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2 **Bi-objective finite horizon optimal control problems with Bolza and** 3 **maximum running costs**

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9 **ARTICLE HISTORY**

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11 **Abstract**

12 In this paper, we investigate optimal control problems with two objective functions
13 of different nature that need to be minimized simultaneously. One objective is in
14 the classical Bolza form and the other one is defined as a maximum running cost.
15 In our setting, the existence of Pareto solutions is not guaranteed. We first analyze
16 some notions of ε -Pareto solutions. Then we consider a relaxed problem for which
17 the Pareto front exists and show that in any neighborhood of this front, there exists
18 a (weak) ε -Pareto value of the original problem. We also give a precise characteri-
19 zation of the Pareto front of the relaxed problem and to the ε -Pareto front of the
20 original problem. Our approach is based in the Hamilton-Jacobi-Bellman framework.
21 A numerical example is considered to show the relevance of our approach.

22 **KEYWORDS**

23 Multi-objective programming; Hamilton-Jacobi-Bellman approach; Optimal
24 control problems; Approximate Pareto solutions.

25 **AMS CLASSIFICATION**

26 49J20; 49L25; 65M15.

27 **1. Introduction**

28 Optimal control problems for ordinary differential equations (ODE) consist of control-
29 ling the evolution of the dynamical systems, along a certain period of time, in a manner
30 that the pair of control-and-state is optimal with respect to some objective function.
31 The form of this objective function is of great importance when developing the theory.
32 It can be defined, for example, in a Bolza form [1,2] or in a max-type [3,4].

33 Different approaches have been developed to characterize and compute optimal so-
34 lutions for continuous control problems. In particular, the approach based on the Dy-
35 namic Programming principle consists on the analysis of the value function that asso-
36 ciates, to every initial data, the optimal value of the control problem, [5]. The value
37 function can be characterized as the unique solution of a partial differential equation,
38 called Hamilton-Jacobi-Bellman (HJB) equation [6]. Moreover, the value function is

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1 very useful to reconstruct the optimal trajectories and the corresponding control strate-
2 gies [7,8]. In general when the set of trajectories is not closed, it is not possible to
3 guarantee the existence of a minimizer for the optimal control problem. A very known
4 approach to deal with this issue is to consider a relaxed optimal control problem over a
5 compactified set of trajectories; see for instance [9,10]. Under suitable assumptions, the
6 value function of the original problem coincides with the value function of the relaxed
7 problem, [10].

8 In the present paper, we consider finite horizon optimal control problems with two
9 objective functions, of different nature, that need to be minimized simultaneously.
10 Namely, in the vector objective function, one component is a Bolza cost and another
11 one is a maximum running cost.

12 Multi-objective optimization problems arise in various interesting real-world applica-
13 tions. In these contexts, generally, it is not possible to minimize all the criteria simulta-
14 neously. For this reason, several solution concepts have been proposed in the literature.
15 In the famous work "Cours d'Economie Politique" [11], the pioneering economist V.
16 Pareto introduced the notion of efficient or Pareto solution. At a Pareto solution it is
17 not possible to improve one criterion without worsening at least one of the other ones.
18 For any given problem, the set of Pareto solutions may be infinite and unbounded.
19 A larger set is given by weak Pareto solutions, at which it is not possible to improve
20 all the objective functions simultaneously. The image of the set of all (weak) Pareto
21 solutions by the objective function is called (weak) Pareto front. It is useful for prac-
22 titioners for finding a trade-off between conflicting criteria. Several papers and mono-
23 graphs have been devoted to the analysis and numerical methods for multi-objective
24 problems, see for instance [12,13] and the reference therein. One of the most common
25 approach for solving multi-objective optimization problem is to relate it with a fam-
26 ily of mono-objective optimization problems, in such a way that the solutions of the
27 multi-objective problem can be obtained by solving a sequence of classical nonlinear
28 programming problems. The most popular scalarization techniques are the weighted
29 sum method, the weighted Chebyshev method and the epsilon-constraint method [14].
30 As the scalarization usually depends on certain auxiliary parameters, some numerical
31 difficulties may appear if the single objective optimization problem has feasible so-
32 lutions only with very few parameter values. Moreover, a weakness of the weighting
33 method is that all the Pareto optimal points cannot be found if the Pareto front is
34 non convex [15]. For multi-objective optimal control problems several numerical algo-
35 rithms based on scalarization techniques have been developed (see for instance [16,17]
36 and the references therein). For some discrete time optimal control problems with
37 mixed state-control constraints the Dynamical Programming Principle has been used
38 for characterizing the Pareto fronts in [18]. The HJB approach have also been a tool
39 to investigate multi-objective optimal control problems. A method that combines the
40 HJB framework and the weighted sum method to find some points of the Pareto front
41 was introduced in [19]. In [20], an approach based on HJB theory is investigated in
42 the context of exit time problems. In that paper, the Pareto front is characterized by
43 using the value function associated to an auxiliary control problem where one of the
44 cost objectives is chosen as primary cost and the other objectives are transformed into
45 auxiliary variables subject to state constraints. This idea was extended to a class of
46 hybrid control problems, see [21]. In [22] the set-valued function is characterized as a
47 unique generalized solution of an HJB equation. In [23], the idea of introducing an aux-
48 iliary problem to deal with mono-objective optimal control problems [24], is extended
49 to work with multi-objective optimal control problems.

50 In practice, for some problems, it is difficult to calculate the sets of (weak) Pareto

1 solutions and what it could be obtained is just an approximate set of solutions. The
 2 notion of approximate optimal solutions for multi-objective problems was introduced
 3 by [25]. Several notions of ε -Pareto solutions can be considered, see [26]. We discuss in
 4 this paper three of these concepts.

5 Usually, the multi-objective control problems are investigated in the case when the
 6 cost functions are of the same nature (Bolza with free or fixed final time horizon).
 7 In this work, we use a HJB approach to characterize the Pareto front for a finite
 8 horizon bi-objective optimal control problem with objectives of different nature. In
 9 the considered problem it is not possible to guarantee that the set of trajectories is
 10 closed, so we introduce a relaxed (convexified) problem. We prove that if a feasible
 11 pair (\mathbf{y}, \mathbf{u}) is a Pareto optimal solution for the relaxed problem, then there exists an
 12 ε -Pareto optimal solution of the original problem that is in the neighborhood of (\mathbf{y}, \mathbf{u}) .
 13 Following some ideas developed in [23], we define an adequate auxiliary control problem
 14 and show that the Pareto front of the relaxed problem is a subset of the zero level set of
 15 the corresponding value function. Moreover, with a geometrical approach we establish
 16 a characterization of the Pareto fronts.

17 This paper is organized as follows. Section 2 introduces a bi-objective control prob-
 18 lem with cost functions of different nature. Section 3 discuss the concepts and proper-
 19 ties of solutions and ε -solutions of bi-objective problems. In Section 4 a relaxed control
 20 problem is considered and some results about the relation of solutions of the original
 21 and relaxed problems are proved. Moreover an auxiliary control problem is considered,
 22 the HJB equation and some properties of the auxiliary value function are derived. Sec-
 23 tion 5 studies the link between the 0-level set of the auxiliary value function and the
 24 Pareto front. Moreover the characterization of the Pareto Front for the relaxed problem
 25 is obtained. In Section 6 we shown how to obtain ε -Pareto optimal solutions for the
 26 original problem. We discuss the reconstruction of (approximate) Pareto trajectories
 27 on Section 7. An illustrative numerical example is investigated in Section 8. Section 9
 28 concludes the paper with final remarks.

29 **Notations:** Throughout this paper, \mathbb{R} denotes the set of real numbers, $|\cdot|$ is the
 30 Euclidean norm. For a set $S \subseteq \mathbb{R}^n$, \bar{S} and $\text{bdry}(S)$ denote its closure and boundary,
 31 respectively. We denote by $\vec{1}$ the vector in \mathbb{R}^2 of ones and by $B(a, R)$ the open ball
 32 with center at $a \in \mathbb{R}^n$ with radius $R > 0$, $\{x \in \mathbb{R}^n : |x - a| < R\}$. Moreover, for every
 33 $t, T \in \mathbb{R}$ and $T > t$ we denote by $W^{1,1}([t, T]; \mathbb{R}^m)$ (for any $m > 0$) the space of all
 34 integrable functions $f : [t, T] \rightarrow \mathbb{R}^m$, such that f admits a weak derivative that is
 35 also integrable on $[t, T]$.

36 We will use the standard convention that $\inf \emptyset = +\infty$.

37 For $z, z' \in \mathbb{R}^M$, we will use the following notations for different partial order rela-
 38 tions:

$$z < z' \Leftrightarrow [z_i < z'_i \quad \forall i = 1, \dots, M]; \quad z \leq z' \Leftrightarrow [z_i \leq z'_i \quad \forall i = 1, \dots, M].$$

1 **2. Problem statement**

Let U be a given compact non-empty subset of \mathbb{R}^M (for $M \geq 1$) and $T \in \mathbb{R}$ ($T > 0$) a given finite horizon. Consider the set of all admissible controls defined by \mathcal{U} :

$$\mathcal{U} = \left\{ \mathbf{u} : [0, T] \rightarrow \mathbb{R}^M \text{ measurable, } \mathbf{u}(s) \in U \text{ a.e.} \right\}.$$

2 Consider the dynamical system:

$$\begin{cases} \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)) & s \geq t, \\ \mathbf{y}(t) = x. \end{cases} \quad (1)$$

3 The dynamics f satisfies the following hypothesis:

(**H₁**) $f : \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ is continuous. For any $R > 0$, $\exists L_f(R) > 0$ such that for every $u \in U$:

$$|f(x, u) - f(x', u)| \leq L_f(R)(|x - x'|) \quad \forall x, x' \in \mathbb{R}^N \text{ with } |x| \leq R, |x'| \leq R.$$

4 Moreover, there exists $c_f > 0$ such that for any $x \in \mathbb{R}^N$ we have: $\max\{|f(x, u)| : u \in U\} \leq c_f(1 + |x|)$.

