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1 Multistage Matchings

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14 — Abstract —

15 We consider a *multistage* version of the PERFECT MATCHING problem which models the
16 scenario where the costs of edges change over time and we seek to obtain a solution that achieves
17 low total cost, while minimizing the number of changes from one instance to the next. Formally,
18 we are given a sequence of edge-weighted graphs on the same set of vertices V , and are asked to
19 produce a perfect matching in each instance so that the total edge cost plus the transition cost
20 (the cost of exchanging edges), is minimized. This model was introduced by Gupta et al. (ICALP
21 2014), who posed as an open problem its approximability for bipartite instances. We completely
22 resolve this question by showing that *Minimum Multistage Perfect Matching* (MIN-MPM) does
23 not admit an $n^{1-\epsilon}$ -approximation, even on bipartite instances with only two time steps.

24 Motivated by this negative result, we go on to consider two variations of the problem. In
25 *Metric Minimum Multistage Perfect Matching* problem (METRIC-MIN-MPM) we are promised
26 that edge weights in each time step satisfy the triangle inequality. We show that this problem
27 admits a 3-approximation when the number of time steps is 2 or 3. On the other hand, we
28 show that even the metric case is APX-hard already for 2 time steps. We then consider the
29 complementary maximization version of the problem, *Maximum Multistage Perfect Matching*
30 problem (MAX-MPM), where we seek to maximize the total profit of all selected edges plus the
31 total number of non-exchanged edges. We show that MAX-MPM is also APX-hard, but admits
32 a constant factor approximation algorithm for any number of time steps.

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41 **1 Introduction**

42 In classical Combinatorial Optimization, given an instance of a problem the goal is to find a
 43 solution optimizing the value of the objective function. However, in many applications the
 44 instance may change over time and the goal is to find a tradeoff between the quality of the
 45 solution in each time step and the stability of the solution in consecutive time steps. As
 46 an example, consider an instance of an assignment problem, where the goal is to compute
 47 the best assignment of tasks to workers, assuming that we know the cost c_{ij} of performing
 48 task j by worker i . In the classical setting, it is possible to choose the assignment that
 49 minimizes the total cost in polynomial time. When the costs change over time (as for
 50 instance when a worker is not able to do some long task on a very busy day (infinite cost))
 51 the optimal solutions of each time step may differ, inducing a transition cost for setting new
 52 task-worker pairs between two consecutive solutions. Hence, the naïve approach of finding a
 53 new optimal solution in each time step has the drawback that it does not take care of the
 54 penalty (transition cost) that is induced by the changes in the solution.

55 In this paper we study a multistage version of the *Perfect Matching problem* that follows
 56 this motivation and was originally introduced by Gupta, Talwar, and Wieder [11]. In this
 57 problem we are given a time horizon: $t = 1, 2, \dots, T$ where for each time t we are given an
 58 instance G_t of *Perfect Matching* (that is, an edge-weighted graph) on the same set of vertices
 59 V . The goal is to determine a sequence of solutions $S = (M_1, M_2, \dots, M_T)$ that both (1) are
 60 near-optimal (*quality*), and (2) induce small transition costs (*stability*). In other words, the
 61 goal is to determine a sequence of perfect matchings, one for each stage (time step) t , such
 62 that their total cost is small and the solution does not change too radically from one step to
 63 the next.

64 It was shown in [11] that this multistage problem is significantly harder than classical
 65 *Perfect Matching*. In fact, it is NP-hard to even approximate the optimal solution within
 66 $n^{1-\epsilon}$, for instances with only 8 times steps. Gupta et al. then posed as an explicit question
 67 whether the problem becomes easier for bipartite instances. Their work suggests also the
 68 question whether this hardness also applies for fewer than 8 steps. The bipartite restriction
 69 is especially interesting because Gupta et al. showed that related matroid-based optimization
 70 problems remain tractable for $T = 2$, and bipartite *Perfect Matching* can be seen as a matroid
 71 intersection problem. One could therefore hope that the matroid structure might make the
 72 bipartite case tractable for some small values of T , or at least approximable.

73 Our main contribution in this paper is to settle this question from [11] in the negative:
 74 we show that *Minimum Multistage Perfect Matching* (MIN-MPM) is $n^{1-\epsilon}$ -inapproximable,
 75 even for $T = 2$ time steps, unless $P = NP$. Motivated by this very negative result, we
 76 then investigate two other version of the problem: the *Metric Minimum Multistage Perfect*
 77 *Matching* problem (METRIC-MIN-MPM), where the input is guaranteed to satisfy the triangle
 78 inequality, and the *Maximum Multistage Perfect Matching* problem (MAX-MPM), where we
 79 consider the complementary optimization objective.

