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CONTINUITY THEOREM FOR NON-LOCAL FUNCTIONALS INDEXED BY YOUNG MEASURES AND STOCHASTIC HOMOGENIZATION

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Abstract. We establish a continuity theorem for non-local functionals indexed by Young measures that we use to deal with homogenization of stochastic non-diffusive reaction differential equations. Non-local effects induced by homogenization of such stochastic differential equations are studied.

Contents

Key words and phrases. Γ-convergence, Stochastic homogenization, Non-local functionals, Non-diffusive reaction differential equations, Non-local effects, Young measures.

1. Introduction

Let $T > 0$, let $N \geqslant 1$ be an integer and let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $\mathcal{X}$ be a polish subspace of a suitable metric space (see §2.1 for more details) and let $\mathcal{Y}(\Omega;\mathcal{X})$ be the class of Young measures on $\Omega \times \mathcal{X}$. Let us set

$$\mathcal{H}_0^1 := \Big\{ u \in H^1(]0,T[;L^2(\Omega;\mathbb{R})) : u(0,\cdot) = 0 \Big\}$$

and for each $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ and each $u \in \mathcal{H}_0^1$, let us consider the following two classes of functions:

$$\mathcal{H}_{0,\mu}^1 := \Big\{ U \in H^1\big(]0,T[;L^2_\mu(\Omega\times\mathcal{X};\mathbb{R})\big) : U(0,\cdot,\cdot) = 0 \Big\};$$

$$\mathcal{S}_\mu(u) := \left\{ U \in \mathcal{H}_{0,\mu}^1 : \int_{\mathcal{X}} U(t,x,\xi)d\mu_x(\xi) = u(t,x) \text{ for } dt\otimes dx\text{-a.a. } (t,x) \in]0,T[\times\Omega \right\},$$

where $\{\mu_x\}_{x\in\Omega}$ denotes the desintegration of μ on $\Omega\times\mathcal{X}$. In this paper we are concerned with non-local functionals $\Phi_\mu : \mathcal{H}_0^1 \to \mathbb{R}$ defined by

$$\Phi_\mu(u) := \inf\Big\{ \phi_\mu(U) : U \in \mathcal{S}_\mu(u) \Big\}$$

with $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$, where $\phi_\mu : \mathcal{S}_\mu(u) \to \mathbb{R}$ is given by

$$\phi_\mu(U) := \int_{]0,T[\times\Omega\times\mathcal{X}} \xi\big(t,x,U(t,x,\xi),\dot{U}(t,x,\xi)\big)dt\otimes d\mu(x,\xi) + \int_{\Omega\times\mathcal{X}} \theta(U(T,x,\xi))d\mu(x,\xi)$$

with $\theta : \mathbb{R} \to [0,\infty[$ a convex function of 2-polynomial growth (see (C_1), (C_2), (C_3) and (C_4) in §2.1 for more details). Let us set $\mathcal{F} := \{\Phi_\mu : \mu \in \mathcal{Y}(\Omega;\mathcal{X})\}$. Our main result is to prove that the map

$$\mu \mapsto \Phi_\mu$$

is continuous from $\mathcal{Y}(\Omega;\mathcal{X})$ endowed with the narrow convergence to $\mathcal{F}$ endowed with the Γ-convergence (see Theorem 2.7 and also Corollaries 2.8 and 2.9), i.e.,

$$\mu_\varepsilon \xrightarrow{\text{nar}} \mu \Rightarrow \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_\varepsilon} = \Phi_\mu.$$

The interest of this result comes from the convergence, when ε goes to zero, of non-diffusive reaction differential equations of the type

$$(R_\varepsilon) \quad \begin{cases} -\dfrac{\partial u_\varepsilon}{\partial t}(t,x) = \dfrac{\partial \psi_\varepsilon}{\partial s}(t,x,u_\varepsilon(t)) \text{ for } dt\otimes dx\text{-a.a. } (t,x) \in]0,T[\times\Omega \\ \\ u_\varepsilon(0,\cdot) = 0 \end{cases}$$

with $\psi_\varepsilon : [0,T]\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$. In fact, under suitable assumptions (see §2.3 for more details) every (R_ε) admits a unique solution $u_\varepsilon \in \mathcal{H}_0^1$ which is also a minimizer of Φ_{μ_ε} with:

$$\begin{aligned} &\mu_\varepsilon = dx\otimes\delta_{f_\varepsilon(x)}; \\ &f_\varepsilon(x)(t,y,s,\dot{s}) = \psi_\varepsilon(t,x,s) + \psi_\varepsilon^*(t,x,-\dot{s}); \\ &\theta(s) = \frac{1}{2}|s|^2, \end{aligned}$$

where ψ_ε^* denotes the Legendre-Fenchel conjugate of ψ_ε with respect to the third variable (see Proposition 2.11). When there exists $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ such that $dx \otimes \delta_{f_\varepsilon(x)} \xrightarrow{\text{nar}} \mu$, Theorem 2.7 implies that any cluster point of $\{u_\varepsilon\}_\varepsilon$ with respect to the weak convergence in $\mathcal{H}_0^1$ is a minimizer of Φ_μ with the same $\theta : \mathbb{R} \to [0,\infty[$ given above (see Corollary 2.12).

This latter result point out the appearance of non-local effects induced by homogenization of non-diffusive reaction equations of type (R_ε) and, in some way, can be seen as a generalization of previous non-local results obtained by Mascarenhas (see [Mas93]) and Toader (see [Toa99]). The fact that non-local effects appear when dealing with homogenization of equations of type (R_ε) is illustrated by studying stochastic homogenization of such equations in the framework of a Poisson point process (see Theorem 4.6 and also Theorems 4.3 and 4.4). In particular, we show that the weak limit of solutions of stochastic non-diffusive reaction differential equations of type (R_ε) is characterized by two integro-differential equations (see §4.3).

The plan of the paper is as follows. In Section 2 we state the main result of the paper (see Theorem 2.7) and direct consequences of this result (see Corollaries 2.8 and 2.9) together with results related to homogenization of non-diffusive reaction equations (see Proposition 2.11, Corollary 2.12 and Proposition 2.13). Propositions 2.11 and 2.13 are proved in Section 2 while the proof of Theorem 2.7 is given in Section 3. Section 4 is devoted to stochastic homogenization. In §4.1, as a consequence of Theorem 2.7, we state and prove a non-local stochastic homogenization theorem (see Theorem 4.2) that we apply in §4.2, in the case of a Poisson point process, to homogenization of stochastic integral functionals as well as to homogenization of stochastic non-diffusive reaction equations (see Theorems 4.3, 4.4 and 4.6). Theorems 4.3 and 4.4 and Theorem 4.6 are respectively proved in §4.2 and §4.3. Finally, in the appendix, we recall some standard results on Young measures (see Appendix A) and inf-convolution and parallel sum (see Appendix B).

Notation. Throughout the paper, for any function w, $\dot{w}$ denotes the partial derivative of w with respect to t, i.e., $\dot{w} = \frac{\partial w}{\partial t}$.

2. Main results

2.1. **A continuity theorem for a class of non-local functionals indexed by Young measures.** Let $N \geqslant 1$ be an integer and let $T > 0$. Let $\alpha, \beta \in]0,\infty[$ be such $\alpha \leqslant \beta$ and let $\gamma : [0,\infty[\to [0,\infty[$ be such that $\lim_{r\to 0} \gamma(r) = 0$. In what follows, we denote by $\mathcal{M}_{\alpha,\beta,\gamma}$ the class of functions $\xi : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following three conditions:

- (C_1) $\xi(t,x,\cdot,\cdot)$ is convex for all $(t,x) \in [0,T] \times \mathbb{R}^N$;
- (C_2) $\alpha(|(s,\dot{s})|^2 - 1) \leqslant \xi(t,x,s,\dot{s}) \leqslant \beta(|(s,\dot{s})|^2 + 1)$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$ and all $(s,\dot{s}) \in \mathbb{R} \times \mathbb{R}$;
- (C_3) $|\xi(t_1,x_1,s,\dot{s}) - \xi(t_2,x_2,s,\dot{s})| \leqslant \gamma(|t_1 - t_2| + |x_1 - x_2|)$ for all $(t_1,x_1),(t_2,x_2) \in [0,T] \times \mathbb{R}^N$ and all $(s,\dot{s}) \in \mathbb{R} \times \mathbb{R}$.

From now on, to simplify notation we will sometimes write S to denote the couple $(s,\dot{s}) \in \mathbb{R}^2$. From (C_1), (C_2) and (C_3) it is easy to see that there exists $L > 0$ such that

$$|\xi(t_1,x_1,S_1) - \xi(t_2,x_2,S_2)| \leqslant \gamma(|t_1 - t_2| + |x_1 - x_2|) + L|S_1 - S_2|(1 + |S_1| + |S_2|) \tag{2.1}$$

for all $\xi \in \mathcal{M}_{\alpha,\beta,\gamma}$, all $(t_2,x_2) \in [0,T] \times \mathbb{R}^N$ and all $(S_1,S_2) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Proposition 2.1. *Under* (C_1), (C_2) *and* (C_3) *the space* $\mathcal{M}_{\alpha,\beta,\gamma}$ *is a compact metric space with respect to the metric* $d : \mathcal{M}_{\alpha,\beta,\gamma} \times \mathcal{M}_{\alpha,\beta,\gamma} \to [0, \infty[$ *defined by*

$$d(\xi_1, \xi_2) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \inf\Big\{1, \|\xi_1 - \xi_2\|_{\infty,n}\Big\}, \tag{2.2}$$

where $\|\cdot\|_{\infty,n}$ *is the uniform norm on* $[0,T] \times \overline{B}_n(0)$*, i.e., for each* $\xi \in \mathcal{M}_{\alpha,\beta,\gamma}$*,*

$$\|\xi\|_{\infty,n} = \sup\Big\{|\xi(t,x,S)| : t \in [0,T] \text{ and } (x,S) \in \overline{B}_n(0)\Big\}$$

with $\overline{B}_n(0) \subset \mathbb{R}^{N+2}$ *being the* $(N+2)$*-dimensional closed ball centered at* 0 *with radius* n*. Moreover, the evaluation map* $\mathcal{E} : \mathcal{M}_{\alpha,\beta,\gamma} \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ *defined by* $\mathcal{E}(\xi,t,x,S) = \xi(t,x,S)$ *is continuous.*

Proof of Proposition 2.1. For each $n \in \mathbb{N}$, we consider the normed space $(X_n, \|\cdot\|_{\infty,n})$ with X_n defined by

$$X_n := \Big\{\xi_{|[0,T]\times\overline{B}_n(0)} : \xi \in \mathcal{M}_{\alpha,\beta,\gamma}\Big\},$$

where $\xi_{|[0,T]\times\overline{B}_n(0)}$ denotes the restriction of ξ to $[0,T] \times \overline{B}_n(0)$. Then $(\mathcal{M}_{\alpha,\beta,\gamma}, d)$ is homeomorphic to the product $\prod_{n\in\mathbb{N}} X_n$. On the other hand, by using (C_2) and (2.1), from Ascoli's compactness theorem we see that $(X_n, \|\cdot\|_{\infty,n})$ is compact for all $n \in \mathbb{N}$. By Tychonov's compactness theorem it follows that $(\mathcal{M}_{\alpha,\beta,\gamma}, d)$ is compact.
Let $(\xi,t,x,S) \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^2$ and let $\{(\xi_j,t_j,x_j,S_j)\}_j \subset \mathcal{M}_{\alpha,\beta,\gamma} \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^2$ be such that $|(t_j,x_j,S_j) - (t,x,S)| \to 0$ and $d(\xi_j,\xi) \to 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $(t,x,S) \in [0,T] \times \overline{B}_{n_0}(0)$ and $(t_j,x_j,S_j) \in [0,T] \times \overline{B}_{n_0}(0)$ for all $j \geqslant 1$. Let $j_0 \geqslant 1$ be such that

$$d(\xi_j,\xi) < \sum_{n=n_0}^{\infty} \frac{1}{2^{n+1}} \text{ for all } j \geqslant j_0. \tag{2.3}$$

Then, for every $j \geqslant j_0$ we have $\inf\big\{1, \|\xi_j - \xi\|_{\infty,n_0}\big\} = \|\xi_j - \xi\|_{\infty,n_0}$. Indeed, fix any $j \geqslant j_0$. If $\inf\big\{1, \|\xi_j - \xi\|_{\infty,n_0}\big\} = 1$ then $\inf\big\{1, \|\xi_j - \xi\|_{\infty,n}\big\} = 1$ for all $n \geqslant n_0$ because $\|\xi_j - \xi\|_{\infty,n_0} \leqslant \|\xi_j - \xi\|_{\infty,n}$ for all $n \geqslant n_0$. Hence

$$d(\xi_j,\xi) \geqslant \sum_{n=n_0}^{\infty} \frac{1}{2^{n+1}},$$

which contradicts (2.3). Using (2.1) we deduce that

$$\begin{aligned}
\big|\mathcal{E}(\xi_j,t_j,x_j,S_j) - \mathcal{E}(\xi,t,x,S)\big| &\leqslant \big|\xi_j(t_j,x_j,S_j) - \xi_j(t,x,S)\big| + \big|(\xi_j - \xi)(t,x,S)\big| \\
&\leqslant \gamma(|t_j - t| + |x_j - x|) + L|S_j - S|(1 + |S_j| + |S|) \\
&\quad + \|\xi_j - \xi\|_{\infty,n_0} \\
&\leqslant \gamma(|t_j - t| + |x_j - x|) + LC_{n_0}|S_j - S| + 2^{n_0+1} d(\xi_j,\xi)
\end{aligned}$$

for all $j \geqslant j_0$ with $C_{n_0} := \sup\{1 + |S_j| + |S| : j \geqslant j_0\} \in]0, \infty[$. Letting $j \to \infty$ we deduce that $|\mathcal{E}(\xi_j,t_j,x_j,S_j) - \mathcal{E}(\xi,t,x,S)| \to 0$. Hence $\mathcal{E}$ is continuous. ■

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. In what follows, we set

$$\mathcal{H}_0^1 := \Big\{ u \in H^1(]0,T[; L^2(\Omega;\mathbb{R})) : u(0,\cdot) = 0 \Big\}.$$

Let $\mathcal{X}$ be a polish subspace of $(\mathcal{M}_{\alpha,\beta,\gamma}, d)$ and let $\mathcal{Y}(\Omega;\mathcal{X})$ be the class of Young measures on $\Omega \times \mathcal{X}$. For each $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ and each $u \in \mathcal{H}_0^1$ we consider the following two classes of functions:

$$\mathcal{H}_{0,\mu}^1 := \Big\{ U \in H^1\big(]0,T[; L^2_\mu(\Omega \times \mathcal{X};\mathbb{R})\big) : U(0,\cdot,\cdot) = 0 \Big\};$$

$$\mathcal{S}_\mu(u) := \left\{ U \in \mathcal{H}_{0,\mu}^1 : \int_{\mathcal{X}} U(t,x,\xi) d\mu_x(\xi) = u(t,x) \text{ for } dt \otimes dx\text{-a.a. } (t,x) \in]0,T[\times\Omega \right\},$$

where $\{\mu_x\}_{x\in\Omega}$ denotes the desintegration of μ on $\Omega \times \mathcal{X}$. Let $\theta : \mathbb{R} \to [0,\infty[$ be a convex function of 2-polynomial growth, i.e.,

(C_4) $\theta(y) \leqslant c(1+|y|^2)$ for all $y \in \mathbb{R}$ and some $c \in]0,\infty[$.

Remark 2.2. Under (C_4) as θ is convex we can assert that there exists $C > 0$ such that

$$|\theta(y_1) - \theta(y_2)| \leqslant C\,|y_1 - y_2|\,(1 + |y_1| + |y_2|) \tag{2.4}$$

for all $y_1, y_2 \in \mathbb{R}$.

To every $u \in \mathcal{H}_0^1$ and every $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ we associate the integral functional $\phi_\mu : \mathcal{S}_\mu(u) \to \mathbb{R}$ defined by

$$\phi_\mu(U) := \int_{]0,T[\times\Omega\times\mathcal{X}} \xi\big(t,x,U(t,x,\xi),\dot{U}(t,x,\xi)\big) dt \otimes d\mu(x,\xi) + \int_{\Omega\times\mathcal{X}} \theta(U(T,x,\xi)) d\mu(x,\xi) \tag{2.5}$$

and we consider the functional $\Phi_\mu : \mathcal{H}_0^1 \to \mathbb{R}$ given by

$$\Phi_\mu(u) := \inf \Big\{ \phi_\mu(U) : U \in \mathcal{S}_\mu(u) \Big\}. \tag{2.6}$$

Remark 2.3. By Proposition 2.1, the evaluation map $\mathcal{E}$ is continuous, and so the map $(t,x,\xi) \mapsto \xi(t,x,U(t,x,\xi),\dot{U}(t,x,\xi))$ is $(\mathcal{B}(]0,T[) \otimes \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathbb{R}))$-measurable. It follows that the integral functional ϕ_μ is well defined.

Remark 2.4. From (C_2) and (C_4) it is easy to see that

$$\alpha \int_{]0,T[\times\Omega\times\mathcal{X}} \Big(|(U,\dot{U})|^2 - 1 \Big)\, dt \otimes d\mu(x,\xi) \leqslant \phi_\mu(U) \leqslant \beta \int_{]0,T[\times\Omega\times\mathcal{X}} \Big(|(U,\dot{U})|^2 + 1 \Big)\, dt \otimes d\mu(x,\xi) + c \int_{\Omega\times\mathcal{X}} \big(1 + |U(T,x)|^2\big)\, d\mu(x,\xi)$$

for all $U \in \mathcal{S}_\mu(u)$ with $u \in \mathcal{H}_0^1$, where $\alpha, \beta, c \in]0,\infty[$ are given by (C_2) and (C_4).

Let us set $\mathcal{F} := \big\{ \Phi_\mu : \mu \in \mathcal{Y}(\Omega;\mathcal{X}) \big\}$.

Remark 2.5. From Remark 2.4 it is easy to see that every $F \in \mathcal{F}$ satisfies the following growth conditions:

$$\alpha \int_{]0,T[\times\Omega} \big(|(u,\dot{u})|^2 - 1\big)\, dt \otimes dx \;\leqslant\; F(u) \;\leqslant\; \beta \int_{]0,T[\times\Omega} \big(|(u,\dot{u})|^2 + 1\big)\, dt \otimes dx + c \int_\Omega \big(1 + |u(T,x)|^2\big)\, dx$$

for all $u \in \mathcal{H}_0^1$.

Let us recall the definition of Γ-convergence (see [DM93, Bra06] for more details).

Definition 2.6 (Γ-convergence)**.** Given $\{F_\varepsilon\}_\varepsilon \subset \mathcal{F}$ and $F \in \mathcal{F}$ we say that $\{F_\varepsilon\}_\varepsilon$ Γ-converges to F at $u \in \mathcal{H}_0^1$ as $\varepsilon \to 0$, and we write $\Big(\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon\Big)\,(u) = F(u)$, if the following two assertions hold:

Γ-*lower bound at* u: for every every $\{u_\varepsilon\}_\varepsilon \subset \mathcal{H}_0^1$, if $u_\varepsilon \xrightarrow{H^1} u$ then

$$\underline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(u_\varepsilon) \geqslant F(u);$$

Γ-*upper bound at* u: there exists $\{v_\varepsilon\}_\varepsilon \subset \mathcal{H}_0^1$ such that $v_\varepsilon \xrightarrow{H^1} u$ and

$$\overline{\lim_{\varepsilon\to 0}}\, F_\varepsilon(v_\varepsilon) \leqslant F(u).$$

When $\Big(\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon\Big)\,(u) = F(u)$ for all $u \in \mathcal{H}_0^1$ we say that $\{F_\varepsilon\}_\varepsilon$ Γ-converges to F as $\varepsilon \to 0$, and we write $\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = F$.

The main result of the paper is the following. (This result is a new and more complete version of [MV02, Theorem 3] in the specific case of the one-dimensional distributional derivative.)

