# Continuity theorem for non-local functionals indexed by Young measures and stochastic homogenization 

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# CONTINUITY THEOREM FOR NON-LOCAL FUNCTIONALS INDEXED BY YOUNG MEASURES AND STOCHASTIC HOMOGENIZATION 

OMAR ANZA HAFSA, JEAN-PHILIPPE MANDALLENA, AND GÉRARD MICHAILLE


#### Abstract

We establish a continuity theorem for non-local functionals indexed by Young measures that we use to deal with homogenization of stochastic non-diffusive reaction differential equations. Non-local effects induced by homogenization of such stochastic differential equations are studied.


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## 1. Introduction

Let $T>0$, let $N \geqslant 1$ be an integer and let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain. Let $\mathcal{X}$ be a polish subspace of a suitable metric space (see $\S 2.1$ for more details) and let $\mathcal{Y}(\Omega ; \mathcal{X})$ be the class of Young measures on $\Omega \times \mathcal{X}$. Let us set

$$
\mathcal{H}_{0}^{1}:=\left\{u \in H^{1}(] 0, T\left[; L^{2}(\Omega ; \mathbb{R})\right): u(0, \cdot)=0\right\}
$$

and for each $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ and each $u \in \mathcal{H}_{0}^{1}$, let us consider the following two classes of functions:

$$
\begin{aligned}
& \mathcal{H}_{0, \mu}^{1}:=\left\{U \in H^{1}(] 0, T\left[; L_{\mu}^{2}(\Omega \times \mathcal{X} ; \mathbb{R})\right): U(0, \cdot, \cdot)=0\right\} \\
& \mathcal{S}_{\mu}(u):=\left\{U \in \mathcal{H}_{0, \mu}^{1}: \int_{\mathcal{X}} U(t, x, \xi) d \mu_{x}(\xi)=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega\},
\end{aligned}
$$

where $\left\{\mu_{x}\right\}_{x \in \Omega}$ denotes the desintegration of $\mu$ on $\Omega \times \mathcal{X}$. In this paper we are concerned with non-local functionals $\Phi_{\mu}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ defined by

$$
\Phi_{\mu}(u):=\inf \left\{\phi_{\mu}(U): U \in \mathcal{S}_{\mu}(u)\right\}
$$

with $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$, where $\phi_{\mu}: \mathcal{S}_{\mu}(u) \rightarrow \mathbb{R}$ is given by

$$
\phi_{\mu}(U):=\int_{] 0, T[\times \Omega \times \mathcal{X}} \xi(t, x, U(t, x, \xi), \dot{U}(t, x, \xi)) d t \otimes d \mu(x, \xi)+\int_{\Omega \times \mathcal{X}} \theta(U(T, x, \xi)) d \mu(x, \xi)
$$

with $\theta: \mathbb{R} \rightarrow\left[0, \infty\left[\right.\right.$ a convex function of 2-polynomial growth (see $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{2}\right),\left(\mathrm{C}_{3}\right)$ and $\left(\mathrm{C}_{4}\right)$ in $\S 2.1$ for more details). Let us set $\mathcal{F}:=\left\{\Phi_{\mu}: \mu \in \mathcal{Y}(\Omega ; \mathcal{X})\right\}$. Our main result is to prove that the map

$$
\mu \mapsto \Phi_{\mu}
$$

is continuous from $\mathcal{Y}(\Omega ; \mathcal{X})$ endowed with the narrow convergence to $\mathcal{F}$ endowed with the $\Gamma$-convergence (see Theorem 2.7 and also Corollaries 2.8 and 2.9), i.e.,

$$
\mu_{\varepsilon} \xrightarrow{\mathrm{nar}} \mu \Rightarrow \Gamma-\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\varepsilon}}=\Phi_{\mu} .
$$

The interest of this result comes from the convergence, when $\varepsilon$ goes to zero, of non-diffusive reaction differential equations of the type

$$
\left(\mathrm{R}_{\varepsilon}\right) \quad\left\{\begin{array}{l}
\left.-\frac{\partial u_{\varepsilon}}{\partial t}(t, x)=\frac{\partial \psi_{\varepsilon}}{\partial s}\left(t, x, u_{\varepsilon}(t)\right) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega \\
u_{\varepsilon}(0, \cdot)=0
\end{array}\right.
$$

with $\psi_{\varepsilon}:[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\varepsilon>0$. In fact, under suitable assumptions (see $\S 2.3$ for more details) every $\left(\mathrm{R}_{\varepsilon}\right)$ admits a unique solution $u_{\varepsilon} \in \mathcal{H}_{0}^{1}$ which is also a minimizer of $\Phi_{\mu_{\varepsilon}}$ with:

$$
\begin{aligned}
& \mu_{\varepsilon}=d x \otimes \delta_{f_{\varepsilon}(x)} \\
& f_{\varepsilon}(x)(t, y, s, \dot{s})=\psi_{\varepsilon}(t, x, s)+\psi_{\varepsilon}^{*}(t, x,-\dot{s}) \\
& \theta(s)=\frac{1}{2}|s|^{2}
\end{aligned}
$$

where $\psi_{\varepsilon}^{*}$ denotes the Legendre-Fenchel conjugate of $\psi_{\varepsilon}$ with respect to the third variable (see Proposition 2.11). When there exists $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ such that $d x \otimes \delta_{f_{\varepsilon}(x)} \xrightarrow{\text { nar }} \mu$, Theorem 2.7 implies that any cluster point of $\left\{u_{\varepsilon}\right\}_{\varepsilon}$ with respect to the weak convergence in $\mathcal{H}_{0}^{1}$ is a minimizer of $\Phi_{\mu}$ with the same $\theta: \mathbb{R} \rightarrow[0, \infty[$ given above (see Corollary 2.12).
This latter result point out the appearance of non-local effects induced by homogenization of non-diffusive reaction equations of type $\left(\mathrm{R}_{\varepsilon}\right)$ and, in some way, can be seen as a generalization of previous non-local results obtained by Mascarenhas (see [Mas93]) and Toader (see [Toa99]). The fact that non-local effects appear when dealing with homogenization of equations of type $\left(\mathrm{R}_{\varepsilon}\right)$ is illustrated by studying stochastic homogenization of such equations in the framework of a Poisson point process (see Theorem 4.6 and also Theorems 4.3 and 4.4). In particular, we show that the weak limit of solutions of stochastic non-diffusive reaction differential equations of type $\left(\mathrm{R}_{\varepsilon}\right)$ is characterized by two integro-differential equations (see $\left.\S 4.3\right)$.

The plan of the paper is as follows. In Section 2 we state the main result of the paper (see Theorem 2.7) and direct consequences of this result (see Corollaries 2.8 and 2.9) together with results related to homogenization of non-diffusive reaction equations (see Proposition 2.11, Corollary 2.12 and Proposition 2.13). Propositions 2.11 and 2.13 are proved in Section 2 while the proof of Theorem 2.7 is given in Section 3. Section 4 is devoted to stochastic homogenization. In $\S 4.1$, as a consequence of Theorem 2.7, we state and prove a non-local stochastic homogenization theorem (see Theorem 4.2) that we apply in $\S 4.2$, in the case of a Poisson point process, to homogenization of stochastic integral functionals as well as to homogenization of stochastic non-diffusive reaction equations (see Theorems 4.3, 4.4 and 4.6). Theorems 4.3 and 4.4 and Theorem 4.6 are respectively proved in $\S 4.2$ and $\S 4.3$. Finally, in the appendix, we recall some standard results on Young measures (see Appendix A) and inf-convolution and parallel sum (see Appendix B).
Notation. Throughout the paper, for any function $w, \dot{w}$ denotes the partial derivative of $w$ with respect to $t$, i.e., $\dot{w}=\frac{\partial w}{\partial t}$.

## 2. Main Results

2.1. A continuity theorem for a class of non-local functionals indexed by Young measures. Let $N \geqslant 1$ be an integer and let $T>0$. Let $\alpha, \beta \in] 0, \infty[$ be such $\alpha \leqslant \beta$ and let $\gamma:\left[0, \infty\left[\rightarrow\left[0, \infty\left[\right.\right.\right.\right.$ be such that $\lim _{r \rightarrow 0} \gamma(r)=0$. In what follows, we denote by $\mathcal{M}_{\alpha, \beta, \gamma}$ the class of functions $\xi:[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following three conditions:
$\left(\mathrm{C}_{1}\right) \xi(t, x, \cdot, \cdot)$ is convex for all $(t, x) \in[0, T] \times \mathbb{R}^{N}$;
$\left(\mathrm{C}_{2}\right) \alpha\left(|(s, \dot{s})|^{2}-1\right) \leqslant \xi(t, x, s, \dot{s}) \leqslant \beta\left(|(s, \dot{s})|^{2}+1\right)$ for all $(t, x) \in[0, T] \times \mathbb{R}^{N}$ and all $(s, \dot{s}) \in \mathbb{R} \times \mathbb{R}$;
$\left(\mathrm{C}_{3}\right)\left|\xi\left(t_{1}, x_{1}, s, \dot{s}\right)-\xi\left(t_{2}, x_{2}, s, \dot{s}\right)\right| \leqslant \gamma\left(\left|t_{1}-t_{2}\right|+\left|x_{1}-x_{2}\right|\right)$ for all $\left(t_{1}, x_{1}\right),\left(t_{2}, x_{2}\right) \in[0, T] \times \mathbb{R}^{N}$ and all $(s, \dot{s}) \in \mathbb{R} \times \mathbb{R}$.
From now on, to simplify notation we will sometimes write $S$ to denote the couple ( $s, \dot{s}$ ) $\in \mathbb{R}^{2}$. From $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{2}\right)$ and $\left(\mathrm{C}_{3}\right)$ it is easy to see that there exists $L>0$ such that

$$
\begin{equation*}
\left|\xi\left(t_{1}, x_{1}, S_{1}\right)-\xi\left(t_{2}, x_{2}, S_{2}\right)\right| \leqslant \gamma\left(\left|t_{1}-t_{2}\right|+\left|x_{1}-x_{2}\right|\right)+L\left|S_{1}-S_{2}\right|\left(1+\left|S_{1}\right|+\left|S_{2}\right|\right) \tag{2.1}
\end{equation*}
$$

for all $\xi \in \mathcal{M}_{\alpha, \beta, \gamma}$, all $\left(t_{2}, x_{2}\right) \in[0, T] \times \mathbb{R}^{N}$ and all $\left(S_{1}, S_{2}\right) \in \mathbb{R}^{2} \times \mathbb{R}^{2}$.

Proposition 2.1. Under $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{2}\right)$ and $\left(\mathrm{C}_{3}\right)$ the space $\mathcal{M}_{\alpha, \beta, \gamma}$ is a compact metric space with respect to the metric $d: \mathcal{M}_{\alpha, \beta, \gamma} \times \mathcal{M}_{\alpha, \beta, \gamma} \rightarrow[0, \infty[$ defined by

$$
\begin{equation*}
d\left(\xi_{1}, \xi_{2}\right)=\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \inf \left\{1,\left\|\xi_{1}-\xi_{2}\right\|_{\infty, n}\right\} \tag{2.2}
\end{equation*}
$$

where $\|\cdot\|_{\infty, n}$ is the uniform norm on $[0, T] \times \bar{B}_{n}(0)$, i.e., for each $\xi \in \mathcal{M}_{\alpha, \beta, \gamma}$,

$$
\|\xi\|_{\infty, n}=\sup \left\{|\xi(t, x, S)|: t \in[0, T] \text { and }(x, S) \in \bar{B}_{n}(0)\right\}
$$

with $\bar{B}_{n}(0) \subset \mathbb{R}^{N+2}$ being the $(N+2)$-dimensional closed ball centered at 0 with radius $n$. Moreover, the evaluation map $\mathcal{E}: \mathcal{M}_{\alpha, \beta, \gamma} \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ defined by $\mathcal{E}(\xi, t, x, S)=$ $\xi(t, x, S)$ is continuous.

Proof of Proposition 2.1. For each $n \in \mathbb{N}$, we consider the normed space $\left(X_{n},\|\cdot\|_{\infty, n}\right)$ with $X_{n}$ defined by

$$
X_{n}:=\left\{\xi_{\mid[0, T] \times \bar{B}_{n}(0)}: \xi \in \mathcal{M}_{\alpha, \beta, \gamma}\right\}
$$

where $\xi_{\mid[0, T] \times \bar{B}_{n}(0)}$ denotes the restriction of $\xi$ to $[0, T] \times \bar{B}_{n}(0)$. Then $\left(\mathcal{M}_{\alpha, \beta, \gamma}, d\right)$ is homeomorphic to the product $\prod_{n \in \mathbb{N}} X_{n}$. On the other hand, by using ( $\mathrm{C}_{2}$ ) and (2.1), from Ascoli's compactness theorem we see that $\left(X_{n},\|\cdot\|_{\infty, n}\right)$ is compact for all $n \in \mathbb{N}$. By Tychonov's compactness theorem it follows that $\left(\mathcal{M}_{\alpha, \beta, \gamma}, d\right)$ is compact.
Let $(\xi, t, x, S) \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R}^{2}$ and let $\left\{\left(\xi_{j}, t_{j}, x_{j}, S_{j}\right)\right\}_{j} \subset \mathcal{M}_{\alpha, \beta, \gamma} \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R}^{2}$ be such that $\left|\left(t_{j}, x_{j}, S_{j}\right)-(t, x, S)\right| \rightarrow 0$ and $d\left(\xi_{j}, \xi\right) \rightarrow 0$. Then, there exists $n_{0} \in \mathbb{N}$ such that $(t, x, S) \in[0, T] \times \bar{B}_{n_{0}}(0)$ and $\left(t_{j}, x_{j}, S_{j}\right) \in[0, T] \times \bar{B}_{n_{0}}(0)$ for all $j \geqslant 1$. Let $j_{0} \geqslant 1$ be such that

$$
\begin{equation*}
d\left(\xi_{j}, \xi\right)<\sum_{n=n_{0}}^{\infty} \frac{1}{2^{n+1}} \text { for all } j \geqslant j_{0} \tag{2.3}
\end{equation*}
$$

Then, for every $j \geqslant j_{0}$ we have $\inf \left\{1,\left\|\xi_{j}-\xi\right\|_{\infty, n_{0}}\right\}=\left\|\xi_{j}-\xi\right\|_{\infty, n_{0}}$. Indeed, fix any $j \geqslant j_{0}$. If $\inf \left\{1,\left\|\xi_{j}-\xi\right\|_{\infty, n_{0}}\right\}=1$ then $\inf \left\{1,\left\|\xi_{j}-\xi\right\|_{\infty, n}\right\}=1$ for all $n \geqslant n_{0}$ because $\left\|\xi_{j}-\xi\right\|_{\infty, n_{0}} \leqslant$ $\left\|\xi_{j}-\xi\right\|_{\infty, n}$ for all $n \geqslant n_{0}$. Hence

$$
d\left(\xi_{j}, \xi\right) \geqslant \sum_{n=n_{0}}^{\infty} \frac{1}{2^{n+1}}
$$

which contradicts (2.3). Using (2.1) we deduce that

$$
\begin{aligned}
\left|\mathcal{E}\left(\xi_{j}, t_{j}, x_{j}, S_{j}\right)-\mathcal{E}(\xi, t, x, S)\right| \leqslant & \left|\xi_{j}\left(t_{j}, x_{j}, S_{j}\right)-\xi_{j}(t, x, S)\right|+\left|\left(\xi_{j}-\xi\right)(t, x, S)\right| \\
\leqslant & \gamma\left(\left|t_{j}-t\right|+\left|x_{j}-x\right|\right)+L\left|S_{j}-S\right|\left(1+\left|S_{j}\right|+|S|\right) \\
& +\left\|\xi_{j}-\xi\right\|_{\infty, n_{0}} \\
\leqslant & \gamma\left(\left|t_{j}-t\right|+\left|x_{j}-x\right|\right)+L C_{n_{0}}\left|S_{j}-S\right|+2^{n_{0}+1} d\left(\xi_{j}, \xi\right)
\end{aligned}
$$

for all $j \geqslant j_{0}$ with $\left.C_{n_{0}}:=\sup \left\{1+\left|S_{j}\right|+|S|: j \geqslant j_{0}\right\} \in\right] 0, \infty$. Letting $j \rightarrow \infty$ we deduce that $\left|\mathcal{E}\left(\xi_{j}, t_{j}, x_{j}, S_{j}\right)-\mathcal{E}(\xi, t, x, S)\right| \rightarrow 0$. Hence $\mathcal{E}$ is continuous.

Let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain. In what follows, we set

$$
\mathcal{H}_{0}^{1}:=\left\{u \in H^{1}(] 0, T\left[; L^{2}(\Omega ; \mathbb{R})\right): u(0, \cdot)=0\right\}
$$

Let $\mathcal{X}$ be a polish subspace of $\left(\mathcal{M}_{\alpha, \beta, \gamma}, d\right)$ and let $\mathcal{Y}(\Omega ; \mathcal{X})$ be the class of Young measures on $\Omega \times \mathcal{X}$. For each $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ and each $u \in \mathcal{H}_{0}^{1}$ we consider the following two classes of functions:

$$
\begin{aligned}
& \mathcal{H}_{0, \mu}^{1}:=\left\{U \in H^{1}(] 0, T\left[; L_{\mu}^{2}(\Omega \times \mathcal{X} ; \mathbb{R})\right): U(0, \cdot, \cdot)=0\right\} \\
& \mathcal{S}_{\mu}(u):=\left\{U \in \mathcal{H}_{0, \mu}^{1}: \int_{\mathcal{X}} U(t, x, \xi) d \mu_{x}(\xi)=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega\},
\end{aligned}
$$

where $\left\{\mu_{x}\right\}_{x \in \Omega}$ denotes the desintegration of $\mu$ on $\Omega \times \mathcal{X}$. Let $\theta: \mathbb{R} \rightarrow[0, \infty[$ be a convex function of 2-polynomial growth, i.e.,
$\left(\mathrm{C}_{4}\right) \theta(y) \leqslant c\left(1+|y|^{2}\right)$ for all $y \in \mathbb{R}$ and some $\left.c \in\right] 0, \infty[$.
Remark 2.2. Under $\left(\mathrm{C}_{4}\right)$ as $\theta$ is convex we can assert that there exists $C>0$ such that

$$
\begin{equation*}
\left|\theta\left(y_{1}\right)-\theta\left(y_{2}\right)\right| \leqslant C\left|y_{1}-y_{2}\right|\left(1+\left|y_{1}\right|+\left|y_{2}\right|\right) \tag{2.4}
\end{equation*}
$$

for all $y_{1}, y_{2} \in \mathbb{R}$.
To every $u \in \mathcal{H}_{0}^{1}$ and every $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ we associate the integral functional $\phi_{\mu}: \mathcal{S}_{\mu}(u) \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\phi_{\mu}(U):=\int_{] 0, T[\times \Omega \times \mathcal{X}} \xi(t, x, U(t, x, \xi), \dot{U}(t, x, \xi)) d t \otimes d \mu(x, \xi)+\int_{\Omega \times \mathcal{X}} \theta(U(T, x, \xi)) d \mu(x, \xi) \tag{2.5}
\end{equation*}
$$

and we consider the functional $\Phi_{\mu}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by

$$
\begin{equation*}
\Phi_{\mu}(u):=\inf \left\{\phi_{\mu}(U): U \in \mathcal{S}_{\mu}(u)\right\} \tag{2.6}
\end{equation*}
$$

Remark 2.3. By Proposition 2.1, the evaluation map $\mathcal{E}$ is continuous, and so the map $(t, x, \xi) \mapsto \xi(t, x, U(t, x, \xi), \dot{U}(t, x, \xi))$ is $(\mathcal{B}(] 0, T[) \otimes \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathbb{R}))$-measurable. It follows that the integral functional $\phi_{\mu}$ is well defined.

Remark 2.4. From $\left(\mathrm{C}_{2}\right)$ and $\left(\mathrm{C}_{4}\right)$ it is easy to see that

$$
\begin{aligned}
\alpha \int_{] 0, T[\times \Omega \times \mathcal{X}}\left(|(U, \dot{U})|^{2}-1\right) d t \otimes d \mu(x, \xi) \leqslant \phi_{\mu}(U) \leqslant \beta & \int_{] 0, T[\times \Omega \times \mathcal{X}}\left(|(U, \dot{U})|^{2}+1\right) d t \otimes d \mu(x, \xi) \\
& +c \int_{\Omega \times \mathcal{X}}\left(1+|U(T, x)|^{2}\right) d \mu(x, \xi)
\end{aligned}
$$

for all $U \in \mathcal{S}_{\mu}(u)$ with $u \in \mathcal{H}_{0}^{1}$, where $\left.\alpha, \beta, c \in\right] 0, \infty\left[\right.$ are given by $\left(\mathrm{C}_{2}\right)$ and $\left(\mathrm{C}_{4}\right)$.
Let us set $\mathcal{F}:=\left\{\Phi_{\mu}: \mu \in \mathcal{Y}(\Omega ; \mathcal{X})\right\}$.

