^+ - v_\nu^+)\, da - \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, da. \tag{4.17}$$

Finally, we combine equality (4.15) with inequalities (4.16), (4.17) and regularity (4.14) to deduce the following variational formulation of Problem 3.

Problem 4. *Find a displacement field* $\boldsymbol{u} \in K$ *such that*

$$\begin{aligned} &\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx + \int_{\Gamma_3} F v_\nu^+\, da - \int_{\Gamma_3} F u_\nu^+\, da \\ &\quad + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, da \geq \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, da \\ &\qquad \text{for all } \ \boldsymbol{v} \in K. \end{aligned} \tag{4.18}$$

In the study of Problem 4 we have the following existence and uniqueness result.

Theorem 4.1. *Assume that* (4.4)–(4.9) *hold and, in addition, assume the smallness condition*

$$\alpha_{j_\nu}\|\gamma\|^2 < m_{\mathcal{A}}. \tag{4.19}$$

Then Problem 4 *has a unique solution* $\boldsymbol{u} \in K$.

The proof of Theorem 4.1 is based on Theorem 2.7. To present it we use arguments of monotonicity, lower semicontinuity and the property of the Clarke subdifferential, which guarantee that the assumptions of Theorem 2.7 are satisfied. We start with some technical results that we gather in the following lemma.

Lemma 4.2. *Assume that* (4.10) *holds and let* $j : V \to \mathbb{R}$ *be the function defined by*

$$j(\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)\, da. \tag{4.20}$$

Then, the following statements hold.

(i) j *is a locally Lipschitz function on* V.

(ii) j *satisfies condition* (2.7) *with* $c_0 = \sqrt{2\,meas(\Gamma_3)}\,\bar{c}_0\|\gamma\|$, $c_1 = \sqrt{2}\,\bar{c}_1\|\gamma\|^2$ *and* $\alpha_j = \alpha_{j_\nu}\|\gamma\|^2$.

(iii) *For all* $\boldsymbol{u}$, $\boldsymbol{v} \in V$, *we have*

$$j^0(\boldsymbol{u};\boldsymbol{v}) \le \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu)\, da. \tag{4.21}$$

Moreover, if

$$\text{either } j_\nu(\boldsymbol{x},\cdot) \text{ or } -j_\nu(\boldsymbol{x},\cdot) \text{ is regular on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \tag{4.22}$$

then j *or* $-j$ *is also regular and* (4.21) *holds with equality.*

(iv) *Under the additional assumption* (4.22) *the following property holds:*

$$\begin{cases} \limsup j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n) \le j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \\ \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \text{ and } \boldsymbol{u}_n \to \boldsymbol{u} \text{ weakly in } V. \end{cases}$$

Note that the points (i)–(iii) of Lemma 4.2 correspond to Lemma 8 in [22, p. 126]. The point (iv) is a direct consequence of Lemma 6 in [22, p. 123]. The details in proof can be found on [22, p. 228] and, for this reason, we omit them.

We now proceed with the proof of Theorem 4.1.

Proof. We use Theorem 2.7 with $X = V$, K given by (4.13), j given by (4.20) and A, φ, $\boldsymbol{f}$ defined as follows:

$$A\colon V \to V^*, \qquad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \tag{4.23}$$

$$\varphi\colon V \to \mathbb{R}, \qquad \varphi(\boldsymbol{v}) = \int_{\Gamma_3} F v_\nu^+\, da, \tag{4.24}$$

$$\boldsymbol{f} \in V^*, \qquad \langle \boldsymbol{f}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, da \tag{4.25}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$. To this end, we check the validity of conditions (2.4)–(2.9).

First, we use the definition (4.13) and assumptions (4.5), (4.9) to see that K is a closed convex subset of V such that $\mathbf{0}_V \in K$ and, therefore, condition (2.4) is satisfied. Next, we use the definition (4.23) and assumption (4.4)(c) to obtain that

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle \geq m_{\mathcal{A}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2 \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{4.26}$$

On the other hand, using assumption (4.4)(b) yields

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{V^*} \leq L_{\mathcal{A}} \|\boldsymbol{u} - \boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{4.27}$$

We conclude by (4.26) and (4.27) that A is a strongly monotone Lipschitz continuous operator on the space V and, therefore, it satisfies condition (2.5). Moreover, using (4.8) and (4.3) it is easy to see that the functional φ defined by (4.24) is a seminorm on the space V and, in addition,

$$\varphi(\boldsymbol{v}) \leq \|\gamma\| \|F\|_{L^2(\Gamma_3)} \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V.$$

It follows from here that φ is a continuous seminorm and, therefore, it satisfies condition (2.6). On the other hand, by Lemma 4.2 we see that condition (2.7) holds. Moreover, assumptions (4.6) and (4.7) imply (2.8) for $\boldsymbol{f}$. Finally, since Lemma 4.2 guarantees that $\alpha_j = \alpha_{j_\nu} \|\gamma\|^2$, we see that assumption (4.19) implies the smallness condition (2.9).

