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Bi-objective finite horizon optimal control problems with Bolza and maximum running cost

Ana Paula Chorobura* and Hasnaa Zidani†

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ABSTRACT

In this paper, we investigate optimal control problems with two objective functions of different nature that need to be minimized simultaneously. One objective is in the classical Bolza form and the other one is defined as a maximum running cost. Our approach is based on the Hamilton-Jacobi-Bellman framework. In the problem considered here the existence of Pareto solutions is not guaranteed. So first, we consider the bi-objective problem to be minimized over the convexified dynamical system. We show that if a vector is (weak) Pareto optimal solution for the convexified problem, then there exists an (weak) ε -Pareto optimal solution of the original problem that is in the neighborhood of this vector. After we define an auxiliary optimal control problem and show that the weak Pareto front of the convexified problem is a subset of the zero level set of the corresponding value function. Moreover, with a geometrical approach we establish a characterization of the Pareto front. It is also proved that the (weak) ε -Pareto front is contained in the negative level set of the auxiliary optimal control problem that is less or equal ε . Some numerical examples are considered to show the relevance of our approach.

KEYWORDS

Multi-objective programming; Hamilton-Jacobi-Bellman approach; Optimal control problem; Approximate Pareto solutions.

1 Introduction

In optimal control problems the aim is to control the dynamical systems evolution along a certain period of time in a manner that the pair of control-and-state is optimal with respect to some criterion, given in the objective function. The type of this objective function is of great importance when developing the theory. It can be defined, for example, in a Bolza form [3, 21, 12, 11, 34] or in a max-type [4, 7].

Continuous optimal control problems have been studied with the Dynamic Programming approach, see [6, 10]. The idea is to work with the value function that results from considering the optimal value of the control problem as a function of the initial state for the dynamical system. Under suitable conditions, such value function satisfies a dynamic programming principle and the value function can be characterized as the unique viscosity solution of a partial differential equation, called Hamilton-Jacobi-Bellman (HJB) equation [2]. Moreover the value function contains all necessary information to reconstruct the optimal trajectories and control strategies [4, 6, 8, 31].

In general when the set of trajectories is not closed, it is not possible to guarantee the existence of a minimizer for the optimal control problem. A very known approach to deal with it is introduce a convexified (relaxed) optimal control problem; see for instance [5, 18]. For this relaxed problem is possible to prove that the set of trajectories is compact. Moreover, under some assumptions, this convexified set of trajectories is the closure of the set of trajectories of the original problem. Therefore this new optimal control problem has an optimal solution and the value function is equal to the value function of the original problem [6, 18].

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This paper presents a novel theory for finite horizon optimal control problems with objectives of different nature that need to be minimized simultaneously. Namely, in the vector objective function, one component can be a Bolza cost and another one a maximum running cost. So the considered bi-objective optimal problem is:

$$\inf_{(\mathbf{y}, \mathbf{u}) \in \mathbb{X}} \left(\varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s)) ds, \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right),$$

for a given final cost $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$, a running cost $\ell : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}$, and a given function $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$, and where \mathbb{X} is the feasible set, T is the final time horizon, $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^M$ is the control variable and $\mathbf{y} : [0, T] \rightarrow \mathbb{R}^N$ is the state variable, solution of a differential system.

Multi-objective optimization is an area of great interest in applications. In this context, generally, it is not possible to minimize all the criteria simultaneously. For this reason, several solution concepts have been proposed in the literature. In the famous work "Cours d'Economie Politique" [30], the pioneering economist V. Pareto introduced the notion of efficient or Pareto solution. At a Pareto solution it is not possible to improve one criterion without worsening at least one of the other ones. For any given problem, the set of Pareto solutions may be infinite and unbounded. A larger set is given by weak Pareto solutions, at which it is not possible to improve all the objective functions simultaneously. The image of the set of all (weak) Pareto solutions by the objective function is called (weak) Pareto front. It is useful for practitioners for finding a trade-off between conflicting criteria. A great number of works are developed in optimization problems with several goals, see for instance [13, 14, 22, 28] and the reference therein.

One of the most common approach for solving multi-objective optimization problem is to relate it with a family of mono-objective optimization problems, in such a way that the solutions of the multi-objective problem can be obtained by solving a sequence of classical nonlinear programming problems. The most popular scalarization techniques are the weighted sum method and the weighted Chebyshev method, [22, 28]. As the scalarization usually depends on certain auxiliary parameters, some numerical difficulties may appear if the single objective optimization problem has feasible solutions only with very few parameter values. Moreover, a weakness of the weighting method is that all the Pareto optimal points cannot be found if the Pareto front is nonconvex [23]. For multi-objective optimal control problems several numerical algorithms based on scalarization techniques have been developed (see for instance [9, 23, 26] and the references therein).

Multi-objective optimal control problems have also been investigated within the HJB framework. A method that combines the HJB approach and the weighted sum method to find some points of the Pareto front was introduced in [29]. In [24] a numerical method for multi-objective optimal control problems under integral constraints is proposed. With the HJB approach for an extending state space the Pareto front can be constructed. These idea was extended to a class of hybrid control problems, see [33]. In [20] the set-valued function is characterized as a unique generalized solution of an HJB equation. In [15], the idea of introduce an auxiliary problem to deal with mono-objective optimal control problems [3], is extended to work with multi-objective optimal control problems.

In practice, for some problems, it is difficult to calculate the sets of (weak) Pareto solutions and what it could be obtained is just an approximate set of solutions. The notion of approximate optimal solutions for multi-objective problems was introduced by [27]. But this notion of ε -Pareto solutions can be investigated with several different definitions, as proposed in [35]. We discuss in this paper three of these concepts, that are the most relevant, being one of them the concept introduced in [27] and also investigate by [16, 17, 19].

All works of multi-objective optimal control refereed here, investigate problems where the objective functions must have the same nature. In this work, we use the HJB approach to characterize the (weak) Pareto front for a finite horizon bi-objective optimal control problem with objectives of different nature. In the considered problem it is not possible to guarantee that the set of trajectories is closed, so we introduce a convexified (relaxed) problem where is possible to guarantee that the set of trajectories is compact. Moreover this convexified set of trajectories is the closure of the set of trajectories of the original problem. Moreover we prove that if a feasible pair (\mathbf{y}, \mathbf{u}) is a Pareto optimal solution for the convexified problem, then there exists an ε -Pareto optimal solution of the original problem that is in the neighborhood of (\mathbf{y}, \mathbf{u}) . After we define an auxiliary control problem that has good properties and can be characterized as a solution of an HJB equation. Then we show that the weak Pareto front of the convexified problem is a subset of the zero level set of the corresponding value function. Moreover, with a geometrical approach we establish a characterization of the Pareto front.

This paper is organized as follows. Section 2 discuss the concepts and properties of solutions of bi-objective problems. In Section 3 the bi-objective problem with objectives of different nature is formulated. Moreover the convexified problem is introduced and some results about the relation of solutions of two problems are proved. In Section 4 an auxiliary control problem is considered, the HJB equation and some properties to the auxiliary value function are derived. Section 5 studies the link between the 0-level set of the auxiliary value function and the (weak) Pareto front. Moreover the characterization of the Pareto Front for the convexified problem is obtained. In Section 6 we shown how to obtain (weak) ε -Pareto optimal solutions for the original problem. We discuss the reconstruction of (approximate) Pareto trajectories on Section 7. Some numerical examples are presented in Section 8. Section 9 concludes the paper with final remarks.

Notations Throughout this paper, \mathbb{R} denotes the set of real numbers, $|\cdot|$ is the maximum norm. For a set $S \subseteq \mathbb{R}^N$, \bar{S} and $\text{bdry}(S)$ denote its closure and boundary, respectively. Moreover, for every $t > 0$ we denote by $W^{1,1}([t, T]; \mathbb{R}^M)$ (for any $M > 0$) the space of integrable functions, from $[t, T]$ in \mathbb{R}^M , that admit a weak derivative integrable also on $[t, T]$.

We will use the standard convention that $\inf \emptyset = +\infty$.

For $z, z' \in \mathbb{R}^N$, we will use the following notations for different partial order relations:

$$z < z' \Leftrightarrow [z_i < z'_i \quad \forall i = 1, \dots, N]; \quad z \leq z' \Leftrightarrow [z_i \leq z'_i \quad \forall i = 1, \dots, N].$$

2 Pareto optimality

Consider the following bi-objective optimization problem:

$$\begin{cases} \text{Minimize } g(x) = (g_1(x), g_2(x)) \\ \text{subject to } x \in X \end{cases} \quad (\text{P})$$

where Y is a Banach space, $g_i : Y \rightarrow \mathbb{R}$ are continuous functions and $X \subset Y$ a feasible nonempty set.

In the multi-objective optimization we want to minimize all the components of the objective function at the same time. If there exists no conflict between the objective functions, then a solution $x^* \in X$ can be found where all objectives attains its optimum, that is

$$g_i(x^*) = \min\{g_i(x), x \in X\}, \quad i = 1, 2.$$

In this case, no special methods are needed. To avoid such trivial cases we assume that does not exist a single solution that minimizes all the objective functions simultaneously. This means that the objective functions are at least partly conflicting and several solution concepts may be associated with the problem (P). In this paper, we use the concept of Pareto optimality [30], to define the notion of solution for the multi-objective optimization problem (P).

Definition 2.1 (Pareto optimal solutions). *Let $x^* \in X$ be an admissible pair.*

- *We will say that x^* is a Pareto optimal solution if and only if there is no admissible pair $x \in X$ such that*

$$(g_1(x), g_2(x)) \leq (g_1(x^*), g_2(x^*)), \text{ and } [g_1(x) < g_1(x^*) \quad \text{or} \quad g_2(x) < g_2(x^*)].$$

The set \mathcal{P} of all Pareto optimal solutions is said to be Pareto optimal solutions set.

- *We will say also that x^* is a weak Pareto optimal solution if and only if there is no admissible pair $x \in X$ such that*

$$(g_1(x), g_2(x)) < (g_1(x^*), g_2(x^*)).$$

The set \mathcal{P}_w of all weak Pareto optimal solutions is said to be weak Pareto optimal solutions set.

With these definitions, we can check that the Pareto set is a subset of the weak Pareto set: $\mathcal{P} \subset \mathcal{P}_w$.

The set of all vectors of objective values at the Pareto (resp. weak Pareto) minima is said to be the *Pareto front* (resp. *weak Pareto front*). More precisely, we have the following definition.

Definition 2.2 (Pareto front). *We will call Pareto front (respectively weak Pareto front) the image of the Pareto optimal solutions set \mathcal{P} (respectively of \mathcal{P}_w) by the multi-objective application $g(x)$:*

$$\mathcal{F} = \{g(x), x \in \mathcal{P}\} \quad \mathcal{F}_w = \{g(x), x \in \mathcal{P}_w\}. \quad (1)$$

Denote the image of the feasible set by \mathbb{Z} and call it the attainable set. It is a subset of the objective space \mathbb{R}^2 . It is known that if a vector $z \in \mathbb{Z}$ is in the (weak) Pareto front then z is necessarily a boundary point of \mathbb{Z} [25]. It is a challenging problem to compute the sets \mathcal{P} and \mathcal{P}_w . In some cases (weak) Pareto solutions may not exist, for instance when the feasible set is not closed. In such a problems what it could be obtained is a set of approximate solutions. The notion of approximate solutions for multi-objective problems can be investigated with several different definitions. One definition was introduced by [27], then in [35] six different concepts of ε -Pareto solutions were proposed for multi-objective problems and some methods for their generation were examined. We discuss in this paper three of these concepts, that are the most relevant and the first one is the concept introduced in [27].

Definition 2.3 (ε -Pareto solutions). *Let $\varepsilon \geq 0$. We define the following sets of ε -Pareto solutions:*

- (i) $\mathcal{P}^{1,\varepsilon} = \{x \in X : \text{there is no } y \in X \text{ such that } g(y) \leq g(x) - \varepsilon 1 \text{ and } g(y) \neq g(x) - \varepsilon 1\}$.
- (ii) $\mathcal{P}^{2,\varepsilon} = \{x \in X : \text{there is } y \in \mathcal{P} \text{ such that } |g(x) - g(y)| \leq \varepsilon\}$.
- (iii) $\mathcal{P}^{3,\varepsilon} = \{x \in X : \text{if } g(y) \neq g(x) \text{ and } g(y) \leq g(x) \text{ for some } y \in X, \text{ then } g(y) \geq g(x) - \varepsilon 1\}$.

An ε -Pareto solution $x^* \in \mathcal{P}^{i,\varepsilon}$, $i = 1, 2, 3$, produces an ε -Pareto outcome $g(x^*)$ and the set of all ε -Pareto outcomes are denoted by $\mathcal{F}^{i,\varepsilon}$, $i = 1, 2, 3$. Note also that, if $\varepsilon_1 \leq \varepsilon_2$ then $\mathcal{F}^{i,\varepsilon_1} \subset \mathcal{F}^{i,\varepsilon_2}$ and $\mathcal{F}_w^{i,\varepsilon_1} \subset \mathcal{F}_w^{i,\varepsilon_2}$, $i = 1, 2, 3$. Moreover, if the feasible set X is a compact set, then the following relation between the ε -Pareto sets was proved by [35]:

$$\mathcal{F} \subset \mathcal{F}^{3,\varepsilon} \subseteq \mathcal{F}^{2,\varepsilon} \subseteq \mathcal{F}^{1,\varepsilon}. \quad (2)$$

Remark 2.4. *The reverse inclusions are not true. Consider the following multi-objective optimization problem:*

$$\begin{cases} \text{Minimize } g(x) = (x_1, x_2) \\ \text{subject to: } x \in X = [-1, 1]^2 \setminus \{x \in [-1, 0]^2 : x_1 + x_2 < -1\}. \end{cases} \quad (3)$$

The feasible set X is given by Figure 1.

