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# Cauchy problem for the nonlinear Schrödinger equation coupled with the Maxwell equation

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## Abstract

In this paper, we study the nonlinear Schrödinger equation coupled with the Maxwell equation. Using energy methods, we obtain a local existence result for the Cauchy problem.

**Key words:** Schrödinger-Maxwell system, Cauchy problem, symmetric hyperbolic system, energy method

**2010 Mathematics Subject Classification.** 35L45, 35Q60, 35L70

## 1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation coupled with Maxwell equation stated in  $\mathbb{R}_+ \times \mathbb{R}^3$  :

$$i\psi_t + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi \cdot \mathbf{A} + ie\psi \operatorname{div}\mathbf{A} - g(|\psi|^2)\psi, \quad (1.1)$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = e \operatorname{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\phi_t - \nabla \operatorname{div}\mathbf{A}, \quad (1.2)$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2 + \operatorname{div}\mathbf{A}_t, \quad (1.3)$$

where  $\psi : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{C}$ ,  $\mathbf{A} : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $\phi : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $e \in \mathbb{R}$  and  $i$  denotes the unit complex number, that is,  $i^2 = -1$ . In this setting,  $\psi$  is an electrically charged field and  $(\phi, \mathbf{A})$  represents a gauge potential of an electromagnetic field. System (1.1)-(1.3) describes the interaction of this Schrödinger wave function  $\psi$  with the Maxwell gauge potential. The constant  $e$  represents the strength of the interaction. For more details and physical backgrounds, we refer to [15].

Since we are interested in the Cauchy Problem, let us consider the following set of initial data:

$$\psi(0, x) = \psi_{(0)}(x), \quad \mathbf{A}(0, x) = \mathbf{A}_{(0)}(x), \quad \mathbf{A}_t(0, x) = \mathbf{A}_{(1)}(x), \quad (1.4)$$

where the regularity of each functions is given in Theorem 1.1. It is known that System (1.1)-(1.3) has a so-called *gauge ambiguity*. Namely if  $(\psi, \mathbf{A}, \phi)$  is a solution of (1.1)-(1.3), then  $(\exp(i\chi)\psi, \mathbf{A} + \nabla\chi, \phi - \chi_t)$  is also a solution of (1.1)-(1.3) for any smooth function  $\chi : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}$ . To push out this ambiguity, we adopt in the sequel the Coulomb gauge:

$$\operatorname{div} \mathbf{A} = 0, \quad (1.5)$$

which is propagated by the set of Equations (1.1)-(1.3). Indeed, if initially

$$\operatorname{div} \mathbf{A}(0, \cdot) = \operatorname{div} \mathbf{A}_t(0, \cdot) = 0,$$

then (1.5) holds for all  $t > 0$ . (See e.g. [12] for the proof.) In this setting, the last Equation (1.3) can be solved explicitly and the solution is given by

$$\phi = \frac{e}{2}(-\Delta)^{-1}|\psi|^2,$$

which imposes that

$$\phi(0, x) = \frac{e}{2}(-\Delta)^{-1}|\psi_{(0)}(x)|^2.$$

From (1.5), we also observe that (1.1) can be written as

$$i\psi_t + L_A\psi - V(x)\psi + g(|\psi|^2)\psi = 0, \quad (1.6)$$

where  $V$  is the non-local potential:  $V(x) = \frac{e^2}{2}(-\Delta)^{-1}|\psi|^2$  and  $L_A$  is the *magnetic Schrödinger operator* which is defined by  $\mathbf{A} = (A_1, A_2, A_3)$  and

$$L_A\psi := \sum_{j=1}^3 \left( \frac{\partial}{\partial x_j} - ieA_j(x) \right)^2 \psi = \Delta\psi - 2ie\nabla\psi \cdot \mathbf{A} - e^2|\mathbf{A}|^2\psi. \quad (1.7)$$

In this context, the two conserved quantities of the Schrödinger-Maxwell system are the charge  $Q$  and the energy  $E$ :

$$Q(\psi) = \int_{\mathbb{R}^3} |\psi|^2 dx, \quad (1.8)$$

$$\begin{aligned}
E(\psi, \mathbf{A}, \phi) = & \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \psi - ie\mathbf{A}\psi|^2 + |\nabla \mathbf{A}|^2 + |\partial_t \mathbf{A}|^2) dx \\
& + \frac{e^2}{4} \int_{\mathbb{R}^3} \phi |\psi|^2 dx - \int_{\mathbb{R}^3} G(|\psi|^2) dx,
\end{aligned} \tag{1.9}$$

where  $G(t) = \int_0^t g(s)ds$ . To prove that (1.8) is formally conserved, one has to multiply Equation (1.1) by  $\bar{\psi}$ , integrate over  $\mathbb{R}^3$  and take the imaginary part of the resulting equation. In a similar way, the conservation of (1.9) can be proved by multiplying (1.1)-(1.3) by  $\partial_t \bar{\psi}$ ,  $\partial_t \mathbf{A}$  and  $\partial_t \phi$  respectively. These conserved quantities play a fundamental role if one wants to investigate the stability properties of such system, which is one of our main motivations. Indeed, in a previous paper [13], we have showed that for small  $e > 0$ , System (1.1)-(1.3) admits a unique orbitally stable ground state of the form:

$$(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega}) := \left( \exp(i\omega t)u_{e,\omega}, \mathbf{0}, \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}|^2 \right). \tag{1.10}$$

In order to investigate the stability of such standing waves  $(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega})$ , it is necessary to prove that the Cauchy Problem (1.1)-(1.3) is almost locally well-posed around  $(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega})$ .

In a previous paper [12], we have proved the local existence of solutions for the nonlinear Klein-Gordon-Maxwell system in Sobolev spaces of high regularity. The method was to convert the Klein-Gordon-Maxwell system into a symmetric hyperbolic system and apply the standard energy estimate. Although our Schrödinger-Maxwell system (1.1)-(1.3) looks similar, especially Equation (1.2) is completely the same, the usual reduction tools does not lead us to a symmetric hyperbolic system, which causes the necessity of a new strategy.

Let us also introduce results concerning the solvability of the Cauchy problem related to (1.1)-(1.3). In [3], [20], the linear Schrödinger equation ( $g \equiv 0$ ) coupled with the Maxwell equations has been studied. Using the Strichartz estimate, the authors obtained the global well-posedness in the energy space. However, it is not clear that their argument can be applied to the nonlinear case  $g \not\equiv 0$ . We also mention the paper [19], where the Cauchy problem of the Schrödinger-Maxwell system in the Lorentz gauge has been studied by using the energy method. On the other hand, a huge attention has been paid in the magnetic Schrödinger equation (1.6). Especially in [18], the local well-posedness for (1.6) in the energy space has been established in the case  $V \equiv 0$ . However, in this situation, the magnetic potential  $\mathbf{A}$  is *given* and was assumed to be  $C^\infty$ , which cannot be expected a priori in our case. We also refer to [14] for the Strichartz estimate for the magnetic Schrödinger operator (1.7) in the case  $\mathbf{A} \in L^2_{loc}(\mathbb{R}^3)$ .

We mention that if we look for the standing wave (1.10), we are led to the following non-local elliptic problem:

$$-\Delta u + \omega u + \left( \frac{e^2}{8\pi|x|} * |u|^2 \right) u = g(|u|^2)u \quad \text{in } \mathbb{R}^3, \quad (1.11)$$

which is referred as the *Schrödinger-Poisson(-Slater)* equation. The existence of ground states of (1.11) as well as their orbital stability have been widely studied (see [2], [5], [4], [7], [17] and references therein). Finally, the orbital stability of standing waves for the magnetic Schrödinger equation (1.6) has been considered in [8], [16]. Our study on the solvability of the Cauchy problem for (1.1)-(1.3) and the result established in [13] enable us to generalize these previous results to the full Schrödinger-Maxwell system.