Remark 2.5. From Remark 2.4 it is easy to see that every $F \in \mathcal{F}$ satisfies the following growth conditions:

$$
\begin{aligned}
\alpha \int_{] 0, T[\times \Omega}\left(|(u, \dot{u})|^{2}-1\right) d t \otimes d x \leqslant F(u) \leqslant & \beta \int_{] 0, T[\times \Omega}\left(|(u, \dot{u})|^{2}+1\right) d t \otimes d x \\
& +c \int_{\Omega}\left(1+|u(T, x)|^{2}\right) d x
\end{aligned}
$$

for all $u \in \mathcal{H}_{0}^{1}$.
Let us recall the definition of $\Gamma$-convergence (see [DM93, Bra06] for more details).
Definition 2.6 ( $\Gamma$-convergence). Given $\left\{F_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{F}$ and $F \in \mathcal{F}$ we say that $\left\{F_{\varepsilon}\right\}_{\varepsilon} \Gamma$-converges to $F$ at $u \in \mathcal{H}_{0}^{1}$ as $\varepsilon \rightarrow 0$, and we write $\left.\left(\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}\right)\right)(u)=F(u)$, if the following two assertions hold:
$\Gamma$-lower bound at $u$ : for every every $\left\{u_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{H}_{0}^{1}$, if $u_{\varepsilon} \xrightarrow{H^{1}} u$ then

$$
\varliminf_{\varepsilon \rightarrow 0} F_{\varepsilon}\left(u_{\varepsilon}\right) \geqslant F(u)
$$

$\Gamma$-upper bound at $u$ : there exists $\left\{v_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{H}_{0}^{1}$ such that $v_{\varepsilon} \xrightarrow{H^{1}} u$ and

$$
\varlimsup_{\varepsilon \rightarrow 0} F_{\varepsilon}\left(v_{\varepsilon}\right) \leqslant F(u)
$$

When $\left.\left(\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}\right)\right)(u)=F(u)$ for all $u \in \mathcal{H}_{0}^{1}$ we say that $\left\{F_{\varepsilon}\right\}_{\varepsilon} \Gamma$-converges to $F$ as $\varepsilon \rightarrow 0$, and we write $\Gamma$ - $\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}=F$.

The main result of the paper is the following. (This result is a new and more complete version of [MV02, Theorem 3] in the specific case of the one-dimensional distributional derivative.)

Theorem 2.7. Under $\left(\mathrm{C}_{1}\right)$, $\left(\mathrm{C}_{2}\right),\left(\mathrm{C}_{3}\right)$ and $\left(\mathrm{C}_{4}\right)$ the map $\mu \mapsto \Phi_{\mu}$ is continuous from $\mathcal{Y}(\Omega ; \mathcal{X})$ endowed with the narrow convergence to $\mathcal{F}$ endowed with the $\Gamma$-convergence, i.e., for every $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ and every $\left\{\mu_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{Y}(\Omega ; \mathcal{X})$, if $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$ then

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\varepsilon}}=\Phi_{\mu}
$$

2.2. From local to non-local functionals. It is easy to see that if $\mu=d x \otimes \delta_{f(x)}$ with $f: \Omega \rightarrow \mathcal{X}$ a $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$-measurable function, then for any $u \in \mathcal{H}_{0}^{1}$ one has

$$
\phi_{\mu}(U)=\int_{] 0, T[\times \Omega} f(x)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\int_{\Omega} \theta(u(T, x)) d x
$$

for all $U \in \mathcal{S}_{\mu}(u)$. Hence

$$
\begin{equation*}
\Phi_{\mu}(u)=\int_{] 0, T[\times \Omega} f(x)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\int_{\Omega} \theta(u(T, x)) d x \tag{2.7}
\end{equation*}
$$

for all $u \in \mathcal{H}_{0}^{1}$. In what follows, when $\mu=d x \otimes \delta_{f(x)}$ we use the notation " $\Phi_{f}$ " instead of " $\Phi_{\mu}$ ". As a direct consequence of Theorem 2.7 we have

Corollary 2.8. Assume that $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{2}\right),\left(\mathrm{C}_{3}\right)$ and $\left(\mathrm{C}_{4}\right)$ hold and, for each $\varepsilon>0$, consider $a(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$-measurable function $f_{\varepsilon}: \Omega \rightarrow \mathcal{X}$. If there exists $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ such that $d x \otimes \delta_{f_{\varepsilon}}(x) \xrightarrow{\text { nar }} \mu$ then

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} \Phi_{f_{\varepsilon}}=\Phi_{\mu}
$$

In corollary 2.8, although the functionals $\Phi_{f_{\varepsilon}}$ are local, i.e., are integral functionals, the $\Gamma$-limit $\Phi_{\mu}$ is in general non-local (here infimum of integral functionals). However, since $d x \otimes \delta_{f_{\varepsilon}}(x) \xrightarrow{\text { nar }} d x \otimes \delta_{f(x)}$ if and only if $\left\{f_{\varepsilon}\right\}_{\varepsilon}$ converges in measure to $f$ (see [ABM14, Proposition 4.3.8] and also [Val90, Proposition 6]), we have

Corollary 2.9. Assume that $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{2}\right),\left(\mathrm{C}_{3}\right)$ and $\left(\mathrm{C}_{4}\right)$ hold and, for each $\varepsilon>0$, consider a $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$-measurable function $f_{\varepsilon}: \Omega \rightarrow \mathcal{X}$. If there exists a $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$-measurable function $f: \Omega \rightarrow \mathcal{X}$ such that $\left\{f_{\varepsilon}\right\}_{\varepsilon}$ converges in measure to $f$, i.e.,

$$
\lim _{\varepsilon \rightarrow 0} \mathcal{L}_{N}\left(\left\{x \in \Omega: d\left(f_{\varepsilon}(x), f(x)\right)>\eta\right\}\right)=0 \quad \text { for all } \eta>0
$$

where $d$ is the metric defined by (2.2), then

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} \Phi_{f_{\varepsilon}}=\Phi_{f}
$$

2.3. Non-diffusive reaction differential equations. For each $\varepsilon>0$, let $\psi_{\varepsilon}:[0, T] \times$ $\mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be a $\left(\mathcal{B}([0, T]) \otimes \mathcal{B}\left(\mathbb{R}^{N}\right) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})\right)$-measurable function such that:
$\left(\mathrm{A}_{1}\right) \psi_{\varepsilon}(t, x, \cdot) \in C^{1}(\mathbb{R})$ for $d t \otimes d x$-a.a. $\left.(t, x) \in\right] 0, T[\times \Omega$;
$\left(\mathrm{A}_{2}\right) \psi_{\varepsilon}(t, x, \cdot)$ is convex for $d t \otimes d x$-a.a. $\left.(t, x) \in\right] 0, T[\times \Omega$;
$\left(\mathrm{A}_{3}\right)$ there exist $c_{1}, C_{1}>0$ (which does not depend on $\varepsilon$ ) such that

$$
c_{1}\left(|s|^{2}-1\right) \leqslant \psi_{\varepsilon}(\cdot, \cdot, s) \leqslant C_{1}\left(|s|^{2}+1\right)
$$

for all $s \in \mathbb{R}$;
$\left(\mathrm{A}_{4}\right)$ there exists $\delta:\left[0, \infty\left[\rightarrow\left[0, \infty\left[\right.\right.\right.\right.$ (which does not depend on $\varepsilon$ ) with $\lim _{r \rightarrow 0} \delta(r)=0$ such that

$$
\left|\psi_{\varepsilon}\left(t_{1}, \cdot, \cdot\right)-\psi_{\varepsilon}\left(t_{2}, \cdot, \cdot\right)\right| \leqslant \delta\left(\left|t_{1}-t_{2}\right|\right)
$$

for all $t_{1}, t_{2} \in[0, T]$;
( $\mathrm{A}_{5}$ ) there exists $L>0$ (which does not depend on $\varepsilon$ ) such that

$$
\left|\frac{\partial \psi_{\varepsilon}}{\partial s}\left(\cdot, \cdot, s_{1}\right)-\frac{\partial \psi_{\varepsilon}}{\partial s}\left(\cdot, \cdot, s_{2}\right)\right| \leqslant L\left|s_{1}-s_{2}\right|
$$

for all $s_{1}, s_{2} \in \mathbb{R}$,
and consider the non-diffusive reaction differential equation defined in $H^{1}(] 0, T\left[; L^{2}(\Omega ; \mathbb{R})\right)$ by

$$
\left(\mathrm{R}_{\varepsilon}\right) \quad\left\{\begin{array}{l}
\left.-\frac{\partial u_{\varepsilon}}{\partial t}(t, x)=\frac{\partial \psi_{\varepsilon}}{\partial s}\left(t, x, u_{\varepsilon}(t, x)\right) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega \\
u_{\varepsilon}(0, \cdot)=0
\end{array}\right.
$$

Remark 2.10. Under $\left(\mathrm{A}_{1}\right),\left(\mathrm{A}_{2}\right),\left(\mathrm{A}_{3}\right),\left(\mathrm{A}_{4}\right)$ and $\left(\mathrm{A}_{5}\right)$, it is well known that every $\left(\mathrm{R}_{\varepsilon}\right)$ has a unique solution $u_{\varepsilon} \in \mathcal{H}_{1}^{0}$ (for a proof, see for instance [AHM18, Theorem 2.3]).

The following proposition makes clear the link between non-diffusive reaction differential equations and integral functionals of type (2.7).
Proposition 2.11. If $\left(\mathrm{A}_{1}\right),\left(\mathrm{A}_{2}\right),\left(\mathrm{A}_{3}\right),\left(\mathrm{A}_{4}\right)$ and $\left(\mathrm{A}_{5}\right)$ hold then, for every $\varepsilon>0$, the nondiffusive reaction differential equation $\left(\mathrm{R}_{\varepsilon}\right)$ admits a unique solution $u_{\varepsilon} \in \mathcal{H}_{0}^{1}$. Moreover, there exist $\alpha, \beta>0$ and $\gamma:\left[0, \infty\left[\rightarrow\left[0, \infty\left[\right.\right.\right.\right.$ with $\lim _{r \rightarrow 0} \gamma(r)=0$ such that every $u_{\varepsilon}$ minimizes the functional $\Phi_{f_{\varepsilon}}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by

$$
\Phi_{f_{\varepsilon}}(u)=\int_{00, T[\times \Omega} f_{\varepsilon}(x)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\int_{\Omega} \theta(u(T, x)) d x
$$

with $f_{\varepsilon}: \Omega \rightarrow \mathcal{M}_{\alpha, \beta, \gamma}$ (here $\mathcal{X}=\mathcal{M}_{\alpha, \beta, \gamma}$ ) and $\theta: \mathbb{R} \rightarrow[0, \infty[$ respectively defined by

$$
\begin{equation*}
f_{\varepsilon}(x)(t, y, s, \dot{s}):=\psi_{\varepsilon}(t, x, s)+\psi_{\varepsilon}^{*}(t, x,-\dot{s}) \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\theta(s):=\frac{1}{2}|s|^{2}, \tag{2.9}
\end{equation*}
$$

where $\psi_{\varepsilon}^{*}$ denotes the Legendre-Fenchel conjugate of $\psi_{\varepsilon}$ with respect to the third variable.
Proof of Proposition 2.11. For $\varepsilon>0$ and define $\Psi_{\varepsilon}:[0, T] \times L^{2}(\Omega ; \mathbb{R}) \rightarrow \mathbb{R}$ by

$$
\Psi_{\varepsilon}(t, v):=\int_{\Omega} \psi_{\varepsilon}(t, x, v(x)) d x
$$

(Note that $\Psi_{\varepsilon}(t, \cdot)$ is convex by $\left(\mathrm{A}_{2}\right)$.) Then $\left(\mathrm{R}_{\varepsilon}\right)$ is equivalent to the following differential inclusion defined in $\mathcal{H}_{0}^{1}$ by

$$
\left.\left(\mathrm{D}_{\varepsilon}\right) \quad-\dot{u}_{\varepsilon}(t, \cdot) \in \partial \Psi_{\varepsilon}\left(t, u_{\varepsilon}(t, \cdot)\right) \text { for } d t \text {-a.a. } t \in\right] 0, T[,
$$

where $\partial \Psi_{\varepsilon}(t, \cdot)$ denotes the subdifferential of $\Psi_{\varepsilon}(t, \cdot)$. According to Fenchel's extremality condition (see [ABM14, Proposition 9.5.1]), we see that $\left(\mathrm{D}_{\varepsilon}\right)$ is equivalent to

$$
\left.\Psi_{\varepsilon}\left(t, u_{\varepsilon}(t, \cdot)\right)+\Psi_{\varepsilon}^{*}\left(t,-\dot{u}_{\varepsilon}(t, \cdot)\right)+\left\langle u_{\varepsilon}(t, \cdot), \dot{u}_{\varepsilon}(t, \cdot)\right\rangle=0 \text { for } d t \text {-a.a. } t \in\right] 0, T[,
$$

where $\Psi_{\varepsilon}^{*}$ denotes the Legendre-Fenchel conjugate of $\Psi_{\varepsilon}$ with respect to the second variable. But, by Legendre-Fenchel's inequality, $\Psi_{\varepsilon}(t, v)+\Psi_{\varepsilon}^{*}(t,-w)+\langle w, v\rangle \geqslant 0$ for $d t$-a.a. $\left.t \in\right] 0, T[$ and all $v, w \in L^{2}(\Omega ; \mathbb{R})$. Consequently, noticing that $\frac{d}{d t}\left\|u_{\varepsilon}(t, \cdot)\right\|_{L^{2}(\Omega ; \mathbb{R})}^{2}=2\left\langle u_{\varepsilon}(t, \cdot), \dot{u}_{\varepsilon}(t, \cdot)\right\rangle$ and defining $F_{\varepsilon}: \mathcal{H}_{0}^{1} \rightarrow[0, \infty]$ by

$$
F_{\varepsilon}(u):=\int_{0}^{T}\left(\Psi_{\varepsilon}(t, u(t, \cdot))+\Psi_{\varepsilon}^{*}(t,-\dot{u}(t, \cdot))+\frac{1}{2} \frac{d}{d t}\|u(t, \cdot)\|_{L^{2}(\Omega ; \mathbb{R})}^{2}\right) d t
$$

it follows that $\left(\mathrm{D}_{\varepsilon}\right)$ is equivalent to $F_{\varepsilon}\left(u_{\varepsilon}\right)=0$. (More precisely, we see that $\left(\mathrm{D}_{\varepsilon}\right)$ is equivalent to $F_{\varepsilon}\left(u_{\varepsilon}\right) \leqslant 0$.) Thus, $u_{\varepsilon}$ solves $\left(\mathrm{R}_{\varepsilon}\right)$ if and only if $u_{\varepsilon}$ minimizes $F_{\varepsilon}$.
On the other hand, as $\psi_{\varepsilon}$ satisfies $\left(\mathrm{A}_{3}\right)$ and $\left(\mathrm{A}_{4}\right)$, so is its Legendre-Fenchel conjugate $\psi_{\varepsilon}^{*}$ with other constants $\hat{c}_{1}, \hat{C}_{1}>0$, instead of $c_{1}, C_{1}>0$, but with the same function $\delta$. Thus $f_{\varepsilon}(x) \in \mathcal{M}_{\alpha, \beta, \gamma}$ for all $x \in \Omega$ (with suitable constants $\alpha, \beta>0$ and a suitable function $\gamma$ ) where, for each $(t, y, s, \dot{s}) \in[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \times \mathbb{R}, f_{\varepsilon}(x)(t, y, s, \dot{s})$ is given by (2.8).
Finally, for each $u \in \mathcal{H}_{0}^{1}$, as $u(0, \cdot)=0$, by using Legendre-Fenchel's calculus, it is easy to see that

$$
F_{\varepsilon}(u)=\int_{] 0, T[\times \Omega} f_{\varepsilon}(x)(t, x, u(x, t), \dot{u}(t, x)) d x \otimes d t+\int_{\Omega} \theta(u(T, x)) d x
$$

where $\theta: \mathbb{R} \rightarrow\left[0, \infty\left[\right.\right.$ is given by (2.9), which means that $F_{\varepsilon}=\Phi_{f_{\varepsilon}}$.
As a direct consequence of Proposition 2.11 and Corollary 2.8 we obtain the following result.
Corollary 2.12. Assume that the assumptions of Proposition 2.11 are satisfied and, for each $\varepsilon>0$, let $u_{\varepsilon} \in \mathcal{H}_{0}^{1}$ be the solution of $\left(\mathrm{R}_{\varepsilon}\right)$. If there exists $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ such that $d x \otimes \delta_{f_{\varepsilon}}(x) \xrightarrow{\text { nar }} \mu$, where $f_{\varepsilon}: \Omega \rightarrow \mathcal{M}_{\alpha, \beta, \gamma}$ is given by (2.8), then any cluster point of $\left\{u_{\varepsilon}\right\}_{\varepsilon}$ with respect to the weak convergence in $\mathcal{H}_{0}^{1}$ is a minimizer of $\Phi_{\mu}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ defined by (2.6) with $\theta: \mathbb{R} \rightarrow[0, \infty[$ given by (2.9).

Thus, the results obtained in [Mas93, Toa99] concerning non-local effects induced by homogenization can be seen as a particular case of Corollary 2.12 (see $\S 5.2$ for more details). Note that if we further assume that
$\left(\mathrm{A}_{6}\right)$ for every $\varepsilon>0, \psi_{\varepsilon}$ does not depend on $t$, and there exists $\psi: \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ such that for $d x$-a.e. $x \in \mathbb{R}^{N}, \psi(x, \cdot) \in C^{1}(\mathbb{R}), \psi(x, \cdot)$ is convex and $\psi(x, s)=\lim _{\varepsilon \rightarrow 0} \psi_{\varepsilon}(x, s)$ for all $s \in \mathbb{R}$,
then the non-local effects disappear. More precisely, we have the following proposition which states a stability result for sequences of non-diffusive reaction differential equations of type $\left(\mathrm{R}_{\varepsilon}\right)$.

Proposition 2.13. Under the assumptions of Proposition 2.11, if moreover $\left(\mathrm{A}_{6}\right)$ holds then (up to a subsequence) the sequence $\left\{u_{\varepsilon}\right\}_{\varepsilon}$ of solutions of $\left(\mathrm{R}_{\varepsilon}\right)$ weakly converges in $\mathcal{H}_{0}^{1}$ to the solution $u$ of the following non-diffusive reaction differential equation:
(R) $\left\{\begin{array}{l}\left.-\frac{\partial u}{\partial t}(t, x)=\frac{\partial \psi}{\partial s}(x, u(t, x)) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega \\ u(0, \cdot)=0 .\end{array}\right.$

Proof of Proposition 2.13. For each $\varepsilon>0$, let $\Psi_{\varepsilon}, \Psi: L^{2}(\Omega ; \mathbb{R}) \rightarrow \mathbb{R}$ be defined by

$$
\left\{\begin{aligned}
\Psi_{\varepsilon}(v) & :=\int_{\Omega} \psi_{\varepsilon}(x, v(x)) d x \\
\Psi(v) & :=\int_{\Omega} \psi(x, v(x)) d x
\end{aligned}\right.
$$

(Note that, by $\left(\mathrm{A}_{6}\right), \psi_{\varepsilon}$ does not depend on $t$.) We first prove that

$$
\begin{equation*}
\mathrm{M}-\lim _{\varepsilon \rightarrow 0} \Psi_{\varepsilon}=\Psi \tag{2.10}
\end{equation*}
$$

where the symbol "M-lim" denotes the Mosco-limit". For any $\varepsilon>0$ and any $\lambda>0$, let $\Psi_{\varepsilon}^{\lambda}, \Psi^{\lambda}: L^{2}(\Omega ; \mathbb{R}) \rightarrow \mathbb{R}$ be the $\lambda$-Moreau-Yosida approximation of $\Psi_{\varepsilon}$ and $\Psi$. By using an

[^1]interchange argument of infimum and integral (see [AHM03]) it easily seen that
\[

\left\{$$
\begin{array}{l}
\Psi_{\varepsilon}^{\lambda}(v)=\int_{\Omega} \psi_{\varepsilon}^{\lambda}(x, v(x)) d x  \tag{2.11}\\
\Psi^{\lambda}(v)=\int_{\Omega} \psi^{\lambda}(x, v(x)) d x
\end{array}
$$\right.
\]

where $\psi_{\varepsilon}^{\lambda}$ and $\psi^{\lambda}$ denotes the $\lambda$-Moreau-Yosida approximation of $\psi_{\varepsilon}$ and $\psi$ with respect to the second variable. From $\left(\mathrm{A}_{1}\right),\left(\mathrm{A}_{2}\right)$ and $\left(\mathrm{A}_{6}\right)$ we deduce that $\Psi$ and $\Psi_{\varepsilon}$ are closed, convex and proper. Consequently, to prove (2.10) it is equivalent to show that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \Psi_{\varepsilon}^{\lambda}(v)=\Psi^{\lambda}(v) \text { for all } \lambda>0 \text { and all } v \in L^{2}(\Omega ; \mathbb{R}) \tag{2.12}
\end{equation*}
$$

(see [Att84, Theorem 3.26]). Taking (2.11) and $\left(\mathrm{A}_{3}\right)$ into account, by Lebesgue's dominated convergence theorem, we see that for establishing (2.10) it suffices to prove that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \psi_{\varepsilon}^{\lambda}(x, s)=\psi^{\lambda}(x, s) \text { for } d x \text {-a.e. } x \in \mathbb{R}^{N}, \text { all } \lambda>0 \text { and all } s \in \mathbb{R} \text {. } \tag{2.13}
\end{equation*}
$$

But $\left(\mathrm{A}_{1}\right),\left(\mathrm{A}_{2}\right)$ and $\left(\mathrm{A}_{6}\right)$ implies that for $d x$-a.e. $x \in \mathbb{R}^{N}, \psi_{\varepsilon}(x, \cdot)$ and $\psi$ are closed, convex and proper, and so (2.13) holds if and only if

$$
\begin{equation*}
\mathrm{M}-\lim _{\varepsilon \rightarrow 0} \psi_{\varepsilon}(x, \cdot)=\psi(x, \cdot) \text { for } d x \text {-a.e. } x \in \mathbb{R}^{N} . \tag{2.14}
\end{equation*}
$$