80 **Problem definition.** Formally, the MIN-MPM problem is defined as follows: We are given
 81 a sequence G_1, \dots, G_T of T undirected graphs, on the same set of vertices V . At each time
 82 step $1 \leq t \leq T$, the graph G_t is given with a cost function c_t on edges: $c_t(e) \in \mathbb{Q}_{\geq 0} \cup \{+\infty\}$.
 83 We are also given a transition cost $M \geq 0$. A solution is a sequence $S = (M_1, \dots, M_t)$ where
 84 M_t is a perfect matching of G_t . Each solution (sequence) has two costs: a *matching cost*
 85 $c(S)$ and a *transition cost* $D(S)$. The goal is to minimize $c(S) + D(S)$. A matching M_t has
 86 a matching cost $c_t(M_t)$ which is equal to the sum of the costs of the edges of the perfect

87 matching. The *matching cost* of S is $c(S) = \sum_{t=1}^T c_t(M_t)$. The *transition cost* is defined
 88 as $D(S) = \sum_{t=1}^{T-1} D_t$, where $D_t = M \cdot |M_{t+1} \setminus M_t|$ is proportional to the number of edges
 89 removed between time t and $t + 1$ – which is equal to the number of added edges since the
 90 matchings are perfect. Notice that by allowing infinite cost on edges we may assume w.l.o.g.
 91 the graphs to be complete.

92 In the METRIC-MIN-MPM, at each stage c_t obeys the triangle inequality: $c_t(u, v) +$
 93 $c_t(v, w) \geq c_t(u, w)$. Finally, in the MAX-MPM version, we consider that $c_t(e)$ is the *profit*
 94 obtained by taking edge e (at time t). Then a solution sequence S has a *matching profit*
 95 $c(S) = \sum_t c_t(M_t)$. We define the *transition profit* $D(S)$ as $D(S) = \sum_{t \leq T-1} D_t$ where
 96 $D_t = M \cdot |M_{t+1} \cap M_t|$ is proportional to the number of edges that *remain* between time t
 97 and $t + 1$. The goal now is to maximize $c(S) + D(S)$. Notice that in MAX-MPM, we may
 98 no longer assume that the graphs are complete, since this assumption modifies the problem
 99 (we get profit by maintaining an edge, even of profit 0, from one time step to the next one).

100 **Related work.** A model that is close to our setting is the reoptimization model of Schieber
 101 et al. [15]. In their work, they are given a starting solution and a new instance and the goal
 102 is to minimize the sum of the cost of the new instance and of the transition cost. The model
 103 of multistage optimization that we use in this work has been studied earlier by Buchbinder et
 104 al. [5] and Buchbinder, Chen and Naor [4] for solving a set of fractional problems. Eisenstat
 105 et al. [7] studied a similar multistage optimization model for facility location problems. Their
 106 main result was a logarithmic approximation algorithm, which was later improved to a
 107 constant factor approximation by An et al. [1]. More broadly, many classical optimization
 108 problems have been considered in online or semi-online settings, where the input changes over
 109 time and the algorithm tries to adjust the solution (re-optimize) by making as few changes
 110 as possible. We refer the reader to [2, 3, 6, 10, 13, 14] and the references therein.

111 As mentioned, Gupta et al. [11] studied the Multistage Maintenance Matroid problem for
 112 both the offline and the online settings. Their main result was a logarithmic approximation
 113 algorithm for this problem, which includes as a special case a natural multistage version of
 114 SPANNING TREE. The same paper also introduced the study of MIN-MPM, which is the
 115 main problem we study here. They showed that the problem becomes hard to approximate
 116 even for a constant number of stages. More precisely, they showed the following result (n
 117 denotes the number of vertices in the graphs).

118 ► **Theorem 1** ([11]). *For any $\epsilon > 0$, MIN-MPM is not $n^{1-\epsilon}$ -approximable unless $P = NP$.
 119 This holds even when the costs are in $\{0, \infty\}$, $M = 1$, and the number of time steps is a
 120 constant.*

121 Theorem 1 is proved for $T = 8$, starting from the fact that 3-colorability is NP-hard in
 122 graphs of maximum degree 4 [8]. The authors leave as an open question the approximability of
 123 the problem in bipartite graphs, and ask for subcases with better approximability behavior.