Before stating the main result of this paper, we introduce the following notations. As usual,  $L^p(\mathbb{R}^3)$  denotes the usual Lebesgue space:

$$L^p(\mathbb{R}^3) = \{u \in \mathcal{S}'(\mathbb{R}^3) ; \|u\|_{L^p} < +\infty\},$$

where

$$\|u\|_{L^p} = \left( \int_{\mathbb{R}^3} |u(x)|^p dx \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < +\infty$$

and

$$\|u\|_{L^\infty} = \text{ess sup } \{|u(x)| ; x \in \mathbb{R}^3\}.$$

We define the Sobolev space  $H^s(\mathbb{R}^3)$  as follows:

$$H^s(\mathbb{R}^3) = \left\{ u \in \mathcal{S}'(\mathbb{R}^3) ; \|u\|_{H^s(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)|^2 d\xi < +\infty \right\},$$

where  $\mathcal{F}(u)(\xi)$  is the Fourier transform of  $u$ . We also introduce the homogeneous Sobolev space  $\dot{H}^1(\mathbb{R}^3)$  as being the completion of  $C_0^\infty(\mathbb{R}^3)$  for the norm  $u \rightarrow \| |\xi| \mathcal{F}(u)(\xi) \|_{L^2(\mathbb{R}^3)}$ . Recall that the space  $\dot{H}^1(\mathbb{R}^3)$  is continuously embedded into  $L^6(\mathbb{R}^3)$ . Finally let  $C(I, E)$  be the space of continuous functions from an interval  $I$  of  $\mathbb{R}$  to a Banach space  $E$ . For  $1 \leq j \leq 3$ , we set  $\partial_{x_j} = \frac{\partial}{\partial x_j}$  and  $\partial_t = \frac{\partial}{\partial t}$ . For  $k \in \mathbb{N}^3$ ,  $k = (k_1, k_2, k_3)$ , we denote  $D^k u = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \partial_{x_3}^{k_3} u$  and for a non-negative integer  $s$ ,  $D^s$  denotes the set of all partial space derivatives of order  $s$ . Different positive constants might be denoted by the same letter  $C$ . We also denote by  $\text{Re}(u)$  and  $\text{Im}(u)$  the real part and the imaginary part of  $u$  respectively.

We assume that  $g$  satisfies

$$g \in C^{m+1}(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad g(0) = 0, \quad (1.12)$$

for some  $m \in \mathbb{N}$  with  $m \geq 2$ , so that the function  $W : \mathbb{C} \rightarrow \mathbb{C}$  defined by  $W(u) := g(|u|^2)u$  satisfies  $W \in C^{m+1}(\mathbb{C}, \mathbb{C})$ ,  $W(0) = W'(0) = 0$ . Some typical examples of the nonlinear term  $g$  are the power nonlinearity  $g(s) = \pm s^{\frac{p-1}{2}}$  with  $[p] \geq 2m + 3$  ( $[p]$  denotes the integer part of  $p$ ), or the *cubic-quintic* nonlinearity  $g(s) = s - \lambda s^2$  for  $\lambda > 0$ , which frequently appears in the study of solitons in physical literatures. (See [22] for example.) In this setting, we prove the following result.

**Theorem 1.1.** *Let  $s$  be any integer larger than  $\frac{3}{2}$  and assume that  $\psi_0 \in H^{s+2}(\mathbb{R}^3, \mathbb{C})$ ,  $\mathbf{A}_{(0)} \in H^{s+2}(\mathbb{R}^3, \mathbb{R}^3)$ ,  $\mathbf{A}_{(1)} \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$  with  $\operatorname{div} \mathbf{A}_{(0)} = 0$ ,  $\operatorname{div} \mathbf{A}_{(1)} = 0$  and  $g$  satisfies (1.12). Then there exist  $T^* > 0$  and a unique solution  $(\psi, \mathbf{A}, \phi)$  to System (1.1)-(1.3) satisfying the initial condition (1.4) such that*

$$\begin{aligned} \psi &\in C([0, T^*]; H^{s+2}(\mathbb{R}^3)) \cap C^1([0, T^*]; H^s(\mathbb{R}^3)), \\ \mathbf{A} &\in C([0, T^*]; H^{s+2}(\mathbb{R}^3)) \cap C^1([0, T^*]; H^{s+1}(\mathbb{R}^3)), \\ \phi &\in C([0, T^*]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)), \quad \nabla \phi \in C([0, T^*]; H^{s+1}(\mathbb{R}^3)), \\ \phi_t &\in C([0, T^*]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)), \quad \nabla \phi_t \in C([0, T^*]; H^{s+1}(\mathbb{R}^3)). \end{aligned}$$

The proof of Theorem 1.1 is based on energy estimates and particularly, on the strategies developed in [9] and [10]. Note also that to overcome the loss of derivatives embedded in Equation (1.2), we use the original idea of Ozawa and Tsutsumi presented in [21].

The paper is organized as follows. In Section 2, we transform System (1.1)-(1.3) into a system to which we can apply the usual energy method. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Transformation of the equations

In this section, we transform the original System (1.1)-(1.3) into a new symmetric system to which we can apply an energy method. In order to overcome the loss of derivatives contained in Equations (1.1)-(1.2), we introduce the following new unknowns (see [21]):

$$\Psi = \partial_t \psi \quad \text{and} \quad \Phi = \partial_t \phi.$$

Let us first derive equations for  $\Psi$  and  $\Phi$ . Differentiating Equation (1.1) with respect to  $t$ , one obtains

$$\begin{aligned} i\Psi_t + \Delta \Psi &= e\Phi\psi + e\phi\Psi + e^2|A|^2\Psi + 2e^2\psi\mathbf{A} \cdot \mathbf{A}_t + 2ie\nabla\Psi \cdot \mathbf{A} + 2ie\nabla\psi \cdot \mathbf{A}_t \\ &\quad - g'(|\psi|^2)(|\psi|^2\Psi + \psi^2\bar{\Psi}) - g(|\psi|^2)\Psi. \end{aligned}$$

Taking advantage of the new unknown  $\Psi$ , we also transform Equation (1.1) into an elliptic version

$$i\Psi + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi \cdot \mathbf{A} - g(|\psi|^2)\psi.$$

Moreover, we derive an equation for  $\Phi$  by applying  $\partial_t$  on Equation (1.3):

$$-\Delta\Phi = \frac{e}{2}(\psi\bar{\Psi} + \Psi\bar{\psi}).$$

Next, in order to ensure the Coulomb condition on  $\mathbf{A}$  for all  $t > 0$ , we introduce the projection operator  $\mathbb{P}$  on divergence free vector fields :

$$\begin{aligned} \mathbb{P} : \left(L^2(\mathbb{R}^3)\right)^3 &\longrightarrow \left(L^2(\mathbb{R}^3)\right)^3 \\ \mathbf{A} &\longmapsto \mathbb{P}\mathbf{A} = (-\Delta)^{-1}\text{rot rot } \mathbf{A}, \end{aligned}$$

so that if  $\text{div } \mathbf{A} = 0$ , then  $\mathbb{P}\mathbf{A} = \mathbf{A}$ . Thus applying  $\mathbb{P}$  on Equation (1.2), we derive

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = \mathbb{P}\left(e \text{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\Phi\right). \quad (2.1)$$