Fix $x \in \mathbb{R}^{N}$ and fix any $\varepsilon>0$. From $\left(\mathrm{A}_{1}\right)$ and $\left(\mathrm{A}_{3}\right)$ we deduce that there exists $C>0$ (which does not depend on $\varepsilon$ ) such that

$$
\begin{equation*}
\left|\psi_{\varepsilon}\left(x, s_{1}\right)-\psi_{\varepsilon}\left(x, s_{2}\right)\right| \leqslant C\left|s_{1}-s_{2}\right|\left(1+\left|s_{1}\right|+\left|s_{2}\right|\right) \tag{2.15}
\end{equation*}
$$

for all $s_{1}, s_{2} \in \mathbb{R}$. Let $s \in \mathbb{R}$ and let $\left\{s_{\varepsilon}\right\}_{\varepsilon} \subset \mathbb{R}$. On one hand, by $\left(\mathrm{A}_{6}\right)$ we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \psi_{\varepsilon}(x, s)=\psi(x, s) \tag{2.16}
\end{equation*}
$$

which gives the M-upper bound of (2.14). On the other hand, from (2.15) it follows that

$$
\psi_{\varepsilon}\left(x, s_{\varepsilon}\right) \leqslant \psi_{\varepsilon}(x, s)-C\left|s_{\varepsilon}-s\right|\left(1+\left|s_{\varepsilon}\right|+|s|\right)
$$

for all $\varepsilon>0$, and so, by using (2.16), if $s_{\varepsilon} \rightarrow s$ in $\mathbb{R}$ then $\underline{\lim }_{\varepsilon \rightarrow 0} \psi_{\varepsilon}\left(x, s_{\varepsilon}\right) \geqslant \psi(x, s)$, which gives the M-lower bound of (2.14). Thus (2.10) is proved.
For each $\varepsilon>0$, let $G_{\varepsilon}, G: L^{2}(] 0, T\left[; L^{2}(\Omega ; \mathbb{R})\right) \rightarrow \mathbb{R}$ be defined by

$$
\left\{\begin{aligned}
G_{\varepsilon}(v) & :=\int_{0}^{T} \Psi_{\varepsilon}(v(t, \cdot)) d t \\
G(v) & :=\int_{0}^{T} \Psi(v(t, \cdot)) d t
\end{aligned}\right.
$$

From (2.10) we deduce that

$$
\begin{equation*}
\mathrm{M}-\lim _{\varepsilon \rightarrow 0} G_{\varepsilon}=G \tag{2.17}
\end{equation*}
$$

(see [ABM14, Lemma 17.4.8]). Hence (see [ABM14, Theorem 17.4.3])

$$
\begin{equation*}
\mathrm{M}-\lim _{\varepsilon \rightarrow 0} G_{\varepsilon}^{*}=G^{*} \tag{2.18}
\end{equation*}
$$

with $G_{\varepsilon}^{*}, G^{*}: L^{2}(] 0, T\left[; L^{2}(\Omega ; \mathbb{R})\right) \rightarrow \mathbb{R}$ denoting the Legendre-Fenchel conjugate of $G_{\varepsilon}$ and $G$, where, by using Legendre-Fenchel's calculus, one has

$$
\left\{\begin{array}{l}
G_{\varepsilon}^{*}(v)=\int_{0}^{T} \Psi_{\varepsilon}^{*}(v(t, \cdot)) d t \\
G^{*}(v)=\int_{0}^{T} \Psi^{*}(v(t, \cdot)) d t
\end{array}\right.
$$

with $\Psi_{\varepsilon}^{*}, \Psi^{*}: L^{2}(\Omega ; \mathbb{R}) \rightarrow \mathbb{R}$ denoting the Legendre-Fenchel conjugate of $\Psi_{\varepsilon}$ and $\Psi$. For each $\varepsilon>0$, let $F_{\varepsilon}, F: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ be defined by

$$
\left\{\begin{aligned}
F_{\varepsilon}(v) & :=G_{\varepsilon}(v)+G_{\varepsilon}^{*}(-\dot{v})+\frac{1}{2}\|v(T, \cdot)\|_{L^{2}(\Omega ; \mathbb{R})}^{2} \\
F(v) & :=G(v)+G^{*}(-\dot{v})+\frac{1}{2}\|v(T, \cdot)\|_{L^{2}(\Omega ; \mathbb{R})}^{2} .
\end{aligned}\right.
$$

Arguing as in the proof of Proposition 2.11 we can assert that:

$$
\begin{align*}
\text { for each } \varepsilon>0, & v \text { solves }\left(\mathrm{R}_{\varepsilon}\right) \text { if and only if } F_{\varepsilon}(v) \leqslant 0  \tag{2.19}\\
& v \text { solves }(\mathrm{R}) \text { if and only if } F(v) \leqslant 0 \tag{2.20}
\end{align*}
$$

Let $\left\{u_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{H}_{0}^{1}$ be a sequence of solutions of $\left(\mathrm{R}_{\varepsilon}\right)$. Taking Remark 2.5 into account, we see that (up to a subsequence) there exists $u \in \mathcal{H}_{0}^{1}$ such that $u_{\varepsilon} \xrightarrow{H^{1}} u$, i.e., $u_{\varepsilon} \xrightarrow{L^{2}} u$ and $\dot{u}_{\varepsilon} \xrightarrow{L^{2}} \dot{u}$. Then, by (2.19), one has $F_{\varepsilon}\left(u_{\varepsilon}\right) \leqslant 0$ for all $\varepsilon>0$. Letting $\varepsilon \rightarrow 0$ we obtain

$$
\begin{equation*}
\varliminf_{\varepsilon \rightarrow 0} F_{\varepsilon}\left(u_{\varepsilon}\right) \leqslant 0 \tag{2.21}
\end{equation*}
$$

On the other hand, from (2.17) and (2.18) we deduce that:

$$
\begin{align*}
& G(u) \leqslant \varliminf_{\varepsilon \rightarrow 0} G_{\varepsilon}\left(u_{\varepsilon}\right)  \tag{2.22}\\
& G^{*}(-\dot{u}) \leqslant \varliminf_{\varepsilon \rightarrow 0} G_{\varepsilon}^{*}\left(-\dot{u}_{\varepsilon}\right) \tag{2.23}
\end{align*}
$$

As, for any $\varepsilon>0, u_{\varepsilon} \in \mathcal{H}_{0}^{1}$ and $u \in \mathcal{H}_{0}^{1}$ we have $u_{\varepsilon}(0, \cdot)=0$ and $u(0, \cdot)=0$, and so $u_{\varepsilon}(T, \cdot)=$ $\int_{0}^{T} \dot{u}_{\varepsilon}(t, \cdot) d t$ and $u(T, \cdot)=\int_{0}^{T} \dot{u}(t, \cdot) d t$. Since $\dot{u}_{\varepsilon} \xrightarrow{L^{2}} \dot{u}$ it follows that $u_{\varepsilon}(T, \cdot) \xrightarrow{L^{2}} u(T, \cdot)$, and consequently

$$
\begin{equation*}
\|u(T, \cdot)\|_{L^{2}(\Omega ; \mathbb{R})} \leqslant \varliminf_{\varepsilon \rightarrow 0}\left\|u_{\varepsilon}(T, \cdot)\right\|_{L^{2}(\Omega ; \mathbb{R})} \tag{2.24}
\end{equation*}
$$

From (2.22), (2.23) and (2.24) we deduce that $F(u) \leqslant \underline{\lim }_{\varepsilon \rightarrow 0} F_{\varepsilon}\left(u_{\varepsilon}\right)$. Hence $F(u) \leqslant 0$ by (2.21). From (2.20) we conclude that $u$ solves (R).

## 3. Proof of the continuity theorem

In this section we prove Theorem 2.7.

Proof of Theorem 2.7. Let $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ and let $\left\{\mu_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{Y}(\Omega ; \mathcal{X})$ be such that $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$. According to the left inequality in Remark 2.5, to prove that

$$
\begin{equation*}
\Gamma-\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\varepsilon}}=\Phi_{\mu} \tag{3.1}
\end{equation*}
$$

it is equivalent to show that every subsequence of $\left\{\Phi_{\mu_{\varepsilon}}\right\}_{\varepsilon}$ contains a further subsequence which $\Gamma$-converges to $\Phi_{\mu}$ (see [DM93, Proposition 8.17]). Let us consider a subsequence that we still denote by $\left\{\Phi_{\mu_{\varepsilon}}\right\}_{\varepsilon}$. Taking the left inequality in Remark 2.5 into account, by compactness, we can assert that, up to subsequence, $\left\{\Phi_{\mu_{\varepsilon}}\right\}_{\varepsilon} \Gamma$-converges (see [DM93, Corollary 8.12]), i.e., $\Gamma$ - $\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\varepsilon}}=\Psi$. So, to establish (3.1) it is sufficient to prove that

$$
\begin{equation*}
\Psi(u)=\Phi_{\mu}(u) \text { for all } u \in \mathcal{H}_{0}^{1} \tag{3.2}
\end{equation*}
$$

For this, we only need to show that for each $u \in \mathcal{H}_{0}^{1}$ the following two assertions hold:
$\left(\mathrm{G}_{1}\right)$ there exists a subsequence $\left\{\Phi_{\mu_{\sigma(\varepsilon)}}\right\}_{\varepsilon}$ and $\left\{v_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{H}_{0}^{1}$ such that $v_{\varepsilon} \xrightarrow{H^{1}} u$ and

$$
\varlimsup_{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(v_{\varepsilon}\right) \leqslant \Phi_{\mu}(u)
$$

$\left(\mathrm{G}_{2}\right)$ for every $\left\{u_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{H}_{0}^{1}$, if $u_{\varepsilon} \xrightarrow{H^{1}} u$ then

$$
\varliminf_{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right) \geqslant \Phi_{\mu}(u)
$$

where $\left\{\Phi_{\mu_{\sigma(\varepsilon)}}\right\}_{\varepsilon}$ is given by $\left(\mathrm{G}_{1}\right)$.
Indeed, let $u \in \mathcal{H}_{0}^{1}$. According to Definition 2.6, from $\left(\mathrm{G}_{1}\right)$ and $\left(\mathrm{G}_{2}\right)$ we see that $\left\{\Phi_{\mu_{\sigma(\varepsilon)}}\right\}_{\varepsilon}$ $\Gamma$-converges to $\Phi_{\mu}$ at $u$ as $\varepsilon \rightarrow 0$, i.e., $\left(\Gamma\right.$ - $\left.\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\right)(u)=\Phi_{\mu}(u)$. On the other hand, as $\left\{\Phi_{\mu_{\sigma(\varepsilon)}}\right\}_{\varepsilon}$ is a subsequence of $\left\{\Phi_{\left.\mu_{\varepsilon}\right)}\right\}_{\varepsilon}$ we have $\Gamma$ - $\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}=\Gamma$ - $\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\varepsilon}}$, and so $\Gamma$ - $\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}=\Psi$. Hence, in particular, $\left(\Gamma-\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\right)(u)=\Psi(u)$. It follows that $\Psi(u)=\Phi_{\mu}(u)$, which proves (3.2).
Let us fix $u \in \mathcal{H}_{0}^{1}$.
Proof of $\left(\mathbf{G}_{1}\right)$. By using the direct method of the calculus of variations, we can assert that there exists $U \in \mathcal{S}_{\mu}(u)$ such that $U$ is a minimizer of $\Phi_{\mu}(u)$, i.e., $\Phi_{\mu}(u)=\phi_{\mu}(U)$. We then have:

$$
\begin{align*}
u(t, x) & =\int_{\mathcal{X}} U(t, x, \xi) d \mu_{x}(\xi)  \tag{3.3}\\
\dot{u}(t, x) & =\int_{\mathcal{X}} \dot{U}(t, x, \xi) d \mu_{x}(\xi) \tag{3.4}
\end{align*}
$$

where $\left\{\mu_{x}\right\}_{x \in \Omega}$ denotes the desintegration of $\mu$ on $\Omega \times \mathcal{X}$. As $C_{c}^{1}(] 0, T\left[; C_{c}(\Omega \times \mathcal{X})\right)$ is strongly dense in $\mathcal{H}_{0, \mu}^{1}$, for each $\left.\left.\delta \in\right] 0,1\right]$ there exists $U_{\delta} \in C_{c}^{1}(] 0, T\left[; C_{c}(\Omega \times \mathcal{X})\right)$ such that:

$$
\begin{align*}
& \int_{] 0, T[\times \Omega \times \mathcal{X}}\left|U_{\delta}(t, x, \xi)-U(t, x, \xi)\right|^{2} d t \otimes d \mu(x, \xi)<\delta ;  \tag{3.5}\\
& \int_{] 0, T[\times \Omega \times \mathcal{X}}\left|\dot{U}_{\delta}(t, x, \xi)-\dot{U}(t, x, \xi)\right|^{2} d t \otimes d \mu(x, \xi)<\delta . \tag{3.6}
\end{align*}
$$

In particular, one has:

$$
\begin{align*}
& U_{\delta} \xrightarrow{L^{2}} U  \tag{3.7}\\
& \dot{U}_{\delta} \xrightarrow{L^{2}} \dot{U} . \tag{3.8}
\end{align*}
$$

For each $\delta \in] 0,1]$ and each $\varepsilon>0$, let $u_{\delta, \varepsilon} \in \mathcal{H}_{0}^{1}$ be given by

$$
\begin{equation*}
u_{\delta, \varepsilon}(t, x):=\int_{\mathcal{X}} U_{\delta}(t, x, \xi) d \mu_{x}^{\varepsilon}(\xi) \tag{3.9}
\end{equation*}
$$

where $\left\{\mu_{x}^{\varepsilon}\right\}_{x \in \Omega}$ denotes the desintegration of $\mu_{\varepsilon}$ on $\Omega \times \mathcal{X}$. Then

$$
\begin{equation*}
\dot{u}_{\delta, \varepsilon}(t, x):=\int_{\mathcal{X}} \dot{U}_{\delta}(t, x, \xi) d \mu_{x}^{\varepsilon}(\xi) . \tag{3.10}
\end{equation*}
$$

(In what follows, we systematically use the fact that $U_{\delta}, U_{\delta}^{2}, \dot{U}_{\delta}$, and $\dot{U}_{\delta}^{2}$ are test functions for the narrow convergence in $\mathcal{Y}(\Omega ; \mathcal{X})$ because these are bounded Carathéodory integrands on $\Omega \times \mathcal{X}$.)
Since $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$, there exists a mapping $\delta \mapsto \eta_{\delta}$ with $\eta_{\delta} \rightarrow 0$ as $\delta \rightarrow 0$ such that for all $\left.\varepsilon \in] 0, \eta_{\delta}\right]$, one has:

$$
\begin{align*}
& \left.\left|\int_{] 0, T[\times \Omega \times \mathcal{X}}\right| U_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu_{\varepsilon}(x, \xi)-\int_{] 0, T[\times \Omega \times \mathcal{X}}\left|U_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu(x, \xi) \mid<1  \tag{3.11}\\
& \left.\left|\int_{] 0, T[\times \Omega \times \mathcal{X}}\right| \dot{U}_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu_{\varepsilon}(x, \xi)-\int_{] 0, T[\times \Omega \times \mathcal{X}}\left|\dot{U}_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu(x, \xi) \mid<1 \tag{3.12}
\end{align*}
$$

Fix any $\delta \in] 0,1]$ and any $\left.\varepsilon \in] 0, \eta_{\delta}\right]$. Using (3.9) and (3.11) we see that

$$
\begin{aligned}
\int_{] 0, T[\times \Omega}\left|u_{\delta, \varepsilon}(t, x)\right|^{2} d t \otimes d x & =\int_{] 0, T[\times \Omega}\left|\int_{\mathcal{X}} U_{\delta}(t, x, \xi) d \mu_{x}^{\varepsilon}(\xi)\right|^{2} d t \otimes d x \\
& \leqslant \int_{] 0, T[\times \Omega \times \mathcal{X}}\left|U_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu_{\varepsilon}(x, \xi) \\
& \leqslant 1+\int_{] 0, T[\times \Omega \times \mathcal{X}}\left|U_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu(x, \xi)
\end{aligned}
$$

But, taking (3.5) and the left inequality in Remark 2.4 into account and recalling that $U \in \mathcal{S}_{\mu}(u)$, we have

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X}}\left|U_{\delta}(t, x, \xi)\right|^{2} d t \otimes d \mu(x, \xi) & \leqslant 2 \delta+2 \int_{] 0, T[\times \Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d t \otimes d \mu(x, \xi) \\
& \leqslant 2\left(1+\frac{1}{\alpha} \phi_{\mu}(U)+T|\Omega|\right)
\end{aligned}
$$

and so, setting $R:=3+\frac{1}{\alpha} \phi_{\mu}(U)+T|\Omega|$, it follows that

$$
\begin{equation*}
\left.\left.\left.\left.\int_{] 0, T[\times \Omega}\left|u_{\delta, \varepsilon}(t, x)\right|^{2} d t \otimes d x \leqslant R \text { for all } \delta \in\right] 0,1\right] \text { and all } \varepsilon \in\right] 0, \eta_{\delta}\right] \tag{3.13}
\end{equation*}
$$

In the same manner, by considering (3.10), (3.12) and (3.6) instead of (3.9), (3.11) and (3.5), we obtain

$$
\begin{equation*}
\left.\left.\left.\left.\int_{] 0, T[\times \Omega}\left|\dot{u}_{\delta, \varepsilon}(t, x)\right|^{2} d t \otimes d x \leqslant R \text { for all } \delta \in\right] 0,1\right] \text { and all } \varepsilon \in\right] 0, \eta_{\delta}\right] \text {. } \tag{3.14}
\end{equation*}
$$

For each $\delta \in] 0,1]$, consider the mapping $\varepsilon \mapsto \lambda_{\delta}(\varepsilon)$ given by

$$
\lambda_{\delta}(\varepsilon):= \begin{cases}\eta_{\delta} & \text { if } \varepsilon>\eta_{\delta} \\ \varepsilon & \text { if } \varepsilon \leqslant \eta_{\delta}\end{cases}
$$

(Note that $\lambda_{\delta}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.) From (3.13) and (3.14) we deduce that

$$
\begin{equation*}
\left\{u_{\delta, \lambda_{\delta}(\varepsilon)}\right\}_{(\delta, \varepsilon) \in] 0,1] \times] 0, \infty[ } \subset \overline{\mathcal{B}}_{R}(0) \tag{3.15}
\end{equation*}
$$

where $\overline{\mathcal{B}}_{R}(0)$ denotes the closed ball in $\mathcal{H}_{0}^{1}$ centered at 0 with radius $R$. As $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$ we have $\mu_{\lambda_{\delta}(\varepsilon)} \xrightarrow{\text { nar }} \mu$ for all $\left.\left.\delta \in\right] 0,1\right]$, and so

$$
\begin{equation*}
\left.\left.u_{\delta, \lambda_{\delta}(\varepsilon)} \stackrel{L^{2}}{\longrightarrow} \int_{\mathcal{X}} U_{\delta}(\cdot, \cdot, \xi) d \mu_{x}(\xi) \text { for all } \delta \in\right] 0,1\right] \tag{3.16}
\end{equation*}
$$

Taking (3.7) and (3.3) into account, from the above it follows that

$$
\begin{equation*}
\int_{\mathcal{X}} U_{\delta}(\cdot, \cdot, \xi) d \mu_{x}(\xi) \xrightarrow{L^{2}} \int_{\mathcal{X}} U(\cdot, \cdot, \xi) d \mu_{x}(\xi)=u \tag{3.17}
\end{equation*}
$$

By the same arguments, using (3.8) and (3.4) instead of (3.7) and (3.3), we obtain:

$$
\begin{align*}
& \left.\left.\dot{u}_{\delta, \lambda_{\delta}(\varepsilon)} \stackrel{L^{2}}{\longrightarrow} \int_{\mathcal{X}} \dot{U}_{\delta}(\cdot, \cdot, \xi) d \mu_{x}(\xi) \text { for all } \delta \in\right] 0,1\right]  \tag{3.18}\\
& \int_{\mathcal{X}} \dot{U}_{\delta}(\cdot, \cdot, \xi) d \mu_{x}(\xi) \xrightarrow{L^{2}} \int_{\mathcal{X}} U(\cdot, \cdot, \xi) d \mu_{x}(\xi)=\dot{u} \tag{3.19}
\end{align*}
$$

Combining (3.16) with (3.18) and (3.17) with (3.19) we see that

$$
\begin{equation*}
u_{\delta, \lambda_{\delta}(\varepsilon)} \stackrel{H^{1}}{\varepsilon \rightarrow 0} \int_{\mathcal{X}} U_{\delta}(\cdot, \cdot, \xi) d \mu_{x}(\xi) \frac{H^{1}}{\delta \rightarrow 0} u \tag{3.20}
\end{equation*}
$$

According to (3.15) and the fact that in $\overline{\mathcal{B}}_{R}(0)$ the weak convergence is metrizable, we can rewrite (3.20) as follows:

$$
\begin{equation*}
d_{w^{-}} \lim _{\delta \rightarrow 0} d_{w^{-}} \lim _{\varepsilon \rightarrow 0} u_{\delta, \lambda_{\delta}(\varepsilon)}=u \tag{3.21}
\end{equation*}
$$

where $d_{w}$ denotes the metric associated with the weak convergence in $\overline{\mathcal{B}}_{R}(0)$.
On the other hand, by using similar arguments together with (2.1) and (2.4), we can assert that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \lim _{\varepsilon \rightarrow 0} \phi_{\mu_{\lambda_{\delta}(\varepsilon)}}\left(U_{\delta}\right)=\phi_{\mu}(U) \tag{3.22}
\end{equation*}
$$

According to (3.21) and (3.22), by diagonalization there exists a mapping $\varepsilon \mapsto \delta_{\varepsilon}$ with $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that:

$$
\begin{align*}
& d_{w^{-}} \lim _{\varepsilon \rightarrow 0} u_{\delta_{\varepsilon}, \lambda_{\delta_{\varepsilon}}(\varepsilon)}=u, \text { i.e., } u_{\delta_{\varepsilon}, \lambda_{\delta_{\varepsilon}}(\varepsilon)} \xrightarrow{H^{1}} u ;  \tag{3.23}\\
& \lim _{\varepsilon \rightarrow 0} \phi_{\mu_{\lambda_{\delta_{\varepsilon}}(\varepsilon)}}\left(U_{\delta_{\varepsilon}}\right)=\phi_{\mu}(U) . \tag{3.24}
\end{align*}
$$

Since $\phi_{\mu}(U)=\Phi_{\mu}(u)$ and, for each $\varepsilon>0, \phi_{\mu_{\lambda_{\delta_{\varepsilon}}(\varepsilon)}}\left(U_{\delta_{\varepsilon}}\right) \geqslant \Phi_{\mu_{\lambda_{\delta_{\varepsilon}}(\varepsilon)}}\left(u_{\delta_{\varepsilon}, \lambda_{\delta_{\varepsilon}}(\varepsilon)}\right)$ because $U_{\delta_{\varepsilon}} \in$ $C_{c}^{1}(] 0, T\left[; C_{c}(\Omega \times \mathcal{X})\right) \subset \mathcal{H}_{0, \mu_{\lambda_{\delta_{\varepsilon}}(\varepsilon)}^{1}}$ and $\int_{\mathcal{X}} U_{\delta_{\varepsilon}}(\cdot, \cdot, \xi) d \mu_{x}^{\lambda_{\delta_{\varepsilon}(\varepsilon)}}(\xi)=u_{\delta_{\varepsilon}, \lambda_{\delta_{\varepsilon}(\varepsilon)}}$, from (3.24) we deduce that

$$
\begin{equation*}
\varlimsup_{\varepsilon \rightarrow 0} \Phi_{\mu_{\lambda_{\varepsilon}(\varepsilon)}}\left(u_{\delta_{\varepsilon}, \lambda_{\delta_{\varepsilon}(\varepsilon)}}\right) \leqslant \Phi_{\mu}(u) \tag{3.25}
\end{equation*}
$$

Taking (3.23) and (3.25) into account and setting $\sigma(\varepsilon):=\lambda_{\delta_{\varepsilon}}(\varepsilon)$ and $v_{\varepsilon}:=u_{\delta_{\varepsilon}, \lambda_{\delta_{\varepsilon}(\varepsilon)}}$ we have $\sigma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and:

$$
\begin{aligned}
& \frac{v_{\varepsilon}}{\lim _{\varepsilon \rightarrow 0}} \Phi_{\mu_{\sigma(\varepsilon)}}\left(v_{\varepsilon}\right) \leqslant \Phi_{\mu}(u),
\end{aligned}
$$

and the proof of $\left(\mathrm{G}_{1}\right)$ is complete.