6 By assumption (**H₁**), for any control input $u \in \mathcal{U}$, the system (1) admits a unique
7 absolutely continuous solution $\mathbf{y}_x^{\mathbf{u}}$ in $W^{1,1}([t, T]; \mathbb{R}^N)$. For every $x \in \mathbb{R}^N$ and $0 \leq t \leq T$,
8 we define $\mathbb{X}_{t,x} \subset W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U}$ as:

$$\mathbb{X}_{t,x} = \{(\mathbf{y}, \mathbf{u}) : \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)), \text{ for a.e. } s \in [t, T]; \quad \mathbf{y}(t) = x \text{ and } \mathbf{u} \in \mathcal{U}\}.$$

9 Introduce the final cost function $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ and the running cost $\ell : \mathbb{R}^N \times U \rightarrow \mathbb{R}$
10 satisfying:

(**H₂**) The function φ is locally Lipschitz continuous on \mathbb{R}^N : for every $R > 0$,

$$\exists L_\varphi(R) > 0 \quad |\varphi(x) - \varphi(x')| \leq L_\varphi(R)|x - x'| \quad \forall x, x' \in \mathbb{R}^N \text{ with } |x| \leq R, |x'| \leq R.$$

11 Moreover there exists $c_\varphi > 0$ and $\lambda_\varphi \geq 1$ such that $\varphi(x) \leq c_\varphi(1 + |x|^{\lambda_\varphi})$ for every
12 $x \in \mathbb{R}^N$.

(**H₃**) The function ℓ is continuous on $\mathbb{R}^N \times U$ and is locally Lipschitz continuous on the first variable uniformly with respect to the second argument: for every $R > 0$,

$$\exists L_\ell(R) > 0 \quad |\ell(x, u) - \ell(x', u)| \leq L_\ell(R)|x - x'| \quad \forall |x| \leq R, |x'| \leq R, \forall u \in U,$$

13 and there exists $c_\ell > 0$ and $\lambda_\ell \geq 1$ such that $\max\{|\ell(x, u)|, u \in U\} \leq c_\ell(1 + |x|^{\lambda_\ell})$.

14 For $x \in \mathbb{R}^N$ and $0 \leq t \leq T$, the objective function in Bolza form $\Phi(t, x; \cdot, \cdot) :$

1 $W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U} \mapsto \mathbb{R}$ is defined as

$$\Phi(t, x; \cdot, \cdot) : W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U} \longrightarrow \mathbb{R}, \quad \Phi(t, x; \mathbf{y}, \mathbf{u}) = \varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s)) ds. \quad (2)$$

2 We are also interested by a second cost function that is measured all along the
3 trajectory by:

$$\Psi(t, x; \cdot) : W^{1,1}([t, T]; \mathbb{R}^N) \longrightarrow \mathbb{R}, \quad \Psi(t, x; \mathbf{y}) = \max_{s \in [t, T]} \psi(\mathbf{y}(s)), \quad (3)$$

4 where the function ψ satisfies:

5 **(H₄)** $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is locally Lipschitz continuous.

6 The multi-objective optimal control problem that will be investigated in this paper
7 is the following:

$$\begin{cases} \inf(\Phi(t, x; \mathbf{y}, \mathbf{u}), \Psi(t, x; \mathbf{y})) \\ \text{s.t } (\mathbf{y}, \mathbf{u}) \in \mathbb{X}_{t,x}. \end{cases} \quad (\text{MOP})$$

8 A simple prey-predator example will be given in section 8 to illustrate the motivation of
9 such bi-objective control problem. The concepts of solution and approximate solutions
10 for multi-objective problems will be made precise in the next section.

11 3. Pareto optimality - General results

12 Consider the following bi-objective optimization problem:

$$\begin{cases} \text{Minimize } g(x) = (g_1(x), g_2(x)) \\ \text{subject to } x \in X, \end{cases} \quad (\text{OP})$$

where Y is a Banach space, $g_i : Y \rightarrow \mathbb{R}$ are continuous functions and $X \subset Y$ a feasible nonempty set. In problem (OP), the aim is to minimize the two components of the objective function g at the same time. If there exists no conflict between the cost functions g_1 and g_2 , then a solution $x^* \in X$ may exist such that:

$$g_i(x^*) = \min\{g_i(x), x \in X\}, \quad i = 1, 2.$$

13 In this paper, we assume that there is no single solution that minimizes all the objec-
14 tive functions simultaneously. This means that the cost functions are at least partly
15 conflicting and several solution concepts may be associated with the problem (OP). A
16 predominant optimality notion for problem (OP) is the one of Pareto solutions.

17 **Definition 3.1** (Pareto optimal solutions). Let $x^* \in X$.

18 • We will say that x^* is a Pareto optimal solution if and only if there does not exist
19 another $x \in X$ such that

$$(g_1(x), g_2(x)) \leq (g_1(x^*), g_2(x^*)), \text{ and } \left[g_1(x) < g_1(x^*) \quad \text{or} \quad g_2(x) < g_2(x^*) \right].$$

1 The set \mathcal{P} of all Pareto optimal solutions is said to be Pareto optimal solutions
2 set.
3 • We will say also that x^* is a weak Pareto optimal solution if and only if there
4 does not exist another $x \in X$ such that

$$(g_1(x), g_2(x)) < (g_1(x^*), g_2(x^*)).$$

5 The set \mathcal{P}_w of all weak Pareto optimal solutions is said to be weak Pareto
6 optimal solutions set.

7 With these definitions, we can check that the Pareto set is a subset of the weak
8 Pareto set: $\mathcal{P} \subset \mathcal{P}_w$. The set of all vectors of objective values at the Pareto (resp. weak
9 Pareto) minima is said to be the *Pareto front* (resp. *weak Pareto front*). More precisely,
10 we have the following definition.

11 **Definition 3.2** (Pareto front). We will call Pareto front (respectively weak Pareto
12 front) the image of the Pareto optimal solutions set \mathcal{P} (respectively of \mathcal{P}_w) by the
13 multi-objective application $g(x)$:

$$\mathcal{F} = \{g(x), x \in \mathcal{P}\} \quad \mathcal{F}_w = \{g(x), x \in \mathcal{P}_w\}. \quad (4)$$

14 Besides, it is known [27] that the (weak) Pareto front is subset of the boundary of
15 the *attainable set* \mathcal{Z} , that is defined as: $\mathcal{Z} := \{g(x), x \in X\} \subset \mathbb{R}^2$.

16 Computing the Pareto fronts and the set of Pareto solutions is a challenging problem.
17 In some cases, these sets may be empty, this is the case, for instance, when the feasible
18 set is not closed. In such a context, it is natural to consider a set of *approximate* Pareto
19 solutions. Different definitions for ε -solutions have been investigated in the literature,
20 see [25,26]. In what follows, we recall three of these concepts.

21 **Definition 3.3** (ε -Pareto solutions). Let $\varepsilon \geq 0$. We define the following sets of ε -
22 Pareto solutions:

- 23 (i) $\mathcal{P}^{1,\varepsilon} = \{x \in X : \text{there is no } y \in X \text{ such that } g(y) \leq g(x) - \varepsilon \vec{1} \text{ and } g(y) \neq$
24 $g(x) - \varepsilon \vec{1}\}$.
25 (ii) $\mathcal{P}^{2,\varepsilon} = \{x \in X : \text{there exists } y \in \mathcal{P} \text{ such that } |g(x) - g(y)|_\infty \leq \varepsilon\}$, where $|\cdot|_\infty$
26 denotes the max norm.
27 (iii) $\mathcal{P}^{3,\varepsilon} = \{x \in X : \text{for any } y \in X, \text{ if } g(y) \neq g(x) \text{ and } g(y) \leq g(x) \text{ then } g(y) \geq$
28 $g(x) - \varepsilon \vec{1}\}$.

29 An ε -Pareto solution $x^* \in \mathcal{P}^{i,\varepsilon}$, $i = 1, 2, 3$, produces an ε -Pareto outcome $g(x^*)$ and
30 the set of all ε -Pareto outcomes are denoted by $\mathcal{F}^{i,\varepsilon}$, $i = 1, 2, 3$. Note that, if $\varepsilon_1 \leq \varepsilon_2$
31 then $\mathcal{F}^{i,\varepsilon_1} \subset \mathcal{F}^{i,\varepsilon_2}$, for $i = 1, 2, 3$. From the above definition, one can easily check that
32 $\mathcal{F}^{3,\varepsilon} \subset \mathcal{F}^{1,\varepsilon}$ (or equivalently $\mathcal{P}^{3,\varepsilon} \subset \mathcal{P}^{1,\varepsilon}$). Moreover, if the feasible set X is a compact
33 set, then the following relation between the ε -Pareto sets is established by [26]:

$$\mathcal{F} \subset \mathcal{F}^{3,\varepsilon} \subseteq \mathcal{F}^{2,\varepsilon} \subseteq \mathcal{F}^{1,\varepsilon}. \quad (5)$$

34 **Remark 1.** We stress on that the inclusions in (5) are true only when the feasible set
35 X is a compact set. However if the feasible set is not closed, there is no guarantee of
36 existence of (weak) Pareto solutions and inclusion 5 is not true anymore. Actually, the

1 weak Pareto set can be empty. Consider, for example, the following problem:

$$\begin{cases} \text{Minimize } g(x) = (x_1, x_2) \\ \text{subject to: } x \in X =]-1, 1[^2 \setminus \{x \in (-1, 0)^2 : x_1 + x_2 < -1\}. \end{cases} \quad (6)$$

2 In this case the feasible set is open, see Figure 1a, where the black dashed lines repre-
 3 sents the boundary of X (this boundary is not included in X). For this example, the
 4 (weak) Pareto set and the set $\mathcal{P}^{2,\varepsilon}$ are empty. However the sets $\mathcal{P}^{1,\varepsilon}$ and $\mathcal{P}^{3,\varepsilon}$ are not
 empty and can be seen in red in Figures 1b and 1c, respectively.

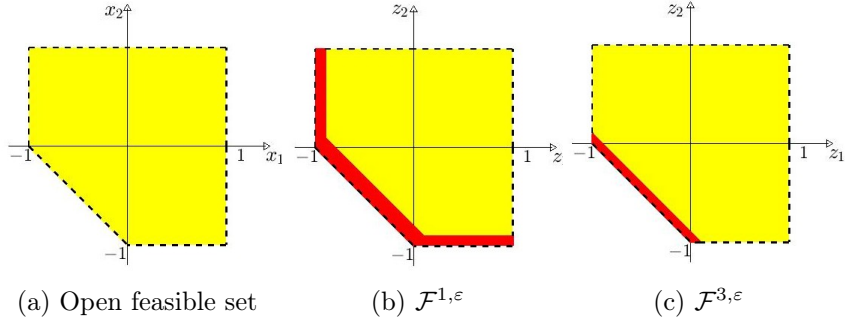


Figure 1.: Feasible set and ε -Pareto fronts for Problem (6), $\varepsilon = 0.1$

5

6 Some concepts of approximate solutions, Definition 3.3, can be also extended to the
 7 notion of weak ε -Pareto solutions (see in [25]).