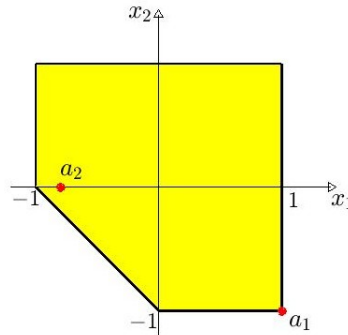


Figure 1: Feasible set X

Note that for all $\varepsilon > 0$ the point $a_1 = (1, -1) \in \mathcal{P}^{1,\varepsilon}$. In fact $g_2(a_1) - \varepsilon = -1 - \varepsilon$, so there is no $x \in X$ such that $g(x) \neq g(a_1)$ and $g(x) \leq g(a_1) - \varepsilon$, therefore $a_1 \in \mathcal{P}^{1,\varepsilon}$. Moreover if $0 < \varepsilon < 1$ then $a_1 \notin \mathcal{P}^{3,\varepsilon} \subset \mathcal{P}^{2,\varepsilon}$,

because $g(0, -1) \neq g(a_1)$, $g(0, -1) \leq g(a_1)$ and $g_1(0, -1) = 0 < 1 - \varepsilon = g_1(a_1) - \varepsilon$. Now, consider $\varepsilon = 0.1$ and the point $a_2 = (-0.8, 0)$. As $(-0.9, -0.1)$ is a Pareto optimal solution, we obtain that $a_2 \in \mathcal{P}^{2,\varepsilon}$, but $a_2 \notin \mathcal{P}^{3,\varepsilon}$. In fact $g(-1, 0) \neq f(a_2)$, $g(-1, 0) \leq f(a_2)$ and $g_1(-1, 0) = -1 < -0.9 = g_1(a_2) - \varepsilon$, therefore $a_2 \notin \mathcal{P}^{3,\varepsilon}$.

The ε -Pareto fronts of this example, for the three different concepts of approximate solutions (Definition 2.3), are represented in red in Figure 2, for $\varepsilon = 0.1$. In Figure 2a the black dashed lines represents that those lines are not in $\mathcal{F}^{1,\varepsilon}$.

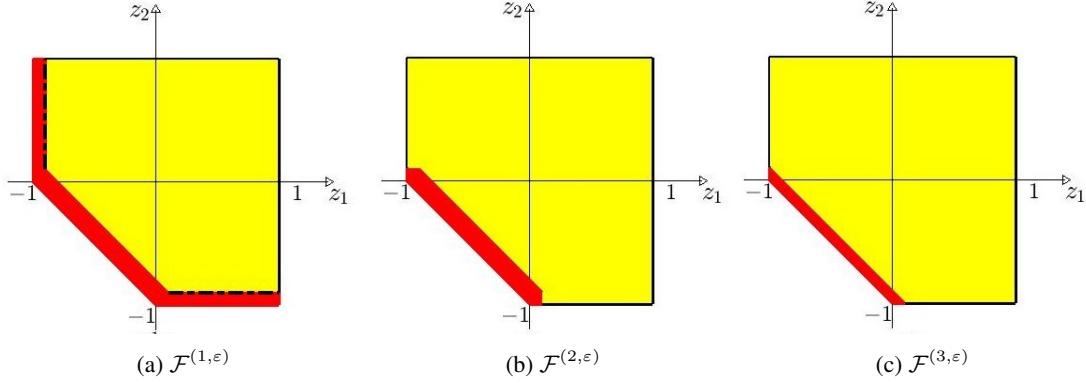


Figure 2: ε -Pareto fronts for Problem 3

Remark 2.5. We stress on that the inclusions in (2) are true only when the feasible set X is a compact subset of \mathbb{R}^N . However if the feasible set it is not closed, there is no guarantee of the existence of (weak) Pareto solutions. Actually, the (weak) Pareto set can be empty. Consider, for example, the problem of

$$\begin{cases} \text{Minimize } g(x) = (x_1, x_2) \\ \text{subject to: } x \in X = (-1, 1)^2 \setminus \{x \in (-1, 0)^2 : x_1 + x_2 \leq -1\}. \end{cases} \quad (4)$$

In this case the feasible set is open, see Figure 3a, where the black dashed lines represents that those lines are not in the set. Then the (weak) Pareto set and the the set $\mathcal{P}^{2,w}$ are empty. However the sets $\mathcal{P}^{1,\varepsilon}$ and $\mathcal{P}^{3,\varepsilon}$ are not empty and can be seen in red in Figures 3b and 3c, respectively.

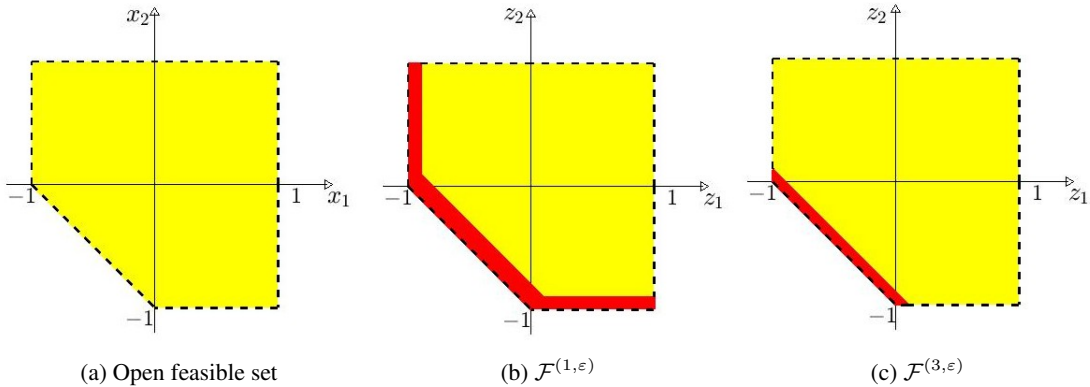


Figure 3: Feasible set and ε -Pareto fronts for Problem 4

The concept of approximate solutions, as defined in 2.3 (i) is extended for weak ε -Pareto solutions in [27].

Definition 2.6. [weak ε -Pareto solution] Let $\varepsilon \geq 0$. We define the following set of ε -Pareto solutions:

$$(i) \mathcal{P}_w^{1,\varepsilon} = \{x \in X : \text{there is no } y \in X \text{ such that } g(y) < g(x) - \varepsilon 1\}.$$

Motivated by the case of Remark 2.5, we are going to consider a problem of minimize the objective functions over the closure of the feasible set X and prove some interesting relations with the (weak) Pareto solutions of this problem 5, and the (weak) ε -Pareto solutions of the original problem P.

$$\begin{cases} \text{Minimize } g(x) = (g_1(x), g_2(x)) \\ \text{subject to } x \in \bar{X} \end{cases} \quad (5)$$

We are going to denote the Pareto set and the weak Pareto set of problem (5) by $\mathcal{P}^\#$ and $\mathcal{P}_w^\#$, respectively.

The following proposition states a relation between the (weak) Pareto optimal solutions of problem (5) and (weak) ε -Pareto optimal solutions of the original problem (P).

Theorem 2.1. Assume that f_i are Lipschitz continuous functions, with Lipschitz constant L_i , $i = 1, 2$.

(i) For any $x^* \in \mathcal{P}^\#$ and for any $\varepsilon > 0$ define $\delta = \min_i(\varepsilon/L_i)$, then there exists $x \in \mathcal{P}^{1,\varepsilon}$ such that

$$|x^* - x| \leq \delta \quad \text{and} \quad |g(x^*) - g(x)| \leq \varepsilon.$$

(ii) For any $x^* \in \mathcal{P}_w^\#$ and for any $\varepsilon > 0$ define $\delta = \min_i(\varepsilon/L_i)$ the there exists $x \in \mathcal{P}_w^{1,\varepsilon}$ such that

$$|x^* - x| \leq \delta \quad \text{and} \quad |g(x^*) - g(x)| \leq \varepsilon.$$

(iii) Given $\varepsilon > 0$, for any $x^* \in \mathcal{P}^{3,\varepsilon}$ there exists $x \in \mathcal{P}^\#$ such that $|g(x) - g(x^*)| \leq 2\varepsilon$.

Proof. (i) Let $x^* \in \mathcal{P}^\#$ and $\varepsilon > 0$. As $x^* \in \bar{X}$, there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Then, given $\varepsilon > 0$, define $\delta = \min_i(\varepsilon/L_i)$, so there exists x_N such that $|x^* - x_N| \leq \delta$. By the Lipschitz continuity of functions g_i , $i = 1, 2$ we obtain that

$$|g_i(x^*) - g_i(x_N)| \leq L_i|x^* - x_N| \leq L_i\delta \leq \varepsilon,$$

which means that $|g(x^*) - g(x_N)| \leq \varepsilon_i$. It remains to prove that $x_N \in \mathcal{P}^{1,\varepsilon}$. Note that, by definition of x^* there is no $x \in X$ such that $g_i(x) \leq g_i(x^*)$, for $i = 1, 2$. This implies that for any $x \in X$ at least for one $i = 1, 2$ the following assertion holds:

$$g_1(x) > g_1(x^*) > g_1(x_N) - \varepsilon \quad \text{or} \quad g_2(x) > g_2(x^*) > g_2(x_N) - \varepsilon$$

Therefore, there is no $x \in X$ such that $g_i(x) \leq g_i(x_N) - \varepsilon$ for $i = 1, 2$, which means that $x_N \in \mathcal{P}^{1,\varepsilon}$ and the assertion is now proved with $x = x_N$.

(ii) Let $x^* \in \mathcal{P}_w^\#$ and $\varepsilon > 0$. As $x^* \in \bar{X}$, there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Then, given $\varepsilon > 0$, define $\delta = \min_i(\varepsilon/L_i)$, so there exists x_N such that such that $|x^* - x_N| \leq \delta$. By the Lipschitz continuity of functions f_i , $i = 1, 2$ we obtain that

$$|g_i(x^*) - g_i(x_N)| \leq L_i|x^* - x_N| \leq L_i\delta \leq \varepsilon,$$

which means that $|g(x^*) - g(x_N)| \leq \varepsilon$. It remains to prove that $x_N \in \mathcal{P}_w^{1,\varepsilon}$. Note that, by definition of x^* there is no $x \in X$ such that $g_i(x) < g_i(x^*)$, for $i = 1, 2$. This implies that for any $x \in X$ at least for one $i = 1, 2$ the following assertion holds:

$$g_i(x) \geq g_1(x^*) > g_1(x_N) - \varepsilon \quad \text{or} \quad g_2(x) \geq g_2(x^*) > g_2(x_N) - \varepsilon.$$

Therefore, there is no $x \in X$ such that $g_i(x) < g_i(x_N) - \varepsilon$ for $i = 1, 2$, which means that $x_N \in \mathcal{P}_w^{1,\varepsilon}$ and the assertion is now proved with $x = x_N$.

(iii) Let $x^* \in \mathcal{P}^{3,\varepsilon}$. Assume that there is no $x \in \mathcal{P}^\#$ such that $|g(x) - g(x^*)| < 2\varepsilon$. Then $x^* \notin \mathcal{P}^\#$, so there exists $y \in \mathcal{P}^\#$ such that $g(y) \neq g(x^*)$ and $g(y) \leq g(x^*)$. As $y \in \bar{X}$, there exists a sequence $\{y_n\} \subset X$ such that $\lim_{n \rightarrow \infty} y_n = y$. Choose y_N such that $g(y_N) \leq g(x^*)$ and $|g(y) - g(y_N)| \leq \varepsilon$. As $x^* \in \mathcal{P}^{3,\varepsilon}$, we must have $g(y_N) \geq g(x^*) - \varepsilon$. Hence $g(x^*) - \varepsilon \leq g(y_N) \leq g(x^*)$, which means that $|g(y_N) - g(x^*)| < \varepsilon$. Then

$$|g(y) - g(x^*)| \leq |g(y) - g(y_N)| + |g(y_N) - g(x^*)| \leq 2\varepsilon,$$

what is a contradiction. □

Remark 2.7. *Theorem 2.1 also means that for any $z^* \in \mathcal{F}^\#$ (resp. $z^* \in \mathcal{F}_w^\#$) and for any $\varepsilon > 0$, there exists $z \in \mathcal{F}^{1,\varepsilon}$ (resp. $z \in \mathcal{F}_w^{1,\varepsilon}$) such that $|z^* - z| \leq \varepsilon$.*

3 Problem statement

Let U be a given compact non-empty subset of \mathbb{R}^M (for $m \geq 1$). A measurable function $\mathbf{u} : [0, +\infty[\rightarrow \mathbb{R}^m$ is said admissible if it satisfies $\mathbf{u}(s) \in U$ for almost every $s \geq 0$. The set of all admissible controls will be denoted by \mathcal{U} :

$$\mathcal{U} = \left\{ \mathbf{u} : [0, +\infty) \rightarrow \mathbb{R}^M \text{ measurable, } \mathbf{u}(s) \in U \text{ a.e.} \right\}.$$

Let $T > 0$ be a fixed finite horizon, and consider the dynamical system:

$$\begin{cases} \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)) & s \geq 0, \\ \mathbf{y}(t) = x. \end{cases} \quad (6)$$

The dynamics f satisfies the following hypothesis:

(H₁) $f : \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ is continuous. For any $R > 0$, $\exists L_f(R) > 0$ such that for every $u \in U$:

$$|f(x, u) - f(x', u)| \leq L_f(R)(|x - x'|) \quad \forall x, x' \in \mathbb{R}^N \text{ with } |x| \leq R, |x'| \leq R.$$

Moreover, there exists $c_f > 0$ such that for any $x \in \mathbb{R}^N$ we have: $\max\{|f(x, u)| : u \in U\} \leq c_f(1 + |x|)$.