Proof of $\left(\mathbf{G}_{2}\right)$. Let $\left\{u_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{H}_{0}^{1}$ be such that $u_{\varepsilon} \xrightarrow{H^{1}} u$. Without loss of generality we can assume that

$$
\begin{equation*}
\varliminf_{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)=\lim _{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)<\infty, \text { and so } \sup _{\varepsilon>0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)<\infty . \tag{3.26}
\end{equation*}
$$

Fix any $\varepsilon>0$. By using the direct method of the calculus of variations, we can assert that there exists $U_{\varepsilon} \in \mathcal{S}_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)$ such that $U_{\varepsilon}$ is a minimizer of $\Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)$, i.e.,

$$
\begin{equation*}
\Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)=\phi_{\mu_{\sigma(\varepsilon)}}\left(U_{\varepsilon}\right) \tag{3.27}
\end{equation*}
$$

with $\phi_{\mu_{\sigma(\varepsilon)}}: \mathcal{S}_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right) \rightarrow \mathbb{R}$ given by (2.5). From (3.26) and the left inequality in $\left(\mathrm{C}_{2}\right)$ we deduce that

$$
\begin{equation*}
\sup _{\varepsilon>0}\left\|\left(U_{\varepsilon}, \dot{U}_{\varepsilon}\right)\right\|_{L_{d t \otimes \mu_{\sigma(\varepsilon)}^{2}}^{2}\left(00, T\left[\times \Omega \times \mathcal{X} ; \mathbb{R}^{2}\right)\right.}<\infty . \tag{3.28}
\end{equation*}
$$

Let $\left.g_{\varepsilon}:\right] 0, T[\times \Omega \times \mathcal{X} \rightarrow] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.$ be defined by

$$
g_{\varepsilon}(t, x, \xi):=\left(t, x, \xi, U_{\varepsilon}(t, x, \xi), \dot{U}_{\varepsilon}(t, x, \xi)\right)
$$

and let $\nu_{\varepsilon}:=g_{\varepsilon}^{\sharp} d t \otimes \mu_{\sigma(\varepsilon)}$. It is clear that $\nu_{\varepsilon} \in \mathcal{Y}(] 0, T\left[\times \Omega ; \mathcal{X} \times \mathbb{R}^{2}\right)$ whose projection measure is $d t \otimes d x$. We claim that $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$ is tight (see Definition A.2). Indeed, given $\eta>0$, as $\left\{\mu_{\sigma(\varepsilon)}\right\}_{\varepsilon}$ is tight, there exists a compact set $\mathcal{K} \subset \mathcal{X}$ such that

$$
\begin{equation*}
\sup _{\varepsilon>0} \mu_{\sigma(\varepsilon)}(\Omega \times(\mathcal{X} \backslash \mathcal{K}))<\frac{\eta}{2 T} \tag{3.29}
\end{equation*}
$$

From (3.28) and Markov's inequality we can assert that there exists $R>0$ such that

$$
\begin{equation*}
\sup _{\varepsilon>0} d t \otimes \mu_{\sigma(\varepsilon)}\left(E_{\varepsilon, R}\right)<\frac{\eta}{2} \tag{3.30}
\end{equation*}
$$

with

$$
E_{\varepsilon, R}:=\{(t, x, \xi) \in] 0, T\left[\times \Omega \times \mathcal{X}:\left(U_{\varepsilon}(t, x, \xi), \dot{U}_{\varepsilon}(t, x, \xi)\right) \in \mathbb{R}^{2} \backslash \bar{B}_{R}(0)\right\}
$$

where $\bar{B}_{R}(0) \subset \mathbb{R}^{2}$ denotes the 2-dimensional closed ball centered at the origin with radius $R$. On the other hand, we have

$$
\begin{aligned}
\nu_{\varepsilon}(] 0, T\left[\times \Omega \times\left(\mathcal{X} \times \mathbb{R}^{2}\right) \backslash\left(\mathcal{K} \times \bar{B}_{R}(0)\right)\right) & \leqslant d t \otimes \mu_{\sigma(\varepsilon)}(] 0, T[\times \Omega \times(\mathcal{X} \backslash \mathcal{K}))+d t \otimes \mu_{\sigma(\varepsilon)}\left(E_{\varepsilon, R}\right) \\
& =T \mu_{\sigma(\varepsilon)}(\Omega \times(\mathcal{X} \backslash \mathcal{K}))+d t \otimes \mu_{\sigma(\varepsilon)}\left(E_{\varepsilon, R}\right)
\end{aligned}
$$

for all $\varepsilon>0$, and consequently

$$
\sup _{\varepsilon>0} \nu_{\varepsilon}(] 0, T\left[\times \Omega \times\left(\mathcal{X} \times \mathbb{R}^{2}\right) \backslash\left(\mathcal{K} \times \bar{B}_{R}(0)\right)\right)<\eta
$$

which proves the claim. Let $h_{\varepsilon}: \Omega \times \mathcal{X} \rightarrow \Omega \times \mathcal{X} \times \mathbb{R}$ defined by

$$
h_{\varepsilon}(x, \xi):=\left(x, \xi, U_{\varepsilon}(T, x, \xi)\right)
$$

and let $\lambda_{\varepsilon}:=h_{\varepsilon}^{\sharp} \mu_{\sigma(\varepsilon)}$. It is clear that $\lambda_{\varepsilon} \in \mathcal{Y}(\Omega ; \mathcal{X} \times \mathbb{R})$. In the same manner we can establish that $\left\{\lambda_{\varepsilon}\right\}_{\varepsilon}$ is tight. From Prokhorov's compactness theorem (see Theorem A.3) we deduce that there exist $\nu \in \mathcal{Y}(] 0, T\left[\times \Omega ; \mathcal{X} \times \mathbb{R}^{2}\right)$ and $\lambda \in \mathcal{Y}(\Omega ; \mathcal{X} \times \mathbb{R})$ such that, up to a subsequence, one has:

$$
\begin{align*}
& \nu_{\varepsilon} \xrightarrow{\mathrm{nar}} \nu  \tag{3.31}\\
& \lambda_{\varepsilon} \xrightarrow{\mathrm{nar}} \lambda \tag{3.32}
\end{align*}
$$

Let $\left.\pi_{] 0, T[\times \Omega \times \mathcal{X}}:\right] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2} \rightarrow\right] 0, T\left[\times \Omega \times \mathcal{X}\left(\right.\right.$ resp. $\left.\pi_{\Omega \times \mathcal{X}}: \Omega \times \mathcal{X} \times \mathbb{R} \rightarrow \Omega \times \mathcal{X}\right)$ be the canonical projection from $] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}(\right.$ resp. $\Omega \times \mathcal{X} \times \mathbb{R}$ ) to $] 0, T[\times \Omega \times \mathcal{X}$ (resp. $\Omega \times \mathcal{X}$ ). From (3.31) and (3.32) it is easy to see that:

$$
\begin{aligned}
& \pi_{] 0, T[\times \Omega \times \mathcal{X}}^{\sharp} \nu=\mathrm{w}-\lim _{\varepsilon \rightarrow 0} \pi_{] 0, T[\times \Omega \times \mathcal{X}}^{\sharp} \nu_{\varepsilon}=d t \otimes \mu ; \\
& \pi_{\Omega \times \mathcal{X}}^{\sharp} \lambda=\mathrm{w}-\lim _{\varepsilon \rightarrow 0} \pi_{] 0, T[\times \Omega \times \mathcal{X}}^{\sharp} \lambda_{\varepsilon}=\mu
\end{aligned}
$$

where "w- lim" denotes the weak limit associated with the weak $\sigma\left(C_{b}^{\prime}, C_{b}\right)$ topology (see [ABM14, Definition 4.2.2]). By using desintegration's theorem (see Theorem A.5) it follows that:

$$
\begin{align*}
& \nu=d t \otimes \mu \otimes \nu_{t, x, \xi}  \tag{3.33}\\
& \lambda=\mu \otimes \lambda_{x, \xi} \tag{3.34}
\end{align*}
$$

where $\left\{\nu_{t, x, \xi}\right\}_{(t, x, \xi) \in] 0, T[\times \Omega \times \mathcal{X}}$ and $\left\{\lambda_{x, \xi}\right\}_{(x, \xi) \in \Omega \times \mathcal{X}}$ are families of probability measures on $\mathbb{R}^{2}$ and $\mathbb{R}$ respectively. We will need the following lemma whose proof is given below.
Lemma 3.1. For each $(t, x, \xi) \in] 0, T[\times \Omega \times \mathcal{X}$, let us denote by $U(t, x, \xi), V(t, x, \xi)$ and $W_{T}(x, \xi)$ the $1^{\text {th }}$ moments of $\nu_{t, x, \xi}$ and $\lambda_{x, \xi}$, i.e.,

$$
\begin{align*}
U(t, x, \xi) & :=\int_{\mathbb{R}^{2}} s d \nu_{t, x, \xi}(s, \dot{s})  \tag{3.35}\\
V(t, x, \xi) & :=\int_{\mathbb{R}^{2}} \dot{s} d \nu_{t, x, \xi}(s, \dot{s})  \tag{3.36}\\
W_{T}(x, \xi) & :=\int_{\mathbb{R}} s_{T} d \lambda_{x, \xi}\left(s_{T}\right) \tag{3.37}
\end{align*}
$$

Then $V=\dot{U}, U \in \mathcal{S}_{\mu}(u)$ and $W_{T}=U(T, \cdot, \cdot)$.

According to (3.27) and the definitions of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$ and $\left\{\lambda_{\varepsilon}\right\}_{\varepsilon}$, we see that

$$
\begin{align*}
\varliminf_{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right)= & \varliminf_{\varepsilon \rightarrow 0} \phi_{\mu_{\sigma(\varepsilon)}}\left(U_{\varepsilon}\right) \\
\geqslant & \varliminf_{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}} \xi\left(t, x, U_{\varepsilon}(t, x, \xi), \dot{U}_{\varepsilon}(t, x, \xi)\right) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) \\
& +\varliminf_{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X}} \theta\left(U_{\varepsilon}(T, x, \xi)\right) d \mu_{\sigma(\varepsilon)}(x, \xi) \\
= & \varliminf_{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \xi(t, x, S) d \nu_{\varepsilon}(t, x, \xi, S) \\
& +\varliminf_{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X} \times \mathbb{R}} \theta\left(s_{T}\right) d \lambda_{\varepsilon}\left(x, \xi, s_{T}\right) . \tag{3.38}
\end{align*}
$$

But, recalling (3.31) and (3.32), by Theorem A.4(i) have:

$$
\begin{align*}
& \varliminf_{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \xi(t, x, S) d \nu_{\varepsilon}(t, x, \xi, S) \geqslant \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \xi(t, x, S) d \nu(t, x, \xi, S)  \tag{3.39}\\
& \varliminf_{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X} \times \mathbb{R}} \theta\left(s_{T}\right) d \lambda_{\varepsilon}\left(x, \xi, s_{T}\right) \geqslant \int_{\Omega \times \mathcal{X} \times \mathbb{R}} \theta\left(s_{T}\right) d \lambda\left(x, \xi, s_{T}\right) \tag{3.40}
\end{align*}
$$

Moreover, by using $\left(\mathrm{C}_{1}\right)$, (3.33) and (3.34), Jensen's inequality and Lemma 3.1, we obtain the following two inequalities:

$$
\begin{align*}
\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \xi(t, x, S) d \nu(t, x, \xi, S) & =\int_{10, T[\times \Omega \times \mathcal{X}}\left(\int_{\mathbb{R}^{2}} \xi(t, x, S) d \nu_{t, x, \xi}(S)\right) d t \otimes d \mu(x, \xi) \\
& \geqslant \int_{] 0, T[\times \Omega \times \mathcal{X}} \xi\left(t, x, \int_{\mathbb{R}^{2}} S d \nu_{t, x, \xi}(S)\right) d t \otimes d \mu(x, \xi) \\
& =\int_{10, T[\times \Omega \times \mathcal{X}} \xi\left(t, x, \int_{\mathbb{R}^{2}} s d \nu_{t, x, \xi}(s, \dot{s}), \int_{\mathbb{R}^{2}} \dot{s} d \nu_{t, x, \xi}(s, \dot{s})\right) d t \otimes d \mu(x, \xi) \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X}} \xi(t, x, U(t, x, \xi), \dot{U}(t, x, \xi)) d t \otimes d \mu(x, \xi) ;  \tag{3.41}\\
\int_{\Omega \times \mathcal{X} \times \mathbb{R}} \theta\left(s_{T}\right) d \lambda\left(x, \xi, s_{T}\right) & =\int_{\Omega \times \mathcal{X}}\left(\int_{\mathbb{R}} \theta\left(s_{T}\right) d \lambda_{x, \xi}\left(s_{T}\right)\right) d \mu(x, \xi) \\
& \geqslant \int_{\Omega \times \mathcal{X}} \theta\left(\int_{\mathbb{R}} s_{T} d \lambda_{x, \xi}\left(s_{T}\right)\right) d \mu(x, \xi) \\
& =\int_{\Omega \times \mathcal{X}} \theta\left(W_{T}(x, \xi)\right) d \mu(x, \xi) \\
& =\int_{\Omega \times \mathcal{X}} \theta(U(T, x, \xi)) d \mu(x, \xi), \tag{3.42}
\end{align*}
$$

where $U \in \mathcal{S}_{\mu}(u)$ by Lemma 3.1. Consequently, combining (3.39) with (3.41) and (3.40) with (3.42), and using (3.38) we conclude that

$$
\begin{aligned}
\varliminf_{\varepsilon \rightarrow 0} \Phi_{\mu_{\sigma(\varepsilon)}}\left(u_{\varepsilon}\right) \geqslant & \int_{] 0, T[\times \Omega \times \mathcal{X}} \xi(t, x, U(t, x, \xi), \dot{U}(t, x, \xi)) d t \otimes d \mu(x, \xi) \\
& +\int_{\Omega \times \mathcal{X}} \theta(U(T, x, \xi)) d \mu(x, \xi) \\
= & \phi_{\mu}(U) \\
\geqslant & \Phi_{\mu}(u)
\end{aligned}
$$

and $\left(\mathrm{G}_{2}\right)$ is proved.
This finishes the proof of Theorem 2.7 (the proof of Lemma 3.1 is given below).
Proof of Lemma 3.1. The proof is divided into six steps.
Step 1: We prove that:

$$
\begin{align*}
& U \in L^{2}(] 0, T\left[; L_{\mu}^{2}(\Omega \times \mathcal{X} ; \mathbb{R})\right)  \tag{3.43}\\
& V \in L^{2}(] 0, T\left[; L_{\mu}^{2}(\Omega \times \mathcal{X} ; \mathbb{R})\right)  \tag{3.44}\\
& W_{T} \in L_{\mu}^{2}(\Omega \times \mathcal{X} ; \mathbb{R}) \tag{3.45}
\end{align*}
$$

Taking (3.35) into account and using Jensen's inequality, we see that

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d t \otimes d \mu(x, \xi) & =\int_{] 0, T[\times \Omega \times \mathcal{X}}\left|\int_{\mathbb{R}^{2}} s d \nu_{t, x, \xi}(s, \dot{s})\right|^{2} d t \otimes d \mu(x, \xi) \\
& \leqslant \int_{00, T[\times \Omega \times \mathcal{X}}\left(\int_{\mathbb{R}^{2}}|s|^{2} d \nu_{t, x, \xi}(s, \dot{s})\right) d t \otimes d \mu(x, \xi)
\end{aligned}
$$

and so

$$
\begin{equation*}
\int_{] 0, T[\times \Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d t \otimes d \mu(x, \xi) \leqslant \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.}|s|^{2} d \nu(t, x, \xi, s, \dot{s}) \tag{3.46}
\end{equation*}
$$

because of (3.33). On the other hand, according to (3.31) and the definition of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$, From Theorem A.4(i) we have

$$
\begin{aligned}
\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.}|s|^{2} d \nu(t, x, \xi, s, \dot{s}) & \leqslant \underline{\lim }_{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.}|s|^{2} d \nu_{\varepsilon}(t, x, \xi, s, \dot{s}) \\
& =\underline{\lim }_{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}}\left|U_{\varepsilon}(t, x, \xi)\right|^{2} d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi)
\end{aligned}
$$

and so

$$
\begin{equation*}
\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.}|s|^{2} d \nu(t, x, \xi, s, \dot{s})<\infty \tag{3.47}
\end{equation*}
$$

by (3.28). From (3.46) and (3.46) we get (3.43). In the same manner, by considering (3.36) instead of (3.35) we obtain (3.44). By using (3.37), (3.34), (3.32) and the definition of $\left\{\lambda_{\varepsilon}\right\}_{\varepsilon}$ instead of (3.35), (3.33), (3.28) and the definition of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$, a similar calculation gives (3.45).