8 **Definition 3.4.** [weak ε -Pareto solution] Let $\varepsilon \geq 0$. We define the following set of
 9 ε -Pareto solutions:

- 10 (i) $\mathcal{P}_w^{1,\varepsilon} = \{x \in X : \text{there is no } y \in X \text{ such that } g(y) < g(x) - \varepsilon \vec{1}\}$.
 11 (ii) $\mathcal{P}_w^{2,\varepsilon} = \{x \in X : \text{there exists } y \in \mathcal{P}_w \text{ such that } |g(x) - g(y)| \leq \varepsilon\}$.

12 Motivated by the observations made in Remark 1, we consider the problem of
 13 minimizing the objective functions over the closure of the feasible set X :

$$\begin{cases} \text{Minimize } g(x) = (g_1(x), g_2(x)) \\ \text{subject to } x \in \overline{X} \end{cases} \quad (7)$$

14 We denote by $\mathcal{P}^\#$ (resp. $\mathcal{P}_w^\#$) the set of Pareto (resp. weak Pareto) solutions of problem
 15 (7). We are interested in the link between the set $\mathcal{P}^\#$ (resp. $\mathcal{P}_w^\#$) and the set of ε -Pareto
 16 (resp. weak ε -Pareto) solutions of the original problem (OP).

17 **Proposition 3.5.** Assume that the functions g_i are Lipschitz continuous, with Lip-
 18 schitz constant L_i , $i = 1, 2$.

- (i) For any $x^* \in \mathcal{P}^\#$ and for any $\varepsilon > 0$, there exists $x_\varepsilon \in \mathcal{P}^{3,\varepsilon}$ such that

$$|x^* - x_\varepsilon| \leq \min_i (\varepsilon/2L_i) \quad \text{and} \quad |g(x^*) - g(x_\varepsilon)| \leq \varepsilon.$$

- (ii) For any $x^* \in \mathcal{P}_w^\#$ and for any $\varepsilon > 0$, there exists $x_\varepsilon \in \mathcal{P}_w^{1,\varepsilon}$ such that

$$|x^* - x_\varepsilon| \leq \min_i (\varepsilon/2L_i) \quad \text{and} \quad |g(x^*) - g(x_\varepsilon)| \leq \varepsilon.$$

1 (iii) For $\varepsilon > 0$, for any $x^* \in \mathcal{P}^{3,\varepsilon}$, there exists $x_\varepsilon \in \mathcal{P}^\#$ such that $|g(x_\varepsilon) - g(x^*)| \leq 2\varepsilon$.

2 *Proof (i)* Let $x^* \in \mathcal{P}^\#$ and $\varepsilon > 0$. As $x^* \in \bar{X}$, there exists a sequence $\{x_n\} \subset X$
3 such that $\lim_{n \rightarrow \infty} x_n = x^*$. Given $\varepsilon > 0$, define $\delta = \min_i(\varepsilon/L_i)$, so there exists x_N such
4 that $|x^* - x_N| \leq \delta$. By the Lipschitz continuity of functions g_i , $i = 1, 2$ we obtain that
5 $|g_i(x^*) - g_i(x_N)| \leq L_i|x^* - x_N| \leq L_i\delta \leq \varepsilon/2$, which means that $|g(x^*) - g(x_N)| \leq \varepsilon$.
6 It remains to prove that $x_N \in \mathcal{P}^{3,\varepsilon}$. Suppose that there exists $g(y) \neq g(x_N)$ such that
7 $g(y) \leq g(x_N) - \varepsilon\vec{1}$. But, as $|g(x^*) - g(x_N)| \leq \varepsilon$, we have that $g(x_N) - \varepsilon\vec{1} \leq g(x^*)$ and
8 then $g(y) \leq g(x^*)$, what is a contradiction.

9 (ii) The proof is similar to (i).

10 (iii) Let $x^* \in \mathcal{P}^{3,\varepsilon}$. Assume that there is no $x \in \mathcal{P}^\#$ such that $|g(x) - g(x^*)| < 2\varepsilon$.
11 Then $x^* \notin \mathcal{P}^\#$, so there exists $y \in \bar{X}$ such that $g(y) \neq g(x^*)$ and $g(y) \leq g(x^*)$. As
12 $y \in \bar{X}$, there exists a sequence $\{y_n\} \subset X$ such that $\lim_{n \rightarrow \infty} y_n = y$. Choose y_N such
13 that $g(y_N) \leq g(x^*)$ and $|g(y) - g(y_N)| \leq \varepsilon$. As $x^* \in \mathcal{P}^{3,\varepsilon}$, we must have $g(y_N) \geq$
14 $g(x^*) - \varepsilon\vec{1}$. Hence $g(x^*) - \varepsilon\vec{1} \leq g(y_N) \leq g(x^*)$, which means that $|g(y_N) - g(x^*)| < \varepsilon$.
15 Then $|g(y) - g(x^*)| \leq |g(y) - g(y_N)| + |g(y_N) - g(x^*)| \leq 2\varepsilon$, what is a contradiction. \square

16

17 **Remark 2.** Assertion (i) of Proposition 3.5 amounts saying also that for any Pareto
18 value $z^* \in \mathcal{F}^\#$ and for every Pareto solution x^* corresponding to it, there exists a
19 ε -Pareto solution $x_\varepsilon \in \mathcal{P}_\varepsilon^{3,\varepsilon}$ and its corresponding value $z_\varepsilon = g(x_\varepsilon) \in \mathcal{F}^{3,\varepsilon}$ such that
20 $|z^* - z_\varepsilon| \leq \varepsilon$ and $|x^* - x_\varepsilon| \leq \min_i(\varepsilon/L_i)$.

21 4. Relaxed control problem. Auxiliary control problem

Now, let us come back to the bi-objective control problem and let us first consider a reformulation of the problem (MOP). Define the set-valued function

$$G(x) = \left\{ \left(\begin{array}{c} f(x, u) \\ -\ell(x, u) - a \end{array} \right), 0 \leq a \leq \mathcal{A}(x, u), u \in U \right\}, \quad \text{for } x \in \mathbb{R}^N$$

where $\mathcal{A}(x, u) = c_\ell(1 + |x|^{\lambda_\ell}) - \ell(x, u)$. Under assumptions **(H₁)** and **(H₃)** the function G is locally Lipschitz continuous in the sense that, for any $R > 0$, there exists $L_G(R) > 0$ such that:

$$G(x') \subset G(x) + |x - y|B(0, L_R) \quad \forall x, x' \in \mathbb{R}^N \text{ with } |x| \leq R, |x'| \leq R.$$

We also define the following set of trajectories:

$$\mathcal{S}_{[t,T]}(x, z) = \{(\mathbf{y}, \zeta) : (\dot{\mathbf{y}}(s), \dot{\zeta}(s))^\top \in G(\mathbf{y}(s)), \text{ for a.e. } s \in [t, T]; \quad (\mathbf{y}(t), \zeta(t)) = (x, z)\},$$

22 and the bi-objective optimal control problem:

$$\begin{cases} \inf \left(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \\ \text{s.t } (\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}(x, 0). \end{cases} \quad (8)$$

23 The introduction of the dynamics G is a classical tool that is usually introduced
24 to recast a cost in Bolza form into a cost in Mayer form. In this reformulation, the
25 vector of state variables is increased by one more component. Let us stress again on

1 that the problem (8) is equivalent to problem (MOP) in the sense that every Pareto
 2 value of (MOP) (if it exists) corresponds to a Pareto value of (8), and the reverse is
 3 true: every Pareto value of (8) corresponds to a Pareto solution of (MOP). So, the two
 4 problems have the same Pareto fronts. The same holds true also for the weak Pareto
 5 fronts. Besides, for every $\varepsilon > 0$, problems (MOP) and (8) have the same weak and
 6 strong ε -Pareto fronts.

Without any additional assumption, the set of trajectories $\mathcal{S}_{[t,T]}(x, 0)$ is not neces-
 sarily closed, and the problem (8) might not have a solution. In this case, the weak and
 strong Pareto fronts might be empty set. One approach to obtain the closure of the
 set of trajectories $\mathcal{S}_{[t,T]}(x, z)$, is to introduce a relaxed (convexified) dynamical system
 [9,10], whose set of solutions is given by:

$$\mathcal{S}_{[t,T]}^\#(x, 0) = \{(\mathbf{y}, \zeta) : (\dot{\mathbf{y}}(s), \dot{\zeta}(s))^\top \in \overline{\text{co}}(G(\mathbf{y}(s))), \text{ for a.e. } s \in [t, T]; \quad (\mathbf{y}(t), \zeta(t)) = (x, z)\},$$

7 where for every subset $S \subset \mathbb{R}^N$, $\overline{\text{co}}(S)$ denotes the closed convex hull of S , that
 8 is the smallest closed convex set containing S . Under assumptions **(H₁)** and **(H₃)**,
 9 following the same arguments of the proof of Filippov-Wazewski Theorem (see for
 10 instance [9]), the closure of $\mathcal{S}_{[t,T]}(x, z)$ in the space of continuous functions $C(t, T)$ is
 11 compact and equal to the set of solutions $\mathcal{S}_{[t,T]}^\#(x, z)$. So we introduce the following
 12 relaxed bi-objective optimal control problem

$$\begin{cases} \min \left(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \\ \text{s.t } (\mathbf{y}, \zeta) \in \mathcal{S}_{[t,T]}^\#(x, 0). \end{cases} \quad (\text{MORP})$$

13 For a fixed $(t, x) \in [0, T] \times \mathbb{R}^N$, we consider $\mathcal{P}^{i,\varepsilon}(t, x)$, $i = 1, 2, 3$, and $\mathcal{P}_w^{i,\varepsilon}(t, x)$, $i =$
 14 $1, 2$, the sets of strong and weak Pareto ε -solutions of problem (8) (according to the
 15 definitions 3.3 and 3.4). Besides, we denote the Pareto front and the weak Pareto front
 16 of problem (MORP) by $\mathcal{F}^\#(t, x)$ and $\mathcal{F}_w^\#(t, x)$, respectively. The next proposition states
 17 the link between the optimal Pareto solution of the relaxed control problem (MORP)
 18 and the ε -Pareto solutions of the original problem (MOP).