Theorem 2.7. *Under* (C_1), (C_2), (C_3) *and* (C_4) *the map* $\mu \mapsto \Phi_\mu$ *is continuous from* $\mathcal{Y}(\Omega;\mathcal{X})$ *endowed with the narrow convergence to* $\mathcal{F}$ *endowed with the* Γ*-convergence, i.e., for every* $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ *and every* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\Omega;\mathcal{X})$*, if* $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$ *then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_\varepsilon} = \Phi_\mu.$$

2.2. **From local to non-local functionals.** It is easy to see that if $\mu = dx \otimes \delta_{f(x)}$ with $f : \Omega \to \mathcal{X}$ a $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$-measurable function, then for any $u \in \mathcal{H}_0^1$ one has

$$\phi_\mu(U) = \int_{]0,T[\times\Omega} f(x)\big(t,x,u(t,x),\dot{u}(t,x)\big)dt \otimes dx + \int_\Omega \theta(u(T,x))dx$$

for all $U \in \mathcal{S}_\mu(u)$. Hence

$$\Phi_\mu(u) = \int_{]0,T[\times\Omega} f(x)\big(t,x,u(t,x),\dot{u}(t,x)\big)dt \otimes dx + \int_\Omega \theta(u(T,x))dx \tag{2.7}$$

for all $u \in \mathcal{H}_0^1$. In what follows, when $\mu = dx \otimes \delta_{f(x)}$ we use the notation "Φ_f" instead of "Φ_μ". As a direct consequence of Theorem 2.7 we have

Corollary 2.8. *Assume that* (C_1), (C_2), (C_3) *and* (C_4) *hold and, for each* $\varepsilon > 0$, *consider a* $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$*-measurable function* $f_\varepsilon : \Omega \to \mathcal{X}$. *If there exists* $\mu \in \mathcal{Y}(\Omega; \mathcal{X})$ *such that* $dx \otimes \delta_{f_\varepsilon}(x) \xrightarrow{\text{nar}} \mu$ *then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \Phi_{f_\varepsilon} = \Phi_\mu.$$

In corollary 2.8, although the functionals Φ_{f_ε} are local, i.e., are integral functionals, the Γ-limit Φ_μ is in general non-local (here infimum of integral functionals). However, since $dx \otimes \delta_{f_\varepsilon}(x) \xrightarrow{\text{nar}} dx \otimes \delta_{f(x)}$ if and only if $\{f_\varepsilon\}_\varepsilon$ converges in measure to f (see [ABM14, Proposition 4.3.8] and also [Val90, Proposition 6]), we have

Corollary 2.9. *Assume that* (C_1), (C_2), (C_3) *and* (C_4) *hold and, for each* $\varepsilon > 0$, *consider a* $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$*-measurable function* $f_\varepsilon : \Omega \to \mathcal{X}$. *If there exists a* $(\mathcal{B}(\Omega), \mathcal{B}(\mathcal{X}))$*-measurable function* $f : \Omega \to \mathcal{X}$ *such that* $\{f_\varepsilon\}_\varepsilon$ *converges in measure to* f, *i.e.,*

$$\lim_{\varepsilon \to 0} \mathcal{L}_N\Big(\big\{x \in \Omega : d(f_\varepsilon(x), f(x)) > \eta\big\}\Big) = 0 \quad \text{for all } \eta > 0,$$

where d *is the metric defined by* (2.2), *then*

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \Phi_{f_\varepsilon} = \Phi_f.$$

2.3. **Non-diffusive reaction differential equations.** For each $\varepsilon > 0$, let $\psi_\varepsilon : [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a $(\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable function such that:

- (A_1) $\psi_\varepsilon(t,x,\cdot) \in C^1(\mathbb{R})$ for $dt \otimes dx$-a.a. $(t,x) \in]0,T[\times \Omega$;
- (A_2) $\psi_\varepsilon(t,x,\cdot)$ is convex for $dt \otimes dx$-a.a. $(t,x) \in]0,T[\times \Omega$;
- (A_3) there exist $c_1, C_1 > 0$ (which does not depend on ε) such that
$$c_1(|s|^2 - 1) \leqslant \psi_\varepsilon(\cdot,\cdot,s) \leqslant C_1(|s|^2 + 1)$$
for all $s \in \mathbb{R}$;
- (A_4) there exists $\delta : [0,\infty[\to [0,\infty[$ (which does not depend on ε) with $\lim_{r \to 0} \delta(r) = 0$ such that
$$|\psi_\varepsilon(t_1,\cdot,\cdot) - \psi_\varepsilon(t_2,\cdot,\cdot)| \leqslant \delta(|t_1 - t_2|)$$
for all $t_1, t_2 \in [0,T]$;
- (A_5) there exists $L > 0$ (which does not depend on ε) such that
$$\left|\frac{\partial \psi_\varepsilon}{\partial s}(\cdot,\cdot,s_1) - \frac{\partial \psi_\varepsilon}{\partial s}(\cdot,\cdot,s_2)\right| \leqslant L|s_1 - s_2|$$
for all $s_1, s_2 \in \mathbb{R}$,

and consider the non-diffusive reaction differential equation defined in $H^1(]0,T[; L^2(\Omega;\mathbb{R}))$ by

$$(R_\varepsilon) \quad \begin{cases} -\dfrac{\partial u_\varepsilon}{\partial t}(t,x) = \dfrac{\partial \psi_\varepsilon}{\partial s}(t,x,u_\varepsilon(t,x)) \text{ for } dt \otimes dx\text{-a.a. } (t,x) \in]0,T[\times \Omega \\ \\ u_\varepsilon(0,\cdot) = 0. \end{cases}$$

Remark 2.10. Under (A_1), (A_2), (A_3), (A_4) and (A_5), it is well known that every (R_ε) has a unique solution $u_\varepsilon \in \mathcal{H}_1^0$ (for a proof, see for instance [AHM18, Theorem 2.3]).

The following proposition makes clear the link between non-diffusive reaction differential equations and integral functionals of type (2.7).

Proposition 2.11. *If* (A_1), (A_2), (A_3), (A_4) *and* (A_5) *hold then, for every* $\varepsilon > 0$, *the non-diffusive reaction differential equation* (R_ε) *admits a unique solution* $u_\varepsilon \in \mathcal{H}^1_0$. *Moreover, there exist* $\alpha, \beta > 0$ *and* $\gamma : [0,\infty[\to [0,\infty[$ *with* $\lim_{r\to 0}\gamma(r) = 0$ *such that every* u_ε *minimizes the functional* $\Phi_{f_\varepsilon} : \mathcal{H}^1_0 \to \mathbb{R}$ *given by*

$$\Phi_{f_\varepsilon}(u) = \int_{]0,T[\times\Omega} f_\varepsilon(x)\big(t,x,u(t,x),\dot{u}(t,x)\big)dt\otimes dx + \int_\Omega \theta(u(T,x))dx$$

with $f_\varepsilon : \Omega \to \mathcal{M}_{\alpha,\beta,\gamma}$ *(here* $\mathcal{X} = \mathcal{M}_{\alpha,\beta,\gamma}$*) and* $\theta : \mathbb{R} \to [0,\infty[$ *respectively defined by*

$$f_\varepsilon(x)(t,y,s,\dot{s}) := \psi_\varepsilon(t,x,s) + \psi^*_\varepsilon(t,x,-\dot{s}) \tag{2.8}$$

and

$$\theta(s) := \frac{1}{2}|s|^2, \tag{2.9}$$

where ψ^*_ε *denotes the Legendre-Fenchel conjugate of* ψ_ε *with respect to the third variable.*

Proof of Proposition 2.11. For $\varepsilon > 0$ and define $\Psi_\varepsilon : [0,T]\times L^2(\Omega;\mathbb{R}) \to \mathbb{R}$ by

$$\Psi_\varepsilon(t,v) := \int_\Omega \psi_\varepsilon(t,x,v(x))dx.$$

(Note that $\Psi_\varepsilon(t,\cdot)$ is convex by (A_2).) Then (R_ε) is equivalent to the following differential inclusion defined in $\mathcal{H}^1_0$ by

$$(D_\varepsilon) \quad -\dot{u}_\varepsilon(t,\cdot) \in \partial\Psi_\varepsilon(t,u_\varepsilon(t,\cdot)) \text{ for } dt\text{-a.a. } t \in]0,T[,$$

where $\partial\Psi_\varepsilon(t,\cdot)$ denotes the subdifferential of $\Psi_\varepsilon(t,\cdot)$. According to Fenchel's extremality condition (see [ABM14, Proposition 9.5.1]), we see that (D_ε) is equivalent to

$$\Psi_\varepsilon(t,u_\varepsilon(t,\cdot)) + \Psi^*_\varepsilon(t,-\dot{u}_\varepsilon(t,\cdot)) + \langle u_\varepsilon(t,\cdot),\dot{u}_\varepsilon(t,\cdot)\rangle = 0 \text{ for } dt\text{-a.a. } t \in]0,T[,$$

where Ψ^*_ε denotes the Legendre-Fenchel conjugate of Ψ_ε with respect to the second variable. But, by Legendre-Fenchel's inequality, $\Psi_\varepsilon(t,v) + \Psi^*_\varepsilon(t,-w) + \langle w,v\rangle \geqslant 0$ for dt-a.a. $t \in]0,T[$ and all $v,w \in L^2(\Omega;\mathbb{R})$. Consequently, noticing that $\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R})} = 2\langle u_\varepsilon(t,\cdot),\dot{u}_\varepsilon(t,\cdot)\rangle$ and defining $F_\varepsilon : \mathcal{H}^1_0 \to [0,\infty]$ by

$$F_\varepsilon(u) := \int_0^T \Big(\Psi_\varepsilon(t,u(t,\cdot)) + \Psi^*_\varepsilon(t,-\dot{u}(t,\cdot)) + \frac{1}{2}\frac{d}{dt}\,\|u(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R})}\Big)dt,$$

it follows that (D_ε) is equivalent to $F_\varepsilon(u_\varepsilon) = 0$. (More precisely, we see that (D_ε) is equivalent to $F_\varepsilon(u_\varepsilon) \leqslant 0$.) Thus, u_ε solves (R_ε) if and only if u_ε minimizes F_ε.
On the other hand, as ψ_ε satisfies (A_3) and (A_4), so is its Legendre-Fenchel conjugate ψ^*_ε with other constants $\hat{c}_1, \hat{C}_1 > 0$, instead of $c_1, C_1 > 0$, but with the same function δ. Thus $f_\varepsilon(x) \in \mathcal{M}_{\alpha,\beta,\gamma}$ for all $x \in \Omega$ (with suitable constants $\alpha,\beta > 0$ and a suitable function γ) where, for each $(t,y,s,\dot{s}) \in [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}$, $f_\varepsilon(x)(t,y,s,\dot{s})$ is given by (2.8).
Finally, for each $u \in \mathcal{H}^1_0$, as $u(0,\cdot) = 0$, by using Legendre-Fenchel's calculus, it is easy to see that

$$F_\varepsilon(u) = \int_{]0,T[\times\Omega} f_\varepsilon(x)(t,x,u(x,t),\dot{u}(t,x))dx\otimes dt + \int_\Omega \theta(u(T,x))dx,$$

where $\theta : \mathbb{R} \to [0, \infty[$ is given by (2.9), which means that $F_\varepsilon = \Phi_{f_\varepsilon}$. ■

As a direct consequence of Proposition 2.11 and Corollary 2.8 we obtain the following result.

Corollary 2.12. *Assume that the assumptions of Proposition* 2.11 *are satisfied and, for each* $\varepsilon > 0$*, let* $u_\varepsilon \in \mathcal{H}_0^1$ *be the solution of* (R_ε)*. If there exists* $\mu \in \mathcal{Y}(\Omega; \mathcal{X})$ *such that* $dx \otimes \delta_{f_\varepsilon}(x) \xrightarrow{\text{nar}} \mu$*, where* $f_\varepsilon : \Omega \to \mathcal{M}_{\alpha,\beta,\gamma}$ *is given by* (2.8)*, then any cluster point of* $\{u_\varepsilon\}_\varepsilon$ *with respect to the weak convergence in* $\mathcal{H}_0^1$ *is a minimizer of* $\Phi_\mu : \mathcal{H}_0^1 \to \mathbb{R}$ *defined by* (2.6) *with* $\theta : \mathbb{R} \to [0, \infty[$ *given by* (2.9)*.*

Thus, the results obtained in [Mas93, Toa99] concerning non-local effects induced by homogenization can be seen as a particular case of Corollary 2.12 (see §5.2 for more details). Note that if we further assume that

(A_6) for every $\varepsilon > 0$, ψ_ε does not depend on t, and there exists $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that for dx-a.e. $x \in \mathbb{R}^N$, $\psi(x, \cdot) \in C^1(\mathbb{R})$, $\psi(x, \cdot)$ is convex and $\psi(x, s) = \lim_{\varepsilon \to 0} \psi_\varepsilon(x, s)$ for all $s \in \mathbb{R}$,

then the non-local effects disappear. More precisely, we have the following proposition which states a stability result for sequences of non-diffusive reaction differential equations of type (R_ε).

Proposition 2.13. *Under the assumptions of Proposition* 2.11*, if moreover* (A_6) *holds then (up to a subsequence) the sequence* $\{u_\varepsilon\}_\varepsilon$ *of solutions of* (R_ε) *weakly converges in* $\mathcal{H}_0^1$ *to the solution* u *of the following non-diffusive reaction differential equation*:

$$(\mathrm{R}) \quad \begin{cases} -\dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial \psi}{\partial s}(x, u(t,x)) \text{ for } dt \otimes dx\text{-a.a. } (t,x) \in]0,T[\times \Omega \\ \\ u(0, \cdot) = 0. \end{cases}$$

Proof of Proposition 2.13. For each $\varepsilon > 0$, let $\Psi_\varepsilon, \Psi : L^2(\Omega; \mathbb{R}) \to \mathbb{R}$ be defined by

$$\begin{cases} \Psi_\varepsilon(v) := \displaystyle\int_\Omega \psi_\varepsilon(x, v(x)) dx \\ \\ \Psi(v) := \displaystyle\int_\Omega \psi(x, v(x)) dx. \end{cases}$$

(Note that, by (A_6), ψ_ε does not depend on t.) We first prove that

$$\text{M-}\lim_{\varepsilon \to 0} \Psi_\varepsilon = \Psi, \tag{2.10}$$

where the symbol "M-lim" denotes the Mosco-limit[1]. For any $\varepsilon > 0$ and any $\lambda > 0$, let $\Psi_\varepsilon^\lambda, \Psi^\lambda : L^2(\Omega; \mathbb{R}) \to \mathbb{R}$ be the λ-Moreau-Yosida approximation of Ψ_ε and Ψ. By using an

[1]Let X be a Hilbert space, let $h : X \to \mathbb{R}$ and, for each $\varepsilon > 0$, let $h_\varepsilon : X \to \mathbb{R}$. We say that $\{h_\varepsilon\}_\varepsilon$ Mosco-converges to h, and we write M-$\lim_{\varepsilon \to 0} h_\varepsilon = h$, if the following two assertions hold:

M-*lower bound*: for every $v \in X$ and every $\{v_\varepsilon\}_\varepsilon \subset X$, if $v_\varepsilon \longrightarrow v$ then $\underline{\lim}_{\varepsilon \to 0} h_\varepsilon(v_\varepsilon) \geqslant h(v)$;

M-*upper bound*: for every $v \in X$ there exists $\{w_\varepsilon\}_\varepsilon \subset X$ such that $w_\varepsilon \to v$ and $\overline{\lim}_{\varepsilon \to 0} h_\varepsilon(w_\varepsilon) \leqslant h(v)$.

interchange argument of infimum and integral (see [AHM03]) it easily seen that

$$\begin{cases} \Psi_\varepsilon^\lambda(v) = \displaystyle\int_\Omega \psi_\varepsilon^\lambda(x, v(x))dx \\ \\ \Psi^\lambda(v) = \displaystyle\int_\Omega \psi^\lambda(x, v(x))dx, \end{cases} \tag{2.11}$$

where ψ_ε^λ and ψ^λ denotes the λ-Moreau-Yosida approximation of ψ_ε and ψ with respect to the second variable. From (A_1), (A_2) and (A_6) we deduce that Ψ and Ψ_ε are closed, convex and proper. Consequently, to prove (2.10) it is equivalent to show that

$$\lim_{\varepsilon\to 0} \Psi_\varepsilon^\lambda(v) = \Psi^\lambda(v) \text{ for all } \lambda > 0 \text{ and all } v \in L^2(\Omega;\mathbb{R}) \tag{2.12}$$

(see [Att84, Theorem 3.26]). Taking (2.11) and (A_3) into account, by Lebesgue's dominated convergence theorem, we see that for establishing (2.10) it suffices to prove that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon^\lambda(x, s) = \psi^\lambda(x, s) \text{ for } dx\text{-a.e. } x \in \mathbb{R}^N, \text{ all } \lambda > 0 \text{ and all } s \in \mathbb{R}. \tag{2.13}$$

But (A_1), (A_2) and (A_6) implies that for dx-a.e. $x \in \mathbb{R}^N$, $\psi_\varepsilon(x, \cdot)$ and ψ are closed, convex and proper, and so (2.13) holds if and only if

$$\text{M-}\lim_{\varepsilon\to 0} \psi_\varepsilon(x, \cdot) = \psi(x, \cdot) \text{ for } dx\text{-a.e. } x \in \mathbb{R}^N. \tag{2.14}$$

Fix $x \in \mathbb{R}^N$ and fix any $\varepsilon > 0$. From (A_1) and (A_3) we deduce that there exists $C > 0$ (which does not depend on ε) such that

$$|\psi_\varepsilon(x, s_1) - \psi_\varepsilon(x, s_2)| \leqslant C|s_1 - s_2|(1 + |s_1| + |s_2|) \tag{2.15}$$

for all $s_1, s_2 \in \mathbb{R}$. Let $s \in \mathbb{R}$ and let $\{s_\varepsilon\}_\varepsilon \subset \mathbb{R}$. On one hand, by (A_6) we have

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(x, s) = \psi(x, s), \tag{2.16}$$

which gives the M-upper bound of (2.14). On the other hand, from (2.15) it follows that

$$\psi_\varepsilon(x, s_\varepsilon) \leqslant \psi_\varepsilon(x, s) - C|s_\varepsilon - s|(1 + |s_\varepsilon| + |s|)$$

for all $\varepsilon > 0$, and so, by using (2.16), if $s_\varepsilon \to s$ in $\mathbb{R}$ then $\underline{\lim}_{\varepsilon\to 0} \psi_\varepsilon(x, s_\varepsilon) \geqslant \psi(x, s)$, which gives the M-lower bound of (2.14). Thus (2.10) is proved.
For each $\varepsilon > 0$, let $G_\varepsilon, G : L^2(]0, T[; L^2(\Omega;\mathbb{R})) \to \mathbb{R}$ be defined by

$$\begin{cases} G_\varepsilon(v) := \displaystyle\int_0^T \Psi_\varepsilon(v(t, \cdot))dt \\ \\ G(v) := \displaystyle\int_0^T \Psi(v(t, \cdot))dt. \end{cases}$$

From (2.10) we deduce that

$$\text{M-}\lim_{\varepsilon\to 0} G_\varepsilon = G \tag{2.17}$$

(see [ABM14, Lemma 17.4.8]). Hence (see [ABM14, Theorem 17.4.3])

$$\text{M-}\lim_{\varepsilon\to 0} G_\varepsilon^* = G^*, \tag{2.18}$$

with $G_\varepsilon^*, G^* : L^2(]0,T[;L^2(\Omega;\mathbb{R})) \to \mathbb{R}$ denoting the Legendre-Fenchel conjugate of G_ε and G, where, by using Legendre-Fenchel's calculus, one has

$$\begin{cases} G_\varepsilon^*(v) = \displaystyle\int_0^T \Psi_\varepsilon^*(v(t,\cdot))dt \\ \\ G^*(v) = \displaystyle\int_0^T \Psi^*(v(t,\cdot))dt \end{cases}$$

with $\Psi_\varepsilon^*, \Psi^* : L^2(\Omega;\mathbb{R}) \to \mathbb{R}$ denoting the Legendre-Fenchel conjugate of Ψ_ε and Ψ. For each $\varepsilon > 0$, let $F_\varepsilon, F : \mathcal{H}_0^1 \to \mathbb{R}$ be defined by

$$\begin{cases} F_\varepsilon(v) := G_\varepsilon(v) + G_\varepsilon^*(-\dot{v}) + \dfrac{1}{2}\left\|v(T,\cdot)\right\|^2_{L^2(\Omega;\mathbb{R})} \\ \\ F(v) := G(v) + G^*(-\dot{v}) + \dfrac{1}{2}\left\|v(T,\cdot)\right\|^2_{L^2(\Omega;\mathbb{R})}. \end{cases}$$