By assumption (H₁), for any control input $u \in \mathcal{U}$, the system (6) admits a unique absolutely continuous solution \mathbf{y}_x^u in $W^{1,1}([t, T]; \mathbb{R}^N)$. For every $x \in \mathbb{R}^N$ and $0 \leq t \leq T$, we define the set, $\mathbb{X}_{t,x} \subset W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U}$ as:

$$\mathbb{X}_{t,x} = \{(\mathbf{y}, \mathbf{u}) : \dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)), \text{ for a.e. } s \in [t, T]; \quad \mathbf{y}(t) = x \text{ and } \mathbf{u} \in \mathcal{U}\}.$$

Now, consider the final cost function $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ and the running cost $\ell : \mathbb{R}^N \times U \rightarrow \mathbb{R}$ satisfying:

(H₂) The function φ is locally Lipschitz continuous on \mathbb{R}^N : for every $R > 0$,

$$\exists L_\varphi(R) > 0 \quad |\varphi_i(x) - \varphi_i(x')| \leq L_\varphi(R)|x - x'| \quad \forall x, x' \in \mathbb{R}^N \text{ with } |x| \leq R, |x'| \leq R.$$

Moreover there exists $c_\varphi > 0$ and $\lambda_\varphi \geq 1$ such that $\varphi(x) \leq c_\varphi(x)(1 + |x|^{\lambda_\varphi})$ for every $x \in \mathbb{R}^N$.

(H₃) The function ℓ is continuous on $\mathbb{R}^N \times U$ and is locally Lipschitz continuous on the first variable uniformly with respect to the second argument: for every $R > 0$,

$$\exists L_\ell(R) > 0 \quad |\ell(x, u) - \ell(x', u)| \leq L_\ell(R)|x - x'| \quad \forall |x| \leq R, |x'| \leq R, \forall u \in U,$$

and there exists $c_\ell > 0$ and $\lambda_\ell \geq 1$ such that $\max\{|\ell(x, u)|, u \in \mathbb{R}\} \leq c_\ell(1 + |x|^{\lambda_\ell})$.

For $x \in \mathbb{R}^N$ and $0 \leq t \leq T$, the objective function in Bolza form $\Phi(t, x; \cdot, \cdot) : W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U} \mapsto \mathbb{R}$ is defined as

$$\Phi(t, x; \cdot, \cdot) : W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U} \longrightarrow \mathbb{R}, \quad \Phi(t, x; \mathbf{y}, \mathbf{u}) = \varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s)) ds. \quad (7)$$

Assume we are interested by another cost function that is measured all along the trajectory by:

$$\Psi(t, x; \cdot, \cdot) : W^{1,1}([t, T]; \mathbb{R}^N) \times \mathcal{U} \longrightarrow \mathbb{R}, \quad \Psi(t, x; \mathbf{y}, \mathbf{u}) = \max_{s \in [t, T]} \psi(\mathbf{y}(s)), \quad (8)$$

where the function ψ satisfies:

(H₄) $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is Lipschitz continuous.

Now, the multi-objective optimal control problem that will be investigated in this paper is the following:

$$\begin{cases} \inf (\Phi(t, x; \mathbf{y}, \mathbf{u}), \Psi(t, x; \mathbf{y}, \mathbf{u})) \\ \text{s.t } (\mathbf{y}, \mathbf{u}) \in \mathbb{X}_{t,x}. \end{cases} \quad (\text{MOP})$$

An example of this kind of problem is to consider the pest control problem, that consider the joint evolution of two species: a nuisance for humans and its predator. The population of these species are the state variables while the control variable is the rate at which a chemical is sprayed to poison the pest (which also kills the predator). The growth of both populations represents the dynamics. The goal is to maintain a certain proportion of both species along the time and also reduce the cost of spraying the chemical. The integral criterion is suitable for the spraying cost and the other objective is naturally defined as a max-type criterion.

Define the set-valued function

$$G(x) = \left\{ \begin{pmatrix} f(x, u) \\ -\ell(x, u) - a \end{pmatrix}, 0 \leq a \leq \mathcal{A}(x, u), u \in U \right\},$$

where $\mathcal{A}(x, u) = c_\ell(1 + |x|^{\lambda_\ell}) - \ell(x, u)$. Under assumptions (H₁) and (H₃) the function G is locally Lipschitz continuous in the sense that, for any $R > 0$, there exists $L_G(R) > 0$ such that:

$$G(x') \subset G(x) + |x - x'|B(0, L_R) \quad \forall x, x' \in \mathbb{R}^N \text{ with } |x| \leq R, |x'| \leq R.$$

We also define the following set of trajectories:

$$\mathcal{S}_{[t, T]}(x, 0) = \{(\mathbf{y}, \mathbf{z}) : (\dot{\mathbf{y}}(s), \dot{\mathbf{z}}(s))^\top \in G(\mathbf{y}(s)), \text{ for a.e. } s \in [t, T]; \quad (\mathbf{y}(t), \mathbf{z}(t)) = (x, 0)\},$$

and the bi-objective optimal control problem:

$$\begin{cases} \inf \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \\ \text{s.t } (\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0). \end{cases} \quad (9)$$

Remark 3.1. From definition of the differential inclusion (3), every trajectory $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0)$ is associated to a control function $\mathbf{u} \in \mathcal{U}$ and $\gamma \in \Gamma(\mathbf{y}(s), \mathbf{u}(s))$, where

$$\Gamma(x, u) = \{\gamma : [0, +\infty) \rightarrow \mathbb{R} \text{ measurable}, \gamma(s) \in [0, \mathcal{A}(x, u)]\}.$$

Let $(\mathbf{y}^*, \mathbf{z}^*)$ a Pareto optimal solution of (9) and \mathbf{u}^* , γ^* the respective controls. Then there exist no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right), \quad (10)$$

$$\text{and } \left[\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \quad \text{or} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) < \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right] \quad (11)$$

Note that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = \varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s))ds + \int_t^T \gamma(s)ds$. As $0 \leq \gamma(s) \leq \Gamma(\mathbf{y}(s), \mathbf{u}(s))$, for $s \geq 0$, and the second objective function of (9) does not depend on control γ we obtain that for all $(\mathbf{y}^*, \mathbf{z}^*)$ that is a Pareto optimal solution of (9) the control function $\gamma^* \equiv 0$. Then by (10), we obtain that there is no $(\mathbf{y}, \mathbf{u}) \in \mathbb{X}_{t,x}$ such that

$$\left(\varphi(\mathbf{y}(T)) + \int_t^T \ell(\mathbf{y}_x(s), \mathbf{u}(s))ds, \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \leq \left(\varphi(\mathbf{y}^*(T)) + \int_t^T \ell(\mathbf{y}^*(s), \mathbf{u}^*(s))ds, \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right), \quad (12)$$

$$\text{and } \left[\varphi(\mathbf{y}^*(T)) + \int_t^T \ell(\mathbf{y}(s), \mathbf{u}(s))ds < \varphi(\mathbf{y}^*(T)) + \int_t^T \ell(\mathbf{y}^*(s), \mathbf{u}^*(s))ds \text{ or } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) < \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right], \quad (13)$$

which means that $(\mathbf{y}^*, \mathbf{u}^*)$ is a Pareto optimal solution for problem (MOP). Moreover as $\eta^* \equiv 0$ the Pareto front for problems (MOP) and (9) are equal. With similar arguments it is possible to prove that if $(\mathbf{y}^*, \mathbf{z}^*)$ is a weak Pareto optimal solution of (9) with respective controls \mathbf{u}^* , and γ^* then $(\mathbf{y}^*, \mathbf{u}^*)$ is a weak Pareto optimal solution for problem (MOP). Moreover the weak Pareto front of two problems coincides.

Without any additional assumption it is not possible to guarantee that the set of trajectories $\mathcal{S}_{[t, T]}(x, 0)$ is closed, so the problem (9) might not have a solution and the Pareto front can not be defined for this problem. One approach to obtain the closeness of the set of trajectories $\mathcal{S}_{[t, T]}(x, z)$, is to introduce a convexified (relaxed) dynamical system [5, 18], whose set of solutions is given by:

$$\mathcal{S}_{[t, T]}^\#(x, 0) = \{(\mathbf{y}, \mathbf{z}) : (\dot{\mathbf{y}}(s), \dot{\mathbf{z}}(s))^\top \in \overline{\text{co}}(G(\mathbf{y}(s))), \text{ for a.e. } s \in [t, T]; \quad (\mathbf{y}(t), \mathbf{z}(t)) = (x, z)\},$$

where $\overline{\text{co}}(S)$ denotes the closed convex hull of the subset S , that is the minimal convex set that contains S .

Under assumptions (\mathbf{H}_1) and (\mathbf{H}_3) , following the same arguments of the proof of Filippov-Wazewski Theorem (see for instance [5]), the closure of $\mathcal{S}_{[t, T]}(x, z)$ in the space of continuous functions $C(t, T)$ is compact and equal to the set of solutions $\mathcal{S}_{[t, T]}^\#(x, z)$. So we introduce the following convexified bi-objective optimal control problem

$$\begin{cases} \min \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \\ \text{s.t } (\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0). \end{cases} \quad (\text{MORP})$$

In the following, for a fixed $(t, x) \in [0, T] \times \mathbb{R}^N$, we are going to denote the Pareto front and the weak Pareto front of bi-objective optimal control problem (9) by $\mathcal{F}(t, x)$ and $\mathcal{F}_w(t, x)$, respectively. We also denote by $\mathcal{P}^{i, \varepsilon}(t, x)$ and $\mathcal{P}_w^{i, \varepsilon}(t, x)$, $i = 1, 2, 3$, the set of (weak) approximate Pareto solutions of problem (9), according with definitions 2.3 and 2.6, respectively. Moreover we denote the Pareto front and the weak Pareto front of problem (MORP) by $\mathcal{F}^\#(t, x)$ and $\mathcal{F}_w^\#(t, x)$, respectively. The next proposition states that all optimal solution of problem (MORP) can be approximated with an ε -optimal solution of problem (9).

Theorem 3.1. Assume that (\mathbf{H}_1) - (\mathbf{H}_4) hold and let $(t, x) \in [0, T] \times \mathbb{R}^N$.

- (i) For any $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^\#(t, x)$ and for any $\varepsilon > 0$ define $R = |(\mathbf{y}^*, \mathbf{z}^*)| + \varepsilon$, then for $\delta = \min(\varepsilon/(L_\varphi(R) + 1), \varepsilon/L_\psi)$ there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{P}^{1, \varepsilon}(t, x)$ such that $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y}, \mathbf{z})| \leq \delta$ and

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \right| \leq \varepsilon.$$

- (ii) For any $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}_w^\#(t, x)$ and for any $\varepsilon > 0$ define $R = |(\mathbf{y}^*, \mathbf{z}^*)| + \varepsilon$, then for $\delta = \min(\varepsilon/(L_\varphi(R) + 1), \varepsilon/L_\psi)$ there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$ such that $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y}, \mathbf{z})| \leq \delta$ and

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \right| \leq \varepsilon.$$

(iii) Given $\varepsilon > 0$, if $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t, x)$, then there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{P}^\#(t, x)$ such that

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \right| \leq 2\varepsilon.$$

Proof. (i) Let $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^\#(t, x)$ and $\varepsilon > 0$. As $(\mathbf{y}^*, \mathbf{z}^*)$ is in the closure of $\mathcal{S}_{[t, T]}(x, 0)$, there exists a sequence $\{(\mathbf{y}_n, \mathbf{z}_n)\} \subset \mathcal{S}_{[t, T]}(x, 0)$ such that

$$\lim_{n \rightarrow \infty} (\mathbf{y}_n, \mathbf{z}_n) = (\mathbf{y}^*, \mathbf{z}^*).$$

Then, given $\varepsilon > 0$, for $R = |(\mathbf{y}^*, \mathbf{z}^*)| + \varepsilon$ define $\delta = \min(\varepsilon/(L_\varphi(R) + 1), \varepsilon/L_\psi)$, so there exists $(\mathbf{y}_N, \mathbf{z}_N)$ such that $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y}_N, \mathbf{z}_N)| \leq \delta$. Moreover, we have that $|(\mathbf{y}^*, \mathbf{z}^*)| \leq R$ and $|(\mathbf{y}_N, \mathbf{z}_N)| \leq R$. By the locally Lipschitz continuity of function φ and the globally Lipschitz continuity of ψ , we obtain that

$$|\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) - \varphi(\mathbf{y}_N(T)) + \mathbf{z}_N(T)| \leq |\varphi(\mathbf{y}^*(T)) - \varphi(\mathbf{y}_N(T))| + |\mathbf{z}^*(T) - \mathbf{z}_N(T)| \leq (L_\varphi(R) + 1)\delta \leq \varepsilon,$$

$$\left| \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) - \max_{s \in [t, T]} \psi(\mathbf{y}_N(s)) \right| \leq \max_{s \in [t, T]} |\psi(\mathbf{y}^*(s)) - \psi(\mathbf{y}_N(s))| \leq L_\psi \delta \leq \varepsilon,$$

which means that

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t, T]} \psi(\mathbf{y}_N(s)) \right) \right| \leq \varepsilon.$$

It remains to prove that $(\mathbf{y}_N, \mathbf{z}_N) \in \mathcal{P}^{1,\varepsilon}(t, x)$. Note that, by definition of $(\mathbf{y}^*, \mathbf{z}^*)$ there is no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0) \subset \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\begin{aligned} & \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right), \\ \text{and } & \left[\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \quad \text{or} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) < \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right]. \end{aligned}$$

This means that for any $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0)$ at least for one of the following assertion holds:

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) > \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \geq \varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T) - \varepsilon, \text{ or}$$

$$\max_{s \in [t, T]} \psi(\mathbf{y}(s)) > \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \geq \max_{s \in [t, T]} \psi(\mathbf{y}_N(s)) - \varepsilon.$$

Therefore, there is no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0)$ such that

$$\begin{aligned} & \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) \leq \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t, T]} \psi(\mathbf{y}_N(s)) \right) - (\varepsilon, \varepsilon), \\ \text{and } & \left[\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T) - \varepsilon \quad \text{or} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) < \max_{s \in [t, T]} \psi(\mathbf{y}_N(s)) - \varepsilon \right], \end{aligned}$$

which means that $(\mathbf{y}_N, \mathbf{z}_N) \in \mathcal{P}^{1,\varepsilon}(t, x)$ and the assertion is now proved for $(\mathbf{y}, \mathbf{z}) = (\mathbf{y}_N, \mathbf{z}_N)$.