19 **Theorem 4.1.** *Assume that **(H₁)**-**(H₄)** hold and let $(t, x) \in [0, T] \times \mathbb{R}^N$.*

- (i) For any $(\mathbf{y}^*, \zeta^*) \in \mathcal{P}^\#(t, x)$ and for any $\varepsilon > 0$, define $R = |(\mathbf{y}^*, \zeta^*)| + \varepsilon$. Then
 for $\delta = \min(\varepsilon/2(L_\varphi(R) + 1), \varepsilon/2L_\psi)$ there exists $(\mathbf{y}, \zeta) \in \mathcal{P}^{1,\varepsilon}(t, x)$ such that
 $|(\mathbf{y}^*, \zeta^*) - (\mathbf{y}, \zeta)| \leq \delta$ and

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \zeta^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \right| \leq \varepsilon.$$

- (ii) For any $(\mathbf{y}^*, \zeta^*) \in \mathcal{P}_w^\#(t, x)$ and for any $\varepsilon > 0$ define $R = |(\mathbf{y}^*, \zeta^*)| + \varepsilon$. Then for
 $\delta = \min(\varepsilon/2(L_\varphi(R) + 1), \varepsilon/2L_\psi)$ there exists $(\mathbf{y}, \zeta) \in \mathcal{P}_w^{1,\varepsilon}(t, x)$ such that
 $|(\mathbf{y}^*, \zeta^*) - (\mathbf{y}, \zeta)| \leq \delta$ and

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \zeta^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \right| \leq \varepsilon.$$

(iii) Given $\varepsilon > 0$, if $(\mathbf{y}^*, \zeta^*) \in \mathcal{P}^{3,\varepsilon}(t, x)$, then there exists $(\mathbf{y}, \zeta) \in \mathcal{P}^\#(t, x)$ such that

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \zeta^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \right| \leq 2\varepsilon.$$

1 *Proof* The proof follows with similar arguments as in Proposition 3.5. □

2

The aim is to characterize the Pareto fronts of the relaxed control problem (MORP) and then to give some useful properties for ε -Pareto fronts of the original problem (8) (see theorem 4.1). The general idea consists of considering a family of scalarized optimal control problems whose optimal values correspond to Pareto values of the bi-objective problem. A predominant method for scalarization is based on the weighted sum problem where the cost function would take the following form: for $\alpha \in [0, 1]$ solve the control problem:

$$\min \left\{ \alpha(\varphi(\mathbf{y}(T)) - \zeta(T)) + (1 - \alpha) \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \mid (\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0) \right\}.$$

3 It is known that this weighted sum scalarization can characterize only a part of the
 4 Pareto front and not the entire front. Another idea would be to consider a control
 5 problem where one of the cost function is chosen as primary cost and the other one is
 6 transformed into an auxiliary variable subject to a state constraint:

$$\min \left\{ \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \mid (\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0) \quad \text{and} \quad \varphi(\mathbf{y}(T)) - \zeta(T) \leq z_1 \right\}, \quad (9)$$

or

$$\min \left\{ \varphi(\mathbf{y}(T)) - \zeta(T) \mid (\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0) \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2 \right\}. \quad (10)$$

7 Following the same ideas as in [20], it is possible to show that all the Pareto values
 8 correspond to the optimal values of problem (9), when z_1 runs through \mathbb{R} . The same
 9 characterization holds if we use (10) and let z_2 runs through \mathbb{R} . However, it should
 10 be noticed that problems (9) and (10) are in presence of state constraints which make
 11 these problems difficult to analyse and to solve, see [28,29] and the references therein.

12 We will follow some ideas introduced in [23], and consider an auxiliary control prob-
 13 lem and its value function $w : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2 \rightarrow \mathbb{R}$, for $z = (z_1, z_2)$, defined as:

$$w(t, x, z) = \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[\left(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1 \right) \bigvee \max_{s \in [t, T]} (\psi(\mathbf{y}(s)) - z_2) \right], \quad (11)$$

14 where the notation $a \bigvee b$ stands for $\max(a, b)$. Let us point out that the additional
 15 state components are very important to get a Dynamical Programming Principle for
 16 the value function w . Moreover, we note that under assumptions **(H₁)**-**(H₄)**, there
 17 exists an admissible pair $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, z)$ solution of the auxiliary control problem
 18 (11). Notice also that from definition of differential inclusion (4) and definition of w ,
 19 for $z = (z_1, z_2)$, it follows that:

$$w(t, x, z) = \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, z_1)} \left[\left(\varphi(\mathbf{y}(T)) - \zeta(T) \right) \bigvee \max_{s \in [t, T]} (\psi(\mathbf{y}(s)) - z_2) \right]. \quad (12)$$

1 Following [23], the value function w satisfies the following property.

2 **Theorem 4.2.** Assume that (\mathbf{H}_1) - (\mathbf{H}_4) hold. The value function w is locally Lipschitz
3 continuous. Moreover, the function w is the unique viscosity solution to the following
4 Hamilton-Jacobi-Bellman equation:

$$\min \left(-\partial_t w(t, x, z) + \mathcal{H}^\#(x, z, D_x w, D_z w), w(t, x, z) - (\psi(x) - z_2) \right) = 0 \quad \text{for } t \in [0, T], (x, z) \in \mathbb{R}^{N+2}, \quad (13a)$$

$$w(T, x, z) = (\varphi(x) - z_1) \vee (\psi(x) - z_2) \quad \text{for } x \in \mathbb{R}^N, z = (z_1, z_2) \in \mathbb{R}^2, \quad (13b)$$

where the function $\mathcal{H}^\#$ is defined by:

$$\mathcal{H}^\#(x, z, p, q) = \max_{(v_x, v_z) \in \overline{\text{co}}(G(x))} \left(-v_x \cdot p - v_z \cdot q_1 \right), \quad \forall p \in \mathbb{R}^N, \forall q = (q_1, q_2) \in \mathbb{R}^2.$$

5 *Proof* With similar arguments as in [5] it is possible to prove that the value function
6 w is locally Lipschitz continuous and satisfies the following Dynamical Programming
7 Principle (DPP) that hold for all $h \geq 0$ such that $t + h < T$ and $(x, z) \in \mathbb{R}^{N+2}$ with
8 $z = (z_1, z_2)$

$$w(t, x, z) = \inf_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, z_1)} \left\{ w(t + h, \mathbf{y}(t + h), \zeta(t + h), z_2) \vee \max_{s \in [t, t+h]} (g(\mathbf{y}(s)) - z_2) \right\}.$$

9 Using this DPP and with similar arguments as in [24] can be shown that the value
10 function w satisfies the HJB equation (13). Moreover the uniqueness follows from [24,
11 Theorem A.1]. \square

12 As in [23] the value function w is monotone with respect to the third argument.

Proposition 4.3. Assume that (\mathbf{H}_1) - (\mathbf{H}_4) hold and let $(t, x) \in [0, T] \times \mathbb{R}^N$. Then

$$\forall z, z' \in \mathbb{R}^2, \left(z \leq z' \Rightarrow w(t, x, z) \geq w(t, x, z') \right).$$

13 *Proof* The proof follows with similar arguments as in [23]. \square

14

15 Notice that from the definition of the Hamiltonian $\mathcal{H}^\#$, we have:

$$\begin{aligned} \mathcal{H}^\#(x, z, p, q) &= \sup_{(v_x, v_z) \in G(x)} \left(-v_x \cdot p - v_z \cdot q_1 \right), \quad \forall p \in \mathbb{R}^N, \forall q = (q_1, q_2) \in \mathbb{R}^2, \\ &= \sup_{\substack{u \in U, \\ \eta \in [0, \mathcal{A}(x, a)]}} \left(-f(x, u) \cdot p + \ell(x, u) \cdot q_1 + \eta \cdot q_1 \right). \end{aligned}$$

16 If $q_1 \leq 0$, then we have $\mathcal{H}^\#(x, z, p, q) = \sup_{u \in U} \left(-f(x, a) \cdot p + \ell(x, a) \cdot q_1 \right)$.

17 Since the value function w is decreasing with respect to the variable z , by using
18 the DPP and using the classical viscosity arguments as in [5,24], one can prove the
19 following:

20 **Theorem 4.4.** Assume that (\mathbf{H}_1) - (\mathbf{H}_4) hold. The value function w is the unique
21 viscosity solution of:

$$\min \left(-\partial_t w(t, x, z) + \mathcal{H}(x, z, D_x w, D_z w), w(t, x, z) - (\psi(x) - z_2) \right) = 0 \quad \text{for } t \in [0, T], x \in \mathbb{R}^N, z \in \mathbb{R}^2 \quad (14a)$$

$$w(T, x, z) = (\varphi(x) - z_1) \vee (\psi(x) - z_2) \quad \text{for } x \in \mathbb{R}^N, z \in \mathbb{R}^2, \quad (14b)$$

where the function \mathcal{H} is defined by:

$$\mathcal{H}(x, z, p, q) = \sup_{u \in U} (-f(x, u) \cdot p + \ell(x, u) \cdot q_1).$$

1 **Remark 3.** If we consider the problem

$$\tilde{w}(t, x, z_1, z_2) = \inf_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[\left(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1 \right) \bigvee \max_{s \in [t, T]} (\psi(\mathbf{y}(s)) - z_2) \right]. \quad (15)$$

2 With similar arguments as in [24], we can prove that the function \tilde{w} can be also
 3 characterized as the unique viscosity solution of the HJB equation (14). By unique-
 4 ness of solution for (14), we conclude that the two function w and \tilde{w} are the same:
 5 $w = \tilde{w}$. This result is not surprising as we know that the set $\mathcal{S}_{[t, T]}^\#(x, 0)$ is the closure of
 6 $\mathcal{S}_{[t, T]}(x, 0)$. However, the problems (MOP) and (MORP) are not the same. More pre-
 7 cisely, (MORP) admits Pareto fronts while for (MOP) the existence of Pareto solutions
 8 is not guaranteed.

9 **5. Characterization of the Pareto fronts of the relaxed bi-objective**
 10 **optimal control problem**

11 In this section, following ideas developed in [23], we relate the Pareto fronts of the
 12 relaxed bi-objective optimal control problem with the zero level set of the value function
 13 w . Moreover a precise characterization of the Pareto front is given.