Step 2: We prove that

$$
\begin{equation*}
\left.\int_{\mathcal{X}} U(t, x, \xi) d \mu_{x}(\xi)=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega \tag{3.48}
\end{equation*}
$$

where $\left\{\mu_{x}\right\}_{x \in \Omega}$ denotes the desintegration of $\mu$ on $\Omega \times \mathcal{X}$. Fix any $\varphi \in C_{c}^{\infty}(] 0, T[\times \Omega)$. As $u_{\varepsilon} \xrightarrow{H^{1}} u$ we have

$$
\begin{equation*}
\int_{] 0, T[\times \Omega} \varphi(t, x) u(t, x) d t \otimes d x=\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega} \varphi(t, x) u_{\varepsilon}(t, x) d t \otimes d x . \tag{3.49}
\end{equation*}
$$

Setting $\psi(t, x, \xi, s, \dot{s}):=\varphi(t, x) s$ and recalling that $U_{\varepsilon} \in \mathcal{S}_{\mu_{\sigma(\varepsilon)}}$ for any $\varepsilon>0$, we see that

$$
\int_{] 0, T[\times \Omega} \varphi(t, x) u_{\varepsilon}(t, x) d t \otimes d x=\int_{] 0, T[\times \Omega \times \mathcal{X}} \psi\left(t, x, \xi, U_{\varepsilon}(t, x, \xi), \dot{U}_{\varepsilon}(t, x, \xi)\right) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi),
$$

and so, according to the definition of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$, one has

$$
\int_{] 0, T[\times \Omega} \varphi(t, x) u_{\varepsilon}(t, x) d t \otimes d x=\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \psi(t, x, \xi, s, \dot{s}) d \nu_{\varepsilon}(t, x, \xi, s, \dot{s})
$$

for all $\varepsilon>0$. Taking (3.31) into account, from Theorem A.4(ii) we deduce that

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega} \varphi(t, x) u_{\varepsilon}(t, x) d t \otimes d x & =\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \psi(t, x, \xi, s, \dot{s}) d \nu(t, x, \xi, s, \dot{s}) \\
& =\int_{00, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \varphi(t, x) s d \nu(t, x, \xi, s, \dot{s}) . \tag{3.50}
\end{align*}
$$

Combining (3.49) with (3.50) and by using desintegration's theorem (see Theorem A.5) and (3.35), we obtain

$$
\begin{aligned}
\int_{] 0, T[\times \Omega} \varphi(t, x) u(t, x) d t \otimes d x & =\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t, x)\left(\int_{\mathbb{R}^{2}} s d \nu_{t, x, \xi}(s, \dot{s})\right) d t \otimes d \mu(x, \xi) \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t, x) U(t, x, \xi) d t \otimes d \mu(x, \xi) \\
& =\int_{] 0, T[\times \Omega} \varphi(t, x)\left(\int_{\mathcal{X}} U(t, x, \xi) d \mu_{x}(\xi)\right) d t \otimes d x
\end{aligned}
$$

which proves (3.48).
Step 3: We prove that

$$
\begin{equation*}
V=\dot{U} \tag{3.51}
\end{equation*}
$$

Fix any $\varphi \in C_{c}^{1}(] 0, T[)$ and any $\phi \in C_{c}(\Omega)$. Set $\psi_{1}(t, x, \xi, s, \dot{s}):=\varphi(t) \phi(x) \dot{s}$. According to (3.36) and the definition of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$, by using desintegration's theorem (see Theorem A.5) and

Theorem A.4(ii) together with (3.31), we see that

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X}}^{\varphi(t)} \phi(x) V(t, x, \xi) d t \otimes d \mu(x, \xi) & =\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t) \phi(x)\left(\int_{\mathbb{R}^{2}} \dot{s} d \nu_{t, x, \xi}(s, \dot{s})\right) d t \otimes d \mu(x, \xi) \\
& =\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \psi_{1}(t, x, \xi, s, \dot{s}) d \nu(t, x, \xi, s, \dot{s}) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{j 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \psi_{1}(t, x, \xi, s, \dot{s}) d \nu_{\varepsilon}(t, x, \xi, s, \dot{s}) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}} \psi_{1}\left(t, x, \xi, U_{\varepsilon}(t, x, \xi), \dot{U}_{\varepsilon}(t, x, \xi)\right) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t) \phi(x) \dot{U}_{\varepsilon}(t, x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) .
\end{aligned}
$$

But, for any $\varepsilon>0$, as $\varphi \in C_{c}^{1}(] 0, T[)$ one has

$$
\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t) \phi(x) \dot{U}_{\varepsilon}(t, x, \xi) d t \otimes d \mu_{\varepsilon}(x, \xi)=-\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) \phi(x) U_{\varepsilon}(t, x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi),
$$

and so, setting $\psi_{2}(t, x, \xi, s, \dot{s}):=\varphi^{\prime}(t) \phi(x) s$ and using the same arguments as in above with (3.35) instead of (3.36), it follows that

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t) \phi(x) V(t, x, \xi) d t \otimes d \mu(x, \xi) & =-\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}} \psi_{2}\left(t, x, \xi, U_{\varepsilon}(t, x, \xi), \dot{U}_{\varepsilon}(t, x, \xi)\right) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) \\
& =-\lim _{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \psi_{2}(t, x, \xi, s, \dot{s}) d \nu_{\varepsilon}(t, x, \xi, s, \dot{s}) \\
& =-\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \psi_{2}(t, x, \xi, s, \dot{s}) d \nu(t, x, \xi, s, \dot{s}) \\
& =-\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \varphi^{\prime}(t) \phi(x) s d \nu(t, x, \xi, s, \dot{s}) \\
& =-\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) \phi(x)\left(\int_{\mathbb{R}^{2}} s d \nu_{t, x, \xi}(s, \dot{s})\right) d t \otimes d \mu(x, \xi) \\
& =-\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) \phi(x) U(t, x, \xi) d t \otimes d \mu(x, \xi),
\end{aligned}
$$

which establishes (3.51).
Step 4: We prove that

$$
\begin{equation*}
U(0, x, \xi)=0 \text { for } \mu \text {-a.a. }(x, \xi) \in \Omega \times \mathcal{X} . \tag{3.52}
\end{equation*}
$$

Let $\varphi \in C^{1}([0, T])$ be such that $\varphi(T)=0$ and $\varphi(0) \neq 0$ and let $\psi \in C_{c}(\Omega \times \mathcal{X})$. Then, we have

$$
\begin{align*}
\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t) \dot{U}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu(x, \xi)= & -\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) U(t, x, \xi) \psi(x, \xi) d \mu(x, \xi) \\
& -\int_{\Omega \times \mathcal{X}} \varphi(0) U(0, x, \xi) \psi(x, \xi) d \mu(x, \xi) . \tag{3.53}
\end{align*}
$$

On the other hand, by (3.51) and (3.36) and the definition of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$, from desintegration's theorem (see Theorem A.5) and Theorem A.4(ii) together with (3.31), we see that

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X}}^{\varphi}(t) \dot{U}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu(x, \xi) & =\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t)\left(\int_{\mathbb{R}^{2}} \dot{s} d \nu_{t, x, \xi}(s, \dot{s})\right) \psi(x, \xi) d t \otimes d \mu(x, \xi) \\
& =\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.}(t) \dot{s} \psi(x, \xi) d \nu(t, x, \xi, s, \dot{s}) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \varphi(t) \dot{s} \psi(x, \xi) d \nu_{\varepsilon}(t, x, \xi, s, \dot{s}) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}}^{\varphi(t) \dot{U}_{\varepsilon}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) .}
\end{aligned}
$$

But, for any $\varepsilon>0$, one has

$$
\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi(t) \dot{U}_{\varepsilon}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi)=-\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) U_{\varepsilon}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi)
$$ because $\varphi(T)=0$ and $U_{\varepsilon}(0, x, \xi)=0$ for $\mu_{\sigma(\varepsilon)-\mathrm{a} . \mathrm{a} .}(x, \xi) \in \Omega \times \mathcal{X}$ since $U_{\varepsilon} \in \mathcal{H}_{0, \mu_{\sigma(\varepsilon)}}^{1}$. Hence, by the same arguments as in above, we get

$$
\begin{align*}
\int_{] 0, T[\times \Omega \times \mathcal{X}}^{\varphi(t) \dot{U}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu(x, \xi)} & =-\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) U_{\varepsilon}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) \\
& =-\lim _{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \varphi^{\prime}(t) s \psi(x, \xi) d \nu_{\varepsilon}(t, x, \xi, s, \dot{s}) \\
& =-\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.}(t) s \psi(x, \xi) d \nu(t, x, \xi, s, \dot{s}) \\
& =-\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t)\left(\int_{\mathbb{R}^{2}} s d \nu_{t, x, \xi}(s, \dot{s})\right) \psi(x, \xi) d t \otimes d \mu(x, \xi) \\
& =-\int_{] 0, T[\times \Omega \times \mathcal{X}} \varphi^{\prime}(t) U(t, x, \xi) \psi(x, \xi) d t \otimes d \mu(x, \xi) . \tag{3.54}
\end{align*}
$$

Combining (3.53) with (3.54) we deduce that

$$
\int_{\Omega \times \mathcal{X}} \varphi(0) U(0, x, \xi) \psi(x, \xi) d \mu(x, \xi)=0
$$

for all $\psi \in C_{c}(\Omega \times \mathcal{X})$, and (3.52) follows because $\varphi(0) \neq 0$.

Step 5: We prove that

$$
\begin{equation*}
W_{T}(x, \xi)=U(T, x, \xi) \text { for } \mu \text {-a.a. }(x, \xi) \in \Omega \times \mathcal{X} \tag{3.55}
\end{equation*}
$$

Fix any $\psi \in C_{c}(\Omega \times \mathcal{X})$. According to (3.37) and the definition of $\left\{\lambda_{\varepsilon}\right\}_{\varepsilon}$, by using desintegration's theorem (see Theorem A.5) and Theorem A.4(ii) together with (3.32), we see that

$$
\begin{align*}
\int_{\Omega \times \mathcal{X}} W_{T}(x, \xi) \psi(x, \xi) d \mu(x, \xi) & =\int_{\Omega \times \mathcal{X}}\left(\int_{\mathbb{R}} s_{T} d \lambda_{x, \xi}\left(s_{T}\right)\right) \psi(x, \xi) d \mu(x, \xi) \\
& =\int_{\Omega \times \mathcal{X} \times \mathbb{R}} s_{T} \psi(x, \xi) d \lambda\left(x, \xi, s_{T}\right) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X} \times \mathbb{R}} s_{T} \psi(x, \xi) d \lambda_{\varepsilon}\left(x, \xi, s_{T}\right) \\
& =\lim _{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X}} U_{\varepsilon}(T, x, \xi) \psi(x, \xi) d \mu_{\sigma(\varepsilon)}(x, \xi) \tag{3.56}
\end{align*}
$$



$$
\begin{align*}
\int_{\Omega \times \mathcal{X}} U_{\varepsilon}(T, x, \xi) \psi(x, \xi) d \mu_{\sigma(\varepsilon)}(x, \xi) & =\int_{\Omega \times \mathcal{X}}\left(\int_{0}^{T} \dot{U}_{\varepsilon}(t, x, \xi) d t\right) \psi(x, \xi) d \mu_{\sigma(\varepsilon)}(x, \xi) \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X}} \dot{U}_{\varepsilon}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) \tag{3.57}
\end{align*}
$$

Moreover, according to (3.51) and (3.36) and the definition of $\left\{\nu_{\varepsilon}\right\}_{\varepsilon}$, by using the same arguments as in above with (3.31) instead of (3.32), we have

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \int_{] 0, T[\times \Omega \times \mathcal{X}} \dot{U}_{\varepsilon}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu_{\sigma(\varepsilon)}(x, \xi) & =\lim _{\varepsilon \rightarrow 0} \int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \dot{s} \psi(x, \xi) d \nu_{\varepsilon}(t, x, \xi, s, \dot{s}) \\
& =\int_{] 0, T\left[\times \Omega \times \mathcal{X} \times \mathbb{R}^{2}\right.} \dot{j} \psi(x, \xi) d \nu(t, x, \xi, s, \dot{s}) \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X}}\left(\int_{\mathbb{R}^{2}} \dot{s} d \nu_{t, x, \xi}(s, \dot{s})\right) \psi(x, \xi) d t \otimes d \mu(x, \xi) \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X}} \dot{U}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu(x, \xi) \tag{3.58}
\end{align*}
$$

From (3.52) we see that

$$
\begin{align*}
\int_{] 0, T[\times \Omega \times \mathcal{X}} \dot{U}(t, x, \xi) \psi(x, \xi) d t \otimes d \mu(x, \xi) & =\int_{\Omega \times \mathcal{X}}\left(\int_{0}^{T} \dot{U}(t, x, \xi) d t\right) \psi(x, \xi) d \mu(x, \xi) \\
& =\int_{\Omega \times \mathcal{X}} U(T, x, \xi) \psi(x, \xi) d \mu(x, \xi) \tag{3.59}
\end{align*}
$$

and consequently, from (3.56), (3.57), (3.58) and (3.59), we conclude that

$$
\int_{\Omega \times \mathcal{X}} W_{T}(x, \xi) \psi(x, \xi) d \mu(x, \xi)=\int_{\Omega \times \mathcal{X}} U(T, x, \xi) \psi(x, \xi) d \mu(x, \xi)
$$

for all $\psi \in C_{c}(\Omega \times \mathcal{X})$, and (3.55) follows.
Step 6: end of the proof. From (3.43), (3.44), (3.51) and (3.52) we deduce that $U \in \mathcal{H}_{0, \mu}^{1}$, and so $U \in \mathcal{S}_{\mu}(u)$ by (3.48), which completes the proof of Lemma 3.1.

## 4. Stochastic homogenization

4.1. A non-local stochastic homogenization theorem. Let $(\Sigma, \mathcal{A}, \mathbb{P})$ be a probability space and let $\left(T_{z}\right)_{z \in \mathbb{Z}^{N}}$ be a group of $\mathbb{P}$-preserving transformations on $(\Sigma, \mathcal{A})$, i.e.,

- (mesurability) $T_{z}$ is $\mathcal{A}$-measurable for all $z \in \mathbb{Z}^{N}$;
- (group property) $T_{z} \mathrm{o} T_{z^{\prime}}=T_{z+z^{\prime}}$ and $T_{-z}=T_{z}^{-1}$ for all $z, z^{\prime} \in \mathbb{Z}^{N}$;
- (mass invariance) $\mathbb{P}\left(T_{z}(A)\right)=\mathbb{P}(A)$ for all $z \in \mathbb{Z}^{N}$ and all $A \in \mathcal{A}$.

Let $f: \Sigma \times \mathbb{R}^{N} \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a $\left(\mathcal{A} \otimes \mathcal{B}\left(\mathbb{R}^{N} \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \times \mathbb{R}\right), \mathcal{B}(\mathbb{R})\right)$ measurable function such that

$$
f(\omega, x):=f(\omega, x, \cdot, \cdot, \cdot, \cdot) \in \mathcal{X}
$$

for all $\omega \in \Sigma$ and all $x \in \mathbb{R}^{N}$. It is easily seen that the map

$$
\begin{aligned}
f: \Sigma & \rightarrow \mathfrak{X}:=\mathcal{X}^{\mathbb{R}^{N}} \\
\omega & \mapsto f(\omega, \cdot)
\end{aligned}
$$

is a $(\mathcal{A}, \mathcal{B})$-random variable with $\mathcal{B}:=\otimes_{\mathbb{R}^{N}} \mathcal{B}(\mathcal{X})$. We futhermore assume that the random variable $f$ is covariant with respect to the dynamical system $\left(\Sigma, \mathcal{A}, \mathbb{P},\left(T_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$, i.e.,

$$
\begin{equation*}
f\left(T_{z} \omega, \cdot\right)=f(\omega, \cdot+z) \tag{4.1}
\end{equation*}
$$

for all $z \in \mathbb{Z}^{N}$ and for $\mathbb{P}$-a.a. $\omega \in \Sigma$. For each $z \in \mathbb{Z}^{N}$, let $\tau_{z}: \mathfrak{X} \rightarrow \mathfrak{X}$ denote the shift map on $\mathfrak{X}$, i.e.,

$$
\begin{equation*}
\tau_{z}(w):=w(\cdot+z) \tag{4.2}
\end{equation*}
$$

for all $w \in \mathfrak{X}$. Then, $f^{\sharp} \mathbb{P}$, i.e., the law of $f$, is invariant under the group $\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}$, i.e.,

$$
\begin{equation*}
\tau_{z}^{\sharp} f^{\sharp} \mathbb{P}=f^{\sharp} \mathbb{P} \tag{4.3}
\end{equation*}
$$

for all $z \in \mathbb{Z}^{N}$. We finally assume that the dynamical $\operatorname{system}\left(\Sigma, \mathcal{A}, \mathbb{P},\left(T_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$, or equivalently $\left(\mathfrak{X}, \mathcal{B}, f^{\sharp} \mathbb{P},\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$, is ergodic, i.e., for each $A \in \mathcal{A}$,

$$
\text { if } T_{z}(A)=A \text { for all } z \in \mathbb{Z}^{N} \text { then } \mathbb{P}(A)=0 \text { or } \mathbb{P}(A)=1
$$

or equivalently, for each $B \in \mathcal{B}$,

$$
\begin{equation*}
\text { if } \tau_{z}(B)=B \text { for all } z \in \mathbb{Z}^{N} \text { then } f^{\sharp} \mathbb{P}(B)=0 \text { or } f^{\sharp} \mathbb{P}(B)=1 \text {. } \tag{4.4}
\end{equation*}
$$

For each $\omega \in \Sigma$ and each $\varepsilon>0$, we consider $F_{\varepsilon}(\omega): \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by

$$
\begin{equation*}
F_{\varepsilon}(\omega)(u):=\Phi_{f(\omega, \dot{\bar{\varepsilon}})}(u)=\int_{] 0, T[\times \Omega} f\left(\omega, \frac{x}{\varepsilon}\right)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\int_{\Omega} \theta(u(T, x)) d x \tag{4.5}
\end{equation*}
$$

For each $\varepsilon>0$, let $\mu_{\varepsilon}: \Sigma \rightarrow \mathcal{Y}(\Omega ; \mathcal{X})$ be the random Young measure defined by

$$
\begin{equation*}
\mu_{\varepsilon}(\omega):=d x \otimes \delta_{f\left(\omega, \frac{x}{\varepsilon}\right)} \tag{4.6}
\end{equation*}
$$

Let $Y:=] 0,1\left[{ }^{N}\right.$ be the unit cell, let $f_{Y}: \Sigma \rightarrow \mathcal{X}^{Y}$ be defined by $f_{Y}(\omega)(y)=f(\omega, y)$ and, for each $\omega \in \Sigma$, let $f_{Y}^{\sharp}(\omega) d y$ be the image by $f_{Y}(\omega)$ of the Lebesgue measure restricted to $Y$ that we denote by $d y$. Let $\mu \in \mathcal{Y}(\Omega ; \mathcal{X})$ be the Young measure defined by

$$
\begin{equation*}
\mu_{f}:=d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y} \tag{4.7}
\end{equation*}
$$

where $d \mathbb{E}_{f_{Y}^{\sharp} d y}$ denotes the expectation of the random probability measure $\omega \mapsto f_{Y}^{\sharp}(\omega) d y$, i.e., $d \mathbb{E}_{f_{Y}^{\sharp} d y}$ is the probability measure on $\mathcal{X}$ given by

$$
\begin{aligned}
d \mathbb{E}_{f_{Y}^{\sharp} d y}(B) & =\int_{\Sigma}\left(\int_{Y} 1_{B}\left(f_{Y}(\omega)(y)\right) d y\right) d \mathbb{P}(\omega) \\
& =\int_{\Sigma \times Y} 1_{B}\left(f_{Y}(\omega)(y)\right) d \mathbb{P}(\omega) \otimes d y
\end{aligned}
$$

for all $B \in \mathcal{B}(\mathcal{X})$, where $1_{B}$ denotes the indicator function of $B$, or equivalently by

$$
\begin{equation*}
\int_{\mathcal{X}} \varphi(\xi) d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi)=\mathbb{E}\left(\int_{Y} \varphi(f(\cdot, y)) d y\right)=\int_{\Sigma \times Y} \varphi(f(\omega, y)) d \mathbb{P}(\omega) \otimes d y \tag{4.8}
\end{equation*}
$$

for all $(\mathcal{B}(\mathcal{X}), \mathcal{B}(\mathbb{R}))$-measurable functions $\varphi: \mathcal{X} \rightarrow \mathbb{R}$ such that $\int_{Y} \varphi(f(\cdot, y)) d y \in L^{1}(\Sigma, \mathcal{A}, \mathbb{P})$.
Proposition 4.1. For $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has

$$
\mu_{\varepsilon}(\omega) \xrightarrow{\text { nar }} \mu_{f}
$$

with $\mu_{f} \in \mathcal{Y}(\Omega ; \mathcal{X})$ given by (4.7).
Proof of Proposition 4.1. It is sufficient to prove that there exists $\widehat{\Sigma} \in \mathcal{A}$ with $\mathbb{P}(\widehat{\Sigma})=1$ such that for every $\omega \in \widehat{\Sigma}$, one has

$$
\lim _{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X}} 1_{A}(x) \varphi(\xi) d \mu_{\varepsilon}(\omega)(x, \xi)=\int_{\Omega \times \mathcal{X}} 1_{A}(x) \varphi(\xi) d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi)
$$

for all $A \in \mathcal{B}(\Omega)$ and all $\varphi \in \mathcal{D}$ with $\mathcal{D}$ a dense subset of $C_{c}(\mathcal{X})$, see [Val90, Val94].
Let $A \in \mathcal{B}(\Omega)$ and let $\varphi \in \mathcal{D}$. Taking (4.8) into account, from the additive ergodic theorem [ABM14, Theorem 12.4.1] (see also [CM94]), we can assert that there exists $N_{\varphi} \in \mathcal{F}$ with $\mathbb{P}\left(N_{\varphi}\right)=0$ such that for every $\omega \in \Sigma \backslash N_{\varphi}$, one has

$$
\begin{equation*}
\varphi\left(f\left(\omega, \frac{\cdot}{\varepsilon}\right)\right) \xrightarrow{L^{1}} \int_{\mathcal{X}} \varphi(\xi) d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi) \tag{4.9}
\end{equation*}
$$

On the other hand, by (4.6) we see that for every $\omega \in \Sigma$, one has

$$
\begin{equation*}
\int_{\Omega \times \mathcal{X}} 1_{A}(x) \varphi(\xi) d \mu_{\varepsilon}(\omega)(x, \xi)=\int_{\Omega} 1_{A}(x) \varphi\left(f\left(\omega, \frac{x}{\varepsilon}\right)\right) d x \text { for all } \varepsilon>0 \tag{4.10}
\end{equation*}
$$

Set $\widehat{\Sigma}:=\cup_{\psi \in \mathcal{D}}\left(\Sigma \backslash N_{\psi}\right)$ (where $N_{\psi}$ corresponds to $N_{\varphi}$ with $\varphi=\psi$ ). Then $\mathbb{P}(\widehat{\Sigma})=0$ and, by using (4.9), from (4.10) we deduce that for every $\omega \in \widehat{\Sigma}$, one has