(ii) Let $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}_w^\#(t, x)$ and $\varepsilon > 0$. Given $\varepsilon > 0$, for $R = |(\mathbf{y}^*, \mathbf{z}^*)| + \varepsilon_1$ define $\delta = \max(\varepsilon/(L_\varphi(R) + 1), \varepsilon/L_\psi)$. With similar arguments as in item (i) we obtain $(\mathbf{y}_N, \mathbf{z}_N) \in \mathcal{S}_{[t, T]}(x, 0)$ such that $|(\mathbf{y}^*, \mathbf{z}^*) - (\mathbf{y}_N, \mathbf{z}_N)| \leq \delta$ and

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t, T]} \psi(\mathbf{y}_N(s)) \right) \right| \leq \varepsilon.$$

It remains to prove that $(\mathbf{y}_N, \mathbf{z}_N) \in \mathcal{P}_w^{1,\varepsilon}(t, x)$. Note that, by definition of $(\mathbf{y}^*, \mathbf{z}^*)$ there is no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0) \subset \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) < \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t, T]} \psi(\mathbf{y}^*(s)) \right).$$

This implies that for any $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t,T]}(x, 0)$ at least for one of the following assertion holds:

$$\begin{aligned} \varphi(\mathbf{y}(T)) - \mathbf{z}(T) &\geq \varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T) \geq \varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T) - \varepsilon, \text{ or} \\ \max_{s \in [t,T]} \psi(\mathbf{y}(s)) &\geq \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \geq \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) - \varepsilon. \end{aligned}$$

Therefore, there is no $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t,T]}(x, 0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) < \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) - (\varepsilon, \varepsilon),$$

which means that $(\mathbf{y}_N, \mathbf{z}_N) \in \mathcal{P}_w^{1,\varepsilon}(t, x)$ and the assertion is now proved for $(\mathbf{y}, \mathbf{z}) = (\mathbf{y}_N, \mathbf{z}_N)$.

(iii) Let $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t, x)$. Assume that there is no $(\mathbf{y}, \mathbf{z}) \in \mathcal{P}^\#(t, x)$ such that

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) \right| \leq 2\varepsilon.$$

Then $(\mathbf{y}^*, \mathbf{z}^*) \notin \mathcal{P}^\#(t, x)$, so there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{P}^\#(t, x)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) \neq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) \quad \text{and}$$

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) \leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right),$$

As (\mathbf{y}, \mathbf{z}) is in the closure of $\mathcal{S}_{[t,T]}(x, 0)$, there exists a sequence $\{(\mathbf{y}_n, \mathbf{z}_n)\} \subset \mathcal{S}_{[t,T]}(x, 0)$ such that $\lim_{n \rightarrow \infty} (\mathbf{y}_n, \mathbf{z}_n) = (\mathbf{y}, \mathbf{z})$. Choose $(\mathbf{y}_N, \mathbf{z}_N)$ such that

$$\begin{aligned} \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) &\leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right), \quad \text{and} \\ \left| \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) \right| &\leq \varepsilon. \end{aligned}$$

As $(\mathbf{y}^*, \mathbf{z}^*) \in \mathcal{P}^{3,\varepsilon}(t, x)$, we must have

$$\left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) \geq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) - (\varepsilon, \varepsilon).$$

Hence

$$\begin{aligned} \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) - (\varepsilon, \varepsilon) &\leq \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) \\ &\leq \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right), \end{aligned}$$

which means that

$$\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) \right| \leq \varepsilon.$$

Then

$$\begin{aligned} &\left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) \right| \\ &\leq \left| \left(\varphi(\mathbf{y}^*(T)) - \mathbf{z}^*(T), \max_{s \in [t,T]} \psi(\mathbf{y}^*(s)) \right) - \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) \right| \\ &\quad + \left| \left(\varphi(\mathbf{y}_N(T)) - \mathbf{z}_N(T), \max_{s \in [t,T]} \psi(\mathbf{y}_N(s)) \right) - \left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t,T]} \psi(\mathbf{y}(s)) \right) \right| \\ &\leq 2\varepsilon. \end{aligned}$$

what is a contradiction. □

4 Auxiliary control problem

In what follows we consider the single objective optimal control problems and the associated value functions:

$$\vartheta_1(t, x) = \min_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \varphi(\mathbf{y}(T)) - \mathbf{z}(T), \quad \vartheta_2(t, x) = \min_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \max_{s \in [t, T]} \psi(\mathbf{y}(s)). \quad (14)$$

Consider the following augmented differential inclusion

$$\widehat{G}(x) = \left\{ \begin{pmatrix} G(x) \\ 0 \end{pmatrix} \right\},$$

and for $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^2$ define the set of trajectories

$$\widehat{\mathcal{S}}_{[t, T]}^\#(x, z) = \{\widehat{\mathbf{y}} = (\mathbf{y}, \mathbf{z}, \mathbf{b}) : \dot{\widehat{\mathbf{y}}}(s) \in \overline{\text{co}}(\widehat{G}(\mathbf{y}(s))), \text{ for a.e. } s \in [t, T]; \widehat{\mathbf{y}}(t) = (x, z)\}.$$

Based on [15], we introduce an auxiliary control problem and its value function $w : [0, T] \times \mathbb{R}^N \times \mathbb{R}^2 \rightarrow \mathbb{R}$. We shall see later that the 0-level set of the auxiliary value function w is very useful for characterizing the Pareto fronts of the problem (MORP).

$$w(t, x, z_1, z_2) = \min_{\widehat{\mathbf{y}} \in \widehat{\mathcal{S}}_{[t, T]}^\#(x, z)} \left[\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \right) \bigvee \max_{s \in [t, T]} (\psi(\mathbf{y}(s)) - z_2) \right], \quad (15)$$

where the notation $a \bigvee b$ stands for $\max(a, b)$. Let us point out that the additional state components are very important to get a Dynamical Programming Principle for the value function w . Moreover, we note that under assumptions (\mathbf{H}_1) - (\mathbf{H}_4) , there exists an admissible pair $(\mathbf{y}, \mathbf{z}, \mathbf{b}) \in \widehat{\mathcal{S}}_{[t, T]}^\#(\mathbf{x}, \mathbf{z})$ that minimizes the auxiliary control problem (15).

Let us remark that from the definition of differential inclusion (3) and from the definition of w , it follows that:

$$w(t, x, z_1, z_2) = \inf_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1 \right) \bigvee \max_{s \in [t, T]} (\psi(\mathbf{y}(s)) - z_2) \right]. \quad (16)$$

Following [15], the value function w satisfies the following property.

Theorem 4.1. *Assume that (\mathbf{H}_1) - (\mathbf{H}_4) hold. The value function w is locally Lipschitz continuous. Moreover, the function w is the unique viscosity solution to the following Hamilton-Jacobi-Bellman equation:*

$$\begin{aligned} \min \left(\partial_t w(t, x, z) + \mathcal{H}^\#(x, D_x w, D_z w), w(t, x, z) - (g(x) - z_2) \right) &= 0 \quad \text{for } t \in [0, T], x \in \mathbb{R}^N, z \in \mathbb{R}^2 \\ w(T, x, z) &= \left(\varphi(x) - z_1 \right) \bigvee \left(g(x) - z_2 \right) \quad \text{for } x \in \mathbb{R}^N, z \in \mathbb{R}^2, \end{aligned}$$

where the function $\mathcal{H}^\#$ is defined by:

$$\mathcal{H}^\#(x, p, q) = \max_{(v_x, v_z) \in \overline{\text{co}}(G(x))} \left(-v_x \cdot p - v_z(t) \cdot q_1 \right),$$

for $p \in \mathbb{R}^N$ and $q \in \mathbb{R}^2$.

Proof. The locally Lipschitz continuity can be obtained by arguments as in [6]. The value function w satisfies the following Dynamical Programming Principle that hold for all $h \geq 0$ such that $t + h < T$ and $(x, z) \in \mathbb{R}^{n+2}$

$$w(t, x, z_1, z_2) = \inf_{\widehat{\mathbf{y}} \in \widehat{\mathcal{S}}_{[t, T]}^\#(x, z)} \left\{ w(t + h, \mathbf{y}(t + h), \mathbf{z}(t + h), z_2) \bigvee \max_{s \in [t, t+h]} (g(\mathbf{y}(s)) - z_2) \right\}.$$

So we can derive the HJB equation

$$\begin{aligned} \min \left(\partial_t w(t, x, z) + \mathcal{H}^\#(x, D_x w, D_z w), w(t, x, z) - (g(x) - z_2) \right) &= 0 \quad \text{for } t \in [0, T], x \in \mathbb{R}^N, z \in \mathbb{R}^2 \\ w(T, x, z) &= \left(\varphi(x) - z_1 \right) \bigvee \left(g(x) - z_2 \right) \quad \text{for } x \in \mathbb{R}^N, z \in \mathbb{R}^2, \end{aligned}$$

The uniqueness result is a consequence of [3, Theorem A.1]. \square

Another interesting property is that w is monotone with respect to the third argument as proved in the following proposition.

Proposition 4.1. *Assume that (\mathbf{H}_1) - (\mathbf{H}_4) hold and let $(t, x) \in [0, T] \times \mathbb{R}^N$. Then*

$$\forall z, z' \in \mathbb{R}^2, \left(z \leq z' \Rightarrow w(t, x, z) \geq w(t, x, z') \right).$$

Proof. Let $z, z' \in \mathbb{R}^2$ such that $z \leq z'$ and one admissible trajectory $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$. Then

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z'_1 \leq \varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z'_2 \leq \max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2,$$

and then

$$(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z'_1) \bigvee \left(\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z'_2 \right) \leq (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee \left(\max_{s \in [t, T]} \psi(\mathbf{y}(s))(t, x; \mathbf{y}, \mathbf{u}) - z_2 \right).$$

Taking the minimum over all $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ it follows from the last inequality that $w(t, x, z') \leq w(t, x, z)$. \square

Remark 4.2. *If we consider the problem*

$$\tilde{w}(t, x, z_1, z_2) = \inf_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee \max_{s \in [t, T]} (\psi(\mathbf{y}(s)) - z_2) \right]. \quad (17)$$

with similar arguments of the proof of Theorem 4.1 the function \tilde{w} can be characterized as the unique viscosity solution of the following HJB equation:

$$\begin{aligned} \min \left(\partial_t w(t, x, z) + \mathcal{H}(x, D_x w, D_z w), w(t, x, z) - (g(x) - z_2) \right) &= 0 \quad \text{for } t \in [0, T], x \in \mathbb{R}^N, z \in \mathbb{R}^2 \\ w(T, x, z) &= (\varphi(x) - z_1) \bigvee (g(x) - z_2) \quad \text{for } x \in \mathbb{R}^N, z \in \mathbb{R}^2, \end{aligned}$$

where the Hamiltonian \mathcal{H} is given by

$$\mathcal{H}(x, p, q) = \max_{u \in U} \left(-f(x, u) \cdot p + \ell(x, u) \cdot q_1 \right),$$

for $p \in \mathbb{R}^N$ and $q \in \mathbb{R}^2$. Moreover, it is possible to prove that for fixed $(t, x, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^2$, the minimum of problem (15) is equal the infimum of problem (17).