14 For every $t \in [0, T]$, $x \in \mathbb{R}^N$ and $i = 1, 2$, define:

$$z_i^*(t, x) = \inf \left\{ \zeta \in \mathbb{R} \mid \exists z \in \mathbb{R}^2 \text{ with } z_i = \zeta, w(t, x, z) \leq 0 \right\}. \quad (16)$$

15 **Proposition 5.1.** Assume that **(H₁)**, **(H₂)**, **(H₃)** and **(H₄)** hold and let $(t, x) \in$
 16 $[0, T] \times \mathbb{R}^N$.

17 **(i)** For every $z \in \mathbb{R}^2$, we have that $w(t, x, z) \leq 0$ if and only if there exists $(\mathbf{y}, \zeta) \in$
 18 $\mathcal{S}_{[t, T]}^\#(x, 0)$ such that:

$$\varphi(\mathbf{y}(T)) - \zeta(T) \leq z_1, \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2. \quad (17)$$

(ii) Moreover, for $i = 1, 2$ and every $(t, x) \in [0, T] \times \mathbb{R}^N$, we have $z_i^*(t, x) = \vartheta_i(t, x)$,
 where

$$\vartheta_1(t, x) = \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \varphi(\mathbf{y}(T)) - \zeta(T), \quad \vartheta_2(t, x) = \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \max_{s \in [t, T]} \psi(\mathbf{y}(s)).$$

Proof Assertion **(i)** Let $z \in \mathbb{R}^2$, then by (11):

$$w(t, x, z) \leq 0 \Leftrightarrow \exists (\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0) \text{ s.t. } \varphi(\mathbf{y}(T)) - \zeta(T) \leq z_1 \text{ and } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2.$$

(ii) Let show that $\vartheta_i(t, x) \leq z_i^*(t, x)$. By assertion (i), for all $z \in \mathbb{R}^2$ such that

$w(t, x, z) \leq 0$, we have:

$$\exists (\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0) \quad \text{s.t.} \quad \varphi(\mathbf{y}(T)) - \zeta(T) \leq z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2.$$

Therefore $\vartheta_i(t, x) \leq z_i$ for all $z \in \mathbb{R}^2$ such that $w(t, x, z) \leq 0$ and then

$$\vartheta_i(t, x) \leq \inf\{\gamma \in \mathbb{R} \mid z \in \mathbb{R}^2, w(t, x, z) \leq 0 \text{ with } z_i = \gamma\} = z_i^*(t, x).$$

1 Let show now that $\vartheta_i(t, x) \geq z_i^*(t, x)$. Without loss of generality, we assume here
 2 that $i = 1$. The proof will be the same for $i = 2$. Assume that $\vartheta_1(t, x) <$
 3 $z_1^*(t, x)$. Then there exists $\delta \in \mathbb{R}$ such that $\vartheta_1(t, x) < \delta < z_1^*(t, x)$. The in-
 4 equality $\vartheta_1(t, x) < \delta$ implies that there exists $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that
 5 $\varphi(\mathbf{y}(T)) - \zeta(T) < \delta$. Then for $z_2 = \max_{s \in [t, T]} \psi(\mathbf{y}(s))$ we have that, $w(t, x, \delta, z_2) \leq$
 6 $(\varphi(\mathbf{y}(T)) - \zeta(T) - \delta) \vee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) = 0$, which implies that $\delta \in \{\gamma \in$
 7 $\mathbb{R} \mid \exists z \in \mathbb{R}^2 \text{ with } z_1 = \gamma, w(t, x, z) \leq 0\}$. However, δ is chosen such that $\delta < z_1^*(t, x)$
 8 which leads to a contradiction. \square

9

10 Note that, as we are considering that there is no feasible solution that
 11 minimizes both objectives simultaneously, it follows from proposition 5.1 that
 12 $w(t, x, z_1^*(t, x), z_2^*(t, x)) > 0$.

13 Let $x \in \mathbb{R}^N$, $t \in [t, T]$ and define:

$$\begin{aligned} \bar{z}_1(t, x) &= \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x, \zeta, z_2^*(t, x)) = 0 \right\} \quad \text{and} \\ \bar{z}_2(t, x) &= \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x, z_1^*(t, x), \zeta) = 0 \right\}. \end{aligned} \quad (18)$$

14 Now denote

$$\Omega = [z_1^*(t, x), \bar{z}_1(t, x)] \times [z_2^*(t, x), \bar{z}_2(t, x)]. \quad (19)$$

15 In the following theorem, we give the first link between the solutions of the multi-
 16 objective problem (MORP) and the function w . This result is an extension of results
 17 obtained in [23] to the case of multi-objective optimal control problems with Bolza and
 18 maximum running costs.

19 **Theorem 5.2.** *Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in*
 20 $[0, T] \times \mathbb{R}^N$. The following assertions hold:

- 21 (i) $\mathcal{F}_w^\#(t, x) \subset \left\{ z \in \mathbb{R}^2 \mid w(t, x, z) = 0 \right\}$.
 22 (ii) $\mathcal{F}^\#(t, x) \subset \mathcal{F}_w^\#(t, x) \cap \Omega \subset \left\{ z \in \Omega \mid w(t, x, z) = 0 \right\}$.
 23 (iii) Let $z \in \Omega$ such that $w(t, x, z) = 0$. If there exists a admissible pair $(\mathbf{y}, \zeta) \in$
 24 $\mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\mathbf{y}(T)) - \zeta(T) = z_1$ and $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) = z_2$, then $z \in$
 25 $\mathcal{F}_w^\#(t, x)$.

Proof (i) Let $z \in \mathcal{F}_w^\#(t, x)$. Then there exists $(\bar{\mathbf{y}}, \bar{\zeta}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\bar{\mathbf{y}}(T)) -$
 $\bar{\zeta}(T) = z_1$, $\max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) = z_2$ and there is no other admissible pair that dominates

(\mathbf{y}, ζ) . This means that for any $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t,T]}^\#(x, 0)$, one of the following assertions holds: $z_1 \leq \varphi(\mathbf{y}(T)) - \zeta(T)$ or $z_2 \leq \max_{s \in [t,T]} \psi(\mathbf{y}(s))$. We can easily check that in the two above cases, we have $(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1) \bigvee (\max_{s \in [t,T]} \psi(\mathbf{y}(s)) - z_2) \geq 0$. Therefore,

$$\begin{aligned} w(t, x, z) &= \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t,T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1) \bigvee (\max_{s \in [t,T]} \psi(\mathbf{y}(s)) - z_2) \right] \\ &= \left[(\varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T) - z_1) \bigvee (\max_{s \in [t,T]} \psi(\bar{\mathbf{y}}(s)) - z_2) \right] = 0. \end{aligned}$$

1 **(ii)** By item (i) we obtain immediately that $\mathcal{F}_w^\#(t, x) \cap \Omega \subset \{z \in \Omega \mid w(t, x, z) =$
2 $0\}$. Moreover $\mathcal{F}^\#(t, x) \subset \mathcal{F}_w^\#(t, x)$. It remains to prove that $\mathcal{F}^\#(t, x) \subset \Omega$. Let $z =$
3 $(z_1, z_2) \in \mathcal{F}^\#(t, x)$. By Proposition 5.1, $z_i^*(t, x) = \vartheta_i(t, x)$, for $i=1, 2$. Then for every
4 $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t,T]}^\#(x, 0)$ we have $z_1^*(t, x) \leq \varphi(\mathbf{y}(T)) - \zeta(T)$ and $z_2^*(t, x) \leq \max_{s \in [t,T]} \psi(\mathbf{y}(s))$.
5 Therefore, $z_1 \geq z_1^*(t, x)$ and $z_2 \geq z_2^*(t, x)$. Now, assume that $z_1 > \bar{z}_1(t, x)$. In this
6 case, by definition of $\bar{z}_1(t, x)$, it would exist $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t,T]}^\#(x, 0)$ such that $\varphi(\mathbf{y}(T)) -$
7 $\zeta(T) \leq \bar{z}_1(t, x) < z_1$ and $\max_{s \in [t,T]} \psi(\mathbf{y}(s)) \leq z_2^*(t, x) \leq z_2$, which contradicts the fact
8 that $z \in \mathcal{F}^\#(t, x)$. We conclude that $z_1 \leq \bar{z}_1(t, x)$. The same argument shows also that
9 $z_2 \leq \bar{z}_2(t, x)$, and then z belongs to Ω .

(iii) Let $z \in \Omega$ such that $w(t, x, z) = 0$ and there exists an admissible pair $(\bar{\mathbf{y}}, \bar{\zeta}) \in \mathcal{S}_{[t,T]}^\#(x, 0)$ such that $\varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T) = z_1$ and $\max_{s \in [t,T]} \psi(\bar{\mathbf{y}}(s)) = z_2$. By definition of w ,

$$\min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t,T]}^\#(x, 0)} (\varphi(\mathbf{y}(T)) - \zeta(T) - z_1) \bigvee \left(\max_{s \in [t,T]} \psi(\mathbf{y}(s)) - z_2 \right) = 0.$$

10 That means, there exists no admissible pair (\mathbf{y}, ζ) such that $\varphi(\mathbf{y}(T)) - \zeta(T) < \varphi(\bar{\mathbf{y}}(T)) -$
11 $\bar{\zeta}(T) = z_1$ and $\max_{s \in [t,T]} \psi(\mathbf{y}(s)) < \max_{s \in [t,T]} \psi(\bar{\mathbf{y}}(s)) = z_2$. Therefore, by definition of weak
12 Pareto solution $z \in \mathcal{F}_w^\#(t, x)$. □

13

14 **Remark 4.** With similar arguments as in [23] it can be proved that outside the
15 set Ω only some trivial parts of the weak Pareto front might exist, that is $z \in$
16 $\mathcal{F}_w^\#(t, x) \cap \Omega^C \Leftrightarrow z_1 = z_1^*(t, x)$ and $z_2 > \bar{z}_2(t, x)$, or $z_2 = z_2^*(t, x)$ and $z_1 >$
17 $\bar{z}_1(t, x)$.

18 Now we give a precise characterization of the Pareto front of the bi-objective optimal
19 control problem (MORP) using the value function w . As in [23] consider the functions
20

$$\eta_1 : [z_1^*, \bar{z}_1] \rightarrow [z_2^*, \bar{z}_2], \quad \eta_1(\zeta_1) = \inf\{\gamma \mid w(t, x, \zeta_1, \gamma) \leq 0\}, \quad (20a)$$

$$\eta_2 : [z_2^*, \bar{z}_2] \rightarrow [z_1^*, \bar{z}_1], \quad \eta_2(\zeta_2) = \inf\{\gamma \mid w(t, x, \gamma, \zeta_2) \leq 0\}. \quad (20b)$$

21 **Lemma 5.3.** Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in $[0, T] \times$
22 \mathbb{R}^N . Then for $j = 1, 2$ the functions $\eta_j(\cdot)$ are decreasing.