Therefore, we are in a position to use Theorem 2.7. In this way we deduce that there exists a unique element $\boldsymbol{u} \in K$ such that

$$\begin{aligned} \langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) + j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \\ \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \text{ for all } \ \boldsymbol{v} \in K. \end{aligned} \tag{4.28}$$

We now combine (4.28) with inequality (4.21) and notation (4.23)–(4.25) to see that the solution $\boldsymbol{u}$ of (4.28) satisfies (4.18), which proves the existence part of the theorem.

We now turn to the proof of the uniqueness part. To this end, let $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in K$ be solutions to inequality (4.18). Then, the inequalities below hold, for all $\boldsymbol{v} \in K$:

$$\begin{aligned} &\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1))\, dx + \int_{\Gamma_3} F(u_{1\nu})(v_\nu^+ - u_{1\nu}^+)\, da \\ &+ \int_{\Gamma_3} j_\nu^0(u_{1\nu}; v_\nu - u_{1\nu})\, da \geq \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u}_1)\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u}_1)\, da, \end{aligned}$$

$$\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}_2))\,dx+\int_{\Gamma_3}F(u_{2\nu})(v_\nu^+-u_{2\nu}^+)\,da$$
$$+\int_{\Gamma_3}j_\nu^0(u_{2\nu};v_\nu-u_{2\nu})\,da\geq\int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v}-\boldsymbol{u}_2)\,dx+\int_{\Gamma_2}\boldsymbol{f}_2\cdot(\boldsymbol{v}-\boldsymbol{u}_2)\,da.$$

We take $\boldsymbol{v}=\boldsymbol{u}_2$ in the first inequality and $\boldsymbol{v}=\boldsymbol{u}_1$ in the second one, then we add the resulting inequalities, use the hypotheses (4.4)(b), (4.10)(c) and inequality (4.3) to obtain that

$$(m_{\mathcal{A}}-\alpha_{j_\nu}\|\gamma\|^2)\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2\leq 0.$$

Finally, we use the smallness condition (4.19) to deduce that $\boldsymbol{u}_1=\boldsymbol{u}_2$, which concludes the proof. □

Note that Problem 4 is formulated in terms of the displacement field. The solution of this problem is called a weak solution for the contact problem 3. The question of finding the stress function $\boldsymbol{\sigma}$ and the interface functions η_ν and ξ_ν is left open.

5 A first convergence result

In this section we illustrate the use of the abstract result in Theorem 2.8 in the study of Problem 4. To this end we use arguments of monotonicity and the properties of the projection operators in order to construct an appropriate penalty operator for the set of admissible displacement fields (4.13). We start by considering a normal compliance function p_ν which satisfies

$$\begin{cases} p_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+ \text{ is such that}\\ \quad\text{(a) there exists } L_{p_\nu}>0 \text{ such that}\\ \qquad |p_\nu(\boldsymbol{x},r_1)-p_\nu(\boldsymbol{x},r_2)|\leq L_{p_\nu}|r_1-r_2|\\ \qquad\quad \text{for all } r_1,\ r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3,\\ \quad\text{(b) } (p_\nu(\boldsymbol{x},r_1)-p_\nu(\boldsymbol{x},r_2))\,(r_1-r_2)\geq 0\\ \qquad\quad \text{for all } r_1,\ r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3,\\ \quad\text{(c) } p_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R},\\ \quad\text{(d) } p_\nu(\boldsymbol{x},r)=0 \text{ if and only if } r\leq 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases} \tag{5.1}$$

A typical example of function satisfying (5.1) is $p_\nu(\boldsymbol{x},r)=r^+$ for all $r\in\mathbb{R}$, a.e. $\boldsymbol{x}\in\Gamma_3$. Moreover, we use assumption (4.5) and denote by $P_B:\mathbb{S}^d\to B$ the projection operator on the set B. Then, for every penalty parameter $\lambda>0$, we consider the following frictionless contact problem without unilateral constraint.

Problem 5. *Find a displacement field* $\boldsymbol{u}_\lambda\colon \Omega \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma}_\lambda\colon \Omega \to \mathbb{S}^d$ *and two interface function* $\eta_{\lambda\nu}\colon \Gamma_3 \to \mathbb{R}$*,* $\xi_{\lambda\nu}\colon \Gamma_3 \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma}_\lambda = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda) + \frac{1}{\lambda}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda)) \qquad \text{in } \Omega, \tag{5.2}$$

$$\operatorname{Div}\boldsymbol{\sigma}_\lambda + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{5.3}$$

$$\boldsymbol{u}_\lambda = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{5.4}$$

$$-\sigma_\lambda\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{5.5}$$

$$\left.\begin{aligned} &-\sigma_{\lambda\nu} = \tfrac{1}{\lambda}\, p_\nu(u_{\lambda\nu} - g) + \eta_{\lambda\nu} + \xi_{\lambda\nu}, \\ &0 \le \eta_\nu(t) \le F, \\ &\eta_\nu(t) = \begin{cases} 0 & \text{if} \quad u_\nu(t) < 0, \\ F & \text{if} \quad u_\nu(t) > 0, \end{cases} \\ &\xi_\nu(t) \in \partial j_\nu(u_\nu(t)) \end{aligned}\right\} \qquad \text{on } \Gamma_3, \tag{5.6}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on } \Gamma_3. \tag{5.7}$$

Here and below $u_{\lambda\nu}$ and $\sigma_{\lambda\nu}$ denote the normal components of the unknowns $\boldsymbol{u}_\lambda$ and $\boldsymbol{\sigma}_\lambda$, and $\boldsymbol{\sigma}_{\lambda\tau}$ represents the tangential part of the tensor $\boldsymbol{\sigma}_\lambda$, respectively.