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \int_{\Omega \times \mathcal{X}} 1_{A}(x) \varphi(\xi) d \mu_{\varepsilon}(\omega)(x, \xi) & =\lim _{\varepsilon \rightarrow 0} \int_{\Omega} 1_{A}(x) \varphi\left(f\left(\omega, \frac{x}{\varepsilon}\right)\right) d x \\
& =\int_{\Omega} 1_{A}(x)\left(\int_{\mathcal{X}} \varphi(\xi) d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi)\right) d x \\
& =\int_{\Omega \times \mathcal{X}} 1_{A}(x) \varphi(\xi) d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi),
\end{aligned}
$$

and the proof is complete.
Let $F_{\text {hom }}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
F_{\mathrm{hom}}(u):=\inf \left\{\int_{\Sigma \times Y} \Phi_{f(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y: v \in \mathcal{S}_{\text {hom }}(u)\right\} \tag{4.11}
\end{equation*}
$$

where $\mathcal{S}_{\text {hom }}(u)$ is given by
$\mathcal{S}_{\text {hom }}(u):=\left\{v \in \mathcal{H}_{0, d x \otimes \mathbb{P} \otimes d y}^{1}: \int_{\Sigma \times Y} v(t, x, \omega, y) d \mathbb{P}(\omega) \otimes d y=u(t, x)\right.$ for $d t \otimes d x$-a.a. $\left.(t, x) \in\right] 0, T[\times \Omega\}$
with

$$
\mathcal{H}_{0, d x \otimes \mathbb{P} \otimes d y}^{1}:=\left\{v \in H^{1}(] 0, T\left[; L_{d x \otimes \mathbb{P} \otimes d y}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right): v(0, \cdot, \cdot, \cdot)=0\right\}
$$

and

$$
\begin{align*}
\Phi_{f(\omega, y)}(v(\cdot, \cdot, \omega, y))= & \int_{] 0, T[\times \Omega} f(\omega, y)(t, x, v(t, x, \omega, y), \dot{v}(t, x, \omega, y)) d t \otimes d x \\
& +\int_{\Omega} \theta(v(T, x, \omega, y)) d x \tag{4.12}
\end{align*}
$$

for all $\omega \in \Sigma$ and all $y \in Y$. The following result is a consequence of Corollary 2.8 and Proposition 4.1.
Theorem 4.2. For $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}(\omega)=F_{\mathrm{hom}}
$$

with $F_{\text {hom }}$ given by (4.11)-(4.12).
Proof of Theorem 4.2. From Corollary 2.8 and Proposition 4.1 we deduce that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}(\omega)=\Phi_{\mu_{f}}
$$

with $\Phi_{\mu_{f}}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by (2.6). (Since $\mu_{f}$ does not depend on $\omega$, so is for $\Phi_{\mu_{f}}$.) Taking (4.7) and (4.8) into account we see that for any $u \in \mathcal{H}_{0}^{1}$, one has

$$
\begin{aligned}
\mathcal{S}_{\mu_{f}}(u) & =\left\{U \in \mathcal{H}_{0, \mu_{f}}^{1}: \int_{\mathcal{X}} U(t, x, \xi) d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi)=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega\} \\
& =\left\{U \in \mathcal{H}_{0, \mu_{f}}^{1}: \int_{\Sigma \times Y} U(t, x, f(\omega, y)) d \mathbb{P}(\omega) \otimes d y=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega\}
\end{aligned}
$$

and

$$
\Phi_{\mu_{f}}(u)=\inf \left\{\phi_{\mu_{f}}(U): U \in \mathcal{S}_{\mu_{f}}(u)\right\}
$$

with

$$
\begin{align*}
\phi_{\mu_{f}}(U)= & \int_{] 0, T[\times \Omega \times \mathcal{X}} \xi(t, x, U(t, x), \dot{U}(t, x, \xi)) d t \otimes d \mu_{f}(x, \xi)+\int_{\Omega \times \mathcal{X}} \theta(U(T, x, \xi)) d \mu_{f}(x, \xi) \\
= & \left.\int_{] 0, T[\times \Omega \times \mathcal{X}} \xi(t, x, U(t, x, \xi), \dot{U}(t, x, \xi)) d t \otimes d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi)+\int_{\Omega \times \mathcal{X}} \theta(T, x, \xi)\right) d x \otimes d \mathbb{E}_{f_{Y}^{\sharp}}(\xi) \\
= & \int_{] 0, T[\times \Omega \times \Sigma \times Y} f(\omega, y)(t, x, U(t, x, f(\omega, y)), \dot{U}(t, x, f(\omega, y))) d t \otimes d x \otimes d \mathbb{P}(\omega) \otimes d y \\
& +\int_{\Omega \times \Sigma \times Y} \theta(U(T, x, f(\omega, y))) d x \otimes d \mathbb{P}(\omega) \otimes d y \tag{4.13}
\end{align*}
$$

So, it remains to prove that $\Phi_{\mu_{f}}=F_{\text {hom }}$, i.e., for every $u \in \mathcal{H}_{0}^{1}$, one has:

$$
\begin{align*}
& \Phi_{\mu_{f}}(u) \geqslant F_{\mathrm{hom}}(u)  \tag{4.14}\\
& \Phi_{\mu_{f}}(u) \leqslant F_{\mathrm{hom}}(u) \tag{4.15}
\end{align*}
$$

Let us fix $u \in \mathcal{H}_{0}^{1}$.
Proof of (4.14). Let $U \in \mathcal{S}_{\mu_{f}}(u)$. Then:

$$
\begin{align*}
& U \in L^{2}(] 0, T\left[; L_{\mu_{f}}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right) ;  \tag{4.16}\\
& \dot{U} \in L^{2}(] 0, T\left[; L_{\mu_{f}}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right) ;  \tag{4.17}\\
& U(0, \cdot, \cdot)=0  \tag{4.18}\\
& \left.\int_{\Sigma \times Y} U(t, x, f(\omega, y)) d \mathbb{P}(\omega) \otimes d y=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega \tag{4.19}
\end{align*}
$$

Set $v(t, x, \omega, y):=U(t, x, f(\omega, y))$. By (4.18) we have $v(0, \cdot, \cdot, \cdot)=0$ and from (4.19) it is clear that

$$
\left.\int_{\Sigma \times Y} v(t, x, \omega, y) d \mathbb{P}(\omega) \otimes d y=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega
$$

On the other hand, taking (4.8) into account, we see that

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d x \otimes d \mathbb{P}(\omega) \otimes d y & =\int_{] 0, T[\times \Omega}\left(\int_{\Sigma \times Y}|U(t, x, f(\omega, y))|^{2} d \mathbb{P}(\omega) \otimes d y\right) d t \otimes d x \\
& =\int_{] 0, T[ }\left(\int_{\Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d x \otimes d \mathbb{E}_{f_{Y}^{\sharp}}(\xi)\right) d t \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d t \otimes d \mu_{f}(x, \xi)
\end{aligned}
$$

and so, from (4.16) we deduce that $v \in L^{2}(] 0, T\left[; L_{d x \otimes \mathbb{P} \otimes d y}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right)$. In the same way, by using (4.17) instead of (4.16) we obtain $\dot{v} \in L^{2}(] 0, T\left[; L_{d x \otimes \mathbb{P} \otimes d y}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right)$. It follows that $v \in \mathcal{S}_{\mu_{f}}(u)$.

We have thus proved that for each $U \in \mathcal{S}_{\mu_{f}}(u)$ one has $v \in \mathcal{S}_{\text {hom }}(u)$. According to (4.11)-(4.12) and (4.13) we conclude that

$$
\phi_{\mu_{f}}(U)=\int_{\Sigma \times Y} \Phi_{f(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \geqslant F_{\mathrm{hom}}(u)
$$

for all $U \in \mathcal{S}_{\mu_{f}}(u)$, and (4.14) follows.
Proof of (4.15). Consider $g: \Sigma \times Y \rightarrow \mathcal{X} \times \Sigma \times Y$ defined by $g(\omega, y):=(f(\omega, y), \omega, y)$ and set

$$
\begin{equation*}
\lambda:=g^{\sharp}(\mathbb{P} \otimes d y) . \tag{4.20}
\end{equation*}
$$

Then $\pi_{\mathcal{X}}^{\sharp} \lambda=d \mathbb{E}_{f_{Y}^{\sharp} d y}$ with $\pi_{\mathcal{X}}: \mathcal{X} \times \Sigma \times Y \rightarrow \mathcal{X}$ denoting the canonical projection from $\mathcal{X} \times \Sigma \times Y$ to $\mathcal{X}$. So, from desintegration's theorem (see Theorem A.5) we can assert that there exists a familiy $\left(\lambda_{\xi}\right)_{\xi \in \mathcal{X}}$ of probability measures on $\Sigma \times Y$ such that

$$
\begin{equation*}
\lambda=d \mathbb{E}_{f_{Y}^{\sharp} d y} \otimes \lambda_{\xi} . \tag{4.21}
\end{equation*}
$$

Fix any $v \in \mathcal{S}_{\text {hom }}(u)$ and set

$$
\begin{equation*}
U(t, x, \xi):=\int_{\Sigma \times Y} v(t, x, \omega, y) d \lambda_{\xi}(\omega, y) \tag{4.22}
\end{equation*}
$$

(In particular, $U(T, x, \xi):=\int_{\Sigma \times Y} v(T, x, \omega, y) d \lambda_{\xi}(\omega, y)$.) Then, it is easy to see that

$$
\begin{equation*}
\dot{U}(t, x, \xi):=\int_{\Sigma \times Y} \dot{v}(t, x, \omega, y) d \lambda_{\xi}(\omega, y) \tag{4.23}
\end{equation*}
$$

We claim that $U \in \mathcal{S}_{\mu_{f}}(u)$. Indeed, as $v \in \mathcal{S}_{\text {hom }}(u)$ we have:

$$
\begin{align*}
& v \in L^{2}(] 0, T\left[; L_{d x \otimes \mathbb{P} \otimes d y}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right) ;  \tag{4.24}\\
& \dot{v} \in L^{2}(] 0, T\left[; L_{d x \otimes \mathbb{P} \otimes d y}^{2}(\Omega \times \Sigma \times Y ; \mathbb{R})\right) ;  \tag{4.25}\\
& v(0, \cdot, \cdot, \cdot)=0 ;  \tag{4.26}\\
& \left.\int_{\Sigma \times Y} v(t, x, \omega, y) d \mathbb{P}(\omega) \otimes d y=u(t, x) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega . \tag{4.27}
\end{align*}
$$

Firstly, by (4.26) it is clear that $U(0, \cdot, \cdot)=0$. Secondly, by using (4.8), (4.22), (4.21), (4.20) and (4.27) we see that

$$
\begin{aligned}
\int_{\Sigma \times Y} U(t, x, f(\omega, y)) d \mathbb{P}(\omega) \otimes d y & =\int_{\mathcal{X}} U(t, x, \xi) d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi) \\
& =\int_{\mathcal{X}}\left(\int_{\Sigma \times Y} v(t, x, \omega, y) d \lambda_{\xi}(\omega, y)\right) d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi) \\
& =\int_{\mathcal{X} \times \Sigma \times Y} v(t, x, \omega, y) d \lambda(\xi, \omega, y) \\
& =\int_{\Sigma \times Y} v(t, x, \omega, y) d \mathbb{P}(\omega) \otimes d y \\
& =u(t, x)
\end{aligned}
$$

for $d t \otimes d x$-a.a. $\quad(t, x) \in] 0, T[\times \Omega$. Thirdly, taking (4.22) into account and using Jensen's inequality, we have

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d t \otimes d \mu_{f}(x, \xi) & =\int_{] 0, T[\times \Omega \times \mathcal{X}}\left|\int_{\Sigma \times Y} v(t, x, \omega, y) d \lambda_{\xi}(\omega, y)\right|^{2} d t \otimes d \mu_{f}(x, \xi) \\
& \leqslant \int_{] 0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d \mu_{f}(x, \xi) \otimes d \lambda_{\xi}(\omega, y)
\end{aligned}
$$

But, by (4.7), (4.21) and (4.20) we have

$$
\begin{aligned}
\int_{] 0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d \mu_{f}(x, \xi) \otimes d \lambda_{\xi}(\omega, y) & =\int_{] 0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi) \otimes d \lambda_{\xi}(\omega, y) \\
& =\int_{] 0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d x \otimes d \lambda(\xi, \omega, y) \\
& =\int_{] 0, T[\times \Omega \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d x \otimes d \mathbb{P}(\omega) \otimes d y
\end{aligned}
$$

hence

$$
\int_{] 0, T[\times \Omega \times \mathcal{X}}|U(t, x, \xi)|^{2} d t \otimes d \mu_{f}(x, \xi) \leqslant \int_{] 0, T[\times \Omega \times \Sigma \times Y}|v(t, x, \omega, y)|^{2} d t \otimes d x \otimes d \mathbb{P}(\omega) \otimes d y
$$

and so, from (4.24) we deduce that $U \in L^{2}(] 0, T\left[; L_{\mu_{f}}^{2}(\Omega \times \mathcal{X} ; \mathbb{R})\right)$. In the same way, by using (4.25) instead of (4.24) we obtain $\dot{U} \in L^{2}(] 0, T\left[; L_{\mu_{f}}^{2}(\Omega \times \mathcal{X} ; \mathbb{R})\right)$, and the claim is proved. On the other hand, taking (4.22) and (4.23) into account and using Jensen's inequality, we have

$$
\begin{aligned}
\phi_{\mu_{f}}(U)= & \int_{] 0, T[\times \Omega \times \mathcal{X}} \xi\left(t, x, \int_{\Sigma \times Y} v(t, x, \omega, y) d \lambda_{\xi}(\omega, y), \int_{\Sigma \times Y} \dot{v}(t, x, \omega, y) d \lambda_{\xi}(\omega, y)\right) d t \otimes d \mu_{f}(x, \xi) \\
& +\int_{\Omega \times \mathcal{X}} \theta\left(\int_{\Sigma \times Y} v(T, x, \omega, y) d \lambda_{\xi}(\omega, y)\right) d \mu_{f}(x, \xi) \\
\leqslant & \int_{] 0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y} \xi(t, x, v(t, x, \omega, y), \dot{v}(t, x, \omega, y)) d t \otimes d \mu_{f}(x, \xi) \otimes d \lambda_{\xi}(\omega, y) \\
& +\int_{\Omega \times \mathcal{X} \times \Sigma \times Y} \theta(v(T, x, \omega, y)) d \mu_{f}(x, \xi) \otimes d \lambda_{\xi}(\omega, y)
\end{aligned}
$$

hence, by using (4.7), (4.21), we obtain

$$
\begin{aligned}
\phi_{\mu_{f}}(U) \leqslant & \int_{0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y} \xi(t, x, v(t, x, \omega, y), \dot{v}(t, x, \omega, y)) d t \otimes d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi) \otimes d \lambda_{\xi}(\omega, y) \\
& +\int_{\Omega \times \mathcal{X} \times \Sigma \times Y} \theta(v(T, x, \omega, y)) d x \otimes d \mathbb{E}_{f_{Y}^{\sharp} d y}(\xi) \otimes d \lambda_{\xi}(\omega, y) \\
\leqslant & \int_{] 0, T[\times \Omega \times \mathcal{X} \times \Sigma \times Y} \xi(t, x, v(t, x, \omega, y), \dot{v}(t, x, \omega, y)) d t \otimes d x \otimes d \lambda(\xi, \omega, y) \\
& +\int_{\Omega \times \mathcal{X} \times \Sigma \times Y} \theta(v(T, x, \omega, y)) d x \otimes d \lambda(\xi, \omega, y)
\end{aligned}
$$

and consequently, from (4.20) we conclude that

$$
\begin{aligned}
\phi_{\mu_{f}}(U) \leqslant & \int_{] 0, T[\times \Omega \times \Sigma \times Y} f(\omega, y)(t, x, v(t, x, \omega, y), \dot{v}(t, x, \omega, y)) d t \otimes d x \otimes d \mathbb{P}(\omega) \otimes d y \\
& +\int_{\Omega \times \Sigma \times Y} \theta(v(T, x, \omega, y)) d x \otimes d \mathbb{P}(\omega) \otimes d y \\
= & \int_{\Sigma \times Y} \Phi_{f(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y
\end{aligned}
$$

We have thus prove that for each $v \in \mathcal{S}_{\text {hom }}(u)$ there exists $U \in \mathcal{S}_{\mu_{f}}(u)$ such that

$$
\phi_{\mu_{f}}(U) \leqslant \int_{\Sigma \times Y} \Phi_{f(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y
$$

which implies (4.15).
This finishes the proof of Theorem 4.2.
4.2. Non-local stochastic homogenization in the setting of a Poisson point process. Let $D: \Sigma \rightrightarrows \mathbb{R}^{N}$ be a $\left(\mathcal{A}, \mathcal{B}\left(\mathbb{R}^{N}\right)\right)$-measurable multifunction such that for every $\omega \in \Sigma$, the set $D(\omega)$ is countable and without cluster point and let $N: \Sigma \times \mathcal{B}\left(\mathbb{R}^{N}\right) \rightarrow \mathbb{N} \cup\{\infty\}$ be defined by

$$
N(\omega, B):=\sum_{z \in D(\omega)} \delta_{z}(B)=\operatorname{card}(D(\omega) \cap B)
$$

where $\delta_{z}$ denotes the Dirac measure at the point $z \in \mathbb{R}^{N}$. (Note that for each $\omega \in \Sigma, N(\omega, \cdot)$ is a counting measure.) From now on, we assume that $\{N(\cdot, B)\}_{B \in \mathcal{B}\left(\mathbb{R}^{N}\right)}$ is a Poisson point process with intensity $\lambda>0$, i.e.,

- for every bounded set $B \in \mathcal{B}\left(\mathbb{R}^{N}\right)$ and every $k \in \mathbb{N}$, one has

$$
\begin{equation*}
\mathbb{P}([N(\cdot, B)=k])=|B|^{k} \lambda^{k} \frac{\mathrm{e}^{-\lambda|B|}}{k!} \tag{4.28}
\end{equation*}
$$

where $[N(\cdot, B)=k]:=\{\omega \in \Sigma: N(\omega, B)=k\}$ and $|B|$ denotes the Lebesgue measure of $B$;

- for every disjoint and bounded sets $A, B \in \mathcal{B}\left(\mathbb{R}^{N}\right), N(\cdot, A)$ and $N(\cdot, B)$ are independant.
Fix $r>0$ and $g, h \in \mathcal{X}$ and consider the $\left(\mathcal{A}, \mathcal{B}:=\otimes_{\mathbb{R}^{N}} \mathcal{B}(\mathcal{X})\right)$-measurable function $f_{\mathrm{p}}: \Sigma \rightarrow$ $\mathfrak{X}:=\mathcal{X}^{\mathbb{R}^{N}}$ given by

$$
\begin{align*}
f_{\mathrm{p}}(\omega, x) & := \begin{cases}g & \text { if } x \in \underset{z \in D(\omega)}{\cup} B_{r}(z) \\
h & \text { otherwise }\end{cases} \\
& =h+(g-h) \min \left\{1, N\left(\omega, B_{r}(x)\right)\right\} \\
& = \begin{cases}g & \text { if } N\left(\omega, B_{r}(x)\right) \geqslant 1 \\
h & \text { if } N\left(\omega, B_{r}(x)\right)=0 .\end{cases} \tag{4.29}
\end{align*}
$$

Then, $f_{\mathrm{p}}$ is covariant with respect to the dynamical system $\left(\Sigma, \mathcal{A}, \mathbb{P},\left(T_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$, i.e., (4.1) hols with $f=f_{\mathrm{p}}$, and the law of $f_{\mathrm{p}}$ is invariant under the group $\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}$, i.e., (4.3) holds with
$f=f_{\mathrm{p}}$ and $\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}$ given by (4.2), and moreover the dynamical system ( $\left.\mathfrak{X}, \mathcal{B}, f_{\mathrm{p}}^{\sharp} \mathbb{P},\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$ is ergodic, i.e., (4.4) holds with $f=f_{\mathrm{p}}$ (see [MM94, §5]). (The ergodic dynamical system $\left(\mathcal{X}, \mathcal{B}, f_{\mathrm{p}}^{\sharp} \mathbb{P},\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$ models environments whose heterogeneities are independently distributed with a frequency $\lambda$.)
For each $\omega \in \Sigma$ and each $\varepsilon>0$, we consider $F_{\varepsilon}^{\mathrm{p}}(\omega): \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by (4.5) with $f=f_{\mathrm{p}}$, i.e.,

$$
F_{\varepsilon}^{\mathrm{p}}(\omega)(u):=\Phi_{f_{\mathrm{p}}(\omega, \dot{\bar{\varepsilon}})}(u)=\int_{] 0, T[\times \Omega} f_{\mathrm{p}}\left(\omega, \frac{x}{\varepsilon}\right)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\int_{\Omega} \theta(u(T, x)) d x
$$

Set $\sigma:=\mathbb{P}\left(\left[N\left(\cdot, B_{r}(0)\right) \geqslant 1\right]\right)=1-\mathrm{e}^{-\lambda\left|B_{r}(0)\right|}$ and consider the integral functionals $G, H$ : $\mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by:

$$
\begin{align*}
& G(u):=\sigma\left[\int_{] 0, T[\times \Omega} g\left(t, x, \frac{u(t, x)}{\sigma}, \frac{\dot{u}(t, x)}{\sigma}\right) d t \otimes d x+\int_{\Omega} \theta\left(\frac{u(T, x)}{\sigma}\right) d x\right]  \tag{4.30}\\
& H(u):=(1-\sigma)\left[\int_{] 0, T[\times \Omega} h\left(t, x, \frac{u(t, x)}{1-\sigma}, \frac{\dot{u}(t, x)}{1-\sigma}\right) d t \otimes d x+\int_{\Omega} \theta\left(\frac{u(T, x)}{1-\sigma}\right) d x\right] . \tag{4.31}
\end{align*}
$$

Let $G \square H: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ be the inf-convolution of $G$ and $H$, i.e.,

$$
G \square H(u):=\inf \left\{G\left(u_{1}\right)+H\left(u_{2}\right): u_{1}, u_{2} \in \mathcal{H}_{0}^{1} \text { and } u_{1}+u_{2}=u\right\} .
$$

The following result is a consequence of Theorem 4.2.
Theorem 4.3. For $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}^{\mathrm{p}}(\omega)=G \square H .
$$

Proof of Theorem 4.3. Applying Theorem 4.2 with $f=f_{\mathrm{p}}$ we deduce that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}^{\mathrm{p}}(\omega)=F_{\mathrm{hom}}
$$

with $F_{\text {hom }}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by (4.11)-(4.12) with $f=f_{\mathrm{p}}$. So, it remains to prove that $F_{\text {hom }}=G \square H$, i.e., for every $u \in \mathcal{H}_{0}^{1}$, one has:

$$
\begin{align*}
& F_{\mathrm{hom}}(u) \geqslant G \square H(u) ;  \tag{4.32}\\
& F_{\mathrm{hom}}(u) \leqslant G \square H(u) . \tag{4.33}
\end{align*}
$$

Let us fix $u \in \mathcal{H}_{0}^{1}$ and let us set:

$$
\begin{aligned}
& {\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]:=\left\{(\omega, y) \in \Sigma \times Y: N\left(\omega, B_{r}(y)\right) \geqslant 1\right\}} \\
& {\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]:=\left\{(\omega, y) \in \Sigma \times Y: N\left(\omega, B_{r}(y)\right)=0\right\}}
\end{aligned}
$$