Arguing as in the proof of Proposition 2.11 we can assert that:

$$\text{for each } \varepsilon > 0,\ v \text{ solves } (\mathrm{R}_\varepsilon) \text{ if and only if } F_\varepsilon(v) \leqslant 0; \tag{2.19}$$

$$v \text{ solves } (\mathrm{R}) \text{ if and only if } F(v) \leqslant 0; \tag{2.20}$$

Let $\{u_\varepsilon\}_\varepsilon \subset \mathcal{H}_0^1$ be a sequence of solutions of (R_ε). Taking Remark 2.5 into account, we see that (up to a subsequence) there exists $u \in \mathcal{H}_0^1$ such that $u_\varepsilon \xrightarrow{H^1} u$, i.e., $u_\varepsilon \xrightarrow{L^2} u$ and $\dot{u}_\varepsilon \xrightarrow{L^2} \dot{u}$. Then, by (2.19), one has $F_\varepsilon(u_\varepsilon) \leqslant 0$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ we obtain

$$\underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqslant 0. \tag{2.21}$$

On the other hand, from (2.17) and (2.18) we deduce that:

$$G(u) \leqslant \underline{\lim}_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon); \tag{2.22}$$

$$G^*(-\dot{u}) \leqslant \underline{\lim}_{\varepsilon\to 0} G_\varepsilon^*(-\dot{u}_\varepsilon). \tag{2.23}$$

As, for any $\varepsilon > 0$, $u_\varepsilon \in \mathcal{H}_0^1$ and $u \in \mathcal{H}_0^1$ we have $u_\varepsilon(0,\cdot) = 0$ and $u(0,\cdot) = 0$, and so $u_\varepsilon(T,\cdot) = \int_0^T \dot{u}_\varepsilon(t,\cdot)dt$ and $u(T,\cdot) = \int_0^T \dot{u}(t,\cdot)dt$. Since $\dot{u}_\varepsilon \xrightarrow{L^2} \dot{u}$ it follows that $u_\varepsilon(T,\cdot) \xrightarrow{L^2} u(T,\cdot)$, and consequently

$$\left\|u(T,\cdot)\right\|_{L^2(\Omega;\mathbb{R})} \leqslant \underline{\lim}_{\varepsilon\to 0} \left\|u_\varepsilon(T,\cdot)\right\|_{L^2(\Omega;\mathbb{R})}. \tag{2.24}$$

From (2.22), (2.23) and (2.24) we deduce that $F(u) \leqslant \underline{\lim}_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)$. Hence $F(u) \leqslant 0$ by (2.21). From (2.20) we conclude that u solves (R). ■

3. Proof of the continuity theorem

In this section we prove Theorem 2.7.

Proof of Theorem 2.7. Let $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ and let $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\Omega;\mathcal{X})$ be such that $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$. According to the left inequality in Remark 2.5, to prove that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_\varepsilon} = \Phi_\mu \tag{3.1}$$

it is equivalent to show that every subsequence of $\{\Phi_{\mu_\varepsilon}\}_\varepsilon$ contains a further subsequence which Γ-converges to Φ_μ (see [DM93, Proposition 8.17]). Let us consider a subsequence that we still denote by $\{\Phi_{\mu_\varepsilon}\}_\varepsilon$. Taking the left inequality in Remark 2.5 into account, by compactness, we can assert that, up to subsequence, $\{\Phi_{\mu_\varepsilon}\}_\varepsilon$ Γ-converges (see [DM93, Corollary 8.12]), i.e., $\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_\varepsilon} = \Psi$. So, to establish (3.1) it is sufficient to prove that

$$\Psi(u) = \Phi_\mu(u) \text{ for all } u \in \mathcal{H}^1_0. \tag{3.2}$$

For this, we only need to show that for each $u \in \mathcal{H}^1_0$ the following two assertions hold:

(G_1) there exists a subsequence $\{\Phi_{\mu_{\sigma(\varepsilon)}}\}_\varepsilon$ and $\{v_\varepsilon\}_\varepsilon \subset \mathcal{H}^1_0$ such that $v_\varepsilon \xrightarrow{H^1} u$ and

$$\overline{\lim_{\varepsilon\to 0}}\, \Phi_{\mu_{\sigma(\varepsilon)}}(v_\varepsilon) \leqslant \Phi_\mu(u);$$

(G_2) for every $\{u_\varepsilon\}_\varepsilon \subset \mathcal{H}^1_0$, if $u_\varepsilon \xrightarrow{H^1} u$ then

$$\underline{\lim_{\varepsilon\to 0}}\, \Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) \geqslant \Phi_\mu(u),$$

where $\{\Phi_{\mu_{\sigma(\varepsilon)}}\}_\varepsilon$ is given by (G_1).

Indeed, let $u \in \mathcal{H}^1_0$. According to Definition 2.6, from (G_1) and (G_2) we see that $\{\Phi_{\mu_{\sigma(\varepsilon)}}\}_\varepsilon$ Γ-converges to Φ_μ at u as $\varepsilon \to 0$, i.e., $(\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_{\sigma(\varepsilon)}})(u) = \Phi_\mu(u)$. On the other hand, as $\{\Phi_{\mu_{\sigma(\varepsilon)}}\}_\varepsilon$ is a subsequence of $\{\Phi_{\mu_\varepsilon}\}_\varepsilon$ we have $\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_{\sigma(\varepsilon)}} = \Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_\varepsilon}$, and so $\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_{\sigma(\varepsilon)}} = \Psi$. Hence, in particular, $(\Gamma\text{-}\lim_{\varepsilon\to 0} \Phi_{\mu_{\sigma(\varepsilon)}})(u) = \Psi(u)$. It follows that $\Psi(u) = \Phi_\mu(u)$, which proves (3.2).

Let us fix $u \in \mathcal{H}^1_0$.

Proof of (G_1). By using the direct method of the calculus of variations, we can assert that there exists $U \in \mathcal{S}_\mu(u)$ such that U is a minimizer of $\Phi_\mu(u)$, i.e., $\Phi_\mu(u) = \phi_\mu(U)$. We then have:

$$u(t,x) = \int_{\mathcal{X}} U(t,x,\xi) d\mu_x(\xi); \tag{3.3}$$

$$\dot{u}(t,x) = \int_{\mathcal{X}} \dot{U}(t,x,\xi) d\mu_x(\xi), \tag{3.4}$$

where $\{\mu_x\}_{x\in\Omega}$ denotes the desintegration of μ on $\Omega\times\mathcal{X}$. As $C^1_c(]0,T[;C_c(\Omega\times\mathcal{X}))$ is strongly dense in $\mathcal{H}^1_{0,\mu}$, for each $\delta \in]0,1]$ there exists $U_\delta \in C^1_c(]0,T[;C_c(\Omega\times\mathcal{X}))$ such that:

$$\int_{]0,T[\times\Omega\times\mathcal{X}} |U_\delta(t,x,\xi) - U(t,x,\xi)|^2 \, dt \otimes d\mu(x,\xi) < \delta; \tag{3.5}$$

$$\int_{]0,T[\times\Omega\times\mathcal{X}} \left|\dot{U}_\delta(t,x,\xi) - \dot{U}(t,x,\xi)\right|^2 dt \otimes d\mu(x,\xi) < \delta. \tag{3.6}$$

In particular, one has:

$$U_\delta \xrightarrow{L^2} U; \tag{3.7}$$

$$\dot{U}_\delta \xrightarrow{L^2} \dot{U}. \tag{3.8}$$

For each $\delta \in]0,1]$ and each $\varepsilon > 0$, let $u_{\delta,\varepsilon} \in \mathcal{H}_0^1$ be given by

$$u_{\delta,\varepsilon}(t,x) := \int_{\mathcal{X}} U_\delta(t,x,\xi) d\mu_x^\varepsilon(\xi), \tag{3.9}$$

where $\{\mu_x^\varepsilon\}_{x\in\Omega}$ denotes the desintegration of μ_ε on $\Omega \times \mathcal{X}$. Then

$$\dot{u}_{\delta,\varepsilon}(t,x) := \int_{\mathcal{X}} \dot{U}_\delta(t,x,\xi) d\mu_x^\varepsilon(\xi). \tag{3.10}$$

(In what follows, we systematically use the fact that U_δ, U_δ^2, $\dot{U}_\delta$, and $\dot{U}_\delta^2$ are test functions for the narrow convergence in $\mathcal{Y}(\Omega;\mathcal{X})$ because these are bounded Carathéodory integrands on $\Omega \times \mathcal{X}$.)
Since $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$, there exists a mapping $\delta \mapsto \eta_\delta$ with $\eta_\delta \to 0$ as $\delta \to 0$ such that for all $\varepsilon \in]0,\eta_\delta]$, one has:

$$\left| \int_{]0,T[\times\Omega\times\mathcal{X}} |U_\delta(t,x,\xi)|^2 dt \otimes d\mu_\varepsilon(x,\xi) - \int_{]0,T[\times\Omega\times\mathcal{X}} |U_\delta(t,x,\xi)|^2 dt \otimes d\mu(x,\xi) \right| < 1; \tag{3.11}$$

$$\left| \int_{]0,T[\times\Omega\times\mathcal{X}} |\dot{U}_\delta(t,x,\xi)|^2 dt \otimes d\mu_\varepsilon(x,\xi) - \int_{]0,T[\times\Omega\times\mathcal{X}} |\dot{U}_\delta(t,x,\xi)|^2 dt \otimes d\mu(x,\xi) \right| < 1. \tag{3.12}$$

Fix any $\delta \in]0,1]$ and any $\varepsilon \in]0,\eta_\delta]$. Using (3.9) and (3.11) we see that

$$\begin{aligned} \int_{]0,T[\times\Omega} |u_{\delta,\varepsilon}(t,x)|^2 dt \otimes dx &= \int_{]0,T[\times\Omega} \left| \int_{\mathcal{X}} U_\delta(t,x,\xi) d\mu_x^\varepsilon(\xi) \right|^2 dt \otimes dx \\ &\leqslant \int_{]0,T[\times\Omega\times\mathcal{X}} |U_\delta(t,x,\xi)|^2 dt \otimes d\mu_\varepsilon(x,\xi) \\ &\leqslant 1 + \int_{]0,T[\times\Omega\times\mathcal{X}} |U_\delta(t,x,\xi)|^2 dt \otimes d\mu(x,\xi). \end{aligned}$$

But, taking (3.5) and the left inequality in Remark 2.4 into account and recalling that $U \in \mathcal{S}_\mu(u)$, we have

$$\begin{aligned} \int_{]0,T[\times\Omega\times\mathcal{X}} |U_\delta(t,x,\xi)|^2 dt \otimes d\mu(x,\xi) &\leqslant 2\delta + 2 \int_{]0,T[\times\Omega\times\mathcal{X}} |U(t,x,\xi)|^2 dt \otimes d\mu(x,\xi) \\ &\leqslant 2\left(1 + \frac{1}{\alpha}\phi_\mu(U) + T|\Omega|\right), \end{aligned}$$

and so, setting $R := 3 + \frac{1}{\alpha}\phi_\mu(U) + T|\Omega|$, it follows that

$$\int_{]0,T[\times\Omega} |u_{\delta,\varepsilon}(t,x)|^2 dt \otimes dx \leqslant R \text{ for all } \delta \in]0,1] \text{ and all } \varepsilon \in]0,\eta_\delta]. \tag{3.13}$$

In the same manner, by considering (3.10), (3.12) and (3.6) instead of (3.9), (3.11) and (3.5), we obtain

$$\int_{]0,T[\times\Omega} |\dot{u}_{\delta,\varepsilon}(t,x)|^2 dt \otimes dx \leqslant R \text{ for all } \delta \in]0,1] \text{ and all } \varepsilon \in]0,\eta_\delta]. \tag{3.14}$$

For each $\delta \in]0,1]$, consider the mapping $\varepsilon \mapsto \lambda_\delta(\varepsilon)$ given by

$$\lambda_\delta(\varepsilon) := \begin{cases} \eta_\delta & \text{if } \varepsilon > \eta_\delta \\ \varepsilon & \text{if } \varepsilon \leqslant \eta_\delta. \end{cases}$$

(Note that $\lambda_\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.) From (3.13) and (3.14) we deduce that

$$\{u_{\delta,\lambda_\delta(\varepsilon)}\}_{(\delta,\varepsilon)\in]0,1]\times]0,\infty[} \subset \overline{\mathcal{B}}_R(0), \tag{3.15}$$

where $\overline{\mathcal{B}}_R(0)$ denotes the closed ball in $\mathcal{H}_0^1$ centered at 0 with radius R. As $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$ we have $\mu_{\lambda_\delta(\varepsilon)} \xrightarrow{\text{nar}} \mu$ for all $\delta \in]0,1]$, and so

$$u_{\delta,\lambda_\delta(\varepsilon)} \xrightarrow{L^2} \int_{\mathcal{X}} U_\delta(\cdot,\cdot,\xi) d\mu_x(\xi) \text{ for all } \delta \in]0,1]. \tag{3.16}$$

Taking (3.7) and (3.3) into account, from the above it follows that

$$\int_{\mathcal{X}} U_\delta(\cdot,\cdot,\xi) d\mu_x(\xi) \xrightarrow{L^2} \int_{\mathcal{X}} U(\cdot,\cdot,\xi) d\mu_x(\xi) = u. \tag{3.17}$$

By the same arguments, using (3.8) and (3.4) instead of (3.7) and (3.3), we obtain:

$$\dot{u}_{\delta,\lambda_\delta(\varepsilon)} \xrightarrow{L^2} \int_{\mathcal{X}} \dot{U}_\delta(\cdot,\cdot,\xi) d\mu_x(\xi) \text{ for all } \delta \in]0,1]; \tag{3.18}$$

$$\int_{\mathcal{X}} \dot{U}_\delta(\cdot,\cdot,\xi) d\mu_x(\xi) \xrightarrow{L^2} \int_{\mathcal{X}} U(\cdot,\cdot,\xi) d\mu_x(\xi) = \dot{u}. \tag{3.19}$$

Combining (3.16) with (3.18) and (3.17) with (3.19) we see that

$$u_{\delta,\lambda_\delta(\varepsilon)} \xrightarrow[\varepsilon\to 0]{H^1} \int_{\mathcal{X}} U_\delta(\cdot,\cdot,\xi) d\mu_x(\xi) \xrightarrow[\delta\to 0]{H^1} u. \tag{3.20}$$

According to (3.15) and the fact that in $\overline{\mathcal{B}}_R(0)$ the weak convergence is metrizable, we can rewrite (3.20) as follows:

$$d_w\text{-}\lim_{\delta\to 0} d_w\text{-}\lim_{\varepsilon\to 0} u_{\delta,\lambda_\delta(\varepsilon)} = u, \tag{3.21}$$

where d_w denotes the metric associated with the weak convergence in $\overline{\mathcal{B}}_R(0)$.
On the other hand, by using similar arguments together with (2.1) and (2.4), we can assert that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \phi_{\mu_{\lambda_\delta(\varepsilon)}}(U_\delta) = \phi_\mu(U). \tag{3.22}$$

According to (3.21) and (3.22), by diagonalization there exists a mapping $\varepsilon \mapsto \delta_\varepsilon$ with $\delta \to 0$ as $\varepsilon \to 0$ such that:

$$d_w\text{-}\lim_{\varepsilon\to 0} u_{\delta_\varepsilon,\lambda_{\delta_\varepsilon}(\varepsilon)} = u, \text{ i.e., } u_{\delta_\varepsilon,\lambda_{\delta_\varepsilon}(\varepsilon)} \xrightarrow{H^1} u; \tag{3.23}$$

$$\lim_{\varepsilon\to 0} \phi_{\mu_{\lambda_{\delta_\varepsilon}(\varepsilon)}}(U_{\delta_\varepsilon}) = \phi_\mu(U). \tag{3.24}$$

Since $\phi_\mu(U) = \Phi_\mu(u)$ and, for each $\varepsilon > 0$, $\phi_{\mu_{\lambda_{\delta_\varepsilon}(\varepsilon)}}(U_{\delta_\varepsilon}) \geqslant \Phi_{\mu_{\lambda_{\delta_\varepsilon}(\varepsilon)}}(u_{\delta_\varepsilon,\lambda_{\delta_\varepsilon}(\varepsilon)})$ because $U_{\delta_\varepsilon} \in C_c^1(]0,T[;C_c(\Omega\times\mathcal{X})) \subset \mathcal{H}^1_{0,\mu_{\lambda_{\delta_\varepsilon}(\varepsilon)}}$ and $\int_{\mathcal{X}} U_{\delta_\varepsilon}(\cdot,\cdot,\xi)d\mu_x^{\lambda_{\delta_\varepsilon(\varepsilon)}}(\xi) = u_{\delta_\varepsilon,\lambda_{\delta_\varepsilon(\varepsilon)}}$, from (3.24) we deduce that

$$\overline{\lim_{\varepsilon\to 0}}\,\Phi_{\mu_{\lambda_{\delta_\varepsilon}(\varepsilon)}}(u_{\delta_\varepsilon,\lambda_{\delta_\varepsilon(\varepsilon)}}) \leqslant \Phi_\mu(u). \tag{3.25}$$

Taking (3.23) and (3.25) into account and setting $\sigma(\varepsilon) := \lambda_{\delta_\varepsilon}(\varepsilon)$ and $v_\varepsilon := u_{\delta_\varepsilon,\lambda_{\delta_\varepsilon(\varepsilon)}}$ we have $\sigma(\varepsilon)\to 0$ as $\varepsilon\to 0$ and:

$$\begin{aligned} &v_\varepsilon \xrightarrow{H^1} u;\\ &\overline{\lim_{\varepsilon\to 0}}\,\Phi_{\mu_{\sigma(\varepsilon)}}(v_\varepsilon) \leqslant \Phi_\mu(u), \end{aligned}$$

and the proof of (G_1) is complete. ■

Proof of $(\mathbf{G_2})$. Let $\{u_\varepsilon\}_\varepsilon \subset \mathcal{H}^1_0$ be such that $u_\varepsilon \xrightarrow{H^1} u$. Without loss of generality we can assume that

$$\underline{\lim_{\varepsilon\to 0}}\,\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) = \lim_{\varepsilon\to 0}\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) < \infty, \text{ and so } \sup_{\varepsilon>0}\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) < \infty. \tag{3.26}$$

Fix any $\varepsilon > 0$. By using the direct method of the calculus of variations, we can assert that there exists $U_\varepsilon \in \mathcal{S}_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon)$ such that U_ε is a minimizer of $\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon)$, i.e.,

$$\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) = \phi_{\mu_{\sigma(\varepsilon)}}(U_\varepsilon) \tag{3.27}$$

with $\phi_{\mu_{\sigma(\varepsilon)}} : \mathcal{S}_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) \to \mathbb{R}$ given by (2.5). From (3.26) and the left inequality in (C_2) we deduce that

$$\sup_{\varepsilon>0}\left\|(U_\varepsilon,\dot{U}_\varepsilon)\right\|_{L^2_{dt\otimes\mu_{\sigma(\varepsilon)}}(]0,T[\times\Omega\times\mathcal{X};\mathbb{R}^2)} < \infty. \tag{3.28}$$