Note that any solution to (2.1) satisfying

$$\text{div } \mathbf{A}(0, \cdot) = 0 \quad \text{and} \quad \text{div } \mathbf{A}_t(0, \cdot) = 0,$$

obviously satisfies

$$\text{div } \mathbf{A}(t, \cdot) = 0 \quad \text{for all } t > 0.$$

At this step, we have transformed System (1.1)-(1.3) into

$$i\Psi + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi \cdot \mathbf{A} - g(|\psi|^2)\psi, \quad (2.2)$$

$$\begin{aligned} i\Psi_t + \Delta\Psi &= e\Phi\psi + e\phi\Psi + e^2|\mathbf{A}|^2\Psi + 2e^2\psi\mathbf{A} \cdot \mathbf{A}_t + 2ie\nabla\Psi \cdot \mathbf{A} \\ &\quad + 2ie\nabla\psi \cdot \mathbf{A}_t - g'(|\psi|^2)(|\psi|^2\Psi + \psi^2\bar{\Psi}) - g(|\psi|^2)\Psi, \end{aligned} \quad (2.3)$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = \mathbb{P}\left(e \text{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\Phi\right), \quad (2.4)$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2, \quad (2.5)$$

$$-\Delta\Phi = \frac{e}{2}(\psi\bar{\Psi} + \Psi\bar{\psi}). \quad (2.6)$$

In order to take advantage of elliptic regularity properties, we transform Equations (2.2) by adding  $-\alpha\psi$  ( $\alpha > 0$  will be chosen in Lemma 3.3 below) to both sides of the equation to obtain:

$$(-\Delta + \alpha)\psi = i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi - 2ie\nabla\psi \cdot \mathbf{A} + g(|\psi|^2)\psi + \alpha\psi. \quad (2.7)$$

For simplicity, introduce  $\mathcal{U} = (\phi, \Phi)$  and rewrite Equations (2.5) and (2.6) as

$$-\Delta \mathcal{U} = F_1(\psi, \Psi), \quad (2.8)$$

where

$$F_1(\psi, \Psi) = \frac{e}{2} \begin{pmatrix} |\psi|^2 \\ \psi \bar{\Psi} + \Psi \bar{\psi} \end{pmatrix}.$$

Equation (2.3) is then transformed into

$$i\partial_t \Psi + \Delta \Psi = 2ie\nabla \Psi \cdot \mathbf{A} + 2ie\nabla \psi \cdot \mathbf{A}_t + F_2(\mathcal{U}, \psi, \Psi, \mathbf{A}, \mathbf{A}_t), \quad (2.9)$$

where

$$\begin{aligned} F_2(\mathcal{U}, \psi, \Psi, \mathbf{A}, \mathbf{A}_t) = & e\Phi\psi + e\phi\Psi + e^2|\mathbf{A}|^2\Psi + 2e^2\psi\mathbf{A} \cdot \mathbf{A}_t \\ & - g'(|\psi|^2)(|\psi|^2\Psi + \psi^2\bar{\Psi}) - g(|\psi|^2)\Psi. \end{aligned}$$

It is then necessary to work with  $\mathbf{A}_t$  as new unknown. We recall first that  $\mathbf{A} = (a_1, a_2, a_3)$ . To properly write the equations on  $\mathbf{A}$  and  $\mathbf{A}_t$ , for  $j = 1, 2, 3$ ,  $k = 1, 2, 3$  and  $\ell = 1, 2, 3$ , we introduce

$$\begin{aligned} p_{j,k} &= \partial_{x_k} a_j, \\ q_j &= \Delta a_j, \\ r_j &= \partial_t a_j, \\ \lambda_{j,k} &= \partial_{x_k} \Delta^{-1} \partial_t a_j, \\ \mu_{j,k,\ell} &= \partial_{x_\ell} \lambda_{j,k} = \partial_{x_\ell} \partial_{x_k} \Delta^{-1} \partial_t a_j, \\ \nu_{j,k} &= \Delta \lambda_{j,k} = \partial_{x_k} \partial_t a_j, \\ \tau_{j,k} &= \partial_t \lambda_{j,k} = \partial_{x_k} \Delta^{-1} \partial_t^2 a_j, \end{aligned}$$

and set  $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$  with  $\mathcal{A}_j : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^{24}$  and

$$\mathcal{A}_j = {}^t(a_j, p_{j,k}, q_j, r_j, \lambda_{j,k}, \mu_{j,k,\ell}, \nu_{j,k}, \tau_{j,k}).$$

We also need to give some details on the projection operator  $\mathbb{P}$ . For that purpose, we introduce the Riesz transform  $\mathbb{R}_j$  from  $L^2(\mathbb{R}^3)$  to  $L^2(\mathbb{R}^3)$  which is given by

$$\mathbb{R}_j = \partial_{x_j} (-\Delta)^{-\frac{1}{2}} \quad \text{for } j = 1, 2, 3.$$

Then,  $\mathbb{P}$  can be rewritten as  $\mathbb{P} = (\mathbb{P}_{j,m})_{1 \leq j, m \leq 3}$  where

$$\mathbb{P}_{j,m} = \delta_{j,m} + \mathbb{R}_j \mathbb{R}_m.$$



Now we compute the equations for each components of  $\mathcal{A}_j$ . First by the definitions of  $\mathcal{A}_j$ , one finds that

$$\begin{aligned}
\partial_t a_j &= \Delta \Delta^{-1} \partial_t a_j = \sum_{k=1}^3 \partial_{x_k} (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \sum_{k=1}^3 \partial_{x_k} \lambda_{j,k}, \\
\partial_t p_{j,k} &= \partial_t \partial_{x_k} a_j = \Delta (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \Delta \lambda_{j,k} = \sum_{\ell=1}^3 \partial_{x_\ell} (\partial_{x_\ell} \lambda_{j,k}) = \sum_{\ell=1}^3 \partial_{x_\ell} \mu_{j,k,\ell}, \\
\partial_t q_j &= \partial_t \Delta a_j = \sum_{k=1}^3 \partial_{x_k} \Delta (\partial_{x_k} \Delta^{-1} \partial_t a_j) = \sum_{k=1}^3 \partial_{x_k} \Delta \lambda_{j,k} = \sum_{k=1}^3 \partial_{x_k} \nu_{j,k}, \\
\partial_t r_j &= \partial_t^2 a_j = \Delta \Delta^{-1} \partial_t^2 a_j = \sum_{k=1}^3 \partial_{x_k} \partial_t (\partial_{x_k} \Delta^{-1} \partial_t a_j) \\
&= \sum_{k=1}^3 \partial_{x_k} \partial_t \lambda_{j,k} = \sum_{k=1}^3 \partial_{x_k} \tau_{j,k}.
\end{aligned}$$

Next from Equation (2.4), we have

$$\partial_t^2 a_j = \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \right),$$

which provides

$$\begin{aligned}
\partial_t \lambda_{j,k} &= \partial_t \partial_{x_k} \Delta^{-1} \partial_t a_j = \partial_{x_k} \Delta^{-1} \partial_t^2 a_j \\
&= \partial_{x_k} \Delta^{-1} \left( \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \right) \right) \\
&= \partial_{x_k} a_j + h_{j,k}^1(\psi, \mathbf{A}), \\
\partial_t \mu_{j,k,\ell} &= \partial_t \partial_{x_\ell} \partial_{x_k} \Delta^{-1} \partial_t a_j = \partial_{x_\ell} \partial_{x_k} \Delta^{-1} (\partial_t^2 a_j) \\
&= \partial_{x_\ell} \partial_{x_k} \Delta^{-1} \left( \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \right) \right) \\
&= \partial_{x_\ell} p_{j,k} + h_{j,k,\ell}^2(\psi, \mathbf{A}), \\
\partial_t \nu_{j,k} &= \partial_{x_k} \partial_t^2 a_j \\
&= \partial_{x_k} \left( \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \right) \right) \\
&= \partial_{x_k} q_j + h_{j,k}^3(\psi, \mathbf{A}),
\end{aligned}$$

where  $h_{j,k}^1$ ,  $h_{j,k,\ell}^2$ ,  $h_{j,k}^3$  are non-local functions defined as follows:

$$\begin{aligned} h_{j,k}^1(\psi, \mathbf{A}) &= \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi), \\ h_{j,k,\ell}^2(\psi, \mathbf{A}) &= \partial_{x_\ell} \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi), \\ h_{j,k}^3(\psi, \mathbf{A}) &= \partial_{x_k} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi). \end{aligned}$$