5 Characterization of the Pareto fronts of the convexified bi-objective optimal control problem

For every $t \in [0, T]$, $x \in \mathbb{R}^N$ and $i = 1, 2$, we introduce the value:

$$z_i^*(t, x) = \inf \left\{ \zeta \in \mathbb{R} \mid \exists z \in \mathbb{R}^2 \text{ with } z_i = \zeta, w(t, x, z) \leq 0 \right\}. \quad (18)$$

Proposition 5.1. *Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let $(t, x) \in [0, T] \times \mathbb{R}^N$.*

(i) For every $z \in \mathbb{R}^2$, we have that $w(t, x, z) \leq 0$ if and only if there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that:

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \leq z_1, \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2. \quad (19)$$

(ii) Moreover, for $i = 1, 2$ and every $(t, x) \in [0, T] \times \mathbb{R}^N$, we have $z_i^*(t, x) = \vartheta_i(t, x)$.

Proof. Assertion (i) follows directly from (16): Let $z \in \mathbb{R}^2$, then

$$w(t, x, z) \leq 0 \Leftrightarrow \exists (\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0) \text{ s.t. } \varphi(\mathbf{y}(T)) - \mathbf{z}(T) \leq z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2.$$

(ii) Let show that $\vartheta_i(t, x) \leq z_i^*(t, x)$. By item (i) we have that for all $z \in \mathbb{R}^2$ such that $w(t, x, z) \leq 0$

$$\exists (\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0) \text{ s.t. } \varphi(\mathbf{y}(T)) - \mathbf{z}(T) \leq z_1 \text{ and } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2.$$

Therefore $\vartheta_i(t, x) \leq z_i$ for all $z \in \mathbb{R}^2$ such that $w(t, x, z) \leq 0$ and then

$$\vartheta_i(t, x) \leq \inf \{ \gamma \in \mathbb{R} \mid z \in \mathbb{R}^2, w(t, x, z) \leq 0 \text{ with } z_i = \gamma \} = z_i^*(t, x).$$

Let show now that $\vartheta_i(t, x) \geq z_i^*(t, x)$. Without loss of generality, we assume here that $i = 1$. The proof will be the same for $i = 2$. Assume that $\vartheta_1(t, x) < z_1^*(t, x)$. Then there exists $\delta \in \mathbb{R}$ such that $\vartheta_1(t, x) < \delta < z_1^*(t, x)$. The inequality $\vartheta_1(t, x) < \delta$ implies that there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \delta$. Then for $z_2 = \max_{s \in [t, T]} \psi(\mathbf{y}(s))$ we have that,

$$w(t, x, \delta, z_2) \leq (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - \delta) \bigvee \left(\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2 \right) = 0,$$

which implies that $\delta \in \{ \gamma \in \mathbb{R} \mid \exists z \in \mathbb{R}^2 \text{ with } z_1 = \gamma, w(t, x, z) \leq 0 \}$. But, we have chosen δ such that $\delta < z_1^*(t, x)$ which is impossible. \square

Let us also denote

$$\beta^*(t, x) = (z_1^*(t, x), z_2^*(t, x)) \in \mathbb{R}^2. \quad (20)$$

It follows from the proposition 5.1 that β^* is the utopian point associated with the bi-objective control problem for a given $(t, x) \in [0, T] \times \mathbb{R}^N$. If this point is feasible (i.e., there is an admissible pair $(\mathbf{y}_x, \mathbf{z}_0)$ that realizes the minimum of both cost functions $\varphi(\mathbf{y}_x(T)) - \mathbf{z}_0(T)$ and $\max_{s \in [t, T]} \psi(\mathbf{y}_x(s))$), then the Pareto front is reduced to this point. In what follows it is assumed that the utopian point is not feasible. In this case, we have:

$$w(t, x, \beta^*(t, x)) > 0. \quad (21)$$

In the following theorem, we give the first link between the solutions of the multi-objective problem (MORP) and the function w .

Theorem 5.2. Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in $[0, T] \times \mathbb{R}^N$. Then the weak Pareto front $\mathcal{F}_w^\#(t, x)$ for the bi-objective optimal control problem (MORP) with the initial condition (t, x) is a subset of the zero level set of the value function $w(t, x, \cdot, \cdot)$:

$$\mathcal{F}^\#(t, x) \subset \mathcal{F}_w^\#(t, x) \subset \left\{ z \in \mathbb{R}^2 \mid w(t, x, z) = 0 \right\}.$$

Proof. Let $z \in \mathcal{F}_w^\#(t, x)$. Then there exists an admissible pair $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) = z_1, \quad \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) = z_2$$

and there is no other admissible pair that dominates (\mathbf{y}, \mathbf{z}) . This means that for any $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$, one of the following assertions holds:

(i) $z_1 \leq \varphi(\mathbf{y}(T)) - \mathbf{z}(T)$,

(ii) or $z_2 \leq \max_{s \in [t, T]} \psi(\mathbf{y}(s))$.

We can easily check that in the two above cases, we have $(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \vee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \geq 0$.

Therefore,

$$\begin{aligned} w(t, x, z) &= \min_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \vee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \right] \\ &= \left[(\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) - z_1) \vee (\max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) - z_2) \right] = 0. \end{aligned}$$

□

Let $x \in \mathbb{R}^N$ and $t > 0$. We define:

$$\begin{aligned} \bar{z}_1(t, x) &= \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x, \zeta, z_2^*(t, x)) = 0 \right\}, \\ \bar{z}_2(t, x) &= \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x, z_1^*(t, x), \zeta) = 0 \right\}. \end{aligned} \quad (22)$$

By definition, $z_i^*(t, x) \leq \bar{z}_i(t, x) < +\infty$, for $i = 1, 2$. Denote by $(\mathbf{y}_1, \mathbf{z}_1)$ and $(\mathbf{y}_2, \mathbf{z}_2)$ two admissible pairs in $\mathcal{S}_{[t, T]}^\#(x, 0)$ that realize respectively the minimum of $\varphi(\mathbf{y}(T)) - \mathbf{z}(T)$ and $\max_{s \in [t, T]} \psi(\mathbf{y}(s))$:

$$\varphi(\mathbf{y}_1(T)) - \mathbf{z}_1(T) = \vartheta_1(t, x) = z_1^*(t, x), \quad \max_{s \in [t, T]} \psi(\mathbf{y}_2(s)) = \vartheta_2(t, x) = z_2^*(t, x).$$

Then, the values \bar{z}_i can be interpreted as:

$$\bar{z}_1 = \varphi(\mathbf{y}_2(T)) - \mathbf{z}_2(T), \quad \bar{z}_2 = \max_{s \in [t, T]} \psi(\mathbf{y}_1(s)).$$

Let us denote

$$\Omega = [z_1^*(t, x), \bar{z}_1(t, x)] \times [z_2^*(t, x), \bar{z}_2(t, x)]. \quad (23)$$

Theorem 5.3. Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in $[0, T] \times \mathbb{R}^N$. The following assertions hold:

- (i) $\mathcal{F}^\#(t, x) \subset \mathcal{F}_w^\#(t, x) \cap \Omega \subset \{z \in \Omega \mid w(t, x, z) = 0\}$.
- (ii) Let $z \in \Omega$ such that $w(t, x, z) = 0$. If there exists a admissible pair $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = z_1$ and $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) = z_2$, then $z \in \mathcal{F}_w^\#(t, x)$.

Proof. (i) By Theorem 5.2 we obtain immediately that

$$\mathcal{F}_w^\#(t, x) \cap \Omega \subset \{z \in \Omega \mid w(t, x, z) = 0\}.$$

Moreover $\mathcal{F}^\#(t, x) \subset \mathcal{F}_w^\#(t, x)$. It remains to prove that $\mathcal{F}^\#(t, x) \subset \Omega$.

Let $z = (z_1, z_2) \in \mathcal{F}^\#(t, x)$. By Proposition 5.1, $z_i^*(t, x) = \vartheta_i(t, x)$, for $i=1, 2$. Then for every $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ we have

$$z_1^*(t, x) \leq \varphi(\mathbf{y}_1(T)) - \mathbf{z}_1(T), \quad z_2^*(t, x) \leq \max_{s \in [t, T]} \psi(\mathbf{y}(s)).$$

Therefore, $z_1 \geq z_1^*(t, x)$ and $z_2 \geq z_2^*(t, x)$.

Now, assume that $z_1 > \bar{z}_1(t, x)$. In this case, by definition of $\bar{z}_1(t, x)$, it would exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\varphi(\mathbf{y}_1(T)) - \mathbf{z}_1(T) \leq \bar{z}_1(t, x) < z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2^*(t, x) \leq z_2,$$

which contradicts the fact that $z \in \mathcal{F}^\#(t, x)$. We conclude that $z_1 \leq \bar{z}_1(t, x)$. The same argument shows also that $z_2 \leq \bar{z}_2(t, x)$, and then z belongs to Ω .

(ii) Let $z \in \Omega$ such that $w(t, x, z) = 0$ and there exists an admissible pair $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) = z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) = z_2.$$

By definition of w

$$\min_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \vee \left(\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2 \right) = 0.$$

That means, there exists no admissible pair (\mathbf{y}, \mathbf{z}) such that

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < \varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) = z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) < \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) = z_2.$$

Therefore, by definition of the weak Pareto optimal solution $z \in \mathcal{F}_w^\#(t, x)$. \square

Remark 5.4. As proved in [15] outside of the set Ω only some trivial parts of the weak Pareto front might exist, that is

$$z \in \mathcal{F}_w^\#(t, x) \cap \Omega^C \Leftrightarrow z_1 = z_1^*(t, x) \text{ and } z_2 > \bar{z}_2(t, x), \quad \text{or} \quad z_2 = z_2^*(t, x) \text{ and } z_1 > \bar{z}_1(t, x).$$

In this section we give a characterization of the Pareto optimal front of bi-objective optimal control problem (MORP) using the value function w . Assume that the hypothesis $\mathbf{H}_1, \mathbf{i} = 1, 2, 3, 4$ hold and let (t, x) be in $[0, T] \times \mathbb{R}^N$. Let us introduce the operators on \mathbb{R}^2 defined by:

$$\pi_1(z) = z_1, \quad \pi_2(z) = z_2. \quad (24)$$

Let denote

$$\Omega_1 = [z_1^*, \bar{z}_1[; \quad \Omega_2 = [z_2^*, \bar{z}_2[$$

and introduce the following extended functions

$$\eta_1 : [z_1^*, \bar{z}_1] \rightarrow [z_2^*, \bar{z}_2], \quad \eta_1(\zeta_1) = \inf\{\gamma \mid w(t, x, \zeta_1, \gamma) \leq 0\}, \quad (25a)$$

$$\eta_2 : [z_2^*, \bar{z}_2] \rightarrow [z_1^*, \bar{z}_1], \quad \eta_2(\zeta_2) = \inf\{\gamma \mid w(t, x, \gamma, \zeta_2) \leq 0\}. \quad (25b)$$

Proposition 5.5. Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in $[0, T] \times \mathbb{R}^N$. Then for $j = 1, 2$ the functions $\eta_j(\cdot)$ are decreasing:

$$\forall \zeta, \zeta' \in [z_j^*, \bar{z}_j], \quad \left(\zeta \leq \zeta' \Rightarrow \eta_j(\zeta) \geq \eta_j(\zeta') \right)$$

Proof. Let assume that there exists $\xi \leq \zeta$ such that

$$\eta_1(\xi) < \eta_1(\zeta).$$

It follows from definition (25) that $w(t, x, \zeta, \alpha) > 0$, for all $\alpha < \eta_1(\zeta)$.

Let us take $\alpha = \eta_1(\xi)$ and consider the point $z = (\zeta, \eta_1(\xi))$. It is clear that $w(t, x, z) > 0$. On the other hand, by Proposition 4.1 we obtain that

$$w(t, x, z) = w(t, x, \zeta, \eta_1(\xi)) \leq w(t, x, \xi, \eta_1(\xi)) \leq 0$$

because $\xi \leq \zeta$. This is in contradiction with the fact that $w(t, x, z) > 0$ as established before. \square

The following theorem gives a characterization of the Pareto front.

Theorem 5.6. Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold and let (t, x) be in $[0, T] \times \mathbb{R}^N$.

(i) $\mathcal{F}^\#(t, x) = \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\}.$

(ii) For any $z \in \mathcal{F}^\#(t, x)$ let a trajectory $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ that is optimal for the auxiliary problem (16). Then (\mathbf{y}, \mathbf{z}) is a Pareto optimal solution of (MORP).

Proof. (i) **Step 1.** Let us show that $\left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\} \subset \mathcal{F}^\#(t, x)$. Let $z \in \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \Omega_1 \text{ and } \eta_1(\zeta) < +\infty \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \Omega_2 \text{ and } \eta_2(\zeta) < +\infty \right\}$. First, we have to show that such a point is feasible, that is, there exists an admissible pair $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) = z$. By definition of the functions $\eta_i, i = 1, 2$, we have that $w(t, x, z) = 0$. Then there exists at least one admissible pair $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\left((\varphi(\mathbf{y}(T)) - \mathbf{z}(T)) \vee \left(\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2 \right) \right) \leq 0 \Leftrightarrow \varphi(\mathbf{y}(T)) - \mathbf{z}(T) \leq z_1 \text{ and } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2 \quad (26)$$

Assume that z is not feasible. Then for any admissible pair $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ satisfying (26), we have that

$$\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) < z_1 \quad \text{or} \quad \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) < z_2.$$

Let us recall that by choice of z we have that $z_i = \eta_i(\pi_i(z))$. Then, without loss of generality, take

$$\zeta = \varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T), \quad \text{so} \quad w(t, x, \zeta, z_2) \leq 0$$

with $\zeta < \eta_1(\pi_1(z))$ which is in contradiction with the definition of $\eta_1(\pi_1(z))$ (see (25)).