Let $g_\varepsilon :]0,T[\times\Omega\times\mathcal{X} \to]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2$ be defined by

$$g_\varepsilon(t,x,\xi) := (t,x,\xi,U_\varepsilon(t,x,\xi),\dot{U}_\varepsilon(t,x,\xi))$$

and let $\nu_\varepsilon := g_\varepsilon^\sharp dt\otimes\mu_{\sigma(\varepsilon)}$. It is clear that $\nu_\varepsilon \in \mathcal{Y}(]0,T[\times\Omega;\mathcal{X}\times\mathbb{R}^2)$ whose projection measure is $dt\otimes dx$. We claim that $\{\nu_\varepsilon\}_\varepsilon$ is tight (see Definition A.2). Indeed, given $\eta > 0$, as $\{\mu_{\sigma(\varepsilon)}\}_\varepsilon$ is tight, there exists a compact set $\mathcal{K}\subset\mathcal{X}$ such that

$$\sup_{\varepsilon>0}\mu_{\sigma(\varepsilon)}\big(\Omega\times(\mathcal{X}\backslash\mathcal{K})\big) < \frac{\eta}{2T}. \tag{3.29}$$

From (3.28) and Markov's inequality we can assert that there exists $R > 0$ such that

$$\sup_{\varepsilon>0} dt\otimes\mu_{\sigma(\varepsilon)}\left(E_{\varepsilon,R}\right) < \frac{\eta}{2} \tag{3.30}$$

with

$$E_{\varepsilon,R} := \left\{(t,x,\xi)\in]0,T[\times\Omega\times\mathcal{X} : (U_\varepsilon(t,x,\xi),\dot{U}_\varepsilon(t,x,\xi))\in\mathbb{R}^2\backslash\overline{B}_R(0)\right\},$$

where $\overline{B}_R(0) \subset \mathbb{R}^2$ denotes the 2-dimensional closed ball centered at the origin with radius R. On the other hand, we have

$$\begin{aligned}\nu_\varepsilon(]0,T[\times\Omega\times(\mathcal{X}\times\mathbb{R}^2)\backslash(\mathcal{K}\times\overline{B}_R(0))) &\leqslant dt\otimes\mu_{\sigma(\varepsilon)}(]0,T[\times\Omega\times(\mathcal{X}\backslash\mathcal{K})) + dt\otimes\mu_{\sigma(\varepsilon)}(E_{\varepsilon,R})\\ &= T\mu_{\sigma(\varepsilon)}(\Omega\times(\mathcal{X}\backslash\mathcal{K})) + dt\otimes\mu_{\sigma(\varepsilon)}(E_{\varepsilon,R})\end{aligned}$$

for all $\varepsilon > 0$, and consequently

$$\sup_{\varepsilon>0}\nu_\varepsilon(]0,T[\times\Omega\times(\mathcal{X}\times\mathbb{R}^2)\backslash(\mathcal{K}\times\overline{B}_R(0))) < \eta,$$

which proves the claim. Let $h_\varepsilon : \Omega\times\mathcal{X}\to\Omega\times\mathcal{X}\times\mathbb{R}$ defined by

$$h_\varepsilon(x,\xi) := (x,\xi,U_\varepsilon(T,x,\xi))$$

and let $\lambda_\varepsilon := h_\varepsilon^\sharp\mu_{\sigma(\varepsilon)}$. It is clear that $\lambda_\varepsilon\in\mathcal{Y}(\Omega;\mathcal{X}\times\mathbb{R})$. In the same manner we can establish that $\{\lambda_\varepsilon\}_\varepsilon$ is tight. From Prokhorov's compactness theorem (see Theorem A.3) we deduce that there exist $\nu\in\mathcal{Y}(]0,T[\times\Omega;\mathcal{X}\times\mathbb{R}^2)$ and $\lambda\in\mathcal{Y}(\Omega;\mathcal{X}\times\mathbb{R})$ such that, up to a subsequence, one has:

$$\nu_\varepsilon \xrightarrow{\text{nar}} \nu; \tag{3.31}$$

$$\lambda_\varepsilon \xrightarrow{\text{nar}} \lambda. \tag{3.32}$$

Let $\pi_{]0,T[\times\Omega\times\mathcal{X}} :]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2\to]0,T[\times\Omega\times\mathcal{X}$ (resp. $\pi_{\Omega\times\mathcal{X}} : \Omega\times\mathcal{X}\times\mathbb{R}\to\Omega\times\mathcal{X}$) be the canonical projection from $]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2$ (resp. $\Omega\times\mathcal{X}\times\mathbb{R}$) to $]0,T[\times\Omega\times\mathcal{X}$ (resp. $\Omega\times\mathcal{X}$). From (3.31) and (3.32) it is easy to see that:

$$\begin{aligned}\pi^\sharp_{]0,T[\times\Omega\times\mathcal{X}}\nu &= \text{w-}\lim_{\varepsilon\to0}\pi^\sharp_{]0,T[\times\Omega\times\mathcal{X}}\nu_\varepsilon = dt\otimes\mu;\\ \pi^\sharp_{\Omega\times\mathcal{X}}\lambda &= \text{w-}\lim_{\varepsilon\to0}\pi^\sharp_{]0,T[\times\Omega\times\mathcal{X}}\lambda_\varepsilon = \mu,\end{aligned}$$

where "w-lim" denotes the weak limit associated with the weak $\sigma(C_b',C_b)$ topology (see [ABM14, Definition 4.2.2]). By using desintegration's theorem (see Theorem A.5) it follows that:

$$\nu = dt\otimes\mu\otimes\nu_{t,x,\xi}; \tag{3.33}$$

$$\lambda = \mu\otimes\lambda_{x,\xi}, \tag{3.34}$$

where $\{\nu_{t,x,\xi}\}_{(t,x,\xi)\in]0,T[\times\Omega\times\mathcal{X}}$ and $\{\lambda_{x,\xi}\}_{(x,\xi)\in\Omega\times\mathcal{X}}$ are families of probability measures on $\mathbb{R}^2$ and $\mathbb{R}$ respectively. We will need the following lemma whose proof is given below.

Lemma 3.1. *For each $(t,x,\xi)\in]0,T[\times\Omega\times\mathcal{X}$, let us denote by $U(t,x,\xi)$, $V(t,x,\xi)$ and $W_T(x,\xi)$ the 1^{th} moments of $\nu_{t,x,\xi}$ and $\lambda_{x,\xi}$, i.e.,*

$$U(t,x,\xi) := \int_{\mathbb{R}^2} s\,d\nu_{t,x,\xi}(s,\dot{s}); \tag{3.35}$$

$$V(t,x,\xi) := \int_{\mathbb{R}^2} \dot{s}\,d\nu_{t,x,\xi}(s,\dot{s}); \tag{3.36}$$

$$W_T(x,\xi) := \int_{\mathbb{R}} s_T\,d\lambda_{x,\xi}(s_T). \tag{3.37}$$

Then $V = \dot{U}$, $U\in\mathcal{S}_\mu(u)$ and $W_T = U(T,\cdot,\cdot)$.

According to (3.27) and the definitions of $\{\nu_\varepsilon\}_\varepsilon$ and $\{\lambda_\varepsilon\}_\varepsilon$, we see that

$$\begin{aligned}
\underline{\lim}_{\varepsilon\to 0}\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) &= \underline{\lim}_{\varepsilon\to 0}\phi_{\mu_{\sigma(\varepsilon)}}(U_\varepsilon)\\
&\geqslant \underline{\lim}_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}}\xi\big(t,x,U_\varepsilon(t,x,\xi),\dot U_\varepsilon(t,x,\xi)\big)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi)\\
&\quad+\underline{\lim}_{\varepsilon\to 0}\int_{\Omega\times\mathcal{X}}\theta(U_\varepsilon(T,x,\xi))d\mu_{\sigma(\varepsilon)}(x,\xi)\\
&= \underline{\lim}_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\xi(t,x,S)d\nu_\varepsilon(t,x,\xi,S)\\
&\quad+\underline{\lim}_{\varepsilon\to 0}\int_{\Omega\times\mathcal{X}\times\mathbb{R}}\theta(s_T)d\lambda_\varepsilon(x,\xi,s_T). \qquad (3.38)
\end{aligned}$$

But, recalling (3.31) and (3.32), by Theorem A.4(i) have:

$$\underline{\lim}_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\xi(t,x,S)d\nu_\varepsilon(t,x,\xi,S)\geqslant\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\xi(t,x,S)d\nu(t,x,\xi,S); \qquad (3.39)$$

$$\underline{\lim}_{\varepsilon\to 0}\int_{\Omega\times\mathcal{X}\times\mathbb{R}}\theta(s_T)d\lambda_\varepsilon(x,\xi,s_T)\geqslant\int_{\Omega\times\mathcal{X}\times\mathbb{R}}\theta(s_T)d\lambda(x,\xi,s_T). \qquad (3.40)$$

Moreover, by using (C_1), (3.33) and (3.34), Jensen's inequality and Lemma 3.1, we obtain the following two inequalities:

$$\begin{aligned}
\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\xi(t,x,S)d\nu(t,x,\xi,S) &= \int_{]0,T[\times\Omega\times\mathcal{X}}\left(\int_{\mathbb{R}^2}\xi(t,x,S)d\nu_{t,x,\xi}(S)\right)dt\otimes d\mu(x,\xi)\\
&\geqslant\int_{]0,T[\times\Omega\times\mathcal{X}}\xi\left(t,x,\int_{\mathbb{R}^2}Sd\nu_{t,x,\xi}(S)\right)dt\otimes d\mu(x,\xi)\\
&=\int_{]0,T[\times\Omega\times\mathcal{X}}\xi\left(t,x,\int_{\mathbb{R}^2}sd\nu_{t,x,\xi}(s,\dot s),\int_{\mathbb{R}^2}\dot sd\nu_{t,x,\xi}(s,\dot s)\right)dt\otimes d\mu(x,\xi)\\
&=\int_{]0,T[\times\Omega\times\mathcal{X}}\xi\left(t,x,U(t,x,\xi),\dot U(t,x,\xi)\right)dt\otimes d\mu(x,\xi); \qquad (3.41)
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega\times\mathcal{X}\times\mathbb{R}}\theta(s_T)d\lambda(x,\xi,s_T) &= \int_{\Omega\times\mathcal{X}}\left(\int_{\mathbb{R}}\theta(s_T)d\lambda_{x,\xi}(s_T)\right)d\mu(x,\xi)\\
&\geqslant\int_{\Omega\times\mathcal{X}}\theta\left(\int_{\mathbb{R}}s_Td\lambda_{x,\xi}(s_T)\right)d\mu(x,\xi)\\
&=\int_{\Omega\times\mathcal{X}}\theta\left(W_T(x,\xi)\right)d\mu(x,\xi)\\
&=\int_{\Omega\times\mathcal{X}}\theta\left(U(T,x,\xi)\right)d\mu(x,\xi), \qquad (3.42)
\end{aligned}$$

where $U \in \mathcal{S}_\mu(u)$ by Lemma 3.1. Consequently, combining (3.39) with (3.41) and (3.40) with (3.42), and using (3.38) we conclude that

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0}\Phi_{\mu_{\sigma(\varepsilon)}}(u_\varepsilon) &\geqslant \int_{]0,T[\times\Omega\times\mathcal{X}} \xi\left(t,x,U(t,x,\xi),\dot{U}(t,x,\xi)\right)dt\otimes d\mu(x,\xi)\\ &\quad+\int_{\Omega\times\mathcal{X}}\theta\left(U(T,x,\xi)\right)d\mu(x,\xi)\\ &= \phi_\mu(U)\\ &\geqslant \Phi_\mu(u),\end{aligned}$$

and (G_2) is proved. ■

This finishes the proof of Theorem 2.7 (the proof of Lemma 3.1 is given below). ■

Proof of Lemma 3.1. The proof is divided into six steps.

Step 1: We prove that:

$$U \in L^2(]0,T[;L^2_\mu(\Omega\times\mathcal{X};\mathbb{R})); \tag{3.43}$$

$$V \in L^2(]0,T[;L^2_\mu(\Omega\times\mathcal{X};\mathbb{R})); \tag{3.44}$$

$$W_T \in L^2_\mu(\Omega\times\mathcal{X};\mathbb{R}). \tag{3.45}$$

Taking (3.35) into account and using Jensen's inequality, we see that

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}}|U(t,x,\xi)|^2dt\otimes d\mu(x,\xi) &= \int_{]0,T[\times\Omega\times\mathcal{X}}\left|\int_{\mathbb{R}^2}sd\nu_{t,x,\xi}(s,\dot{s})\right|^2dt\otimes d\mu(x,\xi)\\ &\leqslant \int_{]0,T[\times\Omega\times\mathcal{X}}\left(\int_{\mathbb{R}^2}|s|^2d\nu_{t,x,\xi}(s,\dot{s})\right)dt\otimes d\mu(x,\xi),\end{aligned}$$

and so

$$\int_{]0,T[\times\Omega\times\mathcal{X}}|U(t,x,\xi)|^2dt\otimes d\mu(x,\xi)\leqslant\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}|s|^2d\nu(t,x,\xi,s,\dot{s}) \tag{3.46}$$

because of (3.33). On the other hand, according to (3.31) and the definition of $\{\nu_\varepsilon\}_\varepsilon$, From Theorem A.4(i) we have

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}|s|^2d\nu(t,x,\xi,s,\dot{s}) &\leqslant \underline{\lim}_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}|s|^2d\nu_\varepsilon(t,x,\xi,s,\dot{s})\\ &= \underline{\lim}_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}}|U_\varepsilon(t,x,\xi)|^2dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi),\end{aligned}$$

and so

$$\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}|s|^2d\nu(t,x,\xi,s,\dot{s})<\infty \tag{3.47}$$

by (3.28). From (3.46) and (3.46) we get (3.43). In the same manner, by considering (3.36) instead of (3.35) we obtain (3.44). By using (3.37), (3.34), (3.32) and the definition of $\{\lambda_\varepsilon\}_\varepsilon$ instead of (3.35), (3.33), (3.28) and the definition of $\{\nu_\varepsilon\}_\varepsilon$, a similar calculation gives (3.45).

Step 2: We prove that

$$\int_{\mathcal{X}} U(t,x,\xi)d\mu_x(\xi) = u(t,x) \text{ for } dt\otimes dx\text{-a.a. } (t,x)\in]0,T[\times\Omega, \tag{3.48}$$

where $\{\mu_x\}_{x\in\Omega}$ denotes the desintegration of μ on $\Omega\times\mathcal{X}$. Fix any $\varphi\in C_c^\infty(]0,T[\times\Omega)$. As $u_\varepsilon \xrightarrow{H^1} u$ we have

$$\int_{]0,T[\times\Omega}\varphi(t,x)u(t,x)dt\otimes dx = \lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega}\varphi(t,x)u_\varepsilon(t,x)dt\otimes dx. \tag{3.49}$$

Setting $\psi(t,x,\xi,s,\dot{s}) := \varphi(t,x)s$ and recalling that $U_\varepsilon\in\mathcal{S}_{\mu_{\sigma(\varepsilon)}}$ for any $\varepsilon>0$, we see that

$$\int_{]0,T[\times\Omega}\varphi(t,x)u_\varepsilon(t,x)dt\otimes dx = \int_{]0,T[\times\Omega\times\mathcal{X}}\psi(t,x,\xi,U_\varepsilon(t,x,\xi),\dot{U}_\varepsilon(t,x,\xi))dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi),$$

and so, according to the definition of $\{\nu_\varepsilon\}_\varepsilon$, one has

$$\int_{]0,T[\times\Omega}\varphi(t,x)u_\varepsilon(t,x)dt\otimes dx = \int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\psi(t,x,\xi,s,\dot{s})d\nu_\varepsilon(t,x,\xi,s,\dot{s})$$

for all $\varepsilon>0$. Taking (3.31) into account, from Theorem A.4(ii) we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega}\varphi(t,x)u_\varepsilon(t,x)dt\otimes dx &= \int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\psi(t,x,\xi,s,\dot{s})d\nu(t,x,\xi,s,\dot{s})\\ &= \int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\varphi(t,x)sd\nu(t,x,\xi,s,\dot{s}).\end{aligned} \tag{3.50}$$

Combining (3.49) with (3.50) and by using desintegration's theorem (see Theorem A.5) and (3.35), we obtain

$$\begin{aligned}\int_{]0,T[\times\Omega}\varphi(t,x)u(t,x)dt\otimes dx &= \int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t,x)\left(\int_{\mathbb{R}^2}sd\nu_{t,x,\xi}(s,\dot{s})\right)dt\otimes d\mu(x,\xi)\\ &= \int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t,x)U(t,x,\xi)dt\otimes d\mu(x,\xi)\\ &= \int_{]0,T[\times\Omega}\varphi(t,x)\left(\int_{\mathcal{X}}U(t,x,\xi)d\mu_x(\xi)\right)dt\otimes dx,\end{aligned}$$

which proves (3.48).

Step 3: We prove that

$$V = \dot{U}. \tag{3.51}$$

Fix any $\varphi\in C_c^1(]0,T[)$ and any $\phi\in C_c(\Omega)$. Set $\psi_1(t,x,\xi,s,\dot{s}) := \varphi(t)\phi(x)\dot{s}$. According to (3.36) and the definition of $\{\nu_\varepsilon\}_\varepsilon$, by using desintegration's theorem (see Theorem A.5) and

Theorem A.4(ii) together with (3.31), we see that

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t)\phi(x)V(t,x,\xi)dt\otimes d\mu(x,\xi)&=\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t)\phi(x)\left(\int_{\mathbb{R}^2}\dot{s}d\nu_{t,x,\xi}(s,\dot{s})\right)dt\otimes d\mu(x,\xi)\\&=\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\psi_1(t,x,\xi,s,\dot{s})d\nu(t,x,\xi,s,\dot{s})\\&=\lim_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\psi_1(t,x,\xi,s,\dot{s})d\nu_\varepsilon(t,x,\xi,s,\dot{s})\\&=\lim_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}}\psi_1(t,x,\xi,U_\varepsilon(t,x,\xi),\dot{U}_\varepsilon(t,x,\xi))dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi)\\&=\lim_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t)\phi(x)\dot{U}_\varepsilon(t,x,\xi)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi).\end{aligned}$$

But, for any $\varepsilon>0$, as $\varphi\in C_c^1(]0,T[)$ one has

$$\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t)\phi(x)\dot{U}_\varepsilon(t,x,\xi)dt\otimes d\mu_\varepsilon(x,\xi)=-\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi'(t)\phi(x)U_\varepsilon(t,x,\xi)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi),$$

and so, setting $\psi_2(t,x,\xi,s,\dot{s}):=\varphi'(t)\phi(x)s$ and using the same arguments as in above with (3.35) instead of (3.36), it follows that

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi(t)\phi(x)V(t,x,\xi)dt\otimes d\mu(x,\xi)&=-\lim_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}}\psi_2(t,x,\xi,U_\varepsilon(t,x,\xi),\dot{U}_\varepsilon(t,x,\xi))dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi)\\&=-\lim_{\varepsilon\to0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\psi_2(t,x,\xi,s,\dot{s})d\nu_\varepsilon(t,x,\xi,s,\dot{s})\\&=-\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\psi_2(t,x,\xi,s,\dot{s})d\nu(t,x,\xi,s,\dot{s})\\&=-\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2}\varphi'(t)\phi(x)sd\nu(t,x,\xi,s,\dot{s})\\&=-\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi'(t)\phi(x)\left(\int_{\mathbb{R}^2}sd\nu_{t,x,\xi}(s,\dot{s})\right)dt\otimes d\mu(x,\xi)\\&=-\int_{]0,T[\times\Omega\times\mathcal{X}}\varphi'(t)\phi(x)U(t,x,\xi)dt\otimes d\mu(x,\xi),\end{aligned}$$

which establishes (3.51).