124 **Our contribution.** We answer the open question of [11] by showing that the problem is
 125 hard to approximate even for bipartite graphs and for the case of two steps ($T = 2$). Then,
 126 we focus on the case where the edge costs are metric within every time step (METRIC-MIN-
 127 MPM). On the negative side, we prove that the problem remains APX-hard even if $T = 2$.
 128 On the positive side, we show that METRIC-MIN-MPM admits a 3-approximation algorithm
 129 for two and three stages. Finally, for the maximization version of the problem, MAX-MPM,
 130 we prove that it admits a constant factor approximation algorithm but is APX-hard.

131 **2** MIN-MPM for bipartite graphs

132 We answer the open question of [11] about the approximability of bipartite MIN-MPM.

133 ► **Theorem 2.** *For any $\epsilon > 0$, MIN-MPM cannot be approximated within a factor of $n^{1-\epsilon}$,
134 even if the input has $T = 2$ time steps, the input graphs are bipartite, $M = 1$ and the costs
135 of edges are in $\{0, \infty\}$, unless $P=NP$.*

136 Using infinite costs, the same result immediately holds for bipartite complete graphs, as well
137 as for complete graphs.

138 **Proof.** We give a gap-introducing reduction from Perfect 3DM (3-Dimensional Matching),
139 known to be NP-complete [9]. We are given an instance of Perfect 3DM which consists of
140 three sets X, Y, Z , with $|X| = |Y| = |Z| = n$, and a set Q of elements of $X \times Y \times Z$, with
141 $|Q| = m \leq n^3$. We are whether there exists a subset of n pair-wise disjoint elements of Q , or
142 not.

143 We construct an instance of our problem as follows: first, we create four sets of vertices
144 A, B, C, D with $|A| = |B| = n$ and $|C| = |D| = m$. To ease notation suppose that the ele-
145 ments of our sets X, Y, Z, Q, A, B, C, D are labeled as $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}, \{z_1, \dots, z_n\},$
146 $\{q_1, \dots, q_m\}, \{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}, \{c_1, \dots, c_m\}$, and $\{d_1, \dots, d_m\}$ respectively.

147 For any $j \in \{1, \dots, m\}$ we construct a set of $2n^{\lceil \frac{4}{\epsilon} \rceil}$ new vertices. We connect c_j to d_j
148 through a path traversing all these vertices (thus this is a path from c_j to d_j with $2n^{\lceil \frac{4}{\epsilon} \rceil} + 2$
149 vertices). We set the cost of all the internal edges of these paths for both time-steps to 0.

150 For all $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ we do the following: if $x_i \in q_j$ we set the cost of the
151 edge (a_i, c_j) to 0 in time step 1; if $y_i \in q_j$ we set the cost of the edge (a_i, c_j) to 0 in time
152 step 2; if $z_i \in q_j$ we set the cost of the edge (b_i, d_j) to 0 in both time steps. All other edge
153 costs are set to ∞ (or some other sufficiently large value). This completes the construction.
154 Observe that the new graph has $5n + 3m + 2mn^{\lceil \frac{4}{\epsilon} \rceil}$ vertices, so at most $C \cdot n^{\frac{4}{\epsilon} + 4}$ (for some
155 constant C) since $m \leq n^3$. Note also that the new graph is bipartite because the paths
156 that we added from c_j to d_j have odd lengths, hence the bipartition $(A \cup D, B \cup C)$ can be
157 extended to a bipartition of the whole graph.

158 Suppose that the original instance has a set $Q' \subseteq Q$ such that $|Q'| = n$ and no element of
159 $X \cup Y \cup Z$ appears in two elements of Q' . We obtain a multistage matching as follows: For
160 each $q_j \in Q'$ such that $q_e = (x_{i_1}, y_{i_2}, z_{i_3})$ we use the edge (a_{i_1}, c_j) in step 1, the edge (a_{i_2}, c_j)
161 in step 2, and the edge (b_{i_3}, d_j) in both time steps. Note that this fully specifies how the
162 vertices of $A \cup B$ are matched. We now complete the matching by selecting a set of edges
163 from the paths connecting each c_j to d_j : if $q_j \in Q'$, then both c_j, d_j have been matched
164 to $A \cup B$ in both time steps