Now, let us show that $z = (z_1, z_2)$ is Pareto optimal. Assume that there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = \xi_1 \leq z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\mathbf{y}(s)) = \xi_2 \leq z_2,$$

with $(\xi_1, \xi_2) = \xi \neq z$. Consider, without loss of generality, that $\xi_1 < z_1$, then $w(t, x, \xi_1, z_2) \leq 0$. As $\xi_1 < z_1$, by Proposition 4.1 we have that $w(t, x, \xi_1, z_2) \geq w(t, x, z_1, z_2) = 0$. So we conclude that $w(t, x, \xi_1, z_2) = 0$, with $\xi_1 < z_1 = \eta_1(\pi_1(z))$ which is a contradiction.

(i) **Step 2.** Let us show that

$$\mathcal{F}^\#(t, x) \subset \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\}.$$

Assume that $z \in \mathcal{F}^\#(t, x)$ and let $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ be an admissible pair such that

$$z_1 = \varphi(\mathbf{y}(T)) - \mathbf{z}(T) \quad \text{and} \quad z_2 = \max_{s \in [t, T]} \psi(\mathbf{y}(s)).$$

It follows from Theorem 5.2 that $w(t, x, z) = 0$. Then it is obvious that $\eta_1(\pi_1(z)) < +\infty$ and $\eta_2(\pi_2(z)) < +\infty$. If

$$z \notin \left\{ (\zeta, \eta_1(\zeta)), \zeta \in \text{dom}(\eta_1) \right\} \cap \left\{ (\eta_2(\zeta), \zeta), \zeta \in \text{dom}(\eta_2) \right\}$$

then $\exists j$ such that $z_j \neq \eta_j(\pi_j(z))$. Consider, without loss of generality, that $z_1 \neq \eta_1(\pi_1(z))$. As $w(t, x, z) = 0$, we obtain that $z_1 > \eta_1(\pi_1(z))$. Consider

$$\xi = (\eta_1(\pi_1(z)), z_2).$$

By the definition of the function η_1 we have that $w(t, x, \xi) = 0$ and then there exists an admissible pair $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that

$$\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) \leq \xi_1 < z_1 \quad \text{and} \quad \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) \leq \xi_2 = z_2,$$

what is in contradiction with the assumption that z is Pareto optimal.

(ii). Let $z \in \mathcal{F}^\#(t, x)$. Then $w(t, x, z) = 0$ and z is feasible. Take a trajectory $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ that is optimal for the auxiliary control problem (16). Then, as it was be shown (see Proposition 5.1),

$$(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1 \bigvee \max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) = 0 \Leftrightarrow (\varphi(\mathbf{y}(T)) - \mathbf{z}(T) \leq z_1, \text{ and } \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \leq z_2).$$

If $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) < z_1$ then $\xi = (\varphi(\mathbf{y}(T)) - \mathbf{z}(T), z_2)$ is a feasible vector that dominates z which is impossible. In the same manner, if $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) < z_2$ then $\xi' = (z_1, \max_{s \in [t, T]} \psi(\mathbf{y}(s)))$ is a feasible vector that dominates z which is impossible. So, for any trajectory (\mathbf{y}, \mathbf{z}) that is optimal for (16) we have that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) = z$$

that means that the pair (\mathbf{y}, \mathbf{z}) is Pareto optimal for (MORP). □

6 ε -Pareto solutions of the original bi-objective optimal control problem

In this section we return to the original problem presented in this paper. We prove that using the auxiliary value function w it is possible to obtain the region of (weak) ε -Pareto fronts are contained. Moreover (weak) ε -Pareto optimal solutions for problem MOP can be obtained by applying an algorithm of trajectory reconstruction to the auxiliary control problem (15).

Theorem 6.1. *Assume that (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{H}_3) and (\mathbf{H}_4) hold. Let (t, x) be in $[0, T] \times \mathbb{R}^N$ and $\varepsilon > 0$.*

(i) $\mathcal{F}^{1, \varepsilon}(t, x) \subset \mathcal{F}_w^{1, \varepsilon}(t, x) \subset \left\{ z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0 \right\}$.

(ii) Let $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0 \right\}$. If there exists $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary control problem (15). Then $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$ of problem (9).

(iii) Let $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \mid -\varepsilon < w(t, x, z) \leq 0 \right\}$. If there exists $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary control problem (15). Then $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}^{1, \varepsilon}(t, x)$ of problem (9).

Proof. (i) Let $z \in \mathcal{F}_w^{1, \varepsilon}(t, x)$. Then there exists an admissible pair $(\bar{\mathbf{y}}, \bar{\mathbf{z}}) \in \mathcal{S}_{[t, T]}(x, 0)$ such that

$$\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) = z_1, \quad \max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) = z_2,$$

then

$$\begin{aligned} w(t, x, z) &= \min_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \bigvee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \right] \\ &\leq \left[(\varphi(\bar{\mathbf{y}}(T)) - \bar{\mathbf{z}}(T) - z_1) \bigvee (\max_{s \in [t, T]} \psi(\bar{\mathbf{y}}(s)) - z_2) \right] = 0. \end{aligned}$$

Moreover as $z \in \mathcal{F}_w^{1, \varepsilon}(t, x)$, then for any $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0)$, one of the following assertions holds:

(a) $z_1 - \varepsilon \leq \varphi(\mathbf{y}(T)) - \mathbf{z}(T)$,

(b) or $z_2 - \varepsilon \leq \max_{s \in [t, T]} \psi(\mathbf{y}(s))$.

It is possible to check that in the two above cases, we have $(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \vee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \geq \varepsilon$.

As the $\mathcal{S}_{[t, T]}^\#(x, 0)$ is the closure of the $\mathcal{S}_{[t, T]}(x, 0)$, we can conclude that one of assertions above holds for any $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$. Therefore,

$$w(t, x, z) = \min_{(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)} \left[(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_1) \vee (\max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_2) \right] \geq -\varepsilon.$$

(ii) Let $z_\varepsilon \in \left\{ z \in \mathbb{R}^2 \mid -\varepsilon \leq w(t, x, z) \leq 0 \right\}$ and $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary problem $w(t, x, z_\varepsilon)$. Then,

$$-\varepsilon \leq (\varphi(\mathbf{y}_\varepsilon(T)) - \mathbf{z}_\varepsilon(T) - z_{1, \varepsilon}) \vee \max_{s \in [t, T]} \psi(\mathbf{y}_\varepsilon(s)) - z_{2, \varepsilon}$$

Suppose that there exists $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}(x, 0)$ such that

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T), \max_{s \in [t, T]} \psi(\mathbf{y}(s)) \right) < (z_{1, \varepsilon} - \varepsilon, z_{2, \varepsilon} - \varepsilon).$$

Then

$$\left(\varphi(\mathbf{y}(T)) - \mathbf{z}(T) - z_{1, \varepsilon}, \max_{s \in [t, T]} \psi(\mathbf{y}(s)) - z_{2, \varepsilon} \right) < \left(\varphi(\mathbf{y}_\varepsilon(T)) - \mathbf{z}_\varepsilon(T) - z_{1, \varepsilon}, \max_{s \in [t, T]} \psi(\mathbf{y}_\varepsilon(s)) - z_{2, \varepsilon} \right)$$

which is impossible. So, for any trajectory $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon)$ that is optimal for the auxiliary control problem $w(t, x, z_\varepsilon)$ we have that the pair $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$ of problem (9).

(iii) The proof can be obtained with similar arguments of (ii). \square

7 Reconstruction of the Pareto optimal trajectories

Theorems 5.3 and 5.6 provide the characterization of the weak Pareto front and the Pareto front, respectively, of the convexified problem (MORP). Another important concern is to reconstruct an optimal trajectory corresponding to a given Pareto optimal solution. Once the auxiliary value function w is known by Theorem 5.6 we have a characterization of the Pareto front $\mathcal{F}^\#(t, x)$. Now, let z be an optimal Pareto solution. Then a corresponding Pareto trajectory can be obtained by using the value function w . Indeed, by applying an algorithm of trajectory reconstruction to the function w on $[t, T]$ with the initial conditions (x, z) as the algorithm presented in [4], we get an approximation of the optimal trajectory for $w(t, x, z)$. Now, by Theorem 5.6, item (ii), if the trajectory is optimal for the auxiliary problem (15), then is a Pareto optimal trajectory of (MORP).

In the case of weak Pareto solutions, by Theorem 5.3

$$\mathcal{F}_w^\#(t, x) \cap \Omega \subset \{z \in \Omega \mid w(t, x, z) = 0\}.$$

And if there exists an admissible pair $(\mathbf{y}, \mathbf{z}) \in \mathcal{S}_{[t, T]}^\#(x, 0)$ such that $\varphi(\mathbf{y}(T)) - \mathbf{z}(T) = z_1$ and $\max_{s \in [t, T]} \psi(\mathbf{y}(s)) = z_2$, then $z \in \mathcal{F}_w^\#(t, x)$. So we can take $z \in \Omega$ such that $w(t, x, z) = 0$, apply an algorithm of trajectory reconstruction to the function w on $[t, T]$ with the initial (x, z) as in [4], and see if such a trajectory exists. If there is a trajectory in such conditions this is an approximation of a weak Pareto trajectory of (MORP).

Now consider $\varepsilon > 0$ and let $z_\varepsilon \in \mathbb{R}^2$, such that $-\varepsilon \leq w(t, x, z_\varepsilon) \leq 0$. By applying an algorithm of trajectory reconstruction to the function w on $[t, T]$ with the initial conditions (x, z_ε) we get an approximation of the optimal trajectory for $w(t, x, z_\varepsilon)$. Now by Theorem 6.1, if there exists $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary control problem (15). Then $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}_w^{1, \varepsilon}(t, x)$ of problem (9). Moreover if we consider that $-\varepsilon < w(t, x, z_\varepsilon) \leq 0$ and it is possible to obtain $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{S}_{[t, T]}(x, 0)$ that is optimal for the auxiliary control problem (15). Then $(\mathbf{y}_\varepsilon, \mathbf{z}_\varepsilon) \in \mathcal{P}^{1, \varepsilon}(t, x)$ of problem (9).

8 Numerical examples

In this section we present some numerical examples where the method proposed in this paper was applied to compute the (weak) Pareto front. In both examples the resultant HJB equation was solved by a finite difference method implement at C++ HJB-solver "ROC-HJ" [1].

8.1 A problem of business strategy

Denote by $\mathbf{y}(s)$ the quantity of steel produced by an industry, at time s . At every moment, such production can either be reinvested to expand the productive capacity or sold. The initial productive capacity is $x > 0$; such capacity grows as the reinvestment rate. Let the function $\mathbf{u} : [0, T] \rightarrow [0, 1]$, where $\mathbf{u}(s)$ is the fraction of the output at time s that should be reinvested. The objective is to maximize the total sales. As $\mathbf{u}(s)$ is the fraction of the output $\mathbf{y}(s)$ that we reinvest, then $(1 - \mathbf{u}(s))\mathbf{y}(s)$ is the part of $\mathbf{y}(s)$ that we sell by a price P at time s , that is constant over the time horizon. So the first objective is

$$\max \int_0^T P(1 - \mathbf{u}(s))\mathbf{y}(s)ds = \min \int_0^T P(\mathbf{u}(s) - 1)\mathbf{y}(s)ds.$$

However, consider that the owner also wants that the production to be around a quantity c , so the second objective can be modeled as

$$\min \max_{s \in [0, T]} |\mathbf{y}(s) - c|.$$

Hence the bi-objective problem is

$$\left\{ \begin{array}{l} \text{minimize} \quad \left(\int_t^T P(\mathbf{u}(s) - 1)\mathbf{y}(s)ds, \max_{s \in [0, T]} |\mathbf{y}(s) - c| \right) \\ \text{subject to} \quad \dot{\mathbf{y}}(s) = \mathbf{y}(s)\mathbf{u}(s) \\ \mathbf{y}(t) = x_0 \\ 0 \leq \mathbf{u}(s) \leq 1. \end{array} \right.$$

For numerical simulation we considered $P = 0.5$, $c = 0.5$ and $T = 2$. Note that image set of the augmented dynamics is convex. So, by remark 4.2 the auxiliary value function w is the unique viscosity solution to the following HJB equation:

$$\begin{aligned} \min \left(\partial_t w(t, x, z) + \mathcal{H}(x, D_x w, D_z w), w(t, x, z) - (|x - c| - z_2) \right) &= 0, \quad \text{for } t \in [0, T], x \in \mathbb{R}, z \in \mathbb{R}^2 \\ w(T, x, z) &= -z_1 \vee (|x - c| - z_2) \quad \text{for } x \in \mathbb{R}, z \in \mathbb{R}^2. \end{aligned}$$

where the Hamiltonian is given by

$$\mathcal{H}(x, p, q) = \max_{0 \leq u \leq 1} (-xup + P(u - 1)xq_1), \quad \forall x, p \in \mathbb{R}, q \in \mathbb{R}^2$$

The HJB equation was solved on a grid of 300^3 nodes on the domain $[0, e^2] \times [-e^2, 0] \times [0, 7]$. For the initial state $x_0 = 0.4$ and $t = 0$ first we use the formula (18) to obtain the utopian point associated with the bi-objective control problem:

$$\begin{aligned} z_1^*(t, x_0) &= \inf \left\{ \zeta_1 \in \mathbb{R} \mid \exists z_2 \in \mathbb{R} \text{ s.t. } w(t, x_0, \zeta_1, z_2) \leq 0 \right\} \cong -0.539, \\ z_2^*(t, x_0) &= \inf \left\{ \zeta_2 \in \mathbb{R} \mid \exists z_1 \in \mathbb{R} \text{ s.t. } w(t, x_0, z_1, \zeta_2) \leq 0 \right\} \cong 0.0985. \end{aligned}$$

After, following (22), we compute the upper bounds for the set Ω :

$$\begin{aligned} \bar{z}_1(t, x_0) &= \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x_0, \zeta, z_2^*(t, x_0)) = 0 \right\} \cong -0.4359, \\ \bar{z}_2(t, x_0) &= \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x_0, z_1^*(t, x_0), \zeta) = 0 \right\} \cong 0.5711. \end{aligned}$$

So the Pareto front is a subset of the bounded set:

$$\Omega = [z_1^*(t, x_0), \bar{z}_1(t, x_0)] \times [z_2^*(t, x_0), \bar{z}_2(t, x_0)].$$

Figure 4a a part of the negative set of the value function $w(t, x_0, \cdot)$ where is contained $\mathcal{F}_w(t, x_0) \subset \mathcal{F}_w^{1,\varepsilon}(t, x_0)$, for $\varepsilon = 0.05$ that includes the blue region and the red curve. Moreover the red curve is the zero level of the value function $w(t, x_0, \cdot)$ that corresponds of the curve that the weak Pareto front is contained. In this figure the set Ω is represented by a box delimited by black dashed lines. In Figure 4b, we show the intersection of zero level set of $w(t, x_0, \cdot)$ with the set Ω . In that case this set is equal the Pareto front. The possible points in the weak Pareto front are just trivial points and are outside the set Ω .