23 *Proof* The result follows straightforward from the definition □

1
2

The following theorem gives a precise characterization of the Pareto front.

3 **Theorem 5.4.** Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in
4 $[0, T] \times \mathbb{R}^N$.

- 5 (i) $\mathcal{F}^\#(t, x) = \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\}$.
6 (ii) For any $z \in \mathcal{F}^\#(t, x)$ let a trajectory $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ that is optimal for the
7 auxiliary problem (11). Then (\mathbf{y}, ζ) is a Pareto optimal solution of (MORP).

Proof (i) Let $z \in \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\}$. By definition of functions η_i , $i = 1, 2$, we have that $w(t, x, z) = 0$. Then there exists at least one admissible pair $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\begin{aligned} & \left((\varphi(\mathbf{y}(T)) - \zeta(T)) - z_1 \bigvee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \right) \leq 0 \\ \Leftrightarrow & \varphi(\mathbf{y}(T)) - \zeta(T) \leq z_1 \text{ and } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2 \end{aligned} \quad (21)$$

8 Assume that there exists no $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that
9 $\left(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) = z$. Then for $(\bar{\mathbf{y}}, \bar{\zeta}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ satisfying
10 (21), we have that $\varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T) < z_1$ or $\max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) < z_2$. By choice of z we have
11 that $z_j = \eta_i(z_i)$, $j \neq i$. Then, without loss of generality, take $\zeta = \varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T)$,
12 so $w(t, x, \zeta, z_2) \leq 0$ with $\zeta < \eta_2(z_2)$ which is in contradiction with the definition
13 of $\eta_2(z_2)$ (see (20)). Now, to show that $z = (z_1, z_2)$ is in the Pareto front, as-
14 sume that there exists $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\mathbf{y}(T)) - \zeta(T) = \xi_1 \leq z_1$
15 and $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) = \xi_2 \leq z_2$, with $(\xi_1, \xi_2) \neq z$. Consider, without loss of gen-
16 erality, that $\xi_1 < z_1$, then $w(t, x, \xi_1, z_2) \leq 0$. As $\xi_1 < z_1$, by Proposition 4.3,
17 $w(t, x, \xi_1, z_2) \geq w(t, x, z_1, z_2) = 0$. So we conclude that $w(t, x, \xi_1, z_2) = 0$, with
18 $\xi_1 < z_1 = \eta_2(z_2)$ which is a contradiction. Therefore $z \in \mathcal{F}^\#(t, x)$.

Assume now that $z \in \mathcal{F}^\#(t, x)$ and let $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ be an admissible pair such that

$$z_1 = \varphi(\mathbf{y}(T)) - \zeta(T) \quad \text{and} \quad z_2 = \max_{s \in [t, T]} \psi(\mathbf{y}(s)).$$

19 Then $w(t, x, z) = 0$, by Theorem 5.2 and follows that $\eta_1(z_1) < +\infty$ and $\eta_2(z_2) < +\infty$.
20 If $z \notin \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\}$ then $\exists j$ such that
21 $z_j \neq \eta_j(z_j)$. Consider, without loss of generality, that $z_1 \neq \eta_1(z_1)$. As $w(t, x, z) =$
22 0 , we obtain that $z_1 > \eta_1(z_1)$. Consider $\xi = (\eta_1(z_1), z_2)$. By the definition of the
23 function η_1 we have that $w(t, x, \xi) = 0$ and then there exists an admissible pair $(\bar{\mathbf{y}}, \bar{\zeta}) \in$
24 $\mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T) \leq \xi_1 < z_1$ and $\max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) \leq \xi_2 = z_2$, what is
25 in contradiction with the assumption that z is in the Pareto front.

26 (ii). Let $z \in \mathcal{F}^\#(t, x)$. Then $w(t, x, z) = 0$ and z is feasible. Take a trajectory $(\mathbf{y}, \zeta) \in$
27 $\mathcal{S}_{[t, T]}^\#(x, 0)$ that is optimal for the auxiliary control problem (11). Then, as it was be

1 shown (see Proposition 5.1), $(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1 \vee \max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) = 0$ if and
2 only if $(\varphi(\mathbf{y}(T)) - \zeta(T) \leq z_1, \text{ and } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2)$. If $\varphi(\mathbf{y}(T)) - \zeta(T) < z_1$
3 then $\xi = (\varphi(\mathbf{y}(T)) - \zeta(T), z_2)$ is a feasible vector that dominates z which is impossible.
4 In the same manner, if $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) < z_2$ then $\xi' = (z_1, \max_{s \in [t, T]} \psi(\mathbf{y}(s)))$ is a
5 feasible vector that dominates z which is impossible. So, for any trajectory (\mathbf{y}, ζ) that
6 is optimal for (11) we have that $(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s))) = z$ that means
7 the pair (\mathbf{y}, ζ) is Pareto optimal for (MORP). \square

8
9 **Remark 5.** It can be proved that the zero level set of the value function w , that is the
10 set $\{z \in \mathbb{R}^2 \mid w(t, x, z) = 0\}$, is located in a curve. By Theorem 5.2 the weak Pareto
11 front is contained in that curve and we stress on the fact that to guarantee that a point
12 in this curve belongs to the weak Pareto front it is necessary to have an admissible
13 trajectory $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\varphi(\mathbf{y}(T)) - \zeta(T) = z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) = z_2. \quad (22)$$

14 It may happen that, for one $z \in \mathbb{R}^2$, $w(t, x, z) = 0$ and there exists no admissible
15 trajectory that satisfies both equalities in (22).

16 On another hand, Theorem 5.4 gives a characterization of the Pareto front as an
17 intersection of two curves, that can be easily computed, once the value function w
18 is obtained. This intersection aims removing the horizontal and vertical parts of the
19 curve.

20 6. ε -Pareto solutions of the original bi-objective optimal control problem

21 In this section, we return to the original problem presented in this paper. We prove
22 that using the auxiliary value function w it is possible to obtain the region of (weak) ε -
23 Pareto fronts are contained. Moreover, (weak) ε -Pareto optimal solutions for problem
24 MOP can be obtained by applying an algorithm of trajectory reconstruction to the
25 auxiliary control problem (11).

26 **Theorem 6.1.** *Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold. Let (t, x) be in $[0, T] \times$
27 \mathbb{R}^N and $\varepsilon > 0$.*

- 28 (i) $\mathcal{F}^{3, \varepsilon}(t, x) \subset \mathcal{F}^{1, \varepsilon}(t, x) \subset \mathcal{F}_w^{1, \varepsilon}(t, x) \subset \left\{z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0\right\}$.
29 (ii) Let $z_\varepsilon \in \left\{z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0\right\}$. If there exists $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$
30 that is optimal for the auxiliary control problem (11). Then $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$
31 of problem (8).
32 (iii) Let $z_\varepsilon \in \left\{z \in \mathbb{R}^2 \mid -\varepsilon < w(t, x, z) \leq 0\right\}$. If there exists $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$
33 that is optimal for the auxiliary control problem (11). Then $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{P}^{1, \varepsilon}(t, x)$
34 of problem (8).

Proof By definition $\mathcal{F}^{3, \varepsilon}(t, x) \subset \mathcal{F}^{1, \varepsilon}(t, x)$. Let $z \in \mathcal{F}_w^{1, \varepsilon}(t, x)$. Then there exists an
admissible pair $(\bar{\mathbf{y}}, \bar{\zeta}) \in \mathcal{S}_{[t, T]}(x, 0)$ such that $\varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T) = z_1, \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) = z_2,$

and

$$\begin{aligned} w(t, x, z) &= \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1) \bigvee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \right] \\ &\leq \left[(\varphi(\bar{\mathbf{y}}(T)) - \bar{\zeta}(T) - z_1) \bigvee (\max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) - z_2) \right] = 0. \end{aligned}$$

Moreover as $z \in \mathcal{F}_w^{1, \varepsilon}(t, x)$, then for any $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}(x, 0)$, one of the following assertions holds: $z_1 - \varepsilon \leq \varphi(\mathbf{y}(T)) - \zeta(T)$ or $z_2 - \varepsilon \leq \max_{s \in [t, T]} \psi(\mathbf{y}(s))$. It is possible to check that in the two above cases, we have $(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1) \bigvee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \geq -\varepsilon$.

As the $\mathcal{S}_{[t, T]}^\#(x, 0)$ is the closure of the $\mathcal{S}_{[t, T]}(x, 0)$, we can conclude that one of assertions above holds for any $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$. Therefore,

$$w(t, x, z) = \min_{(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \zeta(T) - z_1) \bigvee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \right] \geq -\varepsilon.$$

(ii) Let $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0 \right\}$ and $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary problem $w(t, x, z_\varepsilon)$. Then, $-\varepsilon \leq (\varphi(\mathbf{y}_\varepsilon(T)) - \zeta_\varepsilon(T) - z_{1, \varepsilon}) \bigvee \max_{s \in [t, T]} \psi(\mathbf{y}_\varepsilon(s)) - z_{2, \varepsilon}$. Assume that there exists $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}(x, 0)$ such that $(\varphi(\mathbf{y}(T)) - \zeta(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s))) < (z_{1, \varepsilon} - \varepsilon, z_{2, \varepsilon} - \varepsilon)$. Therefore,

$$\left(\varphi(\mathbf{y}(T)) - \zeta(T) - z_{1, \varepsilon} \bigvee \max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_{2, \varepsilon} \right) < \left(\varphi(\mathbf{y}_\varepsilon(T)) - \zeta_\varepsilon(T) - z_{1, \varepsilon} \bigvee \max_{s \in [t, T]} \psi(\mathbf{y}_\varepsilon(s)) - z_{2, \varepsilon} \right)$$

1 which is impossible. So, for any trajectory $(\mathbf{y}_\varepsilon, \zeta_\varepsilon)$ that is optimal for the auxiliary
2 control problem $w(t, x, z_\varepsilon)$ we have that the pair $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$ of problem (8).