Taking (4.29) into account, as $\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]$ and $\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]$ are disjoints and $\Sigma \times Y=\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right] \cup\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]$, for any $v \in \mathcal{S}_{\text {hom }}(u)$, one has

$$
\begin{aligned}
\int_{\Sigma \times Y} \Phi_{f_{\mathrm{p}}(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y= & \int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \Phi_{f_{\mathrm{p}}(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \\
& +\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \Phi_{f_{\mathrm{p}}(\omega, y)}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \\
= & \int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \Phi_{g}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \\
& +\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \Phi_{h}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y
\end{aligned}
$$

Hence

Fix any $v \in \mathcal{S}_{\text {hom }}(u)$. Taking (4.28) into account we see that

$$
\left\{\begin{array}{l}
\mathbb{P} \otimes d y\left(\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]\right)=\mathbb{P}\left(\left[N\left(\cdot, B_{r}(0) \geqslant 1\right]\right)=\sigma\right.  \tag{4.35}\\
\text { and so } \mathbb{P} \otimes d y\left(\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]\right)=1-\sigma .
\end{array}\right.
$$

Hence, by using Jensen's inequality, one has

$$
\begin{align*}
\int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \Phi_{g}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y & =\sigma \int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \Phi_{g}(v(\cdot, \cdot, \omega, y)) \frac{1}{\sigma} d \mathbb{P}(\omega) \otimes d y \\
& \geqslant \sigma \Phi_{g}\left(\int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} v(\cdot, \cdot \omega, y) \frac{1}{\sigma} d \mathbb{P}(\omega) \otimes d y\right) \\
& =\sigma \Phi_{g}\left(\frac{1}{\sigma} \int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} v(\cdot, \cdot \omega, y) d \mathbb{P}(\omega) \otimes d y\right) \tag{4.36}
\end{align*}
$$

and, in the same way,

$$
\begin{equation*}
\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \Phi_{h}(v(\cdot, \cdot \omega, y)) d \mathbb{P}(\omega) \otimes d y \geqslant(1-\sigma) \Phi_{h}\left(\frac{1}{1-\sigma} \int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]}^{v(\cdot, \cdot \omega, y) d \mathbb{P}(\omega) \otimes d y) .}\right. \tag{4.37}
\end{equation*}
$$

Set

$$
\left\{\begin{aligned}
u_{1}(\cdot, \cdot):=\int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} v(\cdot, \cdot, \omega, y) d \mathbb{P}(\omega) \otimes d y \\
u_{2}(\cdot, \cdot):=\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} v(\cdot, \cdot, \omega, y) d \mathbb{P}(\omega) \otimes d y
\end{aligned}\right.
$$

Then, $u_{1}, u_{2} \in \mathcal{H}_{0}^{1}$ and $u_{1}+u_{2}=u$. Moreover, according to (4.11)-(4.12), from (4.30), (4.31), (4.36) and (4.37) we deduce that:

$$
\begin{aligned}
& \int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \Phi_{g}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \geqslant G\left(u_{1}\right) \\
& \int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \Phi_{h}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \geqslant H\left(u_{2}\right) .
\end{aligned}
$$

Consequently, one has

$$
\int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \underset{g}{ }(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y+\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \underset{h}{ }(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \geqslant G \square H(u)
$$

for all $v \in \mathcal{S}_{\text {hom }}(u)$, and (4.32) follows by taking (4.34) into account. Conversely, for any $u_{1}, u_{2} \in \mathcal{H}_{0}^{1}$ such that $u_{1}+u_{2}=u$, set

$$
v(\cdot, \cdot, \omega, y):= \begin{cases}\frac{u_{1}(\cdot, \cdot)}{\sigma} & \text { if } N\left(\omega, B_{r}(y)\right) \geqslant 1 \\ \frac{u_{2}(\cdot, \cdot)}{1-\sigma} & \text { if } N\left(\omega, B_{r}(y)\right)=0\end{cases}
$$

Then, $v \in \mathcal{S}_{\text {hom }}(u)$ and, from (4.12), (4.30), (4.31), (4.34) and (4.35), we see that

$$
\begin{aligned}
F_{\text {hom }}(u) & \leqslant \int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \Phi_{g}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y+\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \Phi_{h}(v(\cdot, \cdot, \omega, y)) d \mathbb{P}(\omega) \otimes d y \\
& =\int_{\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]} \frac{1}{\sigma} G\left(u_{1}\right) d \mathbb{P}(\omega) \otimes d y+\int_{\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]} \frac{1}{1-\sigma} H\left(u_{2}\right) d \mathbb{P}(\omega) \otimes d y \\
& =\mathbb{P} \otimes d y\left(\left[N\left(\cdot, B_{r}(\cdot)\right) \geqslant 1\right]\right) \frac{1}{\sigma} G\left(u_{1}\right)+\mathbb{P} \otimes d y\left(\left[N\left(\cdot, B_{r}(\cdot)\right)=0\right]\right) \frac{1}{1-\sigma} H\left(u_{2}\right) \\
& =G\left(u_{1}\right)+H\left(u_{2}\right),
\end{aligned}
$$

which gives (4.33), and the proof of Theorem 4.3 is complete.
Assume futhermore that:
$\left(\mathrm{P}_{1}\right)$ for $d t \otimes d x$-a.e. $\left.(t, x) \in\right] 0, T[\times \Omega$, the functions $g(t, x, \cdot, \cdot)$ and $h(t, x, \cdot, \cdot)$ are strictly convex, $g(t, x, \cdot, \cdot) \in C^{1}\left(\mathbb{R}^{2}\right)$ and $h(t, x, \cdot, \cdot) \in C^{1}\left(\mathbb{R}^{2}\right)$;
$\left(\mathrm{P}_{2}\right)$ the function $\theta$ is strictly convex and $\theta \in C^{1}(\mathbb{R})$.
The following result is somehow a version of Theorem 4.3 in terms of weak convergence in $H^{1}$ and Euler-Lagrange's equations.

Theorem 4.4. Assume that $\left(\mathrm{P}_{1}\right)$ and $\left(\mathrm{P}_{2}\right)$ hold and, for each $\omega \in \Sigma$ and each $\varepsilon>0$, let $u_{\varepsilon}(\omega)$ be the minimizer of $F_{\varepsilon}^{\mathrm{p}}(\omega)$. Then, for $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has

$$
u_{\varepsilon}(\omega) \xrightarrow{H^{1}} u
$$

where $u \in \mathcal{H}_{0}^{1}$ is the minimizer of $G \square H$. Moreover, $u$ minimizes $G \square H$ if and only if there exist $v, w \in \mathcal{H}_{0}^{1}$ such that $u=v+w$ with $v$ and $w$ satisfying the following differential system:
$\left(\mathrm{S}_{g, h}\right)\left\{\begin{array}{l}\left.\frac{\partial g}{\partial s}\left(t, x, \frac{v(t, x)}{\sigma}, \frac{\dot{v}(t, x)}{\sigma}\right)-\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \dot{s}}\left(t, x, \frac{v(t, x)}{\sigma}, \frac{\dot{v}(t, x)}{\sigma}\right)\right)=0 d t \otimes d x-\text { a.e. in }\right] 0, T[\times \Omega \\ \left.\frac{\partial h}{\partial s}\left(t, x, \frac{w(t, x)}{1-\sigma}, \frac{\dot{w}(t, x)}{1-\sigma}\right)-\frac{\partial}{\partial t}\left(\frac{\partial h}{\partial \dot{s}}\left(t, x, \frac{w(t, x)}{1-\sigma}, \frac{\dot{w}(t, x)}{1-\sigma}\right)\right)=0 d t \otimes d x-\text { a.e. in }\right] 0, T[\times \Omega\end{array}\right.$
subjected to the time-boundary conditions:

$$
\left(\partial \mathrm{S}_{g, h}\right)\left\{\begin{array}{l}
\frac{d \theta}{d s}\left(\frac{v(T, x)}{\sigma}\right)+\sigma \frac{\partial g}{\partial \dot{s}}\left(T, x, \frac{v(T, x)}{\sigma}, \frac{\dot{v}(T, x)}{\sigma}\right)=0 \quad d x \text {-a.e. in } \Omega \\
\frac{d \theta}{d s}\left(\frac{w(T, x)}{\sigma}\right)+(1-\sigma) \frac{\partial h}{\partial \dot{s}}\left(T, x, \frac{w(T, x)}{1-\sigma}, \frac{\dot{w}(T, x)}{1-\sigma}\right)=0 \quad \text { dx-a.e. in } \Omega \\
v(0, \cdot)=0 \\
w(0, \cdot)=0 .
\end{array}\right.
$$

Proof of Theorem 4.4. The first part of the proposition is a direct consequence of Theorem 4.3. Let us prove the second part of the proposition. First of all, it is easy to see that $G$ and $H$ are convex, and so is $G \square H$ by Propostion B.2. As $G \square H$ is a $\Gamma$-limit (see Theorem 4.3), it follows that $G \square H$ is closed and convex. Consequently, $u$ minimizes $G \square H$ if and only if, one has

$$
\begin{equation*}
0 \in \partial(G \square H)(u) \tag{4.38}
\end{equation*}
$$

(see [ABM14, Proposition 9.5.3]). Taking $\left(\mathrm{P}_{1}\right)$ and $\left(\mathrm{P}_{2}\right)$ into account, we see that $G$ and $H$ are Gâteaux differentiable, and so, from Theorem B. 4 we deduce that

$$
\begin{equation*}
\partial(G \square H)(u)=D G / / D H(u), \tag{4.39}
\end{equation*}
$$

where $D G / / D H$ denotes the parallel sum of $G$ and $H$ (see Definition B.3). Consequently, from (4.39) we can assert that (4.38) holds if and only if there exist $v, w \in \mathcal{H}_{0}^{1}$ such that $u=v+w$ and

$$
\left\{\begin{array}{l}
D G(v)=0 \text {, i.e., } D G(v)(\xi)=0 \text { for all } \xi \in \mathcal{H}_{0}^{1}  \tag{4.40}\\
D H(w)=0, \text { i.e., } D H(w)(\xi)=0 \text { for all } \xi \in \mathcal{H}_{0}^{1} .
\end{array}\right.
$$

By using differential calculus, we see that (4.40) is equivalent to the following integral equation system: for every $\xi \in \mathcal{H}_{0}^{1}$, one has

$$
(\mathrm{I})\left\{\begin{array}{l}
\int_{] 0, T[\times \Omega}\left[\frac{\partial g}{\partial s}\left(t, x, \frac{v(t, x)}{\sigma}, \frac{\dot{v}(t, x)}{\sigma}\right) \xi(t, x)+\frac{\partial g}{\partial \dot{s}}\left(t, x, \frac{v(t, x)}{\sigma}, \frac{\dot{v}(t, x)}{\sigma}\right) \dot{\xi}(t, x)\right] d t \otimes d x \\
+\int_{\Omega} \frac{d \theta}{d s}\left(\frac{v(T, x)}{\sigma}\right) \dot{\xi}(t, x) d t \otimes d x=0 \\
\int_{] 0, T[\times \Omega}\left[\frac{\partial h}{\partial s}\left(t, x, \frac{w(t, x)}{1-\sigma}, \frac{\dot{w}(t, x)}{1-\sigma}\right) \xi(t, x)+\frac{\partial h}{\partial \dot{s}}\left(t, x, \frac{w(t, x)}{\sigma}, \frac{\dot{w}(t, x)}{1-\sigma}\right) \dot{\xi}(t, x)\right] d t \otimes d x \\
+\int_{\Omega} \frac{d \theta}{d s}\left(\frac{w(T, x)}{1-\sigma}\right) \dot{\xi}(t, x) d t \otimes d x=0 .
\end{array}\right.
$$

Finally, in one hand, taking $\xi(t, x)=a(t) b(x)$ with $a \in C_{c}^{\infty}(] 0, T[)$ and $b \in C_{c}^{\infty}(\Omega)$, and integrating by part with respect to the time variable, from (I) we obtain ( $\mathrm{S}_{g, h}$ ). On the other
hand, taking $\xi(t, x)=a(t) b(x)$ with $a(t)=\left(\frac{t}{T}\right)^{n}$ and $b \in C_{c}^{\infty}(\Omega)$, integrating by part with respect to the time variable and letting $n \rightarrow \infty$, from (I) we obtain $\left(\partial \mathrm{S}_{g, h}\right)$.
4.3. Non-local stochastic homogenization of non-diffusive reaction equations. Let $\psi: \Sigma \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be a $\left(\mathcal{A} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}\left(\mathbb{R}^{N}\right) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})\right)$-measurable function such that:
$\left(\mathrm{H}_{0}\right) \psi\left(T_{z} \omega, \cdot, x, \cdot\right)=\psi(\omega, \cdot, x+z, \cdot)$ for $\mathbb{P}$-a.a. $\omega \in \Sigma$, all $x \in \mathbb{R}^{N}$ and all $z \in \mathbb{Z}^{N}$;
$\left(\mathrm{H}_{1}\right) \psi(\omega, t, x, \cdot) \in C^{1}(\mathbb{R})$ for $\mathbb{P} \otimes d t \otimes d x$-a.a. $\left.(\omega, t, x) \in \Sigma \times\right] 0, T\left[\times \mathbb{R}^{N} ;\right.$
$\left(\mathrm{H}_{2}\right) \psi(\omega, t, x, \cdot)$ is strictly convex for $\mathbb{P} \otimes d t \otimes d x$-a.a. $\left.(\omega, t, x) \in \Sigma \times\right] 0, T\left[\times \mathbb{R}^{N}\right.$;
$\left(\mathrm{H}_{3}\right)$ there exist $c_{1}, C_{1}>0$ such that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$
c_{1}\left(|s|^{2}-1\right) \leqslant \psi(\omega, \cdot, \cdot, s) \leqslant C_{1}\left(|s|^{2}+1\right)
$$

for all $s \in \mathbb{R}$;
$\left(\mathrm{H}_{4}\right)$ there exists $\delta:\left[0, \infty\left[\rightarrow\left[0, \infty\left[\right.\right.\right.\right.$ with $\lim _{r \rightarrow 0} \delta(r)=0$ such that for $\mathbb{P} \otimes d x$-a.e. $(\omega, x) \in$ $\Sigma \times \mathbb{R}^{N}$, one has

$$
\left|\psi\left(\omega, t_{1}, \cdot, \cdot\right)-\psi\left(\omega, t_{2}, \cdot, \cdot\right)\right| \leqslant \delta\left(\left|t_{1}-t_{2}\right|\right)
$$

for all $t_{1}, t_{2} \in[0, T]$;
$\left(\mathrm{H}_{5}\right)$ there exists $L>0$ such that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$
\left|\frac{\partial \psi}{\partial s}\left(\omega, \cdot, \cdot, s_{1}\right)-\frac{\partial \psi}{\partial s}\left(\omega, \cdot, \cdot, s_{2}\right)\right| \leqslant L\left|s_{1}-s_{2}\right|
$$

for all $s_{1}, s_{2} \in \mathbb{R}$,
and, for each $\omega \in \Sigma$, consider the non-diffusive reaction differential equation defined in $H^{1}(] 0, T\left[; L^{2}(\Omega ; \mathbb{R})\right)$ by

$$
\left(\mathrm{R}_{\varepsilon}^{\omega}\right)\left\{\begin{array}{l}
\left.-\frac{\partial u_{\varepsilon}}{\partial t}(\omega, t, x)=\frac{\partial \psi}{\partial s}\left(\omega, t, \frac{x}{\varepsilon}, u_{\varepsilon}(\omega, t, x)\right) \text { for } d t \otimes d x \text {-a.a. }(t, x) \in\right] 0, T[\times \Omega \\
u_{\varepsilon}(\omega, 0, \cdot)=0
\end{array}\right.
$$

Then, as in $\S 2.3$, for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and every $\varepsilon>0,\left(\mathrm{R}_{\varepsilon}^{\omega}\right)$ has a unique solution $u_{\varepsilon}(\omega, \cdot, \cdot) \in \mathcal{H}_{0}^{1}$. Moreover, arguing as in the proof of Proposition 2.11, we can assert that there exist $\alpha, \beta>0$ and $\gamma:\left[0, \infty\left[\rightarrow\left[0, \infty\left[\right.\right.\right.\right.$ with $\lim _{r \rightarrow 0} \gamma(r)=0$ such that for each $\omega \in \Sigma, u_{\varepsilon}(\omega, \cdot, \cdot)$ minimizes the functional $\Phi_{f_{\psi}(\omega, \dot{\bar{\varepsilon}})}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by

$$
\Phi_{f_{\psi}(\omega, \dot{\bar{\varepsilon}})}(u)=\int_{] 0, T[\times \Omega} f_{\psi}\left(\omega, \frac{x}{\varepsilon}\right)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\int_{\Omega} \theta(u(T, x)) d x
$$

with $f_{\psi}(\omega, \dot{\bar{\varepsilon}}): \mathbb{R}^{N} \rightarrow \mathcal{M}_{\alpha, \beta, \gamma}$ (here $\left.\mathcal{X}=\mathcal{M}_{\alpha, \beta, \gamma}\right)$ defined by

$$
\begin{equation*}
f_{\psi}\left(\omega, \frac{x}{\varepsilon}\right)(t, y, s, \dot{s}):=\psi\left(\omega, t, \frac{x}{\varepsilon}, s\right)+\psi^{*}\left(\omega, t, \frac{x}{\varepsilon},-\dot{s}\right) \tag{4.41}
\end{equation*}
$$

and $\theta: \mathbb{R} \rightarrow\left[0, \infty\left[\right.\right.$ given by (2.9), where $\psi^{*}$ denotes the Legendre-Fenchel conjugate of $\psi$ with respect to the fourth variable. From $\left(\mathrm{H}_{0}\right)$ we deduce that $f_{\psi}$ is covariant with respect to the dynamical system $\left(\Sigma, \mathcal{A}, \mathbb{P},\left(T_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$, i.e., (4.1) hols with $f=f_{\psi}$, and the law of $f_{\psi}$ is invariant under the group $\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}$, i.e., (4.3) holds with $f=f_{\psi}$ and $\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}$ given by (4.2), and moreover the dynamical system $\left(\mathfrak{X}, \mathcal{B}, f_{\psi}^{\sharp} \mathbb{P},\left(\tau_{z}\right)_{z \in \mathbb{Z}^{N}}\right)$ is ergodic, i.e., (4.4) holds
with $f=f_{\psi}$. On the other hand, from $\left(\mathrm{H}_{2}\right)$ we see that $f_{\psi}$ is strictly convex with respect to $(s, \dot{s})$. Hence $F_{\text {hom }}$ defined by (4.11)-(4.12), with $f=f_{\psi}$ and $\theta$ given by (2.9), has a unique minimizer $u \in \mathcal{H}_{0}^{1}$. So, as a direct consequence of Theorem 4.2 we obtain the following result.