Finally one has

$$\begin{aligned} \partial_t \tau_{j,k} &= \partial_{x_k} \Delta^{-1} \partial_t (\partial_t^2 a_j) \\ &= \partial_{x_k} \Delta^{-1} \partial_t \left( \Delta a_j + \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \operatorname{Im}(\bar{\psi} \partial_{x_m} \psi) - e^2 |\psi|^2 a_m - \partial_{x_m} \Phi \right) \right). \end{aligned}$$

Computing separately each term of the right-hand side of the previous equation, we obtain

$$\begin{aligned} \partial_t (\bar{\psi} \partial_{x_m} \psi) &= \bar{\Psi} \partial_{x_m} \psi + \bar{\psi} \partial_{x_m} \Psi, \\ \partial_t (|\psi|^2 a_m) &= (\Psi \bar{\psi} + \psi \bar{\Psi}) a_m + |\psi|^2 r_m. \end{aligned}$$

Moreover from (2.3) and (2.6), one finds that

$$\begin{aligned} \partial_t (\partial_{x_m} \Phi) &= \partial_t \left( \frac{e}{2} \partial_{x_m} (-\Delta)^{-1} (\psi \bar{\Psi} + \Psi \bar{\psi}) \right) \\ &= \frac{e}{2} \partial_{x_m} (-\Delta)^{-1} (2 |\Psi|^2 + \psi \partial_t \bar{\Psi} + \bar{\psi} \partial_t \Psi) \\ &= e \partial_{x_m} (-\Delta)^{-1} |\Psi|^2 + e \partial_{x_m} (-\Delta)^{-1} \operatorname{Im}(i \partial_t \Psi \bar{\psi}) \\ &= e \partial_{x_m} (-\Delta)^{-1} \left\{ |\Psi|^2 + \operatorname{Im} \left( -\bar{\psi} \Delta \Psi + e \Phi |\psi|^2 + e \phi \bar{\psi} \Psi + e^2 |\mathbf{A}|^2 \bar{\psi} \Psi \right. \right. \\ &\quad \left. \left. + 2e^2 |\psi|^2 \mathbf{A} \cdot \mathbf{A}_t + 2ie \bar{\psi} \nabla \Psi \cdot \mathbf{A} + 2ie \bar{\psi} \nabla \psi \cdot \mathbf{A}_t \right. \right. \\ &\quad \left. \left. - g'(|\psi|^2) |\psi|^2 (\bar{\psi} \Psi + \psi \bar{\Psi}) - g(|\psi|^2) \bar{\psi} \Psi \right) \right\}, \end{aligned}$$

from which we conclude that

$$\partial_t \tau_{j,k} = \partial_{x_k} r_j + h_{j,k}^4(\psi, \Psi, \mathbf{A}, \mathbf{R}),$$

where  $\mathbf{R} = (r_1, r_2, r_3)$  and

$$\begin{aligned}
& h_{j,k}^4(\psi, \Psi, \mathbf{A}, \mathbf{R}) \\
&= \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} \left( e \operatorname{Im}(\bar{\Psi} \partial_{x_m} \psi + \bar{\psi} \partial_{x_m} \Psi) - e^2(|\psi|^2 r_m + (\Psi \bar{\psi} + \psi \bar{\Psi}) a_m) \right) \\
&\quad - \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} \left[ e \partial_{x_m} (-\Delta)^{-1} \left\{ |\Psi|^2 + \operatorname{Im} \left( -\bar{\psi} \Delta \Psi + e \Phi |\psi|^2 + e \phi \bar{\psi} \Psi \right. \right. \right. \\
&\quad \left. \left. \left. + e^2 |\mathbf{A}|^2 \bar{\psi} \Psi + 2e^2 |\psi|^2 \mathbf{A} \cdot \mathbf{R} + 2ie \bar{\psi} \nabla \Psi \cdot \mathbf{A} + 2ie \bar{\psi} \nabla \psi \cdot \mathbf{R} \right. \right. \right. \\
&\quad \left. \left. \left. - g'(|\psi|^2) |\psi|^2 (\bar{\psi} \Psi + \psi \bar{\Psi}) - g(|\psi|^2) \bar{\psi} \Psi \right) \right\} \right].
\end{aligned}$$

The equation on  $\mathcal{A}_j$  can be written as a symmetric system of the form

$$\partial_t \mathcal{A}_j + \mathcal{M}_j(\nabla) \mathcal{A}_j + H_j(\psi, \Psi, \mathbf{A}, \mathbf{R}) = 0 \quad (j = 1, 2, 3),$$

where  $H_j = {}^t(0, 0, 0, 0, h_{j,k}^1, h_{j,k,\ell}^2, h_{j,k}^3, h_{j,k}^4)$ ,  $\mathcal{M}_j(\nabla) = \sum_{k=1}^3 \widetilde{M}_j \partial_{x_k}$  are  $24 \times 24$  symmetric matrices. Recalling that  $\mathcal{A}_j = (a_j, p_j, q_j, r_j, \lambda_j, \mu_j, \nu_j, \tau_j)$ , where  $a_j, q_j, r_j$  are scalar functions,  $p_j, \lambda_j, \nu_j$  and  $\tau_j$  are functions with values in  $\mathbb{R}^3$  and  $\mu_j$  is a function with values in  $\mathbb{R}^9$ ,  $\mathcal{M}_j$  can be simply written by blocks:

$$\mathcal{M}_j(\nabla) = \begin{pmatrix} 0 & 0 & 0 & 0 & \nabla \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nabla \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \nabla \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nabla \cdot \\ \nabla & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N(\nabla) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$N(\nabla) = \begin{pmatrix} \nabla & 0 & 0 \\ 0 & \nabla & 0 \\ 0 & 0 & \nabla \end{pmatrix}.$$

Note that  $\widetilde{M}_j$  are  $24 \times 24$  symmetric matrices whose components are all constants.

Thus from (2.7), (2.8) and (2.9), we have transformed Equations (1.1)-

(1.3) into the following system:

$$-\Delta \mathcal{U} = F_1(\psi, \Psi), \quad (2.10)$$

$$(-\Delta + \alpha)\psi + 2ie\nabla\psi \cdot \mathbf{A} = i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi + g(|\psi|^2)\psi + \alpha\psi, \quad (2.11)$$

$$i\partial_t\Psi + \Delta\Psi - 2ie\nabla\Psi \cdot \mathbf{A} = 2ie\nabla\psi \cdot \mathbf{R} + F_2(\mathcal{U}, \psi, \Psi, \mathbf{A}, \mathbf{R}), \quad (2.12)$$

$$0 = \partial_t\mathcal{A}_j + \mathcal{M}_j(\nabla)\mathcal{A}_j + H_j(\psi, \Psi, \mathbf{A}, \mathbf{R}). \quad (2.13)$$

### 3 Solvability of the Cauchy Problem

The aim of this section is to prove Theorem 1.1. To this end, we use a fix-point argument on a suitable version of System (2.10)-(2.13). In this procedure, the necessary estimates follow from the application of the usual energy methods.