3 (iii) The proof can be obtained with similar arguments of (ii). \square

4

5 We note that if we take a point z_ε in the set $\left\{ z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0 \right\}$ and
6 there exists $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary control problem (11),
7 we could just prove that the optimal pair is in the set $\mathcal{P}^{1, \varepsilon}(t, x)$ that is bigger than the
8 set of ε -Pareto optimal solutions $\mathcal{P}^{3, \varepsilon}(t, x)$. However, this is still relevant, once the
9 definition $\mathcal{P}^{1, \varepsilon}(t, x)$ it is most used in researchs in this subject.

10 7. Reconstruction of the Pareto optimal trajectories

11 The characterization of the weak Pareto front and the Pareto front of the relaxed
12 problem (MORP) is provided in Theorems 5.2 and 5.4, respectively. Another important
13 concern is to reconstruct an optimal trajectory corresponding to a given point in the
14 (weak) Pareto front.

After compute the auxiliary value function w , by Theorem 5.4 we have a characteri-
zation of the Pareto front $\mathcal{F}^\#(t, x)$. Now, let $z = (z_1, z_2) \in \mathcal{F}^\#(t, x)$. Then, an algorithm
of trajectory reconstruction, as the algorithm presented in [3], can be applied to the
function w on $[t, T]$ with the initial conditions (x, z) . To apply the algorithm proposed
in that paper, an initial condition (x, z) should be fixed, and a time step h should be
chosen to discretize the interval $[t, T]$. Then the control values u_k^h are defined by a re-
cursion method based on the minimization of the Dynamical Programming Principle,

that is

$$u_k^h \in \operatorname{argmin}_{u \in U} w^h(s_k, y_k^h + hf(y_k^h), u), \zeta_k^h - \ell(y_k^h, u), z_2) \bigvee \max(g(y_k^h) - z_2),$$

1 and the new state positions are defined by $y_{k+1}^h = y_k^h + hf(y_k^h)$ and $\zeta_{k+1}^h = \zeta_k^h - \ell(y_k^h, u)$.
 2 It is proven that a trajectory generated by this procedure is an approximation of the
 3 optimal trajectory for $w(t, x, z)$. Finally, by Theorem 5.4, item (ii), if the trajectory is
 4 optimal for the auxiliary problem (11), then is a Pareto optimal trajectory of (MORP).

5 In the case of weak Pareto solutions, by Theorem 5.2 $\mathcal{F}_w^\#(t, x) \cap \Omega \subset \{z \in \Omega \mid$
 6 $w(t, x, z) = 0\}$. And if there exists an admissible pair $(\mathbf{y}, \zeta) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that
 7 $\varphi(\mathbf{y}(T)) - \zeta(T) = z_1$ and $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) = z_2$, then $z \in \mathcal{F}_w^\#(t, x)$. So we can take $z \in \Omega$

8 such that $w(t, x, z) = 0$, apply an algorithm of trajectory reconstruction to the function
 9 w on $[t, T]$ with the initial (x, z) as in [3], and see if such a trajectory exists. If there
 10 is a trajectory in such conditions this is an approximation of a weak Pareto trajectory
 11 of (MORP).

12 Now consider $\varepsilon > 0$ and let $z_\varepsilon \in \mathbb{R}^2$, such that $-\varepsilon \leq w(t, x, z_\varepsilon) \leq 0$. By applying an
 13 algorithm of trajectory reconstruction to the function w on $[t, T]$ with the initial condi-
 14 tions (x, z_ε) we get an approximation of the optimal trajectory for $w(t, x, z_\varepsilon)$. Now by
 15 Theorem 6.1, if there exists $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary control
 16 problem (11). Then $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$ of problem (8). Moreover if we consider that
 17 $-\varepsilon < w(t, x, z_\varepsilon) \leq 0$ and it is possible to obtain $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal
 18 for the auxiliary control problem (11). Then $(\mathbf{y}_\varepsilon, \zeta_\varepsilon) \in \mathcal{P}^{1, \varepsilon}(t, x)$ of problem (8).

19 8. A numerical example: Pest control problem

20 In this section we present a numerical example where the method proposed in this
 21 paper was applied to compute the (weak) Pareto front. The resultant HJB equation
 22 was solved by a finite difference method implement at C++ HJB-solver "ROC-HJ"
 23 [30].

Consider a Lotka-Volterra model describing the interaction between two species
 (predator-prey model): a prey, that is a nuisance for humans, and a predator. We
 can act on this model by spraying a chemical to poison the pest (the poison may also
 kill a part of predator population). This model is given by the following dynamical
 system

$$\begin{aligned} \dot{\mathbf{y}}_1(s) &= \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - \mathbf{y}_1(s)c_{\mathbf{y}_1}\mathbf{u}(s) \\ \dot{\mathbf{y}}_2(s) &= -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - \mathbf{y}_2(s)c_{\mathbf{y}_2}\mathbf{u}(s) \end{aligned}$$

where \mathbf{y}_1 represents the pest population, \mathbf{y}_2 the predator, the constants $c_{\mathbf{y}_1}$, $c_{\mathbf{y}_2}$ repre-
 sent the rate of each population that will be killed by the poison. As in [31], the value
 of the constants $c_{\mathbf{y}_1}$, $c_{\mathbf{y}_2}$ are set to 0.4 and 0.2 respectively. The control function $\mathbf{u}(s)$
 is restricted to take values of either 0 or 1. The goal is to keep nuisance expansion under
 control by minimizing the maximum difference of certain proportion of both species
 along the time horizon $T - t$:

$$\max_{s \in [t, T]} 0.25(\mathbf{y}_1(s) - K\mathbf{y}_2(s))^2$$

and also reduce the cost of spraying the chemical:

$$\int_t^T P \mathbf{u}(s) ds,$$

- 1 where $K = 0.7$, $P = 0.3$ and $T = 10$. So, the optimal control problem has two different
 2 objective functions.

Since the control is allowed to take only the values 0 and 1, the set of trajectories is not compact. In the relaxed control problem, the control input may take values in the interval $[0, 1]$. As described in the previous sections, we introduce an auxiliary control problem whose value function w is solution of the following HJB equation:

$$\min \left(-\partial_t w(t, x, z) + \mathcal{H}(x, D_x w, D_z w), w(t, x, z) - (0.25(x_1 - Kx_2)^2 - z_2) \right) = 0,$$

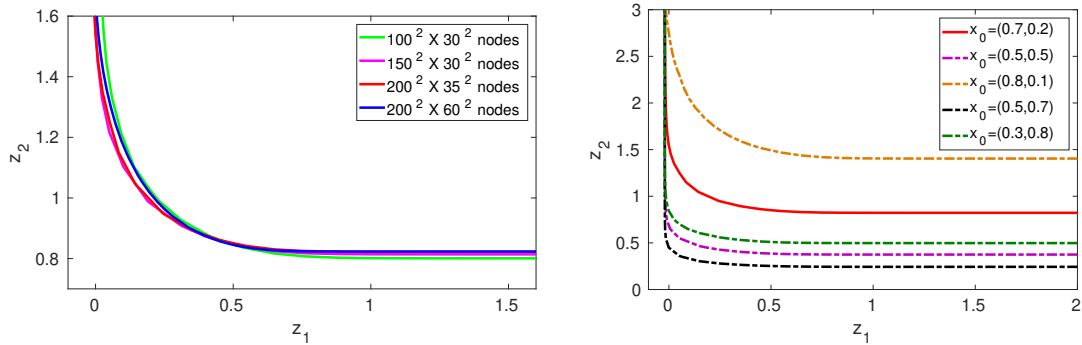
for $t \in [0, T], x, z \in \mathbb{R}^2$

$$w(T, x, z) = -z_1 \sqrt{(0.25(x_1 - Kx_2)^2 - z_2)} \quad \text{for } x \in \mathbb{R}^2, z \in \mathbb{R}^2,$$

where the Hamiltonian is given by

$$\mathcal{H}(x, p, q) = -(x_1 - x_1 x_2) p_1 + (x_2 - x_1 x_2) p_2 + \max(c_{y_1} x_1 p_1 + c_{y_2} x_2 p_2 + P q_1, 0), \forall x, p, q \in \mathbb{R}^2.$$

- 3 The HJB equation is solved on the domain $[0, 3.5] \times [0, 3.5] \times [0, 3] \times [0, 3]$. For this
 4 example, we used an explicit Euler scheme in time and Lax-Friedrich discretization in
 5 space (we refer to [3,32] for more details). This discretization is known to be stable
 6 and convergent under an adequate interplay between the mesh size of the x -grid and
 7 the time step. First, we fix the initial conditions as follows: $x_0 = (0.7, 0.2)$ and $t = 0$.
 8 Figure 2a presents the 0-set-level of the value function $w(0, x_0, \cdot)$ computed on five
 9 different x-grids. This numerical test confirms the (already known) stability of the
 10 numerical scheme. One important feature of the HJB approach to solve bi-objective
 11 control problems is the possibility of obtain the Pareto front for different initial states
 12 with the same auxiliary value function w . Figure 2b shows the Pareto front for different
 13 initial states x_0 computed on a x-grid of $200^2 \times 35^2$ nodes.



(a) 0-level set of $w(0, (0.7, 0.2), \cdot)$ computed on different grids.

(b) 0-level set of $w(0, x_0, \cdot)$ for different initial states x_0 on a grid of $200^2 \times 35^2$.

Figure 2.: 0-level sets on different computational grids and for different initial states.