The variational formulation of Problem 5, obtained by arguments similar to those used in Section 4, is the following.

Problem 6. *Find a displacement field* $\boldsymbol{u}_\lambda \in V$ *such that*

$$\begin{aligned} &\int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda))\,dx \\ &\quad + \frac{1}{\lambda}\int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda))\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda))\,dx \\ &\qquad + \frac{1}{\lambda}\int_{\Gamma_3} p_\nu(u_{\lambda\nu} - g)(v_{\lambda\nu} - u_{\lambda\nu})\,da + \int_{\Gamma_3} F v_\nu^+\,da - \int_{\Gamma_3} F u_{\lambda\nu}^+\,da \\ &\qquad\quad + \int_{\Gamma_3} j_\nu^0(u_{\lambda\nu}; v_\nu - u_{\lambda\nu})\,da \\ &\qquad\quad \ge \int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v} - \boldsymbol{u})\,dx + \int_{\Gamma_2} \boldsymbol{f}_2\cdot(\boldsymbol{v} - \boldsymbol{u})\,da \quad \text{for all} \quad \boldsymbol{v} \in V. \end{aligned}$$

We now state and prove the following existence, uniqueness and convergence result.

Theorem 5.1. *Assume that* (4.4)–(4.9), (4.19), (4.22), (5.1) *hold. Then:*

(i) *For each* $\lambda > 0$*, there exists a unique solution* $\boldsymbol{u}_\lambda \in V$ *to Problem* 6.

(ii) *The solution* $\boldsymbol{u}_\lambda$ *of Problem* 6 *converges to the solution* $\boldsymbol{u}$ *of Problem* 4*, i.e.,* $\boldsymbol{u}_\lambda \to \boldsymbol{u}$ *in* V*, as* $\lambda \to 0$.

In order to present the proof of the theorem we use the Riesz representation theorem to define the operator $P\colon V \to V^*$ by

$$\langle P\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \tag{5.8}$$

$$+ \int_{\Gamma_3} p_\nu(u_\nu - g)v_\nu\, da \quad \text{for all} \quad \boldsymbol{u},\, \boldsymbol{v} \in V.$$

We have the following result.

Lemma 5.2. *The operator* $P\colon V \to V^*$ *is a penalty operator, i.e., it satisfies the conditions in Definition* 2.2 *with* $X = V$ *and* K *given by* (4.13).

Proof. We show that the operator P is bounded, demicontinuous, monotone and $K = \{\, \boldsymbol{u} \in V \mid P\boldsymbol{u} = \boldsymbol{0}_{V^*} \,\}$. Let $\boldsymbol{u} \in V$, $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in V$. Then, using the nonexpansivity of the projection operator it is easy to see that

$$\begin{aligned} &\big|\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big) - \big(\boldsymbol{\varepsilon}(\boldsymbol{v}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{v})\big)\big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\big| \\ &\quad \le 2\,\|(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\|\|\boldsymbol{\varepsilon}(\boldsymbol{w})\| \qquad \text{a.e. in } \Omega, \end{aligned}$$

$$\big((\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})) - (\boldsymbol{\varepsilon}(\boldsymbol{v}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{v}))\big) \cdot \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\big) \ge 0 \quad \text{a.e. in } \Omega.$$

Therefore, we deduce that

$$\int_\Omega \big|\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big) - \big(\boldsymbol{\varepsilon}(\boldsymbol{v}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{v})\big)\big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w})\big|\, dx \le 2\,\|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V, \tag{5.9}$$

$$\int_\Omega \big((\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{v}))\big) \cdot \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v})\big)\, dx \ge 0. \tag{5.10}$$

Next, from (5.1) it is easy to see that

$$\begin{aligned} &|(p_\nu(u_\nu - g) - p_\nu(v_\nu - g))w_\nu| \le L_{p_\nu}|u_\nu - v_\nu||w_\nu| \quad \text{a.e. on } \Gamma_3, \\ &(p_\nu(u_\nu - g) - p_\nu(v_\nu - g))(u_\nu - v_\nu) \ge 0 \quad \text{a.e. on } \Gamma_3. \end{aligned}$$

Therefore, applying the trace inequality (4.3), we have

$$\int_{\Gamma_3} |(p_\nu(u_\nu - g) - p_\nu(v_\nu - g))w_\nu|\, da \le L_{p_\nu}\|\gamma\|^2 \|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V, \tag{5.11}$$

$$\int_{\Gamma_3} (p_\nu(u_\nu - g) - p_\nu(v_\nu - g))(u_\nu - v_\nu)\, da \ge 0. \tag{5.12}$$

We now use the definition (5.8) of the operator P and inequalities (5.9)–(5.12) to deduce that

$$|\langle P\boldsymbol{u} - P\boldsymbol{v}, \boldsymbol{w}\rangle| \le 2\,\|\boldsymbol{u}-\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V + L_{p_\nu}\int_{\Gamma_3} |u_\nu - v_\nu|\,|w_\nu|\,da$$

$$\le (2 + L_{p_\nu}\|\gamma\|^2)\|\boldsymbol{u}-\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V,$$

$$\langle P\boldsymbol{u} - P\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \ge 0.$$

These inequalities show that the operator P is Lipschitz continuous and monotone and, therefore, it is bounded and demicontinuous.