For  $s \in \mathbb{N}$  with  $s > \frac{3}{2}$ , take an initial data

$$\psi_{(0)} \in H^{s+2}(\mathbb{R}^3, \mathbb{C}),$$

$$\mathbf{A}_{(0)} = (a_{1(0)}, a_{2(0)}, a_{3(0)}) \in H^{s+2}(\mathbb{R}^3, \mathbb{R}^3),$$

and

$$\mathbf{A}_{(1)} = (r_{1(0)}, r_{2(0)}, r_{3(0)}) \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^3),$$

satisfying

$$\operatorname{div}\mathbf{A}_{(0)} = 0, \quad \operatorname{div}\mathbf{A}_{(1)} = 0.$$

Let us define  $\Psi_{(0)} \in H^s(\mathbb{R}^3, \mathbb{C})$  by

$$\Psi_{(0)} = i(\Delta\psi_{(0)} - e\phi_{(0)}\psi_{(0)} - e^2|\mathbf{A}_{(0)}|^2\psi_{(0)} - 2ie\nabla\psi_{(0)} \cdot \mathbf{A}_{(0)} - g(|\psi_{(0)}|^2)\psi_{(0)}), \quad (3.1)$$

$$\Phi_{(0)} = \frac{e}{2}(-\Delta)^{-1}(\psi_{(0)}\bar{\Psi}_{(0)} + \Psi_{(0)}\bar{\psi}_{(0)}),$$

where  $\phi_{(0)} = \frac{e}{2}(-\Delta)^{-1}|\psi_{(0)}|^2$ . We also put

$$\mathcal{A}_{(0)} = (\mathcal{A}_{1(0)}, \mathcal{A}_{2(0)}, \mathcal{A}_{3(0)}) \in H^s(\mathbb{R}^3),$$

for  $i, j, k, l = 1, 2, 3$ ,

$$\mathcal{A}_{j(0)} = {}^t(a_{j(0)}, p_{j,k(0)}, q_{j(0)}, r_{j(0)}, \lambda_{j,k(0)}, \mu_{j,k,\ell(0)}, \nu_{j,k(0)}, \tau_{j,k(0)}),$$

$$p_{j,k(0)} = \partial_{x_k} a_{j(0)}, \quad q_{j(0)} = \Delta a_{j(0)}, \quad \lambda_{j,k(0)} = \partial_{x_k} \Delta^{-1} r_{j(0)},$$

$$\mu_{j,k,\ell(0)} = \partial_{x_k} \partial_{x_\ell} \Delta^{-1} r_{j(0)}, \quad \nu_{j,k(0)} = \partial_{x_k} r_{j(0)}$$

and

$$\tau_{j,k(0)} = \partial_{x_k} a_{j(0)} + \partial_{x_k} \Delta^{-1} \sum_{m=1}^3 \mathbb{P}_{j,m} (e \operatorname{Im}(\overline{\psi}_{(0)}) \partial_{x_m} \psi_{(0)} - e^2 |\psi_{(0)}|^2 a_{m(0)} - \partial_{x_m} \Phi_{(0)}).$$

We introduce  $R = 2(\|\psi_{(0)}\|_{H^s} + \|\Psi_{(0)}\|_{H^s} + \|\mathcal{A}_{(0)}\|_{H^s})$  and let  $B(R)$  be the ball of radius  $R$  in  $C([0, T]; (H^s(\mathbb{R}^3))^2)$  for  $T > 0$ .

We prove the existence of a solution  $(\mathcal{U}, \psi, \Psi, \mathcal{A}_j)$  of (2.10)-(2.13) by the following procedure. Take  $(\Psi, \mathcal{A}) \in B(R)$  with  $\operatorname{div} \mathbf{A} = 0$  arbitrarily and construct new functions  $\mathcal{Q}$  and  $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$  as follows.

First we define  $\psi \in C([0, T]; H^s(\mathbb{R}^3, \mathbb{C}))$  by

$$\psi(t, x) := \psi_{(0)}(x) + \int_0^t \Psi(s, x) ds. \quad (3.2)$$

Then by the construction of  $\psi$ , one finds that, for  $T$  small enough,

$$\|\psi\|_{L^\infty[0, T]; H^s} \leq R.$$

Next let  $\mathcal{U} \in C([0, T]; \dot{H}^1(\mathbb{R}^3))$  be a solution to

$$-\Delta \mathcal{U} = F_1(\psi, \Psi). \quad (3.3)$$

We note that  $\mathcal{U} \in C([0, T]; L^\infty(\mathbb{R}^3))$  and  $\nabla \mathcal{U} \in C([0, T]; H^{s+1}(\mathbb{R}^3))$ . (See Lemma 3.2 below.) Next we introduce the solution  $\chi \in C([0, T]; H^{s+2}(\mathbb{R}^3, \mathbb{C}))$  of the following elliptic equation:

$$(-\Delta + \alpha)\chi + 2ie\nabla \chi \cdot \mathbf{A} = i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi + g(|\psi|^2)\psi + \alpha\psi. \quad (3.4)$$

We now consider a linearized version of (2.12)-(2.13). We take  $(\mathcal{Q}, \mathcal{B}) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$  solutions to

$$\begin{cases} i\partial_t \mathcal{Q} + \Delta \mathcal{Q} - 2ie\nabla \mathcal{Q} \cdot \mathbf{A} = 2ie\nabla \chi \cdot \mathbf{R} + F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}), \\ \mathcal{Q}(0, x) = \Psi_{(0)}, \end{cases} \quad (3.5)$$

$$\begin{cases} \partial_t \mathcal{B}_j + \mathcal{M}_j(\nabla) \mathcal{B}_j + H_j(\chi, \Psi, \mathbf{A}, \mathbf{R}) = 0, \\ \mathcal{B}_j(0, x) = \mathcal{A}_{j(0)}. \end{cases} \quad (3.6)$$

Let

$$\mathcal{S} : (\Psi, \mathcal{A}) \mapsto (\mathcal{Q}, \mathcal{B}).$$

Our strategy consists in showing that  $\mathcal{S}$  is a contraction mapping on  $B(R)$ , provided that  $T > 0$  is sufficiently small and to prove that  $\chi = \psi$ , from which we obtain the existence of a solution  $(\mathcal{U}, \psi, \Psi, \mathcal{A}_j)$  of (2.10)-(2.13) and complete the proof of Theorem 1.1.

The proof is divided into 6 steps. We first recall the following classical lemma. (See e.g. [1, Proposition 2.1.1, p. 98] for the proof.)

**Lemma 3.1.** *Let  $u, v \in L^\infty(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$  for  $s \in \mathbb{N}$ . Then for all  $(m_1, m_2) \in \mathbb{N}^3 \times \mathbb{N}^3$  with  $|m_1| + |m_2| = s$ , one has*

$$\|D^{m_1}u D^{m_2}v\|_{L^2} \leq C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}).$$

• Step 1: Solving the elliptic equation (3.3).

**Lemma 3.2.** *There exists a unique solution  $\mathcal{U} \in C([0, T]; \dot{H}^1(\mathbb{R}^3))$  of (3.3). Moreover,  $\mathcal{U} = (\phi, \Phi)$  satisfies the following estimates.*

$$\|\nabla\phi\|_{L^\infty([0, T]; H^{s+1})} \leq C_1(R), \quad \|\phi\|_{L^\infty([0, T]; L^\infty)} \leq C_2(R), \quad (3.7)$$

$$\|\nabla\Phi\|_{L^\infty([0, T]; H^{s+1})} \leq C_3(R), \quad \|\Phi\|_{L^\infty([0, T]; L^\infty)} \leq C_4(R), \quad (3.8)$$

where  $C_1, C_2, C_3$  and  $C_4$  are positive constants depending only on  $R$ .