Corollary 4.5. Assume that $\left(\mathrm{H}_{0}\right),\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right),\left(\mathrm{H}_{3}\right),\left(\mathrm{H}_{4}\right)$ and $\left(\mathrm{H}_{5}\right)$ are satisfied. Then, for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$
u_{\varepsilon}(\omega, \cdot, \cdot) \xrightarrow{H^{1}} u .
$$

In the setting of a Poisson point process, we can precise the (non-local) equations satisfied by $u$ making clear the non-local effects induced by homogenization of stochastic non-difusive reaction differential equations of type $\left(\mathrm{R}_{\varepsilon}^{\omega}\right)$ (see Theorem 4.6(iv)).
4.3.1. The setting of a Poisson point process. Let $\psi_{1}, \psi_{2}:[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be two $\left(\mathcal{B}([0, T]) \otimes \mathcal{B}\left(\mathbb{R}^{N}\right) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})\right)$-measurable functions such that:
$\left(\mathrm{E}_{1}\right)$ for each $i \in\{1,2\}, \psi_{i}(t, x, \cdot), \psi_{i}^{*}(t, x, \cdot) \in C^{1}(\mathbb{R})$ for $d t \otimes d x$-a.a. $\left.(t, x) \in \Sigma \times\right] 0, T\left[\times \mathbb{R}^{N}\right.$;
$\left(\mathrm{E}_{2}\right)$ for each $i \in\{1,2\}, \psi_{i}(t, x, \cdot)$ is strictly convex for $d t \otimes d x$-a.a. $\left.(t, x) \in \Sigma \times\right] 0, T\left[\times \mathbb{R}^{N}\right.$;
$\left(\mathrm{E}_{3}\right)$ there exist $c_{1}, C_{1}>0$ such that for each $i \in\{1,2\}$, one has

$$
c_{1}\left(|s|^{2}-1\right) \leqslant \psi_{i}(\cdot, \cdot, s) \leqslant C_{1}\left(|s|^{2}+1\right)
$$

for all $s \in \mathbb{R}$;
$\left(\mathrm{E}_{4}\right)$ there exists $\delta:\left[0, \infty\left[\rightarrow\left[0, \infty\left[\right.\right.\right.\right.$ with $\lim _{r \rightarrow 0} \delta(r)=0$ such that for each $i \in\{1,2\}$, one has

$$
\left|\psi_{i}\left(t_{1}, \cdot, \cdot\right)-\psi_{i}\left(t_{2}, \cdot, \cdot\right)\right| \leqslant \delta\left(\left|t_{1}-t_{2}\right|\right)
$$

for all $t_{1}, t_{2} \in[0, T]$;
$\left(\mathrm{E}_{5}\right)$ there exists $L>0$ such that for each $i \in\{1,2\}$, one has

$$
\left|\frac{\partial \psi_{i}}{\partial s}\left(\cdot, \cdot, s_{1}\right)-\frac{\partial \psi_{i}}{\partial s}\left(\cdot, \cdot, s_{2}\right)\right| \leqslant L\left|s_{1}-s_{2}\right|
$$

for all $s_{1}, s_{2} \in \mathbb{R}$,
and let $\psi_{\mathrm{p}}: \Sigma \times[0, T] \times \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$
\begin{aligned}
\psi_{\mathrm{p}}(\omega, t, x, s) & := \begin{cases}\psi_{1}(t, x, s) & \text { if } x \in \underset{z \in D(\omega)}{\cup} B_{r}(z) \\
\psi_{2}(t, x, s) & \text { otherwise }\end{cases} \\
& =\psi_{2}(t, x, s)+\left(\psi_{1}(t, x, s)-\psi_{2}(t, x, s)\right) \min \left\{1, N\left(\omega, B_{r}(x)\right)\right\} \\
& = \begin{cases}\psi_{1}(t, x, s) & \text { if } N\left(\omega, B_{r}(x)\right) \geqslant 1 \\
\psi_{2}(t, x, s) & \text { if } N\left(\omega, B_{r}(x)\right)=0 .\end{cases}
\end{aligned}
$$

Then, $\left(\mathrm{H}_{0}\right),\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right),\left(\mathrm{H}_{3}\right),\left(\mathrm{H}_{4}\right)$ and $\left(\mathrm{H}_{5}\right)$ are satisfied with $\psi=\psi_{\mathrm{p}}$. For each $\omega \in \Sigma$ and each $\varepsilon>0$, let $F_{\varepsilon}^{\psi_{\mathrm{p}}}(\omega): \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by (4.5) with $f=f_{\psi_{\mathrm{p}}}$ and $\theta$ given by (2.9), i.e.,
$\left.F_{\varepsilon}^{\psi_{\mathrm{p}}}(\omega)(u): \left.=\Phi_{\left.f_{\psi_{\mathrm{p}}(\omega, \dot{\varepsilon}}\right)}(u)=\int_{0, T[\times \Omega} f_{\psi_{\mathrm{p}}}\left(\omega, \frac{x}{\varepsilon}\right)(t, x, u(t, x), \dot{u}(t, x)) d t \otimes d x+\frac{1}{2} \int_{\Omega} \right\rvert\, u(T, x)\right)\left.\right|^{2} d x$
with $f_{\psi_{\mathrm{p}}}$ given by (4.41) with $\psi=\psi_{\mathrm{p}}$. As in $\S 4.2$, set $\sigma:=\mathbb{P}\left(\left[N\left(\cdot, B_{r}(0)\right) \geqslant 1\right]\right)=1-\mathrm{e}^{-\lambda\left|B_{r}(0)\right|}$ and consider the integral functionals $\Psi_{1}, \Psi_{2}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ given by:

$$
\begin{aligned}
\Psi_{1}(u):= & \sigma \int_{] 0, T[\times \Omega}\left(\psi_{1}\left(t, x, \frac{u(t, x)}{\sigma}\right)+\psi_{1}^{*}\left(t, x,-\frac{\dot{u}(t, x)}{\sigma}\right)\right) d t \otimes d x \\
& +\frac{\sigma}{2} \int_{\Omega}\left|\frac{u(T, x)}{\sigma}\right|^{2} d x ; \\
\Psi_{2}(u):= & (1-\sigma) \int_{] 0, T[\times \Omega}\left(\psi_{2}\left(t, x, \frac{u(t, x)}{1-\sigma}\right)+\psi_{2}^{*}\left(t, x,-\frac{\dot{u}(t, x)}{1-\sigma}\right)\right) d t \otimes d x \\
& +\frac{1-\sigma}{2} \int_{\Omega}\left|\frac{u(T, x)}{1-\sigma}\right|^{2} d x .
\end{aligned}
$$

Let $\Psi_{1} \square \Psi_{2}: \mathcal{H}_{0}^{1} \rightarrow \mathbb{R}$ be the inf-convolution of $\Psi_{1}$ and $\Psi_{2}$, i.e.,

$$
\Psi_{1} \square \Psi_{2}(u):=\inf \left\{\Psi_{1}\left(u_{1}\right)+\Psi_{2}\left(u_{2}\right): u_{1}, u_{2} \in \mathcal{H}_{0}^{1} \text { and } u_{1}+u_{2}=u\right\}
$$

The following result is a consequence of Theorems 4.3 and 4.4 and Corollary 4.5.
Theorem 4.6. Assume that $\left(\mathrm{E}_{1}\right),\left(\mathrm{E}_{2}\right),\left(\mathrm{E}_{3}\right),\left(\mathrm{E}_{4}\right)$ and $\left(\mathrm{E}_{5}\right)$ are satisfied. Then, the following four assertions hold.
(i) For $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has

$$
\Gamma-\lim _{\varepsilon \rightarrow 0} F_{\varepsilon}^{\psi_{\mathrm{p}}}(\omega)=\Psi_{1} \square \Psi_{2}
$$

(ii) For $\mathbb{P}$-a.e. $\omega \in \Sigma$ and every $\varepsilon>0$, let $u(\omega, \cdot, \cdot) \in \mathcal{H}_{0}^{1}$ be the unique solution of $\left(\mathrm{R}_{\varepsilon}^{\omega}\right)$. Then

$$
u_{\varepsilon}(\omega, \cdot, \cdot) \xrightarrow{H^{1}} u
$$

where $u \in \mathcal{H}_{0}^{1}$ is the unique minimizer of $\Psi_{1} \square \Psi_{2}$.
(iii) The function $u$ minimizes $\Psi_{1} \square \Psi_{2}$ if and only if there exist $v, w \in \mathcal{H}_{0}^{1}$ such that $u=v+w$ with $v$ and $w$ satisfying the following differential system:
$\left(\mathrm{S}_{\psi_{\mathrm{p}}}\right)\left\{\begin{array}{l}\left.\frac{\partial \psi_{1}}{\partial s}\left(t, x, \frac{v(t, x)}{\sigma}\right)-\frac{\partial}{\partial t}\left(\frac{\partial \psi_{1}^{*}}{\partial \dot{s}}\left(t, x,-\frac{\dot{v}(t, x)}{\sigma}\right)\right)=0 d t \otimes d x \text {-a.e. in }\right] 0, T[\times \Omega \\ \left.\frac{\partial \psi_{2}}{\partial s}\left(t, x, \frac{w(t, x)}{1-\sigma}\right)-\frac{\partial}{\partial t}\left(\frac{\partial \psi_{2}^{*}}{\partial \dot{s}}\left(t, x,-\frac{\dot{w}(t, x)}{1-\sigma}\right)\right)=0 d t \otimes d x \text {-a.e. in }\right] 0, T[\times \Omega\end{array}\right.$
subjected to the time-boundary conditions:

$$
\left(\partial \mathrm{S}_{\psi_{\mathrm{p}}}\right)\left\{\begin{array}{l}
\frac{v(T, x)}{\sigma}-\sigma \frac{\partial \psi_{1}^{*}}{\partial \dot{s}}\left(T, x,-\frac{\dot{v}(T, x)}{\sigma}\right)=0 \quad \text { dx-a.e. in } \Omega \\
\frac{w(T, x)}{1-\sigma}-(1-\sigma) \frac{\partial \psi_{2}^{*}}{\partial \dot{s}}\left(T, x,-\frac{\dot{\psi}(T, x)}{1-\sigma}\right)=0 \quad \text { dx-a.e. in } \Omega \\
v(0, \cdot)=0 \\
w(0, \cdot)=0
\end{array}\right.
$$

where $\psi_{1}^{*}\left(\right.$ resp. $\left.\psi_{2}^{*}\right)$ denotes the Legendre-Fenchel conjugate of $\psi_{1}$ (resp. $\psi_{2}$ ) with respect to the third variable.
(iv) The function $u$ minimizes $\Psi_{1} \square \Psi_{2}$ if and only if there exist $u_{1}, u_{2} \in \mathcal{H}_{0}^{1}$ such that $u=\sigma u_{1}+(1-\sigma) u_{2}$ with $u_{1}$ and $u_{2}$ being the unique solutions of the following integro-differential equations:

$$
\begin{aligned}
& \left\{\begin{array}{l}
\left.-\frac{\partial u_{1}}{\partial t}(t, x)=\frac{\partial \psi_{1}}{\partial s}\left(t, x, \frac{u_{1}(T, x)}{\sigma}+\int_{t}^{T} \frac{\partial \psi_{1}}{\partial s}\left(\tau, x, \frac{\partial \psi_{1}}{\partial s}\left(\tau, x, u_{1}(\tau, x)\right)\right) d \tau\right) d t \otimes d x-a . e . \text { in }\right] 0, T[\times \Omega \\
u_{1}(0, \cdot)=0
\end{array}\right. \\
& \left\{\begin{array}{c}
\left.-\frac{\partial u_{2}}{\partial t}(t, x)=\frac{\partial \psi_{2}}{\partial s}\left(t, x, \frac{u_{2}(T, x)}{1-\sigma}+\int_{t}^{T} \frac{\partial \psi_{2}}{\partial s}\left(\tau, x, \frac{\partial \psi_{2}}{\partial s}\left(\tau, x, u_{2}(\tau, x)\right)\right) d \tau\right) d t \otimes d x \text {-a.e. in }\right] 0, T[\times \Omega \\
u_{2}(0, \cdot)=0 .
\end{array}\right.
\end{aligned}
$$

Proof of Theorem 4.6. We only need to prove (iv). For this, it suffices to show that $(v, w) \in \mathcal{H}_{0}^{1} \times \mathcal{H}_{0}^{1}$ satisfies the differential system $\left(\mathrm{S}_{\psi_{\mathrm{p}}}\right)-\left(\partial \mathrm{S}_{\psi_{\mathrm{p}}}\right)$ if and only $\left(u_{1}, u_{2}\right)=\left(\frac{v}{\sigma}, \frac{w}{1-\sigma}\right) \in$ $\mathcal{H}_{0}^{1} \times \mathcal{H}_{0}^{1}$ satisfies the integro-differential equations above. First of all, by integrating over ] $t, T$ [ the first equation in $\left(\mathrm{S}_{\psi_{\mathrm{p}}}\right)$, we see that

$$
\begin{equation*}
\frac{\partial \psi_{1}^{*}}{\partial \dot{s}}\left(t, x,-\frac{\dot{v}(t, x)}{\sigma}\right)=\frac{\partial \psi_{1}^{*}}{\partial \dot{s}}\left(T, x,-\frac{\dot{v}(T, x)}{\sigma}\right)+\int_{t}^{T} \frac{\partial \psi_{1}}{\partial s}\left(\tau, x, \frac{v(\tau, x)}{\sigma}\right) d \tau \tag{4.42}
\end{equation*}
$$

But $\frac{\partial \psi_{1}^{*}}{\partial \dot{s}}(t, x, \cdot)=\left(\frac{\partial \psi_{1}}{\partial s}\right)^{-1}(t, x, \cdot)$ (see [ABM14, Theorem 9.5.1]), where $\left(\frac{\partial \psi_{1}}{\partial s}\right)^{-1}$ denotes the inverse of the subdifferential of $\psi_{1}$ with respect to $s$, and so from (4.42) we deduce that

$$
-\frac{1}{\sigma} \frac{\partial v}{\partial t}(t, x)=\frac{\partial \psi_{1}}{\partial s}\left(t, x, \frac{\partial \psi_{1}^{*}}{\partial \dot{s}}\left(T, x,-\frac{\dot{v}(T, x)}{\sigma}\right)+\int_{t}^{T} \frac{\partial \psi_{1}}{\partial s}\left(t, x \frac{\partial \psi_{1}}{\partial s}\left(\tau, x, \frac{v(\tau, x)}{\sigma}\right)\right) d \tau\right) .
$$

Moreover, according to the first equation in $\left(\partial \mathrm{S}_{\psi_{\mathrm{p}}}\right)$, one has

$$
\frac{v(T, x)}{\sigma^{2}}=\frac{\partial \psi_{1}^{*}}{\partial \dot{s}}\left(T, x,-\frac{\dot{v}(T, x)}{\sigma}\right) .
$$

It follows that

$$
-\frac{1}{\sigma} \frac{\partial v}{\partial t}(t, x)=\frac{\partial \psi_{1}}{\partial s}\left(t, x, \frac{v(T, x)}{\sigma^{2}}+\int_{t}^{T} \frac{\partial \psi_{1}}{\partial s}\left(t, x \frac{\partial \psi_{1}}{\partial s}\left(\tau, x, \frac{v(\tau, x)}{\sigma}\right)\right) d \tau\right)
$$

for $d t \otimes d x$-a.e. $(t, x) \in] 0, T\left[\times \Omega\right.$, and setting $u_{1}=\frac{v}{\sigma}$ we obtain the first integro-differential equation. In the same manner, by using the second equations in $\left(\mathrm{S}_{\psi_{\mathrm{p}}}\right)$ and $\left(\partial \mathrm{S}_{\psi_{\mathrm{p}}}\right)$ and by setting $u_{1}=\frac{w}{1-\sigma}$, we obtain the second integro-differential equation.
Conversely, setting $v=\sigma u_{1}$ and $w=(1-\sigma) u_{2}$, by derivating each integro-differential equation with respect to the time variable (resp. by letting $t=T$ in each integro-differential equation), we obtain $\left(\mathrm{S}_{\psi_{\mathrm{p}}}\right)$ (resp. $\left(\partial \mathrm{S}_{\psi_{\mathrm{p}}}\right)$ ).

## Appendix A. Young measures

Let $(\mathcal{V}, d)$ be a polish space, i.e., $(\mathcal{V}, d)$ is a separable and complete metric space, let $k \geqslant 1$ be an integer and let $\mathcal{O} \subset \mathbb{R}^{k}$ be a bounded domain. Let $\mathcal{Y}(\mathcal{O} ; \mathcal{V})$ be the class of Young measures on $\mathcal{O} \times \mathcal{V}$ and let $C \operatorname{th}^{b}(\mathcal{O} ; \mathcal{V})$ be the space of all bounded Carathéodory integrands on $\mathcal{O} \times \mathcal{V}$.

Definition A. 1 (narrow convergence). Let $\left\{\mu_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ and let $\mu \in \mathcal{Y}(\mathcal{O} ; \mathcal{V})$. We say that $\left\{\mu_{\varepsilon}\right\}_{\varepsilon}$ narrow converges to $\mu$ as $\varepsilon \rightarrow 0$, and we write $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$, if

$$
\lim _{\varepsilon \rightarrow 0} \int_{\mathcal{O} \times \mathcal{V}} \psi(y, \xi) d \mu_{\varepsilon}(y, \xi)=\int_{\mathcal{O} \times \mathcal{V}} \psi(y, \xi) d \mu(y, \xi) \quad \text { for all } \psi \in C \operatorname{th}^{b}(\mathcal{O} ; \mathcal{V})
$$

Definition A. 2 (tightness). We say that $\left\{\mu_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ is tight if for every $\eta>0$, there exists a compact set $K \subset \mathcal{V}$ such that $\sup _{\varepsilon>0} \mu_{\varepsilon}(\mathcal{O} \times(\mathcal{V} \backslash K))<\eta$.
A proof of the following compactness result can be found in [Val90, Theorem 11] (see also [Val94, Theorem 7 and Comments 1), 2) and 3)]).
Theorem A. 3 (Prokhorov's compactness theorem). If $\left\{\mu_{\varepsilon}\right\} \subset \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ is tight then there exists $\mu \in \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ such that, up to a subsequence, $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$.
For a proof of the following theorem we also refer to [Val90, Val94].
Theorem A. 4 (lower semicontinuity, continuity). Let $\psi: \mathcal{O} \times \mathcal{V} \rightarrow[0, \infty]$ be a $\mathcal{B}(\mathcal{O}) \times \mathcal{B}(\mathcal{V})$ measurable function such that $\psi(\cdot, y)$ is lower semicontinuous for all $y \in \mathcal{O}$ and let $\Psi$ : $\mathcal{Y}(\mathcal{O} ; \mathcal{V}) \rightarrow[0, \infty]$ be defined by

$$
\Psi(\mu):=\int_{\mathcal{O} \times \mathcal{V}} \psi(y, \xi) d \mu(y, \xi)
$$

(i) The functional $\Psi$ is lower semicontinuous, i.e., for every $\left\{\mu_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ and every $\mu \in \mathcal{Y}(\mathcal{O} ; \mathcal{V})$, if $\mu_{\varepsilon} \xrightarrow{\mathrm{nar}} \mu$ then

$$
\varliminf_{\varepsilon \rightarrow 0} \Psi\left(\mu_{\varepsilon}\right) \geqslant \Psi(\mu)
$$

(ii) If $\psi(\cdot, y)$ is continuous for all $y \in \mathcal{O}$ then for every $\left\{\mu_{\varepsilon}\right\}_{\varepsilon} \subset \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ and every $\mu \in \mathcal{Y}(\mathcal{O} ; \mathcal{V})$ such that $\mu_{\varepsilon} \xrightarrow{\text { nar }} \mu$ and

$$
\lim _{M \rightarrow \infty}\left(\sup _{\varepsilon>0} \int_{[\psi \geqslant M]}|\psi(y, \xi)| d \mu_{\varepsilon}(y, \xi)\right)=0
$$

one has

$$
\lim _{\varepsilon \rightarrow 0} \Psi\left(\mu_{\varepsilon}\right)=\Psi(\mu)
$$

Let $(\mathcal{W}, \delta)$ be a polish space. A proof of the following theorem can be found in [Tor77] (see also [ABM14, Theorem 4.2.4]).
Theorem A. 5 (desintegration). Let $\mu$ be a Borel measure on $\mathcal{V} \times \mathcal{W}$ and let $\nu:=\pi_{\mathcal{V}}^{\sharp} \mu$ where $\pi_{\mathcal{V}}: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V}$ denotes the canonical projection from $\mathcal{V} \times \mathcal{W}$ to $\mathcal{V}$. Then, there exists a unique (up to the equality a.e.) family $\left\{\mu_{\xi}\right\}_{\xi \in \mathcal{V}}$ of probability measures on $\mathcal{W}$ such that for every $\psi \in L_{\mu}^{1}(\mathcal{V} \times \mathcal{W})$, one has:
(i) the function $\xi \mapsto \int_{\mathcal{W}} \psi(\xi, \zeta) d \mu_{\xi}(\zeta)$ is $\nu$-measurable;
(ii) $\int_{\mathcal{V} \times \mathcal{W}} \varphi(\xi, \zeta) d \mu(\xi, \zeta)=\int_{\mathcal{V}}\left(\int_{\mathcal{W}} \psi(\xi, \zeta) d \mu_{\xi}(\zeta)\right) d \nu(\xi)$,

The family $\left\{\mu_{\xi}\right\}_{\xi \in \mathcal{V}}$ of probability measures on $\mathcal{W}$ is called the desintegration of $\mu$ on the product $\mathcal{V} \times \mathcal{W}$. To summarize it, we write $\mu=\nu \otimes \mu_{\xi}$.

## Appendix B. Inf-convolution and parallel sum

Let $E$ be a linear space.
Definition B. 1 (inf-convolution). Let $G, H: E \rightarrow]-\infty, \infty]$. By the inf-convolution of $G$ and $H$ we mean the function $G \square H: E \rightarrow[-\infty, \infty]$ defined by

$$
G \square H(u):=\inf \left\{G\left(u_{1}\right)+H\left(u_{2}\right): u_{1}, u_{2} \in E \text { and } u_{1}+u_{2}=u\right\}
$$

A proof of the following proposition can be found in [ABM14, Proposition 9.2.2].
Proposition B.2. Let $G, H: E \rightarrow]-\infty, \infty]$. If $G$ and $H$ are convex then $G \square H$ is convex.
In what follows, $E$ is a Banach space and $E^{\prime}$ is its dual.
Definition B. 3 (parallel sum). Let $\Gamma, \Lambda: E \rightrightarrows E^{\prime}$ be two multifunctions. By the parallel sum of $\Gamma$ and $\Lambda$ we mean the multifunction $\Gamma / / \Lambda: E \rightrightarrows E^{\prime}$ defined by

$$
\Gamma / / \Lambda(u):=\left\{v \in E^{\prime}: \exists\left(u_{1}, u_{2}\right) \in E \times E \text { such that } u=u_{1}+u_{2} \text { and } v \in \Gamma\left(u_{1}\right) \cap \Lambda\left(u_{2}\right)\right\} .
$$

For a proof of the following theorem, we refer to [Str94, Theorem 3.7] (see also [Str96]).
Theorem B.4. Assume that $E$ is reflexive, $G: E \rightarrow]-\infty, \infty]$ is convex, proper and lower semicontinuous and $H: E \rightarrow]-\infty, \infty[$ is convex and Gâteaux differentiable. Then $G \square H$ is Gâteaux differentiable and $D(G \square H)=\partial G / / D H$.

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[^1]:    ${ }^{1}$ Let $X$ be a Hilbert space, let $h: X \rightarrow \mathbb{R}$ and, for each $\varepsilon>0$, let $h_{\varepsilon}: X \rightarrow \mathbb{R}$. We say that $\left\{h_{\varepsilon}\right\}_{\varepsilon}$ Mosco-converges to $h$, and we write $\mathrm{M}-\lim _{\varepsilon \rightarrow 0} h_{\varepsilon}=h$, if the following two assertions hold:

    M-lower bound: for every $v \in X$ and every $\left\{v_{\varepsilon}\right\}_{\varepsilon} \subset X$, if $v_{\varepsilon} \longrightarrow v$ then $\underline{\lim }_{\varepsilon \rightarrow 0} h_{\varepsilon}\left(v_{\varepsilon}\right) \geqslant h(v)$;
    M-upper bound: for every $v \in X$ there exists $\left\{w_{\varepsilon}\right\}_{\varepsilon} \subset X$ such that $w_{\varepsilon} \rightarrow v$ and $\varlimsup_{\varepsilon \rightarrow 0} h_{\varepsilon}\left(w_{\varepsilon}\right) \leqslant h(v)$.