Assume now that $P\boldsymbol{u} = \boldsymbol{0}_{V^*}$. Then, $\langle P\boldsymbol{u}, \boldsymbol{u}\rangle_V = 0$ which implies that

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})\,dx + \int_{\Gamma_3} p_\nu(u_\nu - g)u_\nu\,da = 0. \tag{5.13}$$

We use the inclusion $\boldsymbol{0}\in B$ and the nonexpansivity of the projection operator P_B to see that

$$\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) \ge 0 \quad \text{a.e. in } \Omega.$$

On the other hand, using (5.1)(b), (d) yields

$$p_\nu(u_\nu - g)u_\nu \ge p_\nu(u_\nu - g)g \ge 0 \quad \text{a.e. on } \Gamma_3.$$

We deduce from above that

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})\,dx \ge 0, \qquad \int_{\Gamma_3} p_\nu(u_\nu - g)u_\nu\,da \ge 0$$

and, therefore, inequality (5.13) implies that

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})\,dx = 0, \qquad \int_{\Gamma_3} p_\nu(u_\nu - g)u_\nu\,da = 0.$$

Since the integrands in these integrals are positive, we deduce that

$$\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) = 0 \quad \text{a.e. in } \Omega, \tag{5.14}$$

$$p_\nu(u_\nu - g)u_\nu = 0 \quad \text{a.e. on } \Gamma_3. \tag{5.15}$$

Next, we use the properties of the projection to see that

$$(P_B\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot\boldsymbol{\tau} \ge (P_B\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot P_B\boldsymbol{\varepsilon}(\boldsymbol{u})$$

for all $\boldsymbol{\tau}\in B$, a.e. in Ω and, therefore, taking $\boldsymbol{\tau} = \boldsymbol{0}$ we find that

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u}))\cdot P_B\boldsymbol{\varepsilon}(\boldsymbol{u}) \ge 0 \quad \text{a.e. in } \Omega. \tag{5.16}$$

We now combine ineequality (5.16) with equality (5.14) to deduce that

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot (P_B\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u})) \geq 0 \quad \text{a.e. in } \Omega.$$

This implies that $\boldsymbol{\varepsilon}(\boldsymbol{u}) = P_B\boldsymbol{\varepsilon}(\boldsymbol{u})$ a.e. in Ω and, therefore, definition (4.12) shows that $\boldsymbol{u} \in W$. On the other hand, equality (5.15) combined with assumption (5.1)(d) implies that $u_\nu \leq g$ a.e. on Γ_3. This shows that $\boldsymbol{u} \in U$. We conclude from above that $\boldsymbol{u} \in U \cap W$, i.e. $\boldsymbol{u} \in K$.

Conversely, if $\boldsymbol{u} \in K$ it follows that $\boldsymbol{\varepsilon}(\boldsymbol{u}) \in B$ a.e. in Ω and $u_\nu \leq g$ a.e. on Γ_3. Using the properties of the projector we have $\boldsymbol{\varepsilon}(\boldsymbol{u}) = P_B\boldsymbol{\varepsilon}(\boldsymbol{u})$ a.e. in Ω and, using assumption (5.1)(d), we deduce that $p(u_\nu - g) = 0$ a.e. on Γ_3. Therefore, from the definition (5.8) of the operator P we deduce that $\langle P\boldsymbol{u}, \boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in V$, which implies that $P\boldsymbol{u} = \mathbf{0}_{V^*}$ and concludes the proof. □

We now have all the ingredients to provide the proof of Theorem 5.1.

Proof. From Lemma 5.2 we see that P is a penalty operator of the set K and, therefore, (2.10) holds. On the other hand, Lemma 4.2 (iv) guarantees that the function (4.20) satisfies condition (2.11) on the space V. We are now in a position to use Theorem 2.8 in order to conclude the proof. □

In addition to the mathematical interest in the convergence result in Theorem 5.1 (ii), it is important from the mechanical point of view, since it provides the link between the weak solutions of two different models of contact. Indeed, note that Problem 5 describes the frictionless contact of an elastic material, (5.2), with a deformable foundation covered by a crust. In contrast, Problem 3 describes the frictionless contact of a locking material with a rigid-deformable foundation covered by a crust.