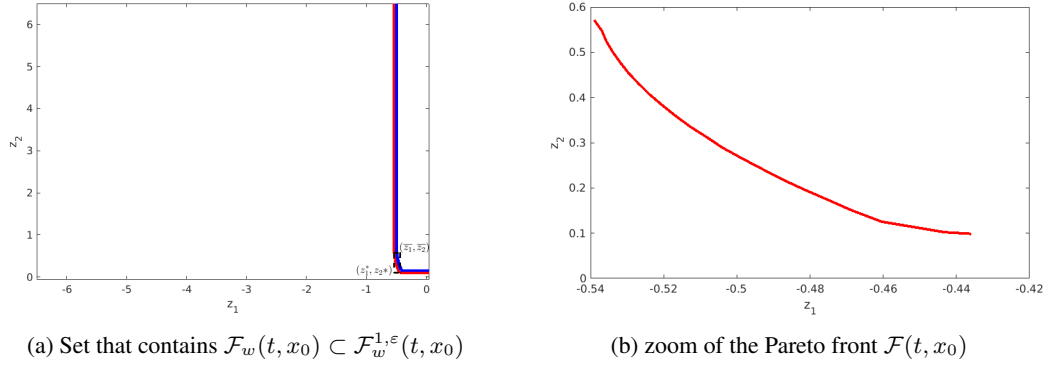


Figure 4: Business strategy: analysis of the zero level set of $w(t, x, \cdot)$, where $x = 0.4$

The value function w also gives sufficiently information to reconstruct Pareto optimal trajectories. We chose four point z_i , $i = 1, 2, 3, 4$ at the Pareto front, that can be seen in Figure 5a, and compute the optimal trajectories for $w(t, x_0, z_i)$, $i = 1, 2, 3, 4$, using the reconstruction algorithm from [4]. Figure 5b shows the corresponding the optimal trajectories for the initial states (x_0, z_i) . By Theorem 5.6 the trajectories are Pareto optimal solutions for the optimal control problem 8.1.

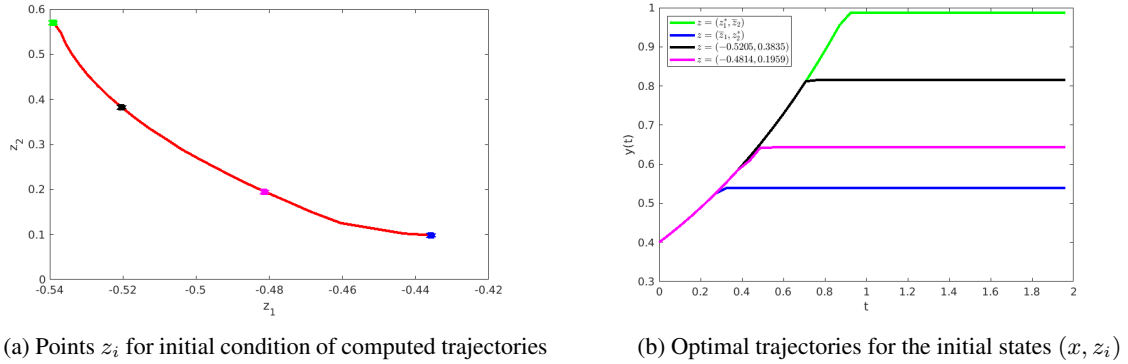


Figure 5: Business strategy: Optimal trajectories as Pareto optimal solutions

One important advantage of the HJB approach to solve bi-objective optimal control problems is the possibility of obtain the Pareto front for different initial states with the same auxiliary value function w . Figure 6a shows the Pareto front for different initial states x_0 . The algorithm is also stable, as can be seen in Figure 6b, that represents the 0-level set for the problem construct with different grids, of 300^3 , 150^3 and 75^3 , respectively. It is possible to see a difference between the zero level set with different discretizations of the state space, however the difference between the curve red and the others is lees than 0.02.

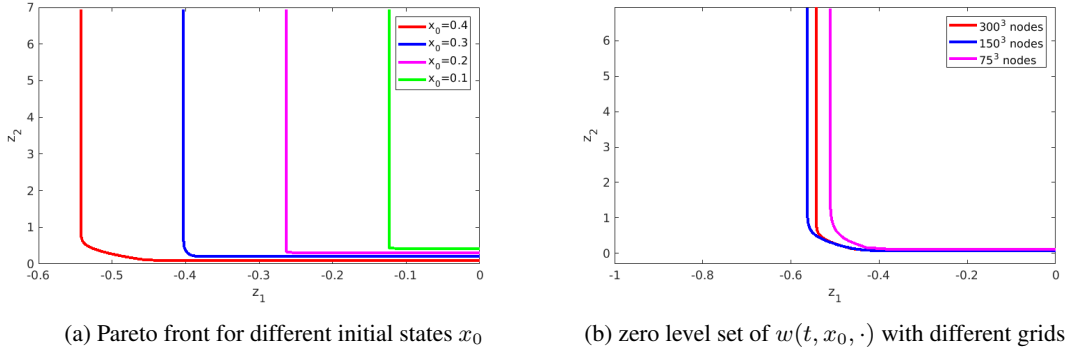


Figure 6: Business strategy: Pareto front for different initial states and grids

8.2 Pest control problem

Consider the Lotka-Volterra equations, also known as predator-prey equations. Those equations are a differential nonlinear system of two equations, and they are used to model biological systems where two species interact: a prey, that is a nuisance for humans, and a predator. Both species grow according to the following system

$$\begin{aligned}\dot{\mathbf{y}}_1(s) &= \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) \\ \dot{\mathbf{y}}_2(s) &= -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s)\end{aligned}$$

where \mathbf{y}_1 represents the pest population, \mathbf{y}_2 the predator. We are going to influence these model of evolution of species spraying a chemical to poison the pest (which also kills the predator). So the dynamical system becomes

$$\begin{aligned}\dot{\mathbf{y}}_1(s) &= \mathbf{y}_1(s) - \mathbf{y}_1(s)\mathbf{y}_2(s) - \mathbf{y}_1(s)c_{\mathbf{y}_1}\mathbf{u}(s) \\ \dot{\mathbf{y}}_2(s) &= -\mathbf{y}_2(s) + \mathbf{y}_1(s)\mathbf{y}_2(s) - \mathbf{y}_2(s)c_{\mathbf{y}_2}\mathbf{u}(s)\end{aligned}$$

where the constants $c_{\mathbf{y}_1}$, $c_{\mathbf{y}_2}$ represent the maximum level of the chemical in a time instant s for each population, that, in our numerical experiments, are going to be 0.4 and 0.2 respectively, as considered in [32]. The control function $u(s)$ is restricted to take values of either 0 or 1.

The goal is to keep nuisance expansion under control by minimizing the maximum difference of certain proportion of both species along the time horizon $T - t$:

$$\max_{s \in [t, T]} 0.25(\mathbf{y}_1(s) - K\mathbf{y}_2(s))^2,$$

and also reduce the cost of spraying the chemical:

$$\int_t^T P\mathbf{u}(s)ds,$$

where, in our experiments, $K = 0.7$, $P = 0.3$ and $T = 10$.

Therefore it is a bi-objective optimal control problem with two different objective functions. Note that in this case the set of trajectories is not compact. To obtain a problem where the set of trajectories is compact we are going to consider that the control function is continuous and can assume any value in the interval $[0, 1]$. So we define the auxiliary control problem where we minimize with controls $\mathbf{u}(s) \in [0, 1]$. The corresponding value function w is solution of the following HJB equation

$$\begin{aligned}\min \left(\partial_t w(t, x, z) + \mathcal{H}(x, D_x w, D_z w), w(t, x, z) - (0.25(x_1 - 0.7x_2)^2 - z_2) \right) &= 0, \quad \text{for } t \in [0, T], x, z \in \mathbb{R}^2 \\ w(T, x, z) &= -z_1 \sqrt{(0.25(x_1 - 0.7x_2)^2 - z_2)} \quad \text{for } x \in \mathbb{R}^2, z \in \mathbb{R}^2.\end{aligned}$$

where the Hamiltonian is given by

$$\mathcal{H}(x, p, q) = \max_{0 \leq u \leq 1} (-(x_1 - x_1 x_2 - 0.4 x_1 u) p_1 - (-x_2 + x_1 x_2 - 0.2 x_2 u) p_2 - 2 + 0.3 u q_1), \quad \forall x, p, q \in \mathbb{R}^2$$

The HJB equation was solved on a grid of 75^4 nodes on the domain $[0, 3.5] \times [0, 3.5] \times [0, 3] \times [0, 3]$. For the initial state $x_0 = (0.7, 0.2)$ and $t = 0$, first we use the formula (18) to obtain the utopian point associated with the bi-objective control problem:

$$z_1^*(t, x_0) = \inf \left\{ \zeta_1 \in \mathbb{R} \mid \exists z_2 \in \mathbb{R} \text{ s.t. } w(t, x_0, \zeta_1, z_2) \leq 0 \right\} \cong 0,$$

$$z_2^*(t, x_0) = \inf \left\{ \zeta_2 \in \mathbb{R} \mid \exists z_1 \in \mathbb{R} \text{ s.t. } w(t, x_0, z_1, \zeta_2) \leq 0 \right\} \cong 0.3245.$$

After, following (22), we compute the upper bounds for the set Ω :

$$\bar{z}_1(t, x_0) = \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x_0, \zeta, z_2^*(t, x_0)) = 0 \right\} \cong 0.9560,$$

$$\bar{z}_2(t, x_0) = \inf \left\{ \zeta \in \mathbb{R} \mid w(t, x_0, z_1^*(t, x_0), \zeta) = 0 \right\} \cong 2.5427.$$

So the Pareto front is a subset of the bounded set:

$$\Omega = [z_1^*(t, x_0), \bar{z}_1(t, x_0)] \times [z_2^*(t, x_0), \bar{z}_2(t, x_0)].$$

Figure 7a shows the zero level set of the value function $w(t, x_0, \cdot)$ in red that contains the (weak) Pareto front $\mathcal{F}^\#(t, x_0) \subset \mathcal{F}_x^\#(t, x_0)$. Moreover it is represented a region in black, including the curve in red, where it is possible to obtain points $z \in \mathbb{R}^2$ such that $z \in \mathcal{F}_w^{1,\varepsilon}(t, x_0)$, for $\varepsilon = 0.05$. In this figure the set Ω is represented by a box delimited by black dashed lines. In Figure 7b, we show the intersection of zero level set of $w(t, x_0, \cdot)$ with the set Ω . In that case this set is equal the set $\mathcal{F}^\#(t, x_0)$. The possible points in the weak Pareto front are just trivial points and are outside the set Ω .

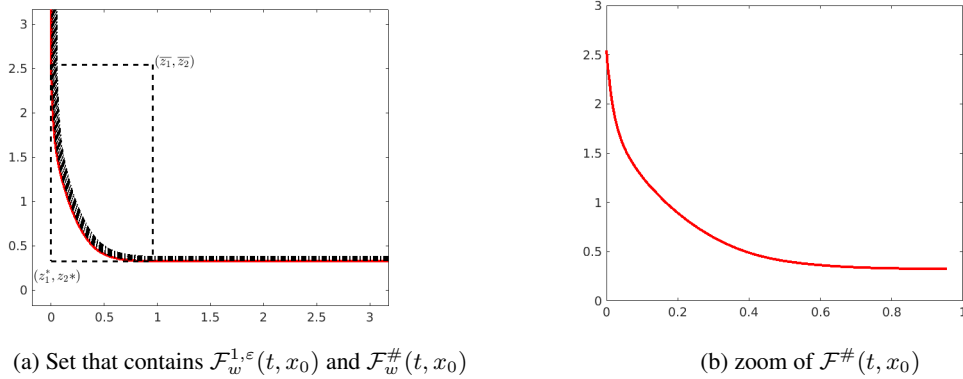


Figure 7: Pest control: analysis of the negative level set of $w(t, x, \cdot)$, where $x_0 = (0.7, 0.2)$

As discussed in Section 7, with the value function w we can also reconstruct optimal Pareto trajectories starting from (x, z) , where $x \in \mathbb{R}^2$ is the initial state and $z \in \mathbb{R}^2$ is a point in the Pareto front. With the algorithm of trajectory reconstruction from [4] we reconstruct the optimal trajectories for $w(t, x, z_1^*(t, x_0), \bar{z}_2(t, x_0))$ and $w(t, x, \bar{z}_1(t, x_0), z_2^*(t, x_0))$ that are also optimal trajectories of the scalar problems $\vartheta_1(x) = z_1^*(t, x_0)$ and $\vartheta_2(x) = z_2^*(t, x_0)$. Those trajectories and the respective optimal control are represented in Figure 8.