- 14 From now, we fix $t = 0$ and $x_0 = (0.7, 0.2)$ and we consider the approximation of the
 15 value function w obtained on x-grid of on a x-grid of $200^2 \times 35^2$ nodes. Formula (16)
 16 allows to give an approximation of the utopian point associated with the bi-objective
 17 control problem:

$$z_1^*(0, x_0) = \inf \left\{ \zeta_1 \in \mathbb{R} \mid \exists z_2 \in \mathbb{R} \text{ s.t. } w(0, x_0, \zeta_1, z_2) \leq 0 \right\} \cong 0,$$

$$z_2^*(0, x_0) = \inf \left\{ \zeta_2 \in \mathbb{R} \mid \exists z_1 \in \mathbb{R} \text{ s.t. } w(0, x_0, z_1, \zeta_2) \leq 0 \right\} \cong 0.839.$$

Following (18), we get also an approximation of the upper bounds of Ω :

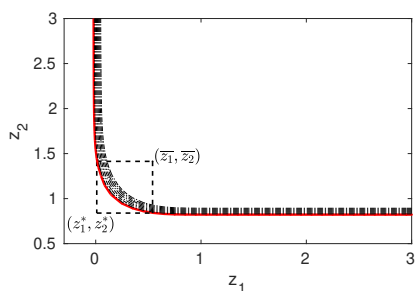
$$\bar{z}_1(0, x_0) = \inf \left\{ \zeta \in \mathbb{R} \mid w(0, x_0, \zeta, z_2^*(0, x_0)) = 0 \right\} \cong 0.54,$$

$$\bar{z}_2(0, x_0) = \inf \left\{ \zeta \in \mathbb{R} \mid w(0, x_0, z_1^*(0, x_0), \zeta) = 0 \right\} \cong 1.414.$$

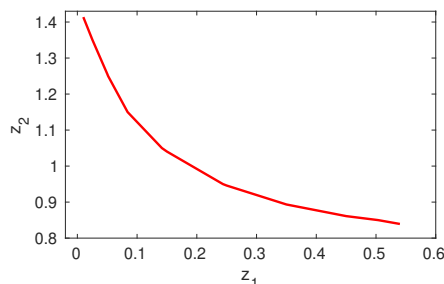
These values lead to an approximation of the set

$$\Omega = [z_1^*(0, x_0), \bar{z}_1(0, x_0)] \times [z_2^*(0, x_0), \bar{z}_2(0, x_0)].$$

- 1 Figure 3a shows the 0-level set of the value function $w(0, x_0, \cdot)$ in red that contains
 - 2 the (weak) Pareto front $\mathcal{F}^\#(0, x_0) \subset \mathcal{F}_x^\#(0, x_0)$. Moreover, the black region in Figure 3a
 - 3 represents a region where it is possible to obtain points in $\mathcal{F}_w^{1,\varepsilon}(0, x_0)$, for $\varepsilon = 0.05$. In
 - 4 this figure the set Ω is represented by a box delimited by black dashed lines. Figure 3b
 - 5 shows the intersection of 0-level set of $w(0, x_0, \cdot)$ with the set Ω . In this example the
 - 6 intersection is equal to the Pareto front $\mathcal{F}^\#(0, x_0)$. The possible points in the weak
- Pareto front are just trivial points and are outside the set Ω .



(a) The 0-level set of $w(0, x_0, \cdot)$ (in red). The set $\{z \mid w(0, x_0, z) \leq -\varepsilon\}$ for $\varepsilon = 0.05$ (in black).



(b) zoom on $\mathcal{F}^\#(0, x_0)$.

Figure 3.: Analysis of the negative level set of $w(0, x_0, \cdot)$, for $x_0 = (0.7, 0.2)$.

7

8 The value function w is also useful to reconstruct Pareto optimal trajectories. By
 9 using an algorithm of trajectory reconstruction from [3], we reconstruct optimal Pareto
 10 trajectories, for five Pareto values $z = (z_1, z_2)$ that are represented by points with
 11 different colors in figure 4a. Figure 4 shows these optimal trajectories for the initial
 12 states (x_0, z) and the color used for each trajectory refers to the corresponding Pareto
 13 value in figure 4.

14 As $z_1^* = 0$, the optimal control law for $w(0, x_0, z_1^*(0, x_0), \bar{z}_2(0, x_0))$ is identically 0
 15 (i.e., $\mathbf{u} \equiv 0$), which means that not only the point $(z_1^*(0, x_0), \bar{z}_2(0, x_0))$ belongs to the
 16 Pareto front of the relaxed problem $\mathcal{F}^\#(0, x_0)$, but it is also optimal Pareto for the
 17 original (non-relaxed) control problem, that is $(z_1^*(0, x_0), \bar{z}_2(0, x_0)) \in \mathcal{F}(0, x_0)$. The
 18 case of the others optimal Pareto trajectories represented in Figure 4 is different. It
 19 is possible to see in Figure 5 and 6 that the optimal control law seems to take other

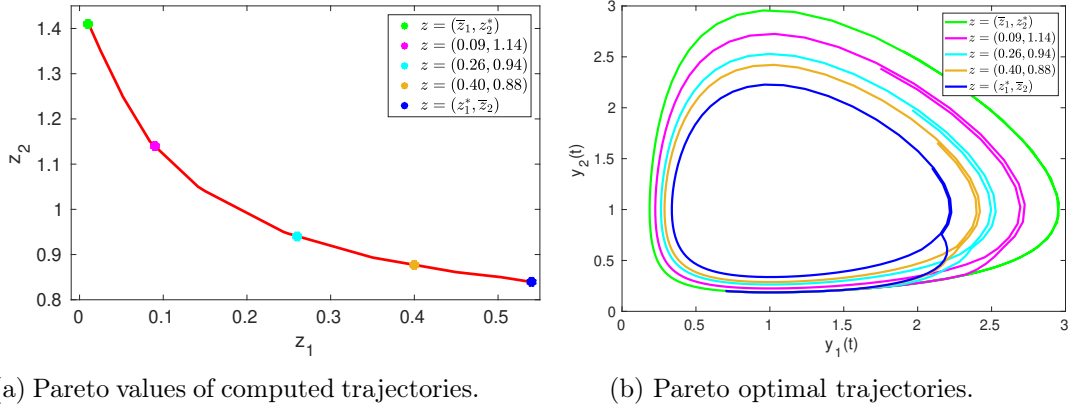


Figure 4.: Optimal Pareto solutions in $\mathcal{P}^\#(0, x_0)$.

1 values than just 0 and 1. So, we cannot guarantee that the Pareto values belong to
 2 $\mathcal{F}(0, x_0)$. However, by Theorem 4.1, there exists a ε -Pareto solution in a neighborhood
 3 of these Pareto values, where the control takes values in the set $\{0, 1\}$. In figure 5
 4 (resp. figure 6), we display the relaxed optimal Pareto trajectory-control corresponding
 5 to $(\bar{z}_1(0, x_0), z_2^*(0, x_0)) \in \mathcal{F}^\#(0, x_0)$ (resp. $(0.09, 1.14) \in \mathcal{F}^\#(0, x_0)$) and an optimal
 6 trajectory-control corresponding to the ε -Pareto value $(\bar{z}_1(0, x_0) + \varepsilon, z_2^*(0, x_0) + \varepsilon)$ (resp.
 7 $(0.1, 1.15)$), for $\varepsilon = 0.02$ (resp. $\varepsilon = 0.01$). As can be seen in this example, the two
 8 trajectories are very close to each other, while the structure of the controls laws are
 different.

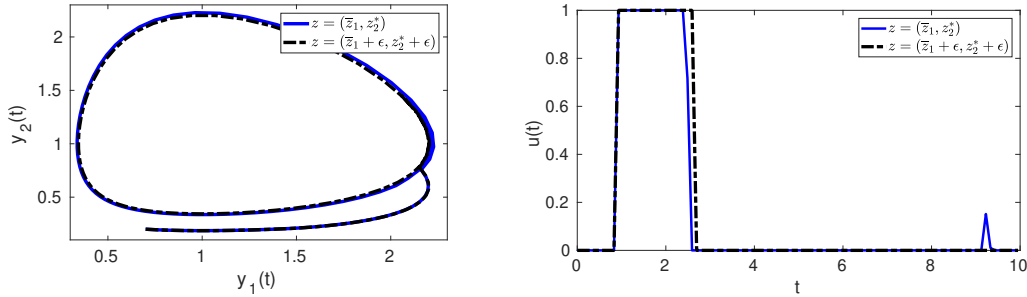


Figure 5.: Relaxed Pareto optimal solution corresponding to $(\bar{z}_1(0, x_0), z_2^*(0, x_0)) \in \mathcal{F}^\#(0, x_0)$ and ε -Pareto solution to the original problem corresponding to $(\bar{z}_1(0, x_0) + \varepsilon, z_2^*(0, x_0) + \varepsilon) \in \mathcal{F}^{1,\varepsilon}(0, x_0)$ with $\varepsilon = 0.02$.

9

10 9. Conclusion

11 In this paper we have investigated a bi-objective optimal control problem with cost
 12 functions of different nature. We considered the situation where the set of trajectories
 13 is not compact and the set of Pareto solutions may be empty. We have studied the
 14 relation of the Pareto front corresponding to the relaxed (convexified) bi-objective
 15 problem with the original one. More precisely, we proved that for any x^* is a (weak)
 16 Pareto optimal solution of the relaxed problem and for any $\varepsilon > 0$ there exist a (weak)

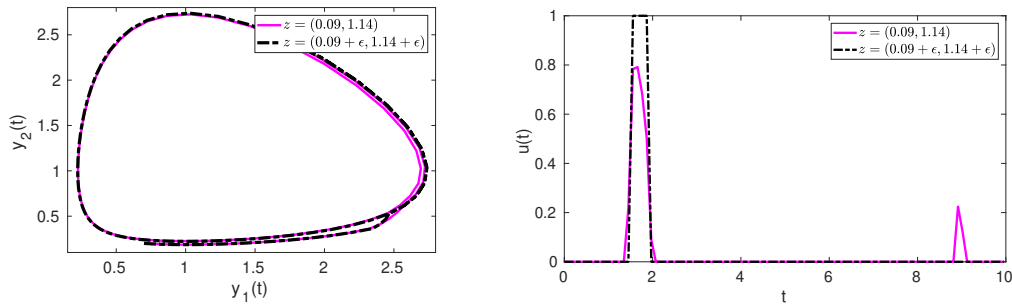


Figure 6.: Relaxed Pareto optimal solution corresponding to $(0.09, 1.14) \in \mathcal{F}^\#(0, x_0)$ and ε -Pareto solution to the original problem corresponding to $(0.1, 1.15) \in \mathcal{F}^{1,\varepsilon}(0, x_0)$ with $\varepsilon = 0.01$.

1 ε -Pareto solution for the original problem that is in a neighborhood of x^* . Moreover,
 2 the distance between the Pareto optimal value achieved by x^* and the optimal value
 3 achieved by x is small and is of order ε . A characterization of the Pareto front of the
 4 relaxed bi-objective problem is derived, along with a characterization of the ε -Pareto
 5 front of the original problem.

6 The relevance of our approach is tested on a simple control problem. The numerical
 7 simulations confirm all the theoretical results. However, it should be noticed that for a
 8 control problem with two state variables and two cost functions, the method requires
 9 to solve a HJB equation in dimension 4. To get an approximation of the Pareto front
 10 in a reasonable time, the dimension of state should remain less than 4 or 5.

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