6 A second convergence result

In this section we state and prove a second convergence result which shows the continuous dependence of the weak solution of Problem 3 with respect the constraints. Our approach is the following: we assume that the convex B is defined by using a bound k and, besides Problem 4 we consider a second problem, constructed by using a perturbation of the data g and k. Then, we use arguments of monotonicity and the properties of generalized directional derivative in order to estimate the difference between the solution of Problem 4 and its perturbation. This estimate leads to a convergence result, Theorem 6.1, which represents the main resut of this section.

We assume in what follows that (4.4), (4.6)–(4.9) and (4.19) hold. Moreover, we assume that the locking constraint set B is defined by equality (3.7), where

$$\mathcal{F} : \mathbb{S}^d \to \mathbb{R} \text{ is a positively homogenous convex continuous function,} \tag{6.1}$$

$$k > 0. \tag{6.2}$$

Then, it is easy to see that condition (4.5) holds. Therefore, using Theorem 4.1 we deduce that Problem 4 has a unique solution $\boldsymbol{u} \in K$. The solution depends on the bounds g and k and, therefore, sometimes we shall denote it by $\boldsymbol{u} = \boldsymbol{u}(g, k)$. The proof of Theorem 4.1 also shows that the solution satisfies the variational-hemivariational inequality (4.28), i.e.,

$$\begin{aligned} \boldsymbol{u} \in K, \qquad & \langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) + j^0(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \\ & \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \text{ for all } \ \boldsymbol{v} \in K. \end{aligned} \tag{6.3}$$

Consider now a perturbation g_n, k_n of the bounds g and k, respectively, such that

$$g_n > 0, \qquad k_n > 0 \qquad \text{for all } \ n \in \mathbb{N}. \tag{6.4}$$

Let $n \in \mathbb{N}$. We use (3.7), (4.11)–(4.13) to define the sets

$$B_n = \{\, \boldsymbol{\tau} \in \mathbb{S}^d \ : \ \mathcal{F}(\boldsymbol{\tau}) \leq k_n \,\}, \tag{6.5}$$

$$U_n = \{\, \boldsymbol{v} \in V : v_\nu \leq g_n \ \text{ a.e. on } \Gamma_3 \,\}, \tag{6.6}$$

$$W_n = \{\, \boldsymbol{v} \in V : \boldsymbol{\varepsilon}(\boldsymbol{v}) \in B_n \ \text{ a.e. in } \Omega \,\}, \tag{6.7}$$

$$K_n = U_n \cap W_n, \tag{6.8}$$

then we consider the problem of finding an element $\boldsymbol{u}_n$ such that

$$\begin{aligned} \boldsymbol{u}_n \in K_n, \qquad & \langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}_n) + j^0(\boldsymbol{u}_n; \boldsymbol{v} - \boldsymbol{u}_n) \\ & \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n\rangle \ \text{ for all } \ \boldsymbol{v} \in K_n. \end{aligned} \tag{6.9}$$

It follows from Theorem 4.1 that, for each $n \in \mathbb{N}$, inequality (6.9) has a unique solution $\boldsymbol{u}_n = \boldsymbol{u}(g_n, k_n)$. Our main result in this section is the following.

Theorem 6.1. *Assume that* (4.4), (4.6)–(4.9), (4.19), (6.1), (6.2) *and* (6.4) *hold. Moreover, assume that*

$$g_n \to g \quad \text{and} \quad k_n \to k. \tag{6.10}$$

Then, the solution $\boldsymbol{u}_n$ *of inequality* (6.9) *converges to the solution* $\boldsymbol{u}$ *of inequality* (6.3)*, i.e.,*

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in} \quad V. \tag{6.11}$$

In order to provide the proof of Theorem 6.1 we recall that the function j satisfies condition (2.7) on the space V with the constant c_0, c_1 and α_j defined in Lemma 4.2 (ii). Moreover, we need the following preliminary result.

Lemma 6.2. *For all $n \in \mathbb{N}$, the solution $\boldsymbol{u}_n$ of the variational-hemivariational inequality* (6.9) *satisfies the bound*

$$\|\boldsymbol{u}_n\|_X \leq \frac{1}{m_A - \alpha_j} \big(\|A\mathbf{0}_V\|_{V^*} + \|\boldsymbol{f}\|_{V^*} + c_0\big). \tag{6.12}$$

Proof. Let $n \in \mathbb{N}$. Taking $\boldsymbol{v} = \mathbf{0}_V \in K$ in (6.9), since $\varphi(\boldsymbol{u}_n) \geq$ and $\varphi(\mathbf{0}_V) = 0$, we deduce that

$$\langle A\boldsymbol{u}_n, \boldsymbol{u}_n\rangle \leq \langle \boldsymbol{f}, \boldsymbol{u}_n\rangle + j^0(\boldsymbol{u}_n; -\boldsymbol{u}_n).$$

We now write $A\boldsymbol{u}_n = A\boldsymbol{u}_n - A\mathbf{0}_V + A\mathbf{0}_V$ and use the strong monotonicity of the operator A, (4.26), to see that

$$m_{\mathcal{A}}\|\boldsymbol{u}_n\|_V^2 \leq (\|A\mathbf{0}_V\|_{V^*} + \|\boldsymbol{f}\|_{V^*})\|\boldsymbol{u}_n\|_V + j^0(\boldsymbol{u}_n; -\boldsymbol{u}_n). \tag{6.13}$$