*Proof.* First we note that the bilinear form

$$a(u, v) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \, dx$$

is continuous and elliptic on  $\dot{H}^1(\mathbb{R}^3, \mathbb{R}) \times \dot{H}^1(\mathbb{R}^3, \mathbb{R})$ . Moreover since  $\psi \in H^s(\mathbb{R}^3)$  and  $\Psi \in H^s(\mathbb{R}^3)$ , a direct computation gives

$$\| |\psi|^2 \|_{L^{\frac{6}{5}}} = \|\psi\|_{L^{\frac{12}{5}}}^2 \leq C\|\psi\|_{H^1}^2,$$

$$\|\psi\bar{\Psi} + \Psi\bar{\psi}\|_{L^{\frac{6}{5}}} \leq C\|\psi\|_{L^3}\|\Psi\|_{L^2} \leq C\|\psi\|_{H^1}\|\Psi\|_{L^2}.$$

Then by the Sobolev embedding  $L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow (\dot{H}^1(\mathbb{R}^3))^*$  and the Lax-Milgram theorem, we deduce that there exists a unique solution  $\mathcal{U} \in C([0, T]; \dot{H}^1(\mathbb{R}^3))$  of (3.3).

Next for  $0 \leq k \leq s$ , we apply  $D^{k+1}$  to the first line of (3.3), multiply the resulting equation by  $D^{k+1}\phi$  and make an integration by parts to obtain

$$\begin{aligned} \|\nabla(D^{k+1}\phi)\|_{L^2}^2 &= \frac{e}{2} \left| \int_{\mathbb{R}^3} D^{k+1}|\psi|^2 D^{k+1}\phi \, dx \right| \\ &\leq C \int_{\mathbb{R}^3} |D^k|\psi|^2| |D^{k+2}\phi| \, dx. \end{aligned}$$

Using the Leibniz rule, Lemma 3.1 and the Schwarz inequality, one has

$$\|\nabla\phi(t, \cdot)\|_{H^{k+1}} \leq C\|\psi(t, \cdot)\|_{H^k}^2 \quad \text{for all } t \in [0, T]. \quad (3.9)$$

Summing inequalities (3.9) from  $k = 0$  to  $s$  and recalling the fact that  $\|\psi\|_{L^\infty([0, T]; H^s)} \leq R$ , we obtain

$$\|\nabla\phi\|_{L^\infty([0, T]; H^{s+1})} \leq C_1(R),$$

where  $C_1(R)$  is a constant depending only on  $R$ .

Finally, the Sobolev embedding  $W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$  provides that

$$\begin{aligned} \|\phi(t, \cdot)\|_{L^\infty} &\leq C \left( \sum_{k=1}^3 \|\partial_{x_k} \phi(t, \cdot)\|_{L^6} + \|\phi(t, \cdot)\|_{L^6} \right) \\ &\leq C \left( \sum_{k=1}^3 \|\nabla(\partial_{x_k} \phi)(t, \cdot)\|_{L^2} + \|\nabla \phi(t, \cdot)\|_{L^2} \right), \end{aligned}$$

from which we deduce that there exists a constant  $C_2(R)$  depending only on  $R$  such that

$$\|\phi\|_{L^\infty([0,T];L^\infty)} \leq C_2(R),$$

which ends the proof of (3.7). The proof of estimates (3.8) is similar and we omit the details.  $\square$

• Step 2: Solving the elliptic equation (3.4).

**Lemma 3.3.** *Suppose that  $\mathbf{A} \in H^s(\mathbb{R}^3, \mathbb{R}^3)$ ,  $s > \frac{3}{2}$  and  $\operatorname{div} \mathbf{A} = 0$ . Then for sufficiently large  $\alpha > 0$ , the bilinear form*

$$b(u, v) := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \bar{v} + \alpha u \bar{v} + 2ie \nabla u \cdot \mathbf{A} \bar{v}) dx$$

*is hermitian, continuous and elliptic on  $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$ .*

*As a consequence, there exists a unique solution  $\chi(t, \cdot) \in H^1(\mathbb{R}^3, \mathbb{C})$  to (3.4) and there exists a constant  $C_5(R)$  such that*

$$\|\chi\|_{L^\infty([0,T];H^{s+2})} \leq C_5(R).$$

*Proof.* First we note that  $b$  is hermitian by the condition  $\operatorname{div} \mathbf{A} = 0$ . Indeed, one has

$$\begin{aligned} 2ie \int_{\mathbb{R}^3} (\nabla u \cdot \mathbf{A}) \bar{v} dx &= -2ie \int_{\mathbb{R}^3} \operatorname{div} \mathbf{A} u \bar{v} dx - 2ie \int_{\mathbb{R}^3} (\nabla \bar{v} \cdot \mathbf{A}) u dx \\ &= \overline{2ie \int_{\mathbb{R}^3} (\nabla v \cdot \mathbf{A}) \bar{u} dx}, \end{aligned}$$

from which it follows directly that  $b(u, v) = \overline{b(v, u)}$ . The continuity is a direct consequence of the Cauchy-Schwarz inequality and the fact that  $\mathbf{A} \in H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ . Finally for all  $u \in H^1(\mathbb{R}^3, \mathbb{C})$ , we have

$$\begin{aligned} \left| 2ie \int_{\mathbb{R}^3} (\nabla u \cdot \mathbf{A}) \bar{u} dx \right| &\leq 2e \|\mathbf{A}\|_{L^\infty} \|\nabla u\|_{L^2} \|u\|_{L^2} \\ &\leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + 2e^2 \|\mathbf{A}\|_{L^\infty}^2 \|u\|_{L^2}^2 \\ &\leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + Ce^2 \|\mathbf{A}\|_{H^s}^2 \|u\|_{L^2}^2. \end{aligned}$$

Taking  $\alpha \geq 2Ce^2\|\mathbf{A}\|_{H^s}^2$ , one gets

$$b(u, u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2.$$

This shows that  $b$  is elliptic on  $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$ .

Now since  $\psi, \Psi \in H^s(\mathbb{R}^3)$  and  $\phi \in \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ , it is obvious that  $i\Psi - e\phi\psi - e^2|\mathbf{A}|^2\psi - g(|\psi|^2)\psi + \alpha\psi$  belongs to  $L^2(\mathbb{R}^3) \hookrightarrow (H^1(\mathbb{R}^3))^*$ . Then the Lax-Milgram theorem ensures the existence of a unique solution  $\chi$  to (3.4) in  $H^1(\mathbb{R}^3)$ . Using the elliptic regularity theory and recalling that

$$(\psi, \Psi) \in C([0, T], H^s(\mathbb{R}^3))^2, \quad \phi \in C([0, T]; \dot{H}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)),$$

$$\nabla\phi \in C([0, T]; H^s(\mathbb{R}^3)), \quad \mathbf{A} \in C([0, T], H^s(\mathbb{R}^3)),$$

one gets

$$\|\chi\|_{L^\infty([0, T]; H^{s+2})} \leq C_5(R),$$

where  $C_5(R)$  is a constant depending only on  $R$ . This ends the proof of Lemma 3.3.  $\square$

• Step 3: Solving the Schrödinger equation (3.5).

For convenience, we introduce the real form of Equation (3.5). Denote  $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) = (\text{Re } \mathcal{Q}, \text{Im } \mathcal{Q})$  and write

$$\partial_t \mathcal{R} + \mathcal{J} \Delta \mathcal{R} - 2e \sum_{j=1}^3 \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R} = \mathcal{L}_1(\nabla \chi, \mathbf{R}) + \mathcal{L}_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}), \quad (3.10)$$

$$\mathcal{R}(0, x) = (\text{Re } \Psi_0(x), \text{Im } \Psi_0(x)),$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{K}_j(\mathbf{A}) = \begin{pmatrix} a_j & 0 \\ 0 & a_j \end{pmatrix},$$

$$\mathcal{L}_1(\nabla \chi, \mathbf{R}) = \begin{pmatrix} \text{Im}(2ie \nabla \chi \cdot \mathbf{R}) \\ -\text{Re}(2ie \nabla \chi \cdot \mathbf{R}) \end{pmatrix},$$

$$\mathcal{L}_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) = \begin{pmatrix} \text{Im } F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) \\ -\text{Re } F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) \end{pmatrix}.$$

Now for  $\varepsilon > 0$ , we consider a long-wave type regularization of (3.10) (see [11]):

$$\partial_t(1 - \varepsilon \Delta) \mathcal{R}_\varepsilon + \mathcal{J} \Delta \mathcal{R}_\varepsilon - 2e \sum_{j=1}^3 \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon = \mathcal{L}_1 + \mathcal{L}_2, \quad (3.11)$$



with  $\mathcal{R}_\varepsilon(0) = (1 - \varepsilon\Delta)^{-1}(\operatorname{Re} \Psi_0, \operatorname{Im} \Psi_0)$ . Since Equation (3.11) is linear and contains differential operator in space of at most zero order, one can show that there exists a unique solution  $\mathcal{R}_\varepsilon \in C([0, T]; H^s(\mathbb{R}^3))$  to Equation (3.11). Furthermore, we have the following estimate.