Step 4: We prove that

$$U(0,x,\xi)=0\text{ for }\mu\text{-a.a. }(x,\xi)\in\Omega\times\mathcal{X}.\tag{3.52}$$

Let $\varphi \in C^1([0,T])$ be such that $\varphi(T) = 0$ and $\varphi(0) \neq 0$ and let $\psi \in C_c(\Omega \times \mathcal{X})$. Then, we have

$$\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi(t)\dot{U}(t,x,\xi)\psi(x,\xi)dt \otimes d\mu(x,\xi) = -\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi'(t)U(t,x,\xi)\psi(x,\xi)d\mu(x,\xi) - \int_{\Omega\times\mathcal{X}} \varphi(0)U(0,x,\xi)\psi(x,\xi)d\mu(x,\xi). \quad (3.53)$$

On the other hand, by (3.51) and (3.36) and the definition of $\{\nu_\varepsilon\}_\varepsilon$, from desintegration's theorem (see Theorem A.5) and Theorem A.4(ii) together with (3.31), we see that

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi(t)\dot{U}(t,x,\xi)\psi(x,\xi)dt \otimes d\mu(x,\xi) &= \int_{]0,T[\times\Omega\times\mathcal{X}} \varphi(t)\left(\int_{\mathbb{R}^2} \dot{s}d\nu_{t,x,\xi}(s,\dot{s})\right)\psi(x,\xi)dt \otimes d\mu(x,\xi)\\ &= \int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2} \varphi(t)\dot{s}\psi(x,\xi)d\nu(t,x,\xi,s,\dot{s})\\ &= \lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2} \varphi(t)\dot{s}\psi(x,\xi)d\nu_\varepsilon(t,x,\xi,s,\dot{s})\\ &= \lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi(t)\dot{U}_\varepsilon(t,x,\xi)\psi(x,\xi)dt \otimes d\mu_{\sigma(\varepsilon)}(x,\xi).\end{aligned}$$

But, for any $\varepsilon > 0$, one has

$$\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi(t)\dot{U}_\varepsilon(t,x,\xi)\psi(x,\xi)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi) = -\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi'(t)U_\varepsilon(t,x,\xi)\psi(x,\xi)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi)$$

because $\varphi(T) = 0$ and $U_\varepsilon(0,x,\xi) = 0$ for $\mu_{\sigma(\varepsilon)}$-a.a. $(x,\xi) \in \Omega \times \mathcal{X}$ since $U_\varepsilon \in \mathcal{H}^1_{0,\mu_{\sigma(\varepsilon)}}$. Hence, by the same arguments as in above, we get

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi(t)\dot{U}(t,x,\xi)\psi(x,\xi)dt \otimes d\mu(x,\xi) &= -\lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi'(t)U_\varepsilon(t,x,\xi)\psi(x,\xi)dt \otimes d\mu_{\sigma(\varepsilon)}(x,\xi)\\ &= -\lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2} \varphi'(t)s\psi(x,\xi)d\nu_\varepsilon(t,x,\xi,s,\dot{s})\\ &= -\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2} \varphi'(t)s\psi(x,\xi)d\nu(t,x,\xi,s,\dot{s})\\ &= -\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi'(t)\left(\int_{\mathbb{R}^2} sd\nu_{t,x,\xi}(s,\dot{s})\right)\psi(x,\xi)dt \otimes d\mu(x,\xi)\\ &= -\int_{]0,T[\times\Omega\times\mathcal{X}} \varphi'(t)U(t,x,\xi)\psi(x,\xi)dt \otimes d\mu(x,\xi). \quad (3.54)\end{aligned}$$

Combining (3.53) with (3.54) we deduce that

$$\int_{\Omega\times\mathcal{X}} \varphi(0)U(0,x,\xi)\psi(x,\xi)d\mu(x,\xi) = 0$$

for all $\psi \in C_c(\Omega \times \mathcal{X})$, and (3.52) follows because $\varphi(0) \neq 0$.

Step 5: We prove that

$$W_T(x,\xi) = U(T,x,\xi) \text{ for } \mu\text{-a.a. } (x,\xi) \in \Omega \times \mathcal{X}. \tag{3.55}$$

Fix any $\psi \in C_c(\Omega \times \mathcal{X})$. According to (3.37) and the definition of $\{\lambda_\varepsilon\}_\varepsilon$, by using desintegration's theorem (see Theorem A.5) and Theorem A.4(ii) together with (3.32), we see that

$$\begin{aligned}
\int_{\Omega\times\mathcal{X}} W_T(x,\xi)\psi(x,\xi)d\mu(x,\xi) &= \int_{\Omega\times\mathcal{X}} \left(\int_{\mathbb{R}} s_T d\lambda_{x,\xi}(s_T)\right)\psi(x,\xi)d\mu(x,\xi) \\
&= \int_{\Omega\times\mathcal{X}\times\mathbb{R}} s_T\psi(x,\xi)d\lambda(x,\xi,s_T) \\
&= \lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{X}\times\mathbb{R}} s_T\psi(x,\xi)d\lambda_\varepsilon(x,\xi,s_T) \\
&= \lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{X}} U_\varepsilon(T,x,\xi)\psi(x,\xi)d\mu_{\sigma(\varepsilon)}(x,\xi). \qquad (3.56)
\end{aligned}$$

But, for any $\varepsilon > 0$, $U_\varepsilon(0,x,\xi) = 0$ for $\mu_{\sigma(\varepsilon)}$-a.a. $(x,\xi) \in \Omega \times \mathcal{X}$ because $U_\varepsilon \in \mathcal{H}^1_{0,\mu_{\sigma(\varepsilon)}}$, and so

$$\begin{aligned}
\int_{\Omega\times\mathcal{X}} U_\varepsilon(T,x,\xi)\psi(x,\xi)d\mu_{\sigma(\varepsilon)}(x,\xi) &= \int_{\Omega\times\mathcal{X}} \left(\int_0^T \dot{U}_\varepsilon(t,x,\xi)dt\right)\psi(x,\xi)d\mu_{\sigma(\varepsilon)}(x,\xi) \\
&= \int_{]0,T[\times\Omega\times\mathcal{X}} \dot{U}_\varepsilon(t,x,\xi)\psi(x,\xi)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi). \qquad (3.57)
\end{aligned}$$

Moreover, according to (3.51) and (3.36) and the definition of $\{\nu_\varepsilon\}_\varepsilon$, by using the same arguments as in above with (3.31) instead of (3.32), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}} \dot{U}_\varepsilon(t,x,\xi)\psi(x,\xi)dt\otimes d\mu_{\sigma(\varepsilon)}(x,\xi) &= \lim_{\varepsilon\to 0}\int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2} \dot{s}\psi(x,\xi)d\nu_\varepsilon(t,x,\xi,s,\dot{s}) \\
&= \int_{]0,T[\times\Omega\times\mathcal{X}\times\mathbb{R}^2} \dot{s}\psi(x,\xi)d\nu(t,x,\xi,s,\dot{s}) \\
&= \int_{]0,T[\times\Omega\times\mathcal{X}} \left(\int_{\mathbb{R}^2} \dot{s}d\nu_{t,x,\xi}(s,\dot{s})\right)\psi(x,\xi)dt\otimes d\mu(x,\xi) \\
&= \int_{]0,T[\times\Omega\times\mathcal{X}} \dot{U}(t,x,\xi)\psi(x,\xi)dt\otimes d\mu(x,\xi). \qquad (3.58)
\end{aligned}$$

From (3.52) we see that

$$\begin{aligned}
\int_{]0,T[\times\Omega\times\mathcal{X}} \dot{U}(t,x,\xi)\psi(x,\xi)dt\otimes d\mu(x,\xi) &= \int_{\Omega\times\mathcal{X}} \left(\int_0^T \dot{U}(t,x,\xi)dt\right)\psi(x,\xi)d\mu(x,\xi) \\
&= \int_{\Omega\times\mathcal{X}} U(T,x,\xi)\psi(x,\xi)d\mu(x,\xi), \qquad (3.59)
\end{aligned}$$

and consequently, from (3.56), (3.57), (3.58) and (3.59), we conclude that

$$\int_{\Omega\times\mathcal{X}} W_T(x,\xi)\psi(x,\xi)d\mu(x,\xi) = \int_{\Omega\times\mathcal{X}} U(T,x,\xi)\psi(x,\xi)d\mu(x,\xi)$$

for all $\psi \in C_c(\Omega \times \mathcal{X})$, and (3.55) follows.

Step 6: end of the proof. From (3.43), (3.44), (3.51) and (3.52) we deduce that $U \in \mathcal{H}^1_{0,\mu}$, and so $U \in \mathcal{S}_\mu(u)$ by (3.48), which completes the proof of Lemma 3.1. ■

4. Stochastic homogenization

4.1. **A non-local stochastic homogenization theorem.** Let $(\Sigma, \mathcal{A}, \mathbb{P})$ be a probability space and let $(T_z)_{z\in\mathbb{Z}^N}$ be a group of $\mathbb{P}$-preserving transformations on $(\Sigma, \mathcal{A})$, i.e.,

- (mesurability) T_z is $\mathcal{A}$-measurable for all $z \in \mathbb{Z}^N$;
- (group property) $T_z \circ T_{z'} = T_{z+z'}$ and $T_{-z} = T_z^{-1}$ for all $z, z' \in \mathbb{Z}^N$;
- (mass invariance) $\mathbb{P}(T_z(A)) = \mathbb{P}(A)$ for all $z \in \mathbb{Z}^N$ and all $A \in \mathcal{A}$.

Let $f : \Sigma \times \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a $(\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable function such that

$$f(\omega, x) := f(\omega, x, \cdot, \cdot, \cdot, \cdot) \in \mathcal{X}$$

for all $\omega \in \Sigma$ and all $x \in \mathbb{R}^N$. It is easily seen that the map

$$\begin{array}{rccc} f & : & \Sigma & \to & \mathfrak{X} := \mathcal{X}^{\mathbb{R}^N} \\ & & \omega & \mapsto & f(\omega, \cdot) \end{array}$$

is a $(\mathcal{A}, \mathcal{B})$-random variable with $\mathcal{B} := \otimes_{\mathbb{R}^N} \mathcal{B}(\mathcal{X})$. We futhermore assume that the random variable f is covariant with respect to the dynamical system $(\Sigma, \mathcal{A}, \mathbb{P}, (T_z)_{z\in\mathbb{Z}^N})$, i.e.,

$$f(T_z\omega, \cdot) = f(\omega, \cdot + z) \tag{4.1}$$

for all $z \in \mathbb{Z}^N$ and for $\mathbb{P}$-a.a. $\omega \in \Sigma$. For each $z \in \mathbb{Z}^N$, let $\tau_z : \mathfrak{X} \to \mathfrak{X}$ denote the shift map on $\mathfrak{X}$, i.e.,

$$\tau_z(w) := w(\cdot + z) \tag{4.2}$$

for all $w \in \mathfrak{X}$. Then, $f^\sharp\mathbb{P}$, i.e., the law of f, is invariant under the group $(\tau_z)_{z\in\mathbb{Z}^N}$, i.e.,

$$\tau_z^\sharp f^\sharp \mathbb{P} = f^\sharp \mathbb{P} \tag{4.3}$$

for all $z \in \mathbb{Z}^N$. We finally assume that the dynamical system $(\Sigma, \mathcal{A}, \mathbb{P}, (T_z)_{z\in\mathbb{Z}^N})$, or equivalently $(\mathfrak{X}, \mathcal{B}, f^\sharp\mathbb{P}, (\tau_z)_{z\in\mathbb{Z}^N})$, is ergodic, i.e., for each $A \in \mathcal{A}$,

$$\text{if } T_z(A) = A \text{ for all } z \in \mathbb{Z}^N \text{ then } \mathbb{P}(A) = 0 \text{ or } \mathbb{P}(A) = 1,$$

or equivalently, for each $B \in \mathcal{B}$,

$$\text{if } \tau_z(B) = B \text{ for all } z \in \mathbb{Z}^N \text{ then } f^\sharp\mathbb{P}(B) = 0 \text{ or } f^\sharp\mathbb{P}(B) = 1. \tag{4.4}$$

For each $\omega \in \Sigma$ and each $\varepsilon > 0$, we consider $F_\varepsilon(\omega) : \mathcal{H}^1_0 \to \mathbb{R}$ given by

$$F_\varepsilon(\omega)(u) := \Phi_{f(\omega, \frac{\cdot}{\varepsilon})}(u) = \int_{]0,T[\times\Omega} f\left(\omega, \frac{x}{\varepsilon}\right)(t, x, u(t,x), \dot{u}(t,x))dt \otimes dx + \int_\Omega \theta(u(T,x))dx. \tag{4.5}$$

For each $\varepsilon > 0$, let $\mu_\varepsilon : \Sigma \to \mathcal{Y}(\Omega; \mathcal{X})$ be the random Young measure defined by

$$\mu_\varepsilon(\omega) := dx \otimes \delta_{f(\omega, \frac{x}{\varepsilon})}. \tag{4.6}$$

Let $Y :=]0,1[^N$ be the unit cell, let $f_Y : \Sigma \to \mathcal{X}^Y$ be defined by $f_Y(\omega)(y) = f(\omega, y)$ and, for each $\omega \in \Sigma$, let $f_Y^\sharp(\omega)dy$ be the image by $f_Y(\omega)$ of the Lebesgue measure restricted to Y that we denote by dy. Let $\mu \in \mathcal{Y}(\Omega;\mathcal{X})$ be the Young measure defined by

$$\mu_f := dx \otimes d\mathbb{E}_{f_Y^\sharp dy}, \tag{4.7}$$

where $d\mathbb{E}_{f_Y^\sharp dy}$ denotes the expectation of the random probability measure $\omega \mapsto f_Y^\sharp(\omega)dy$, i.e., $d\mathbb{E}_{f_Y^\sharp dy}$ is the probability measure on $\mathcal{X}$ given by

$$\begin{aligned} d\mathbb{E}_{f_Y^\sharp dy}(B) &= \int_\Sigma \left(\int_Y 1_B\big(f_Y(\omega)(y)\big)dy \right) d\mathbb{P}(\omega) \\ &= \int_{\Sigma\times Y} 1_B\big(f_Y(\omega)(y)\big) d\mathbb{P}(\omega) \otimes dy \end{aligned}$$

for all $B \in \mathcal{B}(\mathcal{X})$, where 1_B denotes the indicator function of B, or equivalently by

$$\int_{\mathcal{X}} \varphi(\xi) d\mathbb{E}_{f_Y^\sharp dy}(\xi) = \mathbb{E}\left(\int_Y \varphi\left(f(\cdot,y)\right) dy \right) = \int_{\Sigma\times Y} \varphi\left(f(\omega,y)\right) d\mathbb{P}(\omega) \otimes dy \tag{4.8}$$

for all $(\mathcal{B}(\mathcal{X}),\mathcal{B}(\mathbb{R}))$-measurable functions $\varphi : \mathcal{X} \to \mathbb{R}$ such that $\int_Y \varphi\left(f(\cdot,y)\right) dy \in L^1(\Sigma,\mathcal{A},\mathbb{P})$.

Proposition 4.1. *For $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has*

$$\mu_\varepsilon(\omega) \xrightarrow{\text{nar}} \mu_f$$

with $\mu_f \in \mathcal{Y}(\Omega;\mathcal{X})$ given by (4.7).

Proof of Proposition 4.1. It is sufficient to prove that there exists $\widehat{\Sigma} \in \mathcal{A}$ with $\mathbb{P}(\widehat{\Sigma}) = 1$ such that for every $\omega \in \widehat{\Sigma}$, one has

$$\lim_{\varepsilon\to 0} \int_{\Omega\times\mathcal{X}} 1_A(x)\varphi(\xi) d\mu_\varepsilon(\omega)(x,\xi) = \int_{\Omega\times\mathcal{X}} 1_A(x)\varphi(\xi) dx \otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi)$$

for all $A \in \mathcal{B}(\Omega)$ and all $\varphi \in \mathcal{D}$ with $\mathcal{D}$ a dense subset of $C_c(\mathcal{X})$, see [Val90, Val94].
Let $A \in \mathcal{B}(\Omega)$ and let $\varphi \in \mathcal{D}$. Taking (4.8) into account, from the additive ergodic theorem [ABM14, Theorem 12.4.1] (see also [CM94]), we can assert that there exists $N_\varphi \in \mathcal{F}$ with $\mathbb{P}(N_\varphi) = 0$ such that for every $\omega \in \Sigma\backslash N_\varphi$, one has

$$\varphi\left(f\Big(\omega,\frac{\cdot}{\varepsilon}\Big)\right) \xrightarrow{L^1} \int_{\mathcal{X}} \varphi(\xi) d\mathbb{E}_{f_Y^\sharp dy}(\xi) \tag{4.9}$$

On the other hand, by (4.6) we see that for every $\omega \in \Sigma$, one has

$$\int_{\Omega\times\mathcal{X}} 1_A(x)\varphi(\xi) d\mu_\varepsilon(\omega)(x,\xi) = \int_\Omega 1_A(x)\varphi\left(f\Big(\omega,\frac{x}{\varepsilon}\Big)\right) dx \text{ for all } \varepsilon > 0. \tag{4.10}$$

Set $\widehat{\Sigma} := \cup_{\psi\in\mathcal{D}}(\Sigma\backslash N_\psi)$ (where N_ψ corresponds to N_φ with $\varphi=\psi$). Then $\mathbb{P}(\widehat{\Sigma})=0$ and, by using (4.9), from (4.10) we deduce that for every $\omega\in\widehat{\Sigma}$, one has

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega\times\mathcal{X}} 1_A(x)\varphi(\xi)d\mu_\varepsilon(\omega)(x,\xi) &= \lim_{\varepsilon\to 0}\int_\Omega 1_A(x)\varphi\left(f\left(\omega,\frac{x}{\varepsilon}\right)\right)dx\\ &= \int_\Omega 1_A(x)\left(\int_{\mathcal{X}}\varphi(\xi)d\mathbb{E}_{f^\sharp_{Y}dy}(\xi)\right)dx\\ &= \int_{\Omega\times\mathcal{X}} 1_A(x)\varphi(\xi)dx\otimes d\mathbb{E}_{f^\sharp_{Y}dy}(\xi),\end{aligned}$$

and the proof is complete. ■

Let $F_{\rm hom}:\mathcal{H}^1_0\to\mathbb{R}$ be defined by

$$F_{\rm hom}(u) := \inf\left\{\int_{\Sigma\times Y}\Phi_{f(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy : v\in\mathcal{S}_{\rm hom}(u)\right\}, \tag{4.11}$$

where $\mathcal{S}_{\rm hom}(u)$ is given by

$$\mathcal{S}_{\rm hom}(u):=\left\{v\in\mathcal{H}^1_{0,dx\otimes\mathbb{P}\otimes dy}: \int_{\Sigma\times Y} v(t,x,\omega,y)d\mathbb{P}(\omega)\otimes dy = u(t,x) \text{ for } dt\otimes dx\text{-a.a. } (t,x)\in]0,T[\times\Omega\right\}$$

with

$$\mathcal{H}^1_{0,dx\otimes\mathbb{P}\otimes dy} := \left\{v\in H^1(]0,T[;L^2_{dx\otimes\mathbb{P}\otimes dy}(\Omega\times\Sigma\times Y;\mathbb{R})) : v(0,\cdot,\cdot,\cdot)=0\right\}$$

and

$$\begin{aligned}\Phi_{f(\omega,y)}(v(\cdot,\cdot,\omega,y)) &= \int_{]0,T[\times\Omega} f(\omega,y)(t,x,v(t,x,\omega,y),\dot{v}(t,x,\omega,y))dt\otimes dx\\ &\quad+\int_\Omega\theta(v(T,x,\omega,y))dx\end{aligned} \tag{4.12}$$

for all $\omega\in\Sigma$ and all $y\in Y$. The following result is a consequence of Corollary 2.8 and Proposition 4.1.

Theorem 4.2. *For $\mathbb{P}$-a.e. $\omega\in\Sigma$ one has*

$$\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(\omega) = F_{\rm hom}$$

with $F_{\rm hom}$ given by (4.11)-(4.12).