On the other hand, taking $\boldsymbol{v}_1 = \boldsymbol{u}_n$ and $\boldsymbol{v}_2 = \mathbf{0}_V$ in (2.7)(c) we find that

$$j^0(\boldsymbol{u}_n; -\boldsymbol{u}_n) \leq \alpha_j\|\boldsymbol{u}_n\|_V^2 - j^0(\mathbf{0}_V; \boldsymbol{u}_n). \tag{6.14}$$

Moreover, using Proposition 2.6 (ii) we have

$$\begin{aligned} -j^0(\mathbf{0}_V; \boldsymbol{u}_n) \leq |j^0(\mathbf{0}_V; \boldsymbol{u}_n)| &= |\max_{\boldsymbol{\xi}\in\partial j(\mathbf{0}_V)} \langle \boldsymbol{\xi}, \boldsymbol{u}\rangle| \\ &\leq \max_{\boldsymbol{\xi}\in\partial j(\mathbf{0}_V)} |\langle \boldsymbol{\xi}, \boldsymbol{u}_n\rangle| \leq \max_{\boldsymbol{\xi}\in\partial j(\mathbf{0}_V)} \|\boldsymbol{\xi}\|_{X^*}\|\boldsymbol{u}_n\|_V \end{aligned}$$

and, using condition (2.7)(b) with $\boldsymbol{v} = \mathbf{0}_V$ yields

$$-j^0(\mathbf{0}_V; \boldsymbol{u}_n) \leq c_0\|\boldsymbol{u}_n\|_V. \tag{6.15}$$

We now combine inequalities (6.14) and (6.15) to see that

$$j^0(\boldsymbol{u}_n; -\boldsymbol{u}_n) \leq \alpha_j\|\boldsymbol{u}_n\|_V^2 + c_0\|\boldsymbol{u}_n\|_V,$$

then we use this inequality in (6.13) to deduce that

$$(m_{\mathcal{A}} - \alpha_j)\|\boldsymbol{u}_n\|_V \leq (\|A\mathbf{0}_V\|_{V^*} + \|\boldsymbol{f}\|_{V^*}) + c_0.$$

Inequality (6.12) is now a direct consequence of the smallness assumption (4.19) since, recall, $\alpha_j = \alpha_{j_\nu}\|\gamma\|$. □

We now have all the ingredients to provide the proof of Therem 6.1.

Proof. Let $n \in \mathbb{N}$ and let $\alpha_n > 0$, $\beta_n > 0$ be given by

$$\alpha_n = \min\left\{\frac{g_n}{g}, \frac{k_n}{k}\right\}, \quad \beta_n = \min\left\{\frac{g}{g_n}, \frac{k}{k_n}\right\}. \tag{6.16}$$

Using the positive homogeneity of the function $\mathcal{F}$, guaranteed by assumption (6.1), it follows that that

$$\mathcal{F}(\boldsymbol{\varepsilon}(\alpha_n \boldsymbol{u})) = \alpha_n \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \leq \frac{k_n}{k}\, \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$$

and, since $\boldsymbol{u} \in W$, we deduce that $\mathcal{F}(\alpha_n \boldsymbol{\varepsilon}(\boldsymbol{u})) \leq k_n$ a.e. in Ω, which implies that $\alpha_n \boldsymbol{u} \in W_n$. A similar argument shows that $\alpha_n \boldsymbol{u} \in U_n$. We conclude from here that $\alpha_n \boldsymbol{u} \in U_n \cap W_n$. Therefore, using (4.13) yields

$$\alpha_n \boldsymbol{u} \in K_n. \tag{6.17}$$

Using similar arguments it follows that

$$\beta_n \boldsymbol{u}_n \in K. \tag{6.18}$$

Next, the regularity (6.17) allows us to test in (6.9) with $\boldsymbol{v} = \alpha_n \boldsymbol{u}$ to obtain

$$\langle A\boldsymbol{u}_n, \alpha_n \boldsymbol{u} - \boldsymbol{u}_n \rangle + \varphi(\alpha_n \boldsymbol{u}) - \varphi(\boldsymbol{u}_n) + j^0(\boldsymbol{u}_n; \alpha_n \boldsymbol{u} - \boldsymbol{u}_n) \geq \langle \boldsymbol{f}, \alpha_n \boldsymbol{u} - \boldsymbol{u}_n \rangle. \tag{6.19}$$

Moreover, the regularity (6.18) allows us to test in (6.3) with $\boldsymbol{v} = \beta_n \boldsymbol{u}_n$. As a result we find that

$$\langle A\boldsymbol{u}, \beta_n \boldsymbol{u}_n - \boldsymbol{u} \rangle + \varphi(\beta_n \boldsymbol{u}_n) - \varphi(\boldsymbol{u}) + j^0(\boldsymbol{u}; \beta_n \boldsymbol{u}_n - \boldsymbol{u}) \geq \langle \boldsymbol{f}, \beta_n \boldsymbol{u}_n - \boldsymbol{u} \rangle. \tag{6.20}$$