As can be seen in Figure 8 the optimal trajectory for $\vartheta_1(x)$, in green, is obtained with the control $\mathbf{u} \equiv 0$, which means that beyond $(z_1^*(t, x_0), \bar{z}_2(t, x_0)) \in \mathcal{F}^\#(t, x_0)$, we have that $(z_1^*(t, x_0), \bar{z}_2(t, x_0)) \in \mathcal{F}(t, x_0)$. However the control, in blue, related to the optimal trajectory for $\vartheta_2(x)$ do not take just values 0 or 1, which means

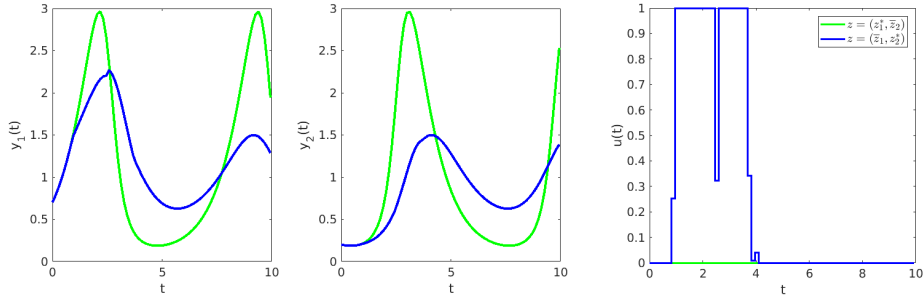


Figure 8: Pest control: optimal trajectories and controls of the scalar problems

that $(\bar{z}_1(t, x_0), z_2^*(t, x_0)) \in \mathcal{F}^\#(t, x_0)$, but $(\bar{z}_1(t, x_0), z_2^*(t, x_0)) \notin \mathcal{F}(t, x_0)$. By Theorem 3.1, there exists an ε -Pareto solution in a neighborhood of (\bar{z}_1, z_2^*) , where the control takes values in the set $\{0, 1\}$. We chose a point $(\bar{z}_1 + 0.01, z_2^* + 0.01)$ that is in the black region of Figure 7a. The optimal trajectory for $w(0, x_0, \bar{z}_1, z_2^*)$ and $w(0, x_0, \bar{z}_1 + 0.01, z_2^* + 0.01)$, that corresponds to a Pareto optimal trajectory for the convexified problem and a ε -Pareto trajectory of the original problem, respectively, are represented, in Figure 9.

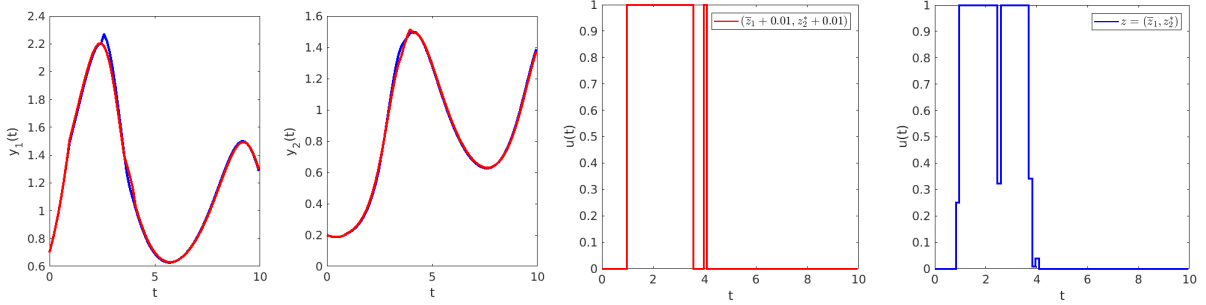
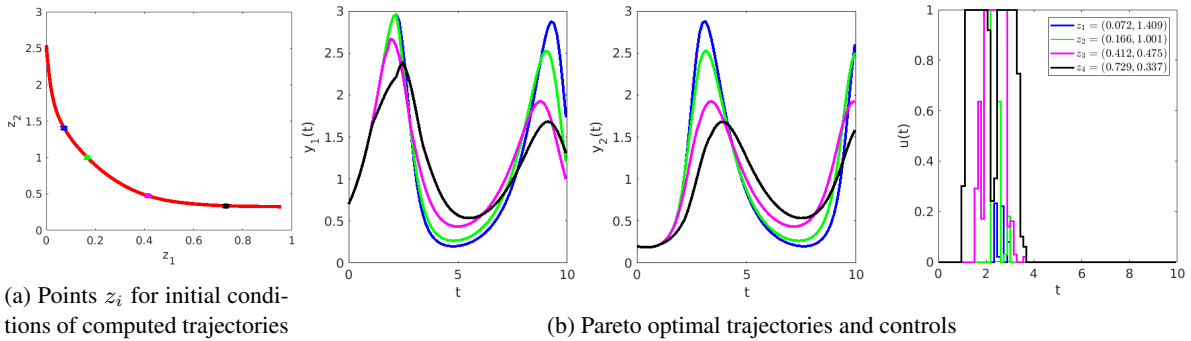


Figure 9: Approximate optimal trajectory and controls of the second scalar problem

To analyze more examples of optimal Pareto trajectories, we chose four point z_i , $i = 1, 2, 3, 4$ at the Pareto front, that can be seen in Figure 10, and compute the optimal trajectories and the respective controls for $w(0, x, z_i)$, $i = 1, 2, 3, 4$, using the reconstruction algorithm from [4]. Figure 5b shows the corresponding optimal trajectories. By Theorem 5.6 the trajectories are the corresponding Pareto optimal solutions of the convexified bi-objective problem.



(a) Points z_i for initial conditions of computed trajectories

(b) Pareto optimal trajectories and controls

Figure 10: Pest control: optimal pairs in $\mathcal{P}^\#(t, x_0)$

We observe that the optimal pairs represented in Figure 10 are in $\mathcal{P}^\#(t, x_0)$ but are not in $\mathcal{P}(t, x_0)$, because the respective control functions for those optimal Pareto trajectories, takes values different from 0 and 1. So we selected two points $z_{1_\varepsilon} = z_1 + (0.01, 0.01)$ and $z_{3_\varepsilon} = z_3 + (0.01, 0.01)$ that are in the black region of Figure 7a to reconstruct the respective optimal trajectories. In Figure 11 are represented Pareto optimal trajectories for the convexified problem for $z_1 = (0.072, 1.409)$, $z_3 = (0.412, 0.4753)$ and ε -Pareto optimal trajectories that the optimal values are in a neighborhood of z_1 and z_3 , when considering $\varepsilon = 0.05$.

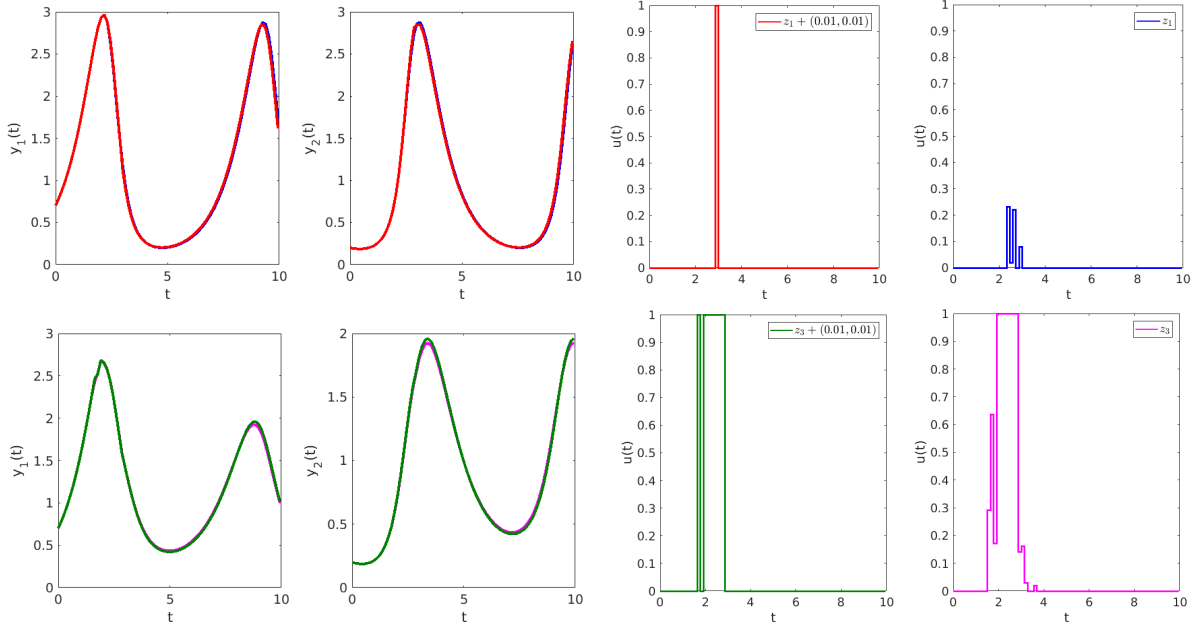
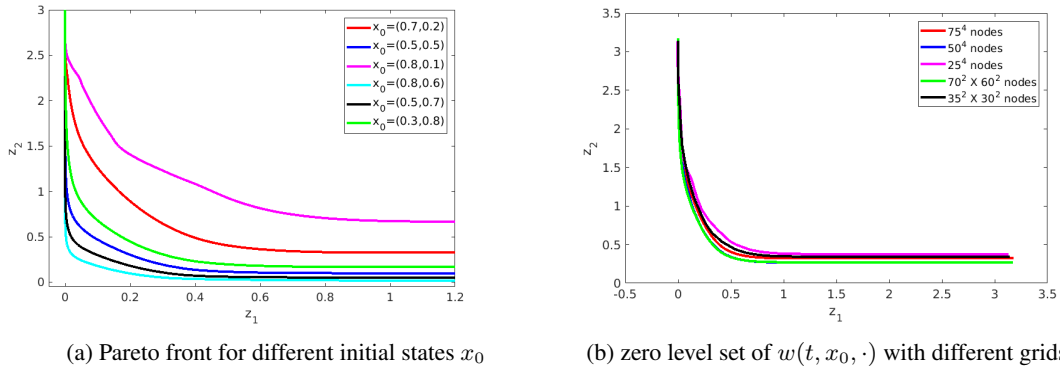


Figure 11: Pest control: comparison between optimal pairs in $\mathcal{P}^\#(t, x_0)$ and $\mathcal{P}^{1,\varepsilon}(t, x_0)$

One important advantage of the HJB approach to solve bi-objective optimal control problems is the possibility of obtain the Pareto front for different initial states with the same auxiliary value function w . Figure 12a shows the Pareto front for different initial states x_0 . Moreover the stability of the method can be seen in Figure 12b, where the zero level set of the auxiliary value problem $w(t, x_0, \cdot)$, with initial condition $x_0 = (0.7, 0.2)$ is represented for different grids. The first grid represented in red, is the one used to obtain the other results of this section. It is possible to see that the zero level set of the grids $70 \times 70 \times 60 \times 60$ and 50^4 nodes, in green and blue, respectively, are practically equal.



(a) Pareto front for different initial states x_0

(b) zero level set of $w(t, x_0, \cdot)$ with different grids

Figure 12: Pest control: Pareto front for different initial states and grids

9 Conclusion

In this paper we have investigated for the first time bi-objective optimal control problems with objectives of different nature. As for the considered problem the existence of Pareto solutions was not guaranteed we consider the bi-objective problem to be minimized over the closure of the set of trajectories. The relation of this relaxed (convexified) problem with the original one is that if x^* is a (weak) Pareto optimal solution of the relaxed problem then for any $\varepsilon > 0$ there exist an (weak) ε -Pareto solution for the original problem that is in a neighborhood of x^* . Moreover the norm of image of x^* and x by the objective function is less or equal ε .

We introduced an auxiliary control problem that the corresponded value function was characterized as a unique viscosity solution of an HJB equation. Then we proved that the weak Pareto front of the convexified problem is contained in the zero level set of the auxiliary control problem. Moreover we could characterize the Pareto front for the convexified bi-objective optimal control problem without any assumption about the convexity of the Pareto front.

A set where the (weak) ε -Pareto front for the original problem is contained could also be determined as a part of the negative level set of value function for the auxiliary control problem. Moreover with the value function is also possible to obtain (weak) Pareto optimal solutions and (weak) ε -Pareto solutions for the original and the relaxed optimal control problems.

The method proposed performs well, as could be seen with the numerical examples. Moreover, some good features are the stability of the method and the fact that after solving the derived HJB equation, the Pareto front for the bi-objective control problem with different initial states can easily be obtained.

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