**Lemma 3.4.** *Let  $\mathcal{R}_\varepsilon$  be the unique solution of Equation (3.11). Then there exist constants  $C_6(R)$ ,  $C_7(R)$  independent of  $\varepsilon$  such that*

$$\|\mathcal{R}_\varepsilon\|_{L^\infty([0, T]; H^s)} \leq e^{C_6(R)T} \|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}}.$$

*Proof.* We first begin with the  $L^2$ -estimate. We multiply (3.11) by  $\mathcal{R}_\varepsilon$  and integrate over  $\mathbb{R}^3$ . Since  $\mathcal{J}$  is skew-symmetric, one obtains

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{1}{2} \int_{\mathbb{R}^3} (|\mathcal{R}_\varepsilon|^2 + \varepsilon |\nabla \mathcal{R}_\varepsilon|^2) dx \right) \\ &= 2e \int_{\mathbb{R}^3} \sum_{j=1}^3 \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon dx + \int_{\mathbb{R}^3} \mathcal{L}_1 \cdot \mathcal{R}_\varepsilon dx + \int_{\mathbb{R}^3} \mathcal{L}_2 \cdot \mathcal{R}_\varepsilon dx. \end{aligned} \quad (3.12)$$

For  $j = 1, 2, 3$ , we have from  $\|\partial_{x_j} a_j\|_{H^s} \leq R$  that

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon \cdot \mathcal{R}_\varepsilon dx \right| &= \left| \frac{1}{2} \int_{\mathbb{R}^3} a_j \partial_{x_j} |\mathcal{R}_\varepsilon|^2 dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_j} a_j |\mathcal{R}_\varepsilon|^2 dx \right| \\ &\leq \frac{1}{2} \|\partial_{x_j} a_j\|_{L^\infty} \|\mathcal{R}_\varepsilon\|_{L^2}^2 \leq C(R) \|\mathcal{R}_\varepsilon\|_{L^2}^2. \end{aligned} \quad (3.13)$$

Since  $\Psi, \mathbf{R} \in H^s$  and  $\mathbf{A} \in H^{s+1}$ , using Lemmas 3.2-3.3, one can also compute as follows :

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \mathcal{L}_1(\nabla \chi, \mathbf{R}) \cdot \mathcal{R}_\varepsilon dx \right| &\leq C(R) \|\mathcal{R}_\varepsilon\|_{L^2}, \\ \left| \int_{\mathbb{R}^3} \mathcal{L}_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}) \cdot \mathcal{R}_\varepsilon dx \right| &\leq C(R) \|\mathcal{R}_\varepsilon\|_{L^2}. \end{aligned} \quad (3.14)$$

Collecting (3.12)-(3.14), we derive

$$\frac{\partial}{\partial t} \|\mathcal{R}_\varepsilon\|_{L^2}^2 \leq C(R) \|\mathcal{R}_\varepsilon\|_{L^2}^2 + C(R).$$

By the Gronwall inequality and from

$$\|\mathcal{R}_\varepsilon(0, \cdot)\|_{L^2} = \|(1 - \varepsilon\Delta)^{-1}(\operatorname{Re} \Psi_0, \operatorname{Im} \Psi_0)\|_{L^2} \leq \|\Psi_0\|_{L^2},$$

it follows that

$$\begin{aligned}
\|\mathcal{R}_\varepsilon(t, \cdot)\|_{L^2}^2 &\leq e^{C(R)t} (\|\mathcal{R}_\varepsilon(0, \cdot)\|_{L^2}^2 + 1 - e^{-C(R)t}) \\
&\leq e^{C(R)T} \|\Psi_0\|_{L^2}^2 + e^{C(R)T} - 1 \\
&\leq \left( e^{\frac{C(R)T}{2}} \|\Psi_0\|_{L^2} + (e^{C(R)T} - 1)^{\frac{1}{2}} \right)^2 \quad \text{for all } t \in [0, T].
\end{aligned}$$

Next we perform the  $H^s$ -estimate. We apply  $D^s$  on (3.11), multiply the resulting equation by  $D^s \mathcal{R}_\varepsilon$ , integrate over  $\mathbb{R}^3$  and use the Gronwall inequality. We limit our attention to non-trivial terms. Recalling that  $\chi \in C([0, T]; H^{s+2})$  and using Lemma 3.3, we obtain

$$\left| \int_{\mathbb{R}^3} D^s \mathcal{L}_1(\nabla \chi, \mathbf{R}) \cdot D^s \mathcal{R}_\varepsilon \, dx \right| \leq C(R) \|\mathcal{R}_\varepsilon\|_{H^s}.$$

Moreover, one gets

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \mathcal{K}_j(\mathbf{A}) \partial_{x_j} D^s \mathcal{R}_\varepsilon \cdot D^s \mathcal{R}_\varepsilon \, dx \right| &= \left| \frac{1}{2} \int_{\mathbb{R}^3} a_j \partial_{x_j} |D^s \mathcal{R}_\varepsilon|^2 \, dx \right| \\
&= \left| \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_j} a_j |D^s \mathcal{R}_\varepsilon|^2 \, dx \right| \\
&\leq \frac{1}{2} \|\partial_{x_j} a_j\|_{L^\infty} \|\mathcal{R}_\varepsilon\|_{H^s}^2.
\end{aligned}$$

Arguing similarly as above, one finds that

$$\|\mathcal{R}_\varepsilon\|_{L^\infty([0, T]; H^s)} \leq e^{C_6(R)T} \|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}},$$

which ends the proof of Lemma 3.4.  $\square$

Now we argue as in [6], [11] and we perform the limit  $\varepsilon \rightarrow 0$ . By Lemma 3.4, we know that  $\mathcal{R}_\varepsilon$  is uniformly bounded in  $L^\infty([0, T], H^s)$ . From (3.11), one also has

$$\begin{aligned}
\partial_t \mathcal{R}_\varepsilon &= -(1 - \varepsilon \Delta)^{-1} \mathcal{J} \Delta \mathcal{R}_\varepsilon + 2e(1 - \varepsilon \Delta)^{-1} \sum_{j=1}^3 \mathcal{K}_j(\mathbf{A}) \partial_{x_j} \mathcal{R}_\varepsilon \\
&\quad + (1 - \varepsilon \Delta)^{-1} \mathcal{L}_1 + (1 - \varepsilon \Delta)^{-1} \mathcal{L}_2.
\end{aligned}$$

This implies that

$$\|\partial_t \mathcal{R}_\varepsilon\|_{H^{s-2}} \leq C(R) \|\mathcal{R}_\varepsilon\|_{H^s} + \|\mathcal{L}_1\|_{H^{s-2}} + \|\mathcal{L}_2\|_{H^{s-2}} \leq C \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in (0, 1].$$

Thus passing to a subsequence, we may assume that

$$\mathcal{R}_\varepsilon \rightarrow \mathcal{R} \in L^\infty([0, T], H^s), \partial_t \mathcal{R}_\varepsilon \rightarrow \partial_t \mathcal{R} \in L^\infty([0, T], H^{s-2}) \text{ in the weak star topology.}$$

From (3.11), one can see that  $\mathcal{R}$  is a solution of Equation (3.10) and satisfies

$$\|\mathcal{R}\|_{L^\infty([0,T];H^s)} \leq e^{C_6(R)T} \|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}}.$$

Moreover since  $\mathcal{R}_\varepsilon(0) \rightarrow (\operatorname{Re} \Psi_0, \operatorname{Im} \Psi_0)$ , we get  $\mathcal{R}(0) = (\operatorname{Re} \Psi_0, \operatorname{Im} \Psi_0)$ . We then deduce the existence of a solution  $\mathcal{Q}$  to Equation (3.5) satisfying

$$\|\mathcal{Q}\|_{L^\infty([0,T];H^s)} \leq e^{C_6(R)T} \|\Psi_0\|_{H^s} + (e^{C_7(R)T} - 1)^{\frac{1}{2}}. \quad (3.15)$$

• Step 4: Solving the symmetric system (3.6).