We now add inequalities (6.19), (6.20) and, after some algebra, using the positive homogeneity of the function φ, we find that

$$\begin{aligned} \langle A\boldsymbol{u}_n - A\boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u} \rangle &\leq \langle A\boldsymbol{u}_n, (\alpha_n - 1)\boldsymbol{u} \rangle + \langle A\boldsymbol{u}, (\beta_n - 1)\boldsymbol{u}_n \rangle \\ &+ (\alpha_n - 1)\varphi(\boldsymbol{u}) + (\beta_n - 1)\varphi(\boldsymbol{u}_n) + j^0(\boldsymbol{u}_n, \alpha_n \boldsymbol{u} - \boldsymbol{u}_n) + j^0(\boldsymbol{u}, \beta_n \boldsymbol{u}_n - \boldsymbol{u}) \\ &+ \langle \boldsymbol{f}, (1 - \alpha_n)\boldsymbol{u} \rangle + \langle \boldsymbol{f}, (1 - \beta_n)\boldsymbol{u}_n \rangle. \end{aligned} \tag{6.21}$$

We now combine the bound (6.12) with the properties of the operator A and the function φ to see that there exists a constant $C > 0$ which does not depend on n such that

$$\|\boldsymbol{u}_n\|_V \leq C, \quad \|A\boldsymbol{u}_n\|_V \leq C, \quad \varphi(\boldsymbol{u}_n) \leq C. \tag{6.22}$$

Using these inequalities in (6.21) yields

$$\begin{aligned} \langle A\boldsymbol{u}_n - A\boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u} \rangle &\leq C|\alpha_n - 1|\|\boldsymbol{u}\|_V + C|\beta_n - 1|\|A\boldsymbol{u}\|_{V^*} \\ &+ C|\alpha_n - 1|\varphi(\boldsymbol{u}) + C(\beta_n - 1) + j^0(\boldsymbol{u}_n, \alpha_n \boldsymbol{u} - \boldsymbol{u}_n) + j^0(\boldsymbol{u}, \beta_n \boldsymbol{u} - \boldsymbol{u}) \\ &+ |\alpha_n - 1|\|\boldsymbol{f}\|_{V^*}\|\boldsymbol{u}\|_V + C|\beta_n - 1|\|\boldsymbol{f}\|_{V^*}. \end{aligned} \tag{6.23}$$

On the other hand, using Proposition 2.6 (i) and condition (2.7)(c) we have

$$\begin{aligned} &j^0(\boldsymbol{u}_n, \alpha_n \boldsymbol{u} - \boldsymbol{u}_n) + j^0(\boldsymbol{u}, \beta_n \boldsymbol{u} - \boldsymbol{u}) \\ &\quad = j^0(\boldsymbol{u}_n, \alpha_n \boldsymbol{u} - \boldsymbol{u} + \boldsymbol{u} - \boldsymbol{u}_n) + j^0(\boldsymbol{u}, \beta_n \boldsymbol{u}_n - \boldsymbol{u}_n + \boldsymbol{u}_n - \boldsymbol{u}) \\ &\quad \leq \alpha_j \|\boldsymbol{u} - \boldsymbol{u}_n\|_V^2 + j^0(\boldsymbol{u}_n, (\alpha_n - 1)\boldsymbol{u}) + j^0(\boldsymbol{u}, (\beta_n - 1)\boldsymbol{u}_n) \end{aligned} \tag{6.24}$$

Moreover, using Proposition 2.6 (ii) and condition (2.7)(b) we have

$$j^0(\boldsymbol{u}_n,(\alpha_n-1)\boldsymbol{u}) \le |j^0(\boldsymbol{u}_n,(\alpha_n-1)\boldsymbol{u})| = |\max_{\boldsymbol{\xi}\in\partial j(\boldsymbol{u}_n)}\langle\boldsymbol{\xi},(\alpha_n-1)\boldsymbol{u}\rangle|$$

$$\le |\alpha_n-1|\max_{\boldsymbol{\xi}\in\partial j(\boldsymbol{u}_n)}|\langle\boldsymbol{\xi},\boldsymbol{u}\rangle| \le |\alpha_n-1|(c_0+c_1\|\boldsymbol{u}_n\|_V)\|\boldsymbol{u}\|_V.$$

Therefore, the bound (6.22) implies that

$$j^0(\boldsymbol{u}_n,(\alpha_n-1)\boldsymbol{u}) \le |\alpha_n-1|(c_0+c_1C)\|\boldsymbol{u}\|_V. \tag{6.25}$$

A similar argument shows that

$$j^0(\boldsymbol{u},(\beta_n-1)\boldsymbol{u}_n) \le C|\beta_n-1|(c_0+c_1\|\boldsymbol{u}\|_V). \tag{6.26}$$