Proof of Theorem 4.2. From Corollary 2.8 and Proposition 4.1 we deduce that for $\mathbb{P}$-a.e. $\omega\in\Sigma$, one has

$$\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(\omega) = \Phi_{\mu_f}$$

with $\Phi_{\mu_f}:\mathcal{H}^1_0\to\mathbb{R}$ given by (2.6). (Since μ_f does not depend on ω, so is for Φ_{μ_f}.) Taking (4.7) and (4.8) into account we see that for any $u\in\mathcal{H}^1_0$, one has

$$\begin{aligned}\mathcal{S}_{\mu_f}(u) &= \left\{U\in\mathcal{H}^1_{0,\mu_f}: \int_{\mathcal{X}}U(t,x,\xi)d\mathbb{E}_{f^\sharp_{Y}dy}(\xi)=u(t,x)\text{for } dt\otimes dx\text{-a.a. } (t,x)\in]0,T[\times\Omega\right\}\\ &= \left\{U\in\mathcal{H}^1_{0,\mu_f}: \int_{\Sigma\times Y}U(t,x,f(\omega,y))d\mathbb{P}(\omega)\otimes dy=u(t,x)\text{for } dt\otimes dx\text{-a.a. } (t,x)\in]0,T[\times\Omega\right\}\end{aligned}$$

and

$$\Phi_{\mu_f}(u) = \inf\Big\{\phi_{\mu_f}(U) : U \in \mathcal{S}_{\mu_f}(u)\Big\}$$

with

$$\begin{aligned}
\phi_{\mu_f}(U) &= \int_{]0,T[\times\Omega\times\mathcal{X}} \xi\big(t,x,U(t,x,\xi),\dot{U}(t,x,\xi)\big)dt\otimes d\mu_f(x,\xi) + \int_{\Omega\times\mathcal{X}} \theta(U(T,x,\xi))d\mu_f(x,\xi)\\
&= \int_{]0,T[\times\Omega\times\mathcal{X}} \xi\big(t,x,U(t,x,\xi),\dot{U}(t,x,\xi)\big)dt\otimes dx\otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi) + \int_{\Omega\times\mathcal{X}} \theta(U(T,x,\xi))dx\otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi)\\
&= \int_{]0,T[\times\Omega\times\Sigma\times Y} f(\omega,y)\left(t,x,U(t,x,f(\omega,y)),\dot{U}(t,x,f(\omega,y))\right)dt\otimes dx\otimes d\mathbb{P}(\omega)\otimes dy\\
&\quad + \int_{\Omega\times\Sigma\times Y} \theta\big(U(T,x,f(\omega,y))\big)dx\otimes d\mathbb{P}(\omega)\otimes dy. \qquad (4.13)
\end{aligned}$$

So, it remains to prove that $\Phi_{\mu_f} = F_{\rm hom}$, i.e., for every $u \in \mathcal{H}_0^1$, one has:

$$\Phi_{\mu_f}(u) \geqslant F_{\rm hom}(u); \qquad (4.14)$$

$$\Phi_{\mu_f}(u) \leqslant F_{\rm hom}(u). \qquad (4.15)$$

Let us fix $u \in \mathcal{H}_0^1$.

Proof of (4.14). Let $U \in \mathcal{S}_{\mu_f}(u)$. Then:

$$U \in L^2(]0,T[; L^2_{\mu_f}(\Omega\times\Sigma\times Y;\mathbb{R})); \qquad (4.16)$$

$$\dot{U} \in L^2(]0,T[; L^2_{\mu_f}(\Omega\times\Sigma\times Y;\mathbb{R})); \qquad (4.17)$$

$$U(0,\cdot,\cdot) = 0; \qquad (4.18)$$

$$\int_{\Sigma\times Y} U(t,x,f(\omega,y))d\mathbb{P}(\omega)\otimes dy = u(t,x) \text{ for } dt\otimes dx\text{-a.a. } (t,x) \in]0,T[\times\Omega. \qquad (4.19)$$

Set $v(t,x,\omega,y) := U(t,x,f(\omega,y))$. By (4.18) we have $v(0,\cdot,\cdot,\cdot) = 0$ and from (4.19) it is clear that

$$\int_{\Sigma\times Y} v(t,x,\omega,y)d\mathbb{P}(\omega)\otimes dy = u(t,x) \text{ for } dt\otimes dx\text{-a.a. } (t,x) \in]0,T[\times\Omega.$$

On the other hand, taking (4.8) into account, we see that

$$\begin{aligned}
\int_{]0,T[\times\Omega\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes dx\otimes d\mathbb{P}(\omega)\otimes dy &= \int_{]0,T[\times\Omega}\left(\int_{\Sigma\times Y}|U(t,x,f(\omega,y))|^2 d\mathbb{P}(\omega)\otimes dy\right)dt\otimes dx\\
&= \int_{]0,T[}\left(\int_{\Omega\times\mathcal{X}}|U(t,x,\xi)|^2 dx\otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi)\right)dt\\
&= \int_{]0,T[\times\Omega\times\mathcal{X}}|U(t,x,\xi)|^2 dt\otimes d\mu_f(x,\xi),
\end{aligned}$$

and so, from (4.16) we deduce that $v \in L^2(]0,T[; L^2_{dx\otimes\mathbb{P}\otimes dy}(\Omega\times\Sigma\times Y;\mathbb{R}))$. In the same way, by using (4.17) instead of (4.16) we obtain $\dot{v} \in L^2(]0,T[; L^2_{dx\otimes\mathbb{P}\otimes dy}(\Omega\times\Sigma\times Y;\mathbb{R}))$. It follows that $v \in \mathcal{S}_{\mu_f}(u)$.

We have thus proved that for each $U \in \mathcal{S}_{\mu_f}(u)$ one has $v \in \mathcal{S}_{\rm hom}(u)$. According to (4.11)-(4.12) and (4.13) we conclude that

$$\phi_{\mu_f}(U) = \int_{\Sigma\times Y} \Phi_{f(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \geqslant F_{\rm hom}(u)$$

for all $U \in \mathcal{S}_{\mu_f}(u)$, and (4.14) follows. ■

Proof of (4.15). Consider $g : \Sigma \times Y \to \mathcal{X} \times \Sigma \times Y$ defined by $g(\omega, y) := (f(\omega, y), \omega, y)$ and set

$$\lambda := g^\sharp(\mathbb{P}\otimes dy). \tag{4.20}$$

Then $\pi^\sharp_{\mathcal{X}}\lambda = d\mathbb{E}_{f^\sharp_Y dy}$ with $\pi_{\mathcal{X}} : \mathcal{X}\times\Sigma\times Y \to \mathcal{X}$ denoting the canonical projection from $\mathcal{X}\times\Sigma\times Y$ to $\mathcal{X}$. So, from desintegration's theorem (see Theorem A.5) we can assert that there exists a familiy $(\lambda_\xi)_{\xi\in\mathcal{X}}$ of probability measures on $\Sigma\times Y$ such that

$$\lambda = d\mathbb{E}_{f^\sharp_Y dy}\otimes\lambda_\xi. \tag{4.21}$$

Fix any $v \in \mathcal{S}_{\rm hom}(u)$ and set

$$U(t,x,\xi) := \int_{\Sigma\times Y} v(t,x,\omega,y)d\lambda_\xi(\omega,y). \tag{4.22}$$

(In particular, $U(T,x,\xi) := \int_{\Sigma\times Y} v(T,x,\omega,y)d\lambda_\xi(\omega,y).$) Then, it is easy to see that

$$\dot U(t,x,\xi) := \int_{\Sigma\times Y} \dot v(t,x,\omega,y)d\lambda_\xi(\omega,y). \tag{4.23}$$

We claim that $U \in \mathcal{S}_{\mu_f}(u)$. Indeed, as $v \in \mathcal{S}_{\rm hom}(u)$ we have:

$$v \in L^2\left(]0,T[; L^2_{dx\otimes\mathbb{P}\otimes dy}(\Omega\times\Sigma\times Y;\mathbb{R})\right); \tag{4.24}$$

$$\dot v \in L^2\left(]0,T[; L^2_{dx\otimes\mathbb{P}\otimes dy}(\Omega\times\Sigma\times Y;\mathbb{R})\right); \tag{4.25}$$

$$v(0,\cdot,\cdot,\cdot) = 0; \tag{4.26}$$

$$\int_{\Sigma\times Y} v(t,x,\omega,y)d\mathbb{P}(\omega)\otimes dy = u(t,x) \text{ for } dt\otimes dx\text{-a.a. } (t,x)\in]0,T[\times\Omega. \tag{4.27}$$

Firstly, by (4.26) it is clear that $U(0,\cdot,\cdot) = 0$. Secondly, by using (4.8), (4.22), (4.21), (4.20) and (4.27) we see that

$$\begin{aligned}
\int_{\Sigma\times Y} U(t,x,f(\omega,y))d\mathbb{P}(\omega)\otimes dy &= \int_{\mathcal{X}} U(t,x,\xi)d\mathbb{E}_{f^\sharp_Y dy}(\xi)\\
&= \int_{\mathcal{X}}\left(\int_{\Sigma\times Y} v(t,x,\omega,y)d\lambda_\xi(\omega,y)\right)d\mathbb{E}_{f^\sharp_Y dy}(\xi)\\
&= \int_{\mathcal{X}\times\Sigma\times Y} v(t,x,\omega,y)d\lambda(\xi,\omega,y)\\
&= \int_{\Sigma\times Y} v(t,x,\omega,y)d\mathbb{P}(\omega)\otimes dy\\
&= u(t,x)
\end{aligned}$$

for $dt \otimes dx$-a.a. $(t,x) \in]0,T[\times\Omega$. Thirdly, taking (4.22) into account and using Jensen's inequality, we have

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}} |U(t,x,\xi)|^2 dt\otimes d\mu_f(x,\xi) &= \int_{]0,T[\times\Omega\times\mathcal{X}} \left|\int_{\Sigma\times Y} v(t,x,\omega,y) d\lambda_\xi(\omega,y)\right|^2 dt\otimes d\mu_f(x,\xi)\\ &\leqslant \int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes d\mu_f(x,\xi)\otimes d\lambda_\xi(\omega,y).\end{aligned}$$

But, by (4.7), (4.21) and (4.20) we have

$$\begin{aligned}\int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes d\mu_f(x,\xi)\otimes d\lambda_\xi(\omega,y) &= \int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes dx\otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi)\otimes d\lambda_\xi(\omega,y)\\ &= \int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes dx\otimes d\lambda(\xi,\omega,y)\\ &= \int_{]0,T[\times\Omega\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes dx\otimes d\mathbb{P}(\omega)\otimes dy\end{aligned}$$

hence

$$\int_{]0,T[\times\Omega\times\mathcal{X}} |U(t,x,\xi)|^2 dt\otimes d\mu_f(x,\xi) \leqslant \int_{]0,T[\times\Omega\times\Sigma\times Y} |v(t,x,\omega,y)|^2 dt\otimes dx\otimes d\mathbb{P}(\omega)\otimes dy,$$

and so, from (4.24) we deduce that $U \in L^2(]0,T[; L^2_{\mu_f}(\Omega\times\mathcal{X};\mathbb{R}))$. In the same way, by using (4.25) instead of (4.24) we obtain $\dot{U} \in L^2(]0,T[; L^2_{\mu_f}(\Omega\times\mathcal{X};\mathbb{R}))$, and the claim is proved. On the other hand, taking (4.22) and (4.23) into account and using Jensen's inequality, we have

$$\begin{aligned}\phi_{\mu_f}(U) &= \int_{]0,T[\times\Omega\times\mathcal{X}} \xi\left(t,x,\int_{\Sigma\times Y} v(t,x,\omega,y) d\lambda_\xi(\omega,y), \int_{\Sigma\times Y} \dot{v}(t,x,\omega,y) d\lambda_\xi(\omega,y)\right) dt\otimes d\mu_f(x,\xi)\\ &\quad + \int_{\Omega\times\mathcal{X}} \theta\left(\int_{\Sigma\times Y} v(T,x,\omega,y) d\lambda_\xi(\omega,y)\right) d\mu_f(x,\xi)\\ &\leqslant \int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} \xi\left(t,x,v(t,x,\omega,y),\dot{v}(t,x,\omega,y)\right) dt\otimes d\mu_f(x,\xi)\otimes d\lambda_\xi(\omega,y)\\ &\quad + \int_{\Omega\times\mathcal{X}\times\Sigma\times Y} \theta\left(v(T,x,\omega,y)\right) d\mu_f(x,\xi)\otimes d\lambda_\xi(\omega,y),\end{aligned}$$

hence, by using (4.7), (4.21), we obtain

$$\begin{aligned}\phi_{\mu_f}(U) &\leqslant \int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} \xi\left(t,x,v(t,x,\omega,y),\dot{v}(t,x,\omega,y)\right) dt\otimes dx\otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi)\otimes d\lambda_\xi(\omega,y)\\ &\quad + \int_{\Omega\times\mathcal{X}\times\Sigma\times Y} \theta\left(v(T,x,\omega,y)\right) dx\otimes d\mathbb{E}_{f_Y^\sharp dy}(\xi)\otimes d\lambda_\xi(\omega,y)\\ &\leqslant \int_{]0,T[\times\Omega\times\mathcal{X}\times\Sigma\times Y} \xi\left(t,x,v(t,x,\omega,y),\dot{v}(t,x,\omega,y)\right) dt\otimes dx\otimes d\lambda(\xi,\omega,y)\\ &\quad + \int_{\Omega\times\mathcal{X}\times\Sigma\times Y} \theta\left(v(T,x,\omega,y)\right) dx\otimes d\lambda(\xi,\omega,y),\end{aligned}$$

and consequently, from (4.20) we conclude that

$$\begin{aligned}\phi_{\mu_f}(U) \;\leqslant\; & \int_{]0,T[\times\Omega\times\Sigma\times Y} f(\omega,y)\,(t,x,v(t,x,\omega,y),\dot v(t,x,\omega,y))\,dt\otimes dx\otimes d\mathbb{P}(\omega)\otimes dy\\ & + \int_{\Omega\times\Sigma\times Y}\theta\,(v(T,x,\omega,y))\,dx\otimes d\mathbb{P}(\omega)\otimes dy\\ = \; & \int_{\Sigma\times Y}\Phi_{f(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy.\end{aligned}$$

We have thus prove that for each $v\in\mathcal{S}_{\rm hom}(u)$ there exists $U\in\mathcal{S}_{\mu_f}(u)$ such that

$$\phi_{\mu_f}(U)\leqslant\int_{\Sigma\times Y}\Phi_{f(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy,$$

which implies (4.15). ■

This finishes the proof of Theorem 4.2. ■

4.2. **Non-local stochastic homogenization in the setting of a Poisson point process.** Let $D:\Sigma\rightrightarrows\mathbb{R}^N$ be a $(\mathcal{A},\mathcal{B}(\mathbb{R}^N))$-measurable multifunction such that for every $\omega\in\Sigma$, the set $D(\omega)$ is countable and without cluster point and let $N:\Sigma\times\mathcal{B}(\mathbb{R}^N)\to\mathbb{N}\cup\{\infty\}$ be defined by

$$N(\omega,B):=\sum_{z\in D(\omega)}\delta_z(B)=\mathrm{card}(D(\omega)\cap B),$$

where δ_z denotes the Dirac measure at the point $z\in\mathbb{R}^N$. (Note that for each $\omega\in\Sigma$, $N(\omega,\cdot)$ is a counting measure.) From now on, we assume that $\{N(\cdot,B)\}_{B\in\mathcal{B}(\mathbb{R}^N)}$ is a Poisson point process with intensity $\lambda>0$, i.e.,

- for every bounded set $B\in\mathcal{B}(\mathbb{R}^N)$ and every $k\in\mathbb{N}$, one has
$$\mathbb{P}\big([N(\cdot,B)=k]\big)=|B|^k\lambda^k\frac{\mathrm{e}^{-\lambda|B|}}{k!}\tag{4.28}$$
where $[N(\cdot,B)=k]:=\{\omega\in\Sigma:N(\omega,B)=k\}$ and $|B|$ denotes the Lebesgue measure of B;
- for every disjoint and bounded sets $A,B\in\mathcal{B}(\mathbb{R}^N)$, $N(\cdot,A)$ and $N(\cdot,B)$ are independant.

Fix $r>0$ and $g,h\in\mathcal{X}$ and consider the $(\mathcal{A},\mathcal{B}:=\otimes_{\mathbb{R}^N}\mathcal{B}(\mathcal{X}))$-measurable function $f_{\rm p}:\Sigma\to\mathfrak{X}:=\mathcal{X}^{\mathbb{R}^N}$ given by

$$\begin{aligned}f_{\rm p}(\omega,x) \;:=\; & \begin{cases} g & \text{if } x\in\underset{z\in D(\omega)}{\cup}B_r(z)\\ h & \text{otherwise}\end{cases}\\ = \; & h+(g-h)\min\Big\{1,N(\omega,B_r(x))\Big\}\\ = \; & \begin{cases} g & \text{if } N(\omega,B_r(x))\geqslant 1\\ h & \text{if } N(\omega,B_r(x))=0.\end{cases}\end{aligned}\tag{4.29}$$

Then, $f_{\rm p}$ is covariant with respect to the dynamical system $(\Sigma,\mathcal{A},\mathbb{P},(T_z)_{z\in\mathbb{Z}^N})$, i.e., (4.1) hols with $f=f_{\rm p}$, and the law of $f_{\rm p}$ is invariant under the group $(\tau_z)_{z\in\mathbb{Z}^N}$, i.e., (4.3) holds with

$f = f_{\rm p}$ and $(\tau_z)_{z\in\mathbb{Z}^N}$ given by (4.2), and moreover the dynamical system $(\mathfrak{X}, \mathcal{B}, f_{\rm p}^\sharp\mathbb{P}, (\tau_z)_{z\in\mathbb{Z}^N})$ is ergodic, i.e., (4.4) holds with $f = f_{\rm p}$ (see [MM94, §5]). (The ergodic dynamical system $(\mathfrak{X}, \mathcal{B}, f_{\rm p}^\sharp\mathbb{P}, (\tau_z)_{z\in\mathbb{Z}^N})$ models environments whose heterogeneities are independently distributed with a frequency λ.)
For each $\omega \in \Sigma$ and each $\varepsilon > 0$, we consider $F_\varepsilon^{\rm p}(\omega) : \mathcal{H}_0^1 \to \mathbb{R}$ given by (4.5) with $f = f_{\rm p}$, i.e.,

$$F_\varepsilon^{\rm p}(\omega)(u) := \Phi_{f_{\rm p}(\omega,\frac{\cdot}{\varepsilon})}(u) = \int_{]0,T[\times\Omega} f_{\rm p}\left(\omega, \frac{x}{\varepsilon}\right)(t, x, u(t, x), \dot{u}(t, x))dt \otimes dx + \int_\Omega \theta(u(T, x))dx.$$

Set $\sigma := \mathbb{P}([N(\cdot, B_r(0)) \geqslant 1]) = 1 - {\rm e}^{-\lambda|B_r(0)|}$ and consider the integral functionals $G, H : \mathcal{H}_0^1 \to \mathbb{R}$ given by:

$$G(u) := \sigma\left[\int_{]0,T[\times\Omega} g\left(t, x, \frac{u(t, x)}{\sigma}, \frac{\dot{u}(t, x)}{\sigma}\right)dt \otimes dx + \int_\Omega \theta\left(\frac{u(T, x)}{\sigma}\right)dx\right]; \tag{4.30}$$

$$H(u) := (1 - \sigma)\left[\int_{]0,T[\times\Omega} h\left(t, x, \frac{u(t, x)}{1-\sigma}, \frac{\dot{u}(t, x)}{1-\sigma}\right)dt \otimes dx + \int_\Omega \theta\left(\frac{u(T, x)}{1-\sigma}\right)dx\right]. \tag{4.31}$$

Let $G\square H : \mathcal{H}_0^1 \to \mathbb{R}$ be the inf-convolution of G and H, i.e.,

$$G\square H(u) := \inf\Big\{G(u_1) + H(u_2) : u_1, u_2 \in \mathcal{H}_0^1 \text{ and } u_1 + u_2 = u\Big\}.$$

The following result is a consequence of Theorem 4.2.