First we note that it is straightforward to prove the existence of a unique solution  $\mathcal{B}_j$  to Equation (3.6). (We refer to [1, Proposition 1.2, P. 115] for the proof.) Furthermore, by using the Fourier transform  $\mathcal{F}$ , one has directly

$$\mathbb{R}_j u = \mathcal{F}^{-1} \left( i \frac{\xi_j}{|\xi|} \mathcal{F}(u) \right) \quad \text{for } j = 1, 2, 3,$$

from which we deduce that  $\mathbb{R}_j$  and hence  $\mathbb{P}_{j,m}$  are bounded from  $L^2(\mathbb{R}^3)$  to  $L^2(\mathbb{R}^3)$ . As a consequence, using the fact  $\Psi \in H^s$ ,  $\mathbf{A} \in H^{s+2}$ ,  $\mathbf{R} \in H^{s+1}$  and  $\chi \in H^{s+2}$ , one can prove that

$$\|H_j(\chi, \Psi, \mathbf{A}, \mathbf{R})\|_{L^\infty([0,T];H^s)} \leq C(R).$$

Thus applying the energy estimate to (3.6), recalling that  $M_j(\nabla) = \sum_{k=1}^3 \widetilde{M}_j \partial_{x_k}$  is symmetric and using the fact  $\widetilde{M}_j$  consists of constant elements, we get

$$\frac{\partial}{\partial t} \|\mathcal{B}_j\|_{H^s}^2 \leq \|\mathcal{B}_j\|_{H^s}^2 + C(R).$$

Then by the Gronwall inequality, we obtain the following estimate.

**Lemma 3.5.** *Let  $\mathcal{B}_j$  be the unique solution of (3.6). Then there exists a constant  $C_8(R) > 0$  such that*

$$\|\mathcal{B}_j\|_{L^\infty([0,T];H^s)} \leq e^{\frac{T}{2}} \|\mathcal{A}_{j(0)}\|_{H^s} + C_8(R)(e^T - 1)^{\frac{1}{2}}. \quad (3.16)$$

Collecting (3.15) and (3.16), we can state the following result.

**Proposition 3.6.** *There exists  $\hat{T} > 0$  such that for  $0 < T \leq \hat{T}$ ,  $\mathcal{S}$  maps  $B(R)$  into itself.*

• Step 5: Contraction mapping.

Now we establish the following result.

**Proposition 3.7.** *There exists  $T^* \in (0, \hat{T}]$  such that  $\mathcal{S}$  is a contraction mapping in the  $L^\infty([0, T^*]; L^2(\mathbb{R}^3))$ -norm.*

*Proof.* The proof is based on the fact that  $s > \frac{3}{2}$  and on the fact that all the functions of Equations (3.3)-(3.6) are Lipschitz with respect to their arguments. The proof is classical and we omit the details.  $\square$

From Propositions 3.6, 3.7 and by the contraction mapping principle, it follows that there exists a unique  $(\Psi, \mathcal{A}) \in B(R)$  such that

$$\mathcal{S}(\Psi, \mathcal{A}) = (\Psi, \mathcal{A}),$$

that is,  $\Psi$  is the unique solution of the Schrödinger equation:

$$\begin{cases} i\partial_t \Psi + \Delta \Psi - 2ie\nabla \Psi \cdot \mathbf{A} = 2ie\nabla \chi \cdot \mathbf{R} + F_2(\mathcal{U}, \chi, \Psi, \mathbf{A}, \mathbf{R}), \\ \Psi(0, x) = \Psi_{(0)}, \end{cases}$$

and  $\mathcal{A}_j$  ( $j = 1, 2, 3$ ) is the unique solution to the symmetric system:

$$\begin{cases} \partial_t \mathcal{A}_j + \mathcal{M}_j(\nabla) \mathcal{A}_j + H_j(\chi, \Psi, \mathbf{A}, \mathbf{R}) = 0, \\ \mathcal{A}_j(0, x) = \mathcal{A}_{j(0)}. \end{cases}$$

Since  $\mathcal{A}(0, x) = \mathcal{A}_{(0)}$ , we also have  $\mathbf{A}(0, x) = \mathbf{A}_{(0)}$  and  $\mathbf{A}_t(0, x) = \mathbf{A}_{(1)}$ .

• Step 6: Proof of Theorem 1.1 completed.

Let us now go back to the original problem (1.1)-(1.3). To this end, we first remark that  $\partial_t \psi = \Psi$  by (3.2). Applying  $\partial_t$  on Equation (3.4), comparing the resulting equation with Equation (2.12) and recalling that  $\mathbf{R} = \partial_t \mathbf{A}$ , one obtains

$$-\Delta(\partial_t \chi - \partial_t \psi) + 2ie\nabla(\partial_t \chi - \partial_t \psi) \cdot \mathbf{A} = 0. \quad (3.17)$$

By Lemma 3.3, we know that the bilinear form

$$b(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \bar{v} + \alpha u \bar{v} + 2ie\nabla u \cdot \mathbf{A} \bar{v}) dx$$

is hermitian, continuous and elliptic on  $H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$ , from which we deduce that Equation (3.17) has a unique solution. Since obviously 0 is a solution to Equation (3.17), one has  $\partial_t \chi = \partial_t \psi$ . Moreover from (3.2), it follows that  $\psi(0, x) = \psi_{(0)}$  and hence  $\phi(0, x) = \phi_{(0)}$  by the uniqueness of the solution of (3.3). Thus substituting  $t = 0$  into (3.4), we get

$$-\Delta \chi(0) + 2ie\nabla \chi(0) \cdot \mathbf{A}_{(0)} = i\Psi_{(0)} - e\phi_{(0)}\psi_{(0)} - e^2|\mathbf{A}_{(0)}|^2\psi_{(0)} + g(|\psi_{(0)}|^2)\psi_{(0)}.$$

By the definition of  $\Psi_{(0)}$  given in (3.1), one finds that

$$-\Delta(\chi(0) - \psi_{(0)}) + 2ie\nabla(\chi(0) - \psi_{(0)}) \cdot \mathbf{A}_{(0)} = 0,$$

yielding that  $\chi(0) = \psi_{(0)}$  by Lemma 3.3.

Since  $\partial_t \chi = \partial_t \psi$  and  $\chi(0) = \psi_{(0)}$ , it follows that  $\chi = \psi$ . As a consequence,  $\psi$  is the unique solution to Equation (1.1). Then from (3.4) and (3.5), we conclude that  $\mathbf{A}$  and  $\phi$  are the unique solutions to (1.2) and (1.3) respectively. Moreover by the uniqueness of (3.3), one finds that  $\partial_t \phi = \Psi$ . Finally by Lemmas 3.2, 3.3 and from (3.15), (3.16),  $(\psi, \mathbf{A}, \phi)$  has the desired regularity as stated in Theorem 1.1.

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