We now combine inequalities (6.23)–(6.26) to find that

$$\langle A\boldsymbol{u}_n-A\boldsymbol{u},\boldsymbol{u}_n-\boldsymbol{u}\rangle \le C|\alpha_n-1|\|\boldsymbol{u}\|_V + C|\beta_n-1|\|A\boldsymbol{u}\|_{V^*}$$
$$+C|\alpha_n-1|\varphi(\boldsymbol{u}) + C(\beta_n-1) + \alpha_j\|\boldsymbol{u}-\boldsymbol{u}_n\|_V^2 + |\alpha_n-1|(c_0+c_1C)\|\boldsymbol{u}\|_V +$$
$$+C|\beta_n-1|(c_0+c_1\|\boldsymbol{u}\|_V) + |\alpha_n-1|\|\boldsymbol{f}\|_{V^*}\|\boldsymbol{u}\|_V + C|\beta_n-1|\|\boldsymbol{f}\|_{V^*}.$$

Next, we use inequality (4.26), equality $\alpha_j=\alpha_{j_\nu}\|\gamma\|^2$ and the smallness assumption (4.19) to see that there exists a positive constant $E(\boldsymbol{u})$, which depends on $\boldsymbol{u}$ but does not depend on n, such that

$$\|\boldsymbol{u}_n-\boldsymbol{u}\|_V^2 \le E(\boldsymbol{u})\big(|\alpha_n-1|+|\beta_n-1|\big). \tag{6.27}$$

On the other hand, it is easy to see that definition (6.16) and assumption (6.10) imply that

$$\alpha_n\to 1, \qquad \beta_n\to 1. \tag{6.28}$$

We now combine inequality (6.27) with the convergence (6.28) to deduce that (6.11) holds, which concludes the proof. □

In addition to the mathematical interest in the convergence result (6.11) it is important from mechanical point of view, since it shows that the weak solution of the contact problem 3 depends continuously both on the thickness of the rigid-plastic layer of the foundation and on the locking constraint.

Theorem 6.1 could be used in the study of some optimization problems associated to inequality (4.18). To provide an example, let $\mathcal{K}\subset\mathbb{R}_+^*\times\mathbb{R}_+^*$ where, here and below, $\mathbb{R}_+^*=(0,+\infty)$, and let $\mathcal{L}:V\to\mathbb{R}$. We consider the following problem.

Problem 7. *Find $(g^*,k^*)\in\mathcal{K}$ such that*

$$\mathcal{L}\big(\boldsymbol{u}(g^*,k^*)\big) = \min_{(g,k)\in\mathcal{K}}\mathcal{L}\big(\boldsymbol{u}(g,k)\big). \tag{6.29}$$

Here, for each $(g,k) \in \mathcal{K}$, $\boldsymbol{u}(g,k)$ represents the solution of the variational-hemivariational inequality (4.18) or, equivalently, the solution of (6.3). In the study of this problem we assume that

$$\mathcal{K} \subset \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ is a compact set of } \mathbb{R}^2. \tag{6.30}$$

$$\mathcal{L} : V \to \mathbb{R} \text{ is a lower semicontinous function.} \tag{6.31}$$

We have the following result.

Corollary 6.3. *Assume the hypothesis of Theorem* 6.1 *and, moreover, assume that* (6.30)–(6.31) *hold. Then, Problem* 7 *hast at least one solution.*

Proof. Note that Theorem 6.1 guarantees that the map $(g,k) \mapsto \boldsymbol{u}(g,k) : \mathbb{R}_+^* \times \mathbb{R}_+^* \to V$ is continuous. Therefore, using assumption (6.31) we deduce that the map $(g,k) \mapsto \mathcal{L}\big(\boldsymbol{u}(g,k)\big) : \mathcal{K} \to \mathbb{R}$ is lower semicontinuous. Recall also that the set $\mathcal{K}$ is a compact subset of $\mathbb{R}^2$, see (6.30). Corollary 6.3 follows now from the well known Weierstrass theorem. □

A first example of Problem 7 can be obtained by taking $\mathcal{K} = [g_0, g_1] \times \{k\}$ where g_0, g_1 and k are strictly positive constants such that $g_0 < g_1$ and

$$\mathcal{L}(\boldsymbol{u}) = \int_{\Gamma_3} (u_\nu - \phi)^2 \, da \tag{6.32}$$

where $\phi \in L^2(\Gamma_3)$ is given. With this choice, the mechanical interpretation of Problem 7 is the following: given a contact process of the form (3.1)–(3.7) with the data $\mathcal{F}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F and k, we are looking for a thickness $g \in [g_0, g_1]$ such that the normal component of the corresponding solution is as close as possible, on Γ_3, to the "desired normal displacement" ϕ.

A second example of Problem 7 can be obtained by taking $\mathcal{K} = \{g\} \times [k_0, k_1]$ where g, k_0, k_1 are strictly positive constants such that $k_0 < k_1$ and

$$\mathcal{L}(\boldsymbol{u}) = \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2 \, dx. \tag{6.33}$$

With this choice, the mechanical interpretation of Problem 7 is the following: given a contact process of the form (3.1)–(3.7), with the data $\mathcal{F}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, F and g, we are looking for a bound $k \in [k_0, k_1]$ such that, the corresponding deformation in the body is as small as possible.

Corollary 6.3 guarantees the existence of the solutions of all these optimization problems.

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