Theorem 4.3. *For $\mathbb{P}$-a.e. $\omega \in \Sigma$ one has*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon^{\rm p}(\omega) = G\square H.$$

Proof of Theorem 4.3. Applying Theorem 4.2 with $f = f_{\rm p}$ we deduce that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon^{\rm p}(\omega) = F_{\rm hom}$$

with $F_{\rm hom} : \mathcal{H}_0^1 \to \mathbb{R}$ given by (4.11)-(4.12) with $f = f_{\rm p}$. So, it remains to prove that $F_{\rm hom} = G\square H$, i.e., for every $u \in \mathcal{H}_0^1$, one has:

$$F_{\rm hom}(u) \geqslant G\square H(u); \tag{4.32}$$

$$F_{\rm hom}(u) \leqslant G\square H(u). \tag{4.33}$$

Let us fix $u \in \mathcal{H}_0^1$ and let us set:

$$\big[N(\cdot, B_r(\cdot)) \geqslant 1\big] := \Big\{(\omega, y) \in \Sigma \times Y : N(\omega, B_r(y)) \geqslant 1\Big\};$$

$$\big[N(\cdot, B_r(\cdot)) = 0\big] := \Big\{(\omega, y) \in \Sigma \times Y : N(\omega, B_r(y)) = 0\Big\}.$$

Taking (4.29) into account, as $[N(\cdot, B_r(\cdot)) \geqslant 1]$ and $[N(\cdot, B_r(\cdot)) = 0]$ are disjoints and $\Sigma \times Y = [N(\cdot, B_r(\cdot)) \geqslant 1] \cup [N(\cdot, B_r(\cdot)) = 0]$, for any $v \in \mathcal{S}_{\rm hom}(u)$, one has

$$\begin{aligned}\int_{\Sigma\times Y} \Phi_{f_{\rm p}(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy &= \int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_{f_{\rm p}(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \\ &\quad + \int_{[N(\cdot,B_r(\cdot))=0]} \Phi_{f_{\rm p}(\omega,y)}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \\ &= \int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_{g}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \\ &\quad + \int_{[N(\cdot,B_r(\cdot))=0]} \Phi_{h}(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy.\end{aligned}$$

Hence

$$F_{\rm hom}(u) = \inf_{v\in\mathcal{S}_{\rm hom}(u)} \left\{ \int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_g(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy + \int_{[N(\cdot,B_r(\cdot))=0]} \Phi_h(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \right\}. \quad (4.34)$$

Fix any $v \in \mathcal{S}_{\rm hom}(u)$. Taking (4.28) into account we see that

$$\left\{ \begin{array}{l} \mathbb{P}\otimes dy([N(\cdot, B_r(\cdot)) \geqslant 1]) = \mathbb{P}([N(\cdot, B_r(0) \geqslant 1]) = \sigma, \\ \text{and so } \mathbb{P}\otimes dy([N(\cdot, B_r(\cdot)) = 0]) = 1-\sigma. \end{array} \right. \quad (4.35)$$

Hence, by using Jensen's inequality, one has

$$\begin{aligned}\int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_g(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy &= \sigma \int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_g(v(\cdot,\cdot,\omega,y))\frac{1}{\sigma}d\mathbb{P}(\omega)\otimes dy \\ &\geqslant \sigma\Phi_g\left(\int_{[N(\cdot,B_r(\cdot))\geqslant 1]} v(\cdot,\cdot,\omega,y)\frac{1}{\sigma}d\mathbb{P}(\omega)\otimes dy\right) \\ &= \sigma\Phi_g\left(\frac{1}{\sigma}\int_{[N(\cdot,B_r(\cdot))\geqslant 1]} v(\cdot,\cdot,\omega,y)d\mathbb{P}(\omega)\otimes dy\right) \quad (4.36)\end{aligned}$$

and, in the same way,

$$\int_{[N(\cdot,B_r(\cdot))=0]} \Phi_h(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \geqslant (1-\sigma)\Phi_h\left(\frac{1}{1-\sigma}\int_{[N(\cdot,B_r(\cdot))=0]} v(\cdot,\cdot,\omega,y)d\mathbb{P}(\omega)\otimes dy\right). \quad (4.37)$$

Set

$$\left\{ \begin{array}{l} u_1(\cdot,\cdot) := \displaystyle\int_{[N(\cdot,B_r(\cdot))\geqslant 1]} v(\cdot,\cdot,\omega,y)d\mathbb{P}(\omega)\otimes dy; \\ \\ u_2(\cdot,\cdot) := \displaystyle\int_{[N(\cdot,B_r(\cdot))=0]} v(\cdot,\cdot,\omega,y)d\mathbb{P}(\omega)\otimes dy. \end{array} \right.$$

Then, $u_1, u_2 \in \mathcal{H}_0^1$ and $u_1 + u_2 = u$. Moreover, according to (4.11)-(4.12), from (4.30), (4.31), (4.36) and (4.37) we deduce that:

$$\begin{aligned}\int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_g(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy &\geqslant G(u_1);\\ \int_{[N(\cdot,B_r(\cdot))=0]} \Phi_h(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy &\geqslant H(u_2).\end{aligned}$$

Consequently, one has

$$\int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_g(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy + \int_{[N(\cdot,B_r(\cdot))=0]} \Phi_h(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy \geqslant G\square H(u)$$

for all $v \in \mathcal{S}_{\rm hom}(u)$, and (4.32) follows by taking (4.34) into account.
Conversely, for any $u_1, u_2 \in \mathcal{H}_0^1$ such that $u_1 + u_2 = u$, set

$$v(\cdot,\cdot,\omega,y) := \begin{cases} \dfrac{u_1(\cdot,\cdot)}{\sigma} & \text{if } N(\omega, B_r(y)) \geqslant 1\\[2ex] \dfrac{u_2(\cdot,\cdot)}{1-\sigma} & \text{if } N(\omega, B_r(y)) = 0.\end{cases}$$

Then, $v \in \mathcal{S}_{\rm hom}(u)$ and, from (4.12), (4.30), (4.31), (4.34) and (4.35), we see that

$$\begin{aligned}F_{\rm hom}(u) &\leqslant \int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \Phi_g(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy + \int_{[N(\cdot,B_r(\cdot))=0]} \Phi_h(v(\cdot,\cdot,\omega,y))d\mathbb{P}(\omega)\otimes dy\\ &= \int_{[N(\cdot,B_r(\cdot))\geqslant 1]} \frac{1}{\sigma}G(u_1)d\mathbb{P}(\omega)\otimes dy + \int_{[N(\cdot,B_r(\cdot))=0]} \frac{1}{1-\sigma}H(u_2)d\mathbb{P}(\omega)\otimes dy\\ &= \mathbb{P}\otimes dy([N(\cdot,B_r(\cdot))\geqslant 1])\frac{1}{\sigma}G(u_1) + \mathbb{P}\otimes dy([N(\cdot,B_r(\cdot))=0])\frac{1}{1-\sigma}H(u_2)\\ &= G(u_1) + H(u_2),\end{aligned}$$

which gives (4.33), and the proof of Theorem 4.3 is complete. ■

Assume futhermore that:

- (P_1) for $dt\otimes dx$-a.e. $(t,x) \in]0,T[\times\Omega$, the functions $g(t,x,\cdot,\cdot)$ and $h(t,x,\cdot,\cdot)$ are strictly convex, $g(t,x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ and $h(t,x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$;
- (P_2) the function θ is strictly convex and $\theta \in C^1(\mathbb{R})$.

The following result is somehow a version of Theorem 4.3 in terms of weak convergence in H^1 and Euler-Lagrange's equations.

Theorem 4.4. *Assume that* (P_1) *and* (P_2) *hold and, for each* $\omega \in \Sigma$ *and each* $\varepsilon > 0$, *let* $u_\varepsilon(\omega)$ *be the minimizer of* $F_\varepsilon^{\rm p}(\omega)$. *Then, for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$ *one has*

$$u_\varepsilon(\omega) \xrightarrow{H^1} u,$$

where $u \in \mathcal{H}_0^1$ is the minimizer of $G\square H$. Moreover, u minimizes $G\square H$ if and only if there exist $v, w \in \mathcal{H}_0^1$ such that $u = v + w$ with v and w satisfying the following differential system:

$$(\mathrm{S}_{g,h}) \begin{cases} \dfrac{\partial g}{\partial s}\left(t,x,\dfrac{v(t,x)}{\sigma},\dfrac{\dot{v}(t,x)}{\sigma}\right) - \dfrac{\partial}{\partial t}\left(\dfrac{\partial g}{\partial \dot{s}}\left(t,x,\dfrac{v(t,x)}{\sigma},\dfrac{\dot{v}(t,x)}{\sigma}\right)\right) = 0 \ dt\otimes dx\text{-a.e. in }]0,T[\times\Omega \\ \dfrac{\partial h}{\partial s}\left(t,x,\dfrac{w(t,x)}{1-\sigma},\dfrac{\dot{w}(t,x)}{1-\sigma}\right) - \dfrac{\partial}{\partial t}\left(\dfrac{\partial h}{\partial \dot{s}}\left(t,x,\dfrac{w(t,x)}{1-\sigma},\dfrac{\dot{w}(t,x)}{1-\sigma}\right)\right) = 0 \ dt\otimes dx\text{-a.e. in }]0,T[\times\Omega \end{cases}$$

subjected to the time-boundary conditions:

$$(\partial\mathrm{S}_{g,h}) \begin{cases} \dfrac{d\theta}{ds}\left(\dfrac{v(T,x)}{\sigma}\right) + \sigma\dfrac{\partial g}{\partial \dot{s}}\left(T,x,\dfrac{v(T,x)}{\sigma},\dfrac{\dot{v}(T,x)}{\sigma}\right) = 0 \quad dx\text{-a.e. in } \Omega \\ \dfrac{d\theta}{ds}\left(\dfrac{w(T,x)}{\sigma}\right) + (1-\sigma)\dfrac{\partial h}{\partial \dot{s}}\left(T,x,\dfrac{w(T,x)}{1-\sigma},\dfrac{\dot{w}(T,x)}{1-\sigma}\right) = 0 \quad dx\text{-a.e. in } \Omega \\ v(0,\cdot) = 0 \\ w(0,\cdot) = 0. \end{cases}$$

Proof of Theorem 4.4. The first part of the proposition is a direct consequence of Theorem 4.3. Let us prove the second part of the proposition. First of all, it is easy to see that G and H are convex, and so is $G\square H$ by Propostion B.2. As $G\square H$ is a Γ-limit (see Theorem 4.3), it follows that $G\square H$ is closed and convex. Consequently, u minimizes $G\square H$ if and only if, one has

$$0 \in \partial(G\square H)(u) \tag{4.38}$$

(see [ABM14, Proposition 9.5.3]). Taking (P_1) and (P_2) into account, we see that G and H are Gâteaux differentiable, and so, from Theorem B.4 we deduce that

$$\partial(G\square H)(u) = DG \mathbin{/\!\!/} DH(u), \tag{4.39}$$

where $DG \mathbin{/\!\!/} DH$ denotes the parallel sum of G and H (see Definition B.3). Consequently, from (4.39) we can assert that (4.38) holds if and only if there exist $v, w \in \mathcal{H}_0^1$ such that $u = v + w$ and

$$\begin{cases} DG(v) = 0, \text{ i.e., } DG(v)(\xi) = 0 \text{ for all } \xi \in \mathcal{H}_0^1 \\ DH(w) = 0, \text{ i.e., } DH(w)(\xi) = 0 \text{ for all } \xi \in \mathcal{H}_0^1. \end{cases} \tag{4.40}$$

By using differential calculus, we see that (4.40) is equivalent to the following integral equation system: for every $\xi \in \mathcal{H}_0^1$, one has

$$(\mathrm{I}) \begin{cases} \displaystyle\int_{]0,T[\times\Omega} \left[\frac{\partial g}{\partial s}\left(t,x,\frac{v(t,x)}{\sigma},\frac{\dot{v}(t,x)}{\sigma}\right)\xi(t,x) + \frac{\partial g}{\partial \dot{s}}\left(t,x,\frac{v(t,x)}{\sigma},\frac{\dot{v}(t,x)}{\sigma}\right)\dot{\xi}(t,x)\right] dt\otimes dx \\ \displaystyle + \int_\Omega \frac{d\theta}{ds}\left(\frac{v(T,x)}{\sigma}\right)\dot{\xi}(t,x)dt\otimes dx = 0 \\ \displaystyle\int_{]0,T[\times\Omega} \left[\frac{\partial h}{\partial s}\left(t,x,\frac{w(t,x)}{1-\sigma},\frac{\dot{w}(t,x)}{1-\sigma}\right)\xi(t,x) + \frac{\partial h}{\partial \dot{s}}\left(t,x,\frac{w(t,x)}{\sigma},\frac{\dot{w}(t,x)}{1-\sigma}\right)\dot{\xi}(t,x)\right] dt\otimes dx \\ \displaystyle + \int_\Omega \frac{d\theta}{ds}\left(\frac{w(T,x)}{1-\sigma}\right)\dot{\xi}(t,x)dt\otimes dx = 0. \end{cases}$$

Finally, in one hand, taking $\xi(t,x) = a(t)b(x)$ with $a \in C_c^\infty(]0,T[)$ and $b \in C_c^\infty(\Omega)$, and integrating by part with respect to the time variable, from (I) we obtain $(\mathrm{S}_{g,h})$. On the other

hand, taking $\xi(t,x) = a(t)b(x)$ with $a(t) = (\frac{t}{T})^n$ and $b \in C_c^\infty(\Omega)$, integrating by part with respect to the time variable and letting $n \to \infty$, from (I) we obtain $(\partial \mathrm{S}_{g,h})$. ■

4.3. **Non-local stochastic homogenization of non-diffusive reaction equations.** Let $\psi : \Sigma \times [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a $(\mathcal{A} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable function such that:

- (H_0) $\psi(T_z\omega, \cdot, x, \cdot) = \psi(\omega, \cdot, x+z, \cdot)$ for $\mathbb{P}$-a.a. $\omega \in \Sigma$, all $x \in \mathbb{R}^N$ and all $z \in \mathbb{Z}^N$;
- (H_1) $\psi(\omega, t, x, \cdot) \in C^1(\mathbb{R})$ for $\mathbb{P} \otimes dt \otimes dx$-a.a. $(\omega, t, x) \in \Sigma \times]0,T[\times \mathbb{R}^N$;
- (H_2) $\psi(\omega, t, x, \cdot)$ is strictly convex for $\mathbb{P} \otimes dt \otimes dx$-a.a. $(\omega, t, x) \in \Sigma \times]0,T[\times \mathbb{R}^N$;
- (H_3) there exist $c_1, C_1 > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has
$$c_1(|s|^2 - 1) \leqslant \psi(\omega, \cdot, \cdot, s) \leqslant C_1(|s|^2 + 1)$$
for all $s \in \mathbb{R}$;
- (H_4) there exists $\delta : [0,\infty[\to [0,\infty[$ with $\lim_{r \to 0} \delta(r) = 0$ such that for $\mathbb{P} \otimes dx$-a.e. $(\omega, x) \in \Sigma \times \mathbb{R}^N$, one has
$$|\psi(\omega, t_1, \cdot, \cdot) - \psi(\omega, t_2, \cdot, \cdot)| \leqslant \delta(|t_1 - t_2|)$$
for all $t_1, t_2 \in [0,T]$;
- (H_5) there exists $L > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Sigma$, one has
$$\left| \frac{\partial \psi}{\partial s}(\omega, \cdot, \cdot, s_1) - \frac{\partial \psi}{\partial s}(\omega, \cdot, \cdot, s_2) \right| \leqslant L|s_1 - s_2|$$
for all $s_1, s_2 \in \mathbb{R}$,

and, for each $\omega \in \Sigma$, consider the non-diffusive reaction differential equation defined in $H^1(]0,T[; L^2(\Omega;\mathbb{R}))$ by

$$(\mathrm{R}_\varepsilon^\omega) \quad \begin{cases} -\dfrac{\partial u_\varepsilon}{\partial t}(\omega, t, x) = \dfrac{\partial \psi}{\partial s}\left(\omega, t, \dfrac{x}{\varepsilon}, u_\varepsilon(\omega, t, x)\right) \text{ for } dt \otimes dx\text{-a.a. } (t,x) \in]0,T[\times \Omega \\ \\ u_\varepsilon(\omega, 0, \cdot) = 0. \end{cases}$$

Then, as in §2.3, for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and every $\varepsilon > 0$, $(\mathrm{R}_\varepsilon^\omega)$ has a unique solution $u_\varepsilon(\omega, \cdot, \cdot) \in \mathcal{H}_0^1$. Moreover, arguing as in the proof of Proposition 2.11, we can assert that there exist $\alpha, \beta > 0$ and $\gamma : [0,\infty[\to [0,\infty[$ with $\lim_{r \to 0} \gamma(r) = 0$ such that for each $\omega \in \Sigma$, $u_\varepsilon(\omega, \cdot, \cdot)$ minimizes the functional $\Phi_{f_\psi(\omega, \frac{\cdot}{\varepsilon})} : \mathcal{H}_0^1 \to \mathbb{R}$ given by

$$\Phi_{f_\psi(\omega, \frac{\cdot}{\varepsilon})}(u) = \int_{]0,T[\times \Omega} f_\psi\left(\omega, \frac{x}{\varepsilon}\right)\big(t, x, u(t,x), \dot{u}(t,x)\big) dt \otimes dx + \int_\Omega \theta(u(T,x)) dx$$

with $f_\psi(\omega, \frac{\cdot}{\varepsilon}) : \mathbb{R}^N \to \mathcal{M}_{\alpha,\beta,\gamma}$ (here $\mathcal{X} = \mathcal{M}_{\alpha,\beta,\gamma}$) defined by

$$f_\psi\left(\omega, \frac{x}{\varepsilon}\right)(t, y, s, \dot{s}) := \psi\left(\omega, t, \frac{x}{\varepsilon}, s\right) + \psi^*\left(\omega, t, \frac{x}{\varepsilon}, -\dot{s}\right) \tag{4.41}$$

and $\theta : \mathbb{R} \to [0,\infty[$ given by (2.9), where ψ^* denotes the Legendre-Fenchel conjugate of ψ with respect to the fourth variable. From (H_0) we deduce that f_ψ is covariant with respect to the dynamical system $(\Sigma, \mathcal{A}, \mathbb{P}, (T_z)_{z \in \mathbb{Z}^N})$, i.e., (4.1) hols with $f = f_\psi$, and the law of f_ψ is invariant under the group $(\tau_z)_{z \in \mathbb{Z}^N}$, i.e., (4.3) holds with $f = f_\psi$ and $(\tau_z)_{z \in \mathbb{Z}^N}$ given by (4.2), and moreover the dynamical system $(\mathfrak{X}, \mathcal{B}, f_\psi^\sharp \mathbb{P}, (\tau_z)_{z \in \mathbb{Z}^N})$ is ergodic, i.e., (4.4) holds

with $f = f_\psi$. On the other hand, from (H_2) we see that f_ψ is strictly convex with respect to $(s, \dot{s})$. Hence F_{hom} defined by (4.11)-(4.12), with $f = f_\psi$ and θ given by (2.9), has a unique minimizer $u \in \mathcal{H}^1_0$. So, as a direct consequence of Theorem 4.2 we obtain the following result.

Corollary 4.5. *Assume that* (H_0), (H_1), (H_2), (H_3), (H_4) *and* (H_5) *are satisfied. Then, for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, one has*

$$u_\varepsilon(\omega, \cdot, \cdot) \xrightarrow{H^1} u.$$

In the setting of a Poisson point process, we can precise the (non-local) equations satisfied by u making clear the non-local effects induced by homogenization of stochastic non-difusive reaction differential equations of type $(\mathrm{R}^\omega_\varepsilon)$ (see Theorem 4.6(iv)).

4.3.1. *The setting of a Poisson point process.* Let $\psi_1, \psi_2 : [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be two $(\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable functions such that:

- (E_1) for each $i \in \{1,2\}$, $\psi_i(t,x,\cdot), \psi_i^*(t,x,\cdot) \in C^1(\mathbb{R})$ for $dt \otimes dx$-a.a. $(t,x) \in \Sigma \times]0,T[\times \mathbb{R}^N$;
- (E_2) for each $i \in \{1,2\}$, $\psi_i(t,x,\cdot)$ is strictly convex for $dt \otimes dx$-a.a. $(t,x) \in \Sigma \times]0,T[\times \mathbb{R}^N$;
- (E_3) there exist $c_1, C_1 > 0$ such that for each $i \in \{1,2\}$, one has
$$c_1(|s|^2 - 1) \leqslant \psi_i(\cdot,\cdot,s) \leqslant C_1(|s|^2 + 1)$$
for all $s \in \mathbb{R}$;
- (E_4) there exists $\delta : [0,\infty[\to [0,\infty[$ with $\lim_{r \to 0} \delta(r) = 0$ such that for each $i \in \{1,2\}$, one has
$$|\psi_i(t_1,\cdot,\cdot) - \psi_i(t_2,\cdot,\cdot)| \leqslant \delta(|t_1 - t_2|)$$
for all $t_1, t_2 \in [0,T]$;
- (E_5) there exists $L > 0$ such that for each $i \in \{1,2\}$, one has
$$\left| \frac{\partial \psi_i}{\partial s}(\cdot,\cdot,s_1) - \frac{\partial \psi_i}{\partial s}(\cdot,\cdot,s_2) \right| \leqslant L|s_1 - s_2|$$
for all $s_1, s_2 \in \mathbb{R}$,

and let $\psi_{\mathrm{p}} : \Sigma \times [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned}
\psi_{\mathrm{p}}(\omega,t,x,s) &:= \begin{cases} \psi_1(t,x,s) & \text{if } x \in \underset{z \in D(\omega)}{\cup} B_r(z) \\ \psi_2(t,x,s) & \text{otherwise} \end{cases} \\
&= \psi_2(t,x,s) + (\psi_1(t,x,s) - \psi_2(t,x,s)) \min\Big\{1, N(\omega, B_r(x))\Big\} \\
&= \begin{cases} \psi_1(t,x,s) & \text{if } N(\omega,B_r(x)) \geqslant 1 \\ \psi_2(t,x,s) & \text{if } N(\omega,B_r(x)) = 0. \end{cases}
\end{aligned}$$

Then, (H_0), (H_1), (H_2), (H_3), (H_4) and (H_5) are satisfied with $\psi = \psi_{\mathrm{p}}$.
For each $\omega \in \Sigma$ and each $\varepsilon > 0$, let $F^{\psi_{\mathrm{p}}}_\varepsilon(\omega) : \mathcal{H}^1_0 \to \mathbb{R}$ given by (4.5) with $f = f_{\psi_{\mathrm{p}}}$ and θ given by (2.9), i.e.,

$$F^{\psi_{\mathrm{p}}}_\varepsilon(\omega)(u) := \Phi_{f_{\psi_{\mathrm{p}}}(\omega, \frac{\cdot}{\varepsilon})}(u) = \int_{]0,T[\times\Omega} f_{\psi_{\mathrm{p}}}\left(\omega, \frac{x}{\varepsilon}\right)(t,x,u(t,x),\dot{u}(t,x)) dt \otimes dx + \frac{1}{2}\int_\Omega \big|u(T,x))\big|^2 dx$$

with $f_{\psi_{\mathrm{p}}}$ given by (4.41) with $\psi = \psi_{\mathrm{p}}$. As in §4.2, set $\sigma := \mathbb{P}([N(\cdot, B_r(0)) \geqslant 1]) = 1 - \mathrm{e}^{-\lambda |B_r(0)|}$ and consider the integral functionals $\Psi_1, \Psi_2 : \mathcal{H}^1_0 \to \mathbb{R}$ given by:

$$\begin{aligned}\Psi_1(u) &:= \sigma\int_{]0,T[\times\Omega}\left(\psi_1\left(t,x,\frac{u(t,x)}{\sigma}\right)+\psi_1^*\left(t,x,-\frac{\dot{u}(t,x)}{\sigma}\right)\right)dt\otimes dx\\ &\quad+\frac{\sigma}{2}\int_\Omega\left|\frac{u(T,x)}{\sigma}\right|^2dx;\\ \Psi_2(u) &:= (1-\sigma)\int_{]0,T[\times\Omega}\left(\psi_2\left(t,x,\frac{u(t,x)}{1-\sigma}\right)+\psi_2^*\left(t,x,-\frac{\dot{u}(t,x)}{1-\sigma}\right)\right)dt\otimes dx\\ &\quad+\frac{1-\sigma}{2}\int_\Omega\left|\frac{u(T,x)}{1-\sigma}\right|^2dx.\end{aligned}$$

Let $\Psi_1\square\Psi_2:\mathcal{H}_0^1\to\mathbb{R}$ be the inf-convolution of Ψ_1 and Ψ_2, i.e.,

$$\Psi_1\square\Psi_2(u):=\inf\Big\{\Psi_1(u_1)+\Psi_2(u_2):u_1,u_2\in\mathcal{H}_0^1\text{ and }u_1+u_2=u\Big\}.$$

The following result is a consequence of Theorems 4.3 and 4.4 and Corollary 4.5.

Theorem 4.6. *Assume that* (E_1), (E_2), (E_3), (E_4) *and* (E_5) *are satisfied. Then, the following four assertions hold.*

(i) *For* $\mathbb{P}$*-a.e.* $\omega\in\Sigma$ *one has*

$$\Gamma\text{-}\lim_{\varepsilon\to0}F_\varepsilon^{\psi_{\mathrm{P}}}(\omega)=\Psi_1\square\Psi_2.$$

(ii) *For* $\mathbb{P}$*-a.e.* $\omega\in\Sigma$ *and every* $\varepsilon>0$*, let* $u(\omega,\cdot,\cdot)\in\mathcal{H}_0^1$ *be the unique solution of* $(\mathrm{R}_\varepsilon^\omega)$*. Then*

$$u_\varepsilon(\omega,\cdot,\cdot)\xrightarrow{H^1}u,$$

where $u\in\mathcal{H}_0^1$ *is the unique minimizer of* $\Psi_1\square\Psi_2$*.*

(iii) *The function* u *minimizes* $\Psi_1\square\Psi_2$ *if and only if there exist* $v,w\in\mathcal{H}_0^1$ *such that* $u=v+w$ *with* v *and* w *satisfying the following differential system*:

$$(\mathrm{S}_{\psi_{\mathrm{P}}})\begin{cases}\dfrac{\partial\psi_1}{\partial s}\left(t,x,\dfrac{v(t,x)}{\sigma}\right)-\dfrac{\partial}{\partial t}\left(\dfrac{\partial\psi_1^*}{\partial\dot{s}}\left(t,x,-\dfrac{\dot{v}(t,x)}{\sigma}\right)\right)=0\ dt\otimes dx\text{-a.e. in }]0,T[\times\Omega\\ \dfrac{\partial\psi_2}{\partial s}\left(t,x,\dfrac{w(t,x)}{1-\sigma}\right)-\dfrac{\partial}{\partial t}\left(\dfrac{\partial\psi_2^*}{\partial\dot{s}}\left(t,x,-\dfrac{\dot{w}(t,x)}{1-\sigma}\right)\right)=0\ dt\otimes dx\text{-a.e. in }]0,T[\times\Omega\end{cases}$$

subjected to the time-boundary conditions:

$$(\partial\mathrm{S}_{\psi_{\mathrm{P}}})\begin{cases}\dfrac{v(T,x)}{\sigma}-\sigma\dfrac{\partial\psi_1^*}{\partial\dot{s}}\left(T,x,-\dfrac{\dot{v}(T,x)}{\sigma}\right)=0\ \ dx\text{-a.e. in }\Omega\\ \dfrac{w(T,x)}{1-\sigma}-(1-\sigma)\dfrac{\partial\psi_2^*}{\partial\dot{s}}\left(T,x,-\dfrac{\dot{w}(T,x)}{1-\sigma}\right)=0\ \ dx\text{-a.e. in }\Omega\\ v(0,\cdot)=0\\ w(0,\cdot)=0,\end{cases}$$

where ψ_1^* *(resp.* ψ_2^**) denotes the Legendre-Fenchel conjugate of* ψ_1 *(resp.* ψ_2*) with respect to the third variable.*

(iv) *The function u minimizes $\Psi_1 \square \Psi_2$ if and only if there exist $u_1, u_2 \in \mathcal{H}_0^1$ such that $u = \sigma u_1 + (1-\sigma)u_2$ with u_1 and u_2 being the unique solutions of the following integro-differential equations*:

$$\begin{cases} -\dfrac{\partial u_1}{\partial t}(t,x) = \dfrac{\partial \psi_1}{\partial s}\left(t, x, \dfrac{u_1(T,x)}{\sigma} + \displaystyle\int_t^T \dfrac{\partial \psi_1}{\partial s}\left(\tau, x, \dfrac{\partial \psi_1}{\partial s}(\tau, x, u_1(\tau,x))\right) d\tau\right) dt \otimes dx\text{-a.e. in }]0,T[\times\Omega \\ u_1(0,\cdot) = 0 \end{cases}$$

$$\begin{cases} -\dfrac{\partial u_2}{\partial t}(t,x) = \dfrac{\partial \psi_2}{\partial s}\left(t, x, \dfrac{u_2(T,x)}{1-\sigma} + \displaystyle\int_t^T \dfrac{\partial \psi_2}{\partial s}\left(\tau, x, \dfrac{\partial \psi_2}{\partial s}(\tau, x, u_2(\tau,x))\right) d\tau\right) dt \otimes dx\text{-a.e. in }]0,T[\times\Omega \\ u_2(0,\cdot) = 0. \end{cases}$$

Proof of Theorem 4.6. We only need to prove (iv). For this, it suffices to show that $(v,w) \in \mathcal{H}_0^1 \times \mathcal{H}_0^1$ satisfies the differential system $(\mathrm{S}_{\psi_\mathrm{p}})$-$(\partial \mathrm{S}_{\psi_\mathrm{p}})$ if and only $(u_1,u_2) = (\frac{v}{\sigma}, \frac{w}{1-\sigma}) \in \mathcal{H}_0^1 \times \mathcal{H}_0^1$ satisfies the integro-differential equations above. First of all, by integrating over $]t,T[$ the first equation in $(\mathrm{S}_{\psi_\mathrm{p}})$, we see that

$$\frac{\partial \psi_1^*}{\partial \dot{s}}\left(t, x, -\frac{\dot{v}(t,x)}{\sigma}\right) = \frac{\partial \psi_1^*}{\partial \dot{s}}\left(T, x, -\frac{\dot{v}(T,x)}{\sigma}\right) + \int_t^T \frac{\partial \psi_1}{\partial s}\left(\tau, x, \frac{v(\tau,x)}{\sigma}\right) d\tau. \tag{4.42}$$

But $\frac{\partial \psi_1^*}{\partial \dot{s}}(t,x,\cdot) = \left(\frac{\partial \psi_1}{\partial s}\right)^{-1}(t,x,\cdot)$ (see [ABM14, Theorem 9.5.1]), where $\left(\frac{\partial \psi_1}{\partial s}\right)^{-1}$ denotes the inverse of the subdifferential of ψ_1 with respect to s, and so from (4.42) we deduce that

$$-\frac{1}{\sigma}\frac{\partial v}{\partial t}(t,x) = \frac{\partial \psi_1}{\partial s}\left(t, x, \frac{\partial \psi_1^*}{\partial \dot{s}}\left(T, x, -\frac{\dot{v}(T,x)}{\sigma}\right) + \int_t^T \frac{\partial \psi_1}{\partial s}\left(t, x \frac{\partial \psi_1}{\partial s}\left(\tau, x, \frac{v(\tau,x)}{\sigma}\right)\right) d\tau\right).$$

Moreover, according to the first equation in $(\partial \mathrm{S}_{\psi_\mathrm{p}})$, one has

$$\frac{v(T,x)}{\sigma^2} = \frac{\partial \psi_1^*}{\partial \dot{s}}\left(T, x, -\frac{\dot{v}(T,x)}{\sigma}\right).$$

It follows that

$$-\frac{1}{\sigma}\frac{\partial v}{\partial t}(t,x) = \frac{\partial \psi_1}{\partial s}\left(t, x, \frac{v(T,x)}{\sigma^2} + \int_t^T \frac{\partial \psi_1}{\partial s}\left(t, x \frac{\partial \psi_1}{\partial s}\left(\tau, x, \frac{v(\tau,x)}{\sigma}\right)\right) d\tau\right).$$

for $dt \otimes dx$-a.e. $(t,x) \in]0,T[\times\Omega$, and setting $u_1 = \frac{v}{\sigma}$ we obtain the first integro-differential equation. In the same manner, by using the second equations in $(\mathrm{S}_{\psi_\mathrm{p}})$ and $(\partial \mathrm{S}_{\psi_\mathrm{p}})$ and by setting $u_1 = \frac{w}{1-\sigma}$, we obtain the second integro-differential equation.
Conversely, setting $v = \sigma u_1$ and $w = (1-\sigma)u_2$, by derivating each integro-differential equation with respect to the time variable (resp. by letting $t = T$ in each integro-differential equation), we obtain $(\mathrm{S}_{\psi_\mathrm{p}})$ (resp. $(\partial \mathrm{S}_{\psi_\mathrm{p}})$). ■

Appendix A. Young measures

Let $(\mathcal{V}, d)$ be a polish space, i.e., $(\mathcal{V}, d)$ is a separable and complete metric space, let $k \geqslant 1$ be an integer and let $\mathcal{O} \subset \mathbb{R}^k$ be a bounded domain. Let $\mathcal{Y}(\mathcal{O}; \mathcal{V})$ be the class of Young measures on $\mathcal{O} \times \mathcal{V}$ and let $C\mathrm{th}^b(\mathcal{O}; \mathcal{V})$ be the space of all bounded Carathéodory integrands on $\mathcal{O} \times \mathcal{V}$.

Definition A.1 (narrow convergence)**.** Let $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\mathcal{O};\mathcal{V})$ and let $\mu \in \mathcal{Y}(\mathcal{O};\mathcal{V})$. We say that $\{\mu_\varepsilon\}_\varepsilon$ narrow converges to μ as $\varepsilon \to 0$, and we write $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$, if

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}\times\mathcal{V}} \psi(y,\xi)d\mu_\varepsilon(y,\xi) = \int_{\mathcal{O}\times\mathcal{V}} \psi(y,\xi)d\mu(y,\xi) \quad \text{for all } \psi \in C\text{th}^b(\mathcal{O};\mathcal{V}).$$

Definition A.2 (tightness)**.** We say that $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\mathcal{O};\mathcal{V})$ is tight if for every $\eta > 0$, there exists a compact set $K \subset \mathcal{V}$ such that $\sup_{\varepsilon>0} \mu_\varepsilon\big(\mathcal{O}\times(\mathcal{V}\backslash K)\big) < \eta$.

A proof of the following compactness result can be found in [Val90, Theorem 11] (see also [Val94, Theorem 7 and Comments 1), 2) and 3)]).

Theorem A.3 (Prokhorov's compactness theorem)**.** *If $\{\mu_\varepsilon\} \subset \mathcal{Y}(\mathcal{O};\mathcal{V})$ is tight then there exists $\mu \in \mathcal{Y}(\mathcal{O};\mathcal{V})$ such that, up to a subsequence, $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$.*

For a proof of the following theorem we also refer to [Val90, Val94].

Theorem A.4 (lower semicontinuity, continuity)**.** *Let $\psi : \mathcal{O}\times\mathcal{V} \to [0,\infty]$ be a $\mathcal{B}(\mathcal{O})\times\mathcal{B}(\mathcal{V})$-measurable function such that $\psi(\cdot,y)$ is lower semicontinuous for all $y \in \mathcal{O}$ and let $\Psi : \mathcal{Y}(\mathcal{O};\mathcal{V}) \to [0,\infty]$ be defined by*

$$\Psi(\mu) := \int_{\mathcal{O}\times\mathcal{V}} \psi(y,\xi)d\mu(y,\xi).$$

(i) *The functional Ψ is lower semicontinuous, i.e., for every $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\mathcal{O};\mathcal{V})$ and every $\mu \in \mathcal{Y}(\mathcal{O};\mathcal{V})$, if $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$ then*

$$\underline{\lim_{\varepsilon\to 0}}\, \Psi(\mu_\varepsilon) \geqslant \Psi(\mu).$$

(ii) *If $\psi(\cdot,y)$ is continuous for all $y \in \mathcal{O}$ then for every $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\mathcal{O};\mathcal{V})$ and every $\mu \in \mathcal{Y}(\mathcal{O};\mathcal{V})$ such that $\mu_\varepsilon \xrightarrow{\text{nar}} \mu$ and*

$$\lim_{M\to\infty}\left(\sup_{\varepsilon>0}\int_{[\psi\geqslant M]} |\psi(y,\xi)|d\mu_\varepsilon(y,\xi)\right) = 0,$$

one has

$$\lim_{\varepsilon\to 0}\Psi(\mu_\varepsilon) = \Psi(\mu).$$

Let $(\mathcal{W},\delta)$ be a polish space. A proof of the following theorem can be found in [Tor77] (see also [ABM14, Theorem 4.2.4]).

Theorem A.5 (desintegration)**.** *Let μ be a Borel measure on $\mathcal{V}\times\mathcal{W}$ and let $\nu := \pi^\sharp_{\mathcal{V}}\mu$ where $\pi_{\mathcal{V}} : \mathcal{V}\times\mathcal{W} \to \mathcal{V}$ denotes the canonical projection from $\mathcal{V}\times\mathcal{W}$ to $\mathcal{V}$. Then, there exists a unique (up to the equality a.e.) family $\{\mu_\xi\}_{\xi\in\mathcal{V}}$ of probability measures on $\mathcal{W}$ such that for every $\psi \in L^1_\mu(\mathcal{V}\times\mathcal{W})$, one has*:

(i) *the function $\xi \mapsto \displaystyle\int_{\mathcal{W}} \psi(\xi,\zeta)d\mu_\xi(\zeta)$ is ν-measurable*;

(ii) $\displaystyle\int_{\mathcal{V}\times\mathcal{W}} \varphi(\xi,\zeta)d\mu(\xi,\zeta) = \int_{\mathcal{V}}\left(\int_{\mathcal{W}} \psi(\xi,\zeta)d\mu_\xi(\zeta)\right)d\nu(\xi),$

The family $\{\mu_\xi\}_{\xi\in\mathcal{V}}$ of probability measures on $\mathcal{W}$ is called the desintegration of μ on the product $\mathcal{V}\times\mathcal{W}$. To summarize it, we write $\mu = \nu\otimes\mu_\xi$.

Appendix B. Inf-convolution and parallel sum

Let E be a linear space.

Definition B.1 (inf-convolution)**.** Let $G, H : E \to]-\infty, \infty]$. By the inf-convolution of G and H we mean the function $G\square H : E \to [-\infty, \infty]$ defined by

$$G\square H(u) \;:=\; \inf\Big\{G(u_1) + H(u_2) : u_1, u_2 \in E \text{ and } u_1 + u_2 = u\Big\}.$$

A proof of the following proposition can be found in [ABM14, Proposition 9.2.2].

Proposition B.2. *Let $G, H : E \to]-\infty, \infty]$. If G and H are convex then $G\square H$ is convex.*

In what follows, E is a Banach space and E' is its dual.

Definition B.3 (parallel sum)**.** Let $\Gamma, \Lambda : E \rightrightarrows E'$ be two multifunctions. By the parallel sum of Γ and Λ we mean the multifunction $\Gamma /\!\!/ \Lambda : E \rightrightarrows E'$ defined by

$$\Gamma /\!\!/ \Lambda(u) := \Big\{v \in E' : \exists (u_1, u_2) \in E \times E \text{ such that } u = u_1 + u_2 \text{ and } v \in \Gamma(u_1) \cap \Lambda(u_2)\Big\}.$$

For a proof of the following theorem, we refer to [Str94, Theorem 3.7] (see also [Str96]).

Theorem B.4. *Assume that E is reflexive, $G : E \to]-\infty, \infty]$ is convex, proper and lower semicontinuous and $H : E \to]-\infty, \infty[$ is convex and Gâteaux differentiable. Then $G\square H$ is Gâteaux differentiable and $D(G\square H) = \partial G /\!\!/ DH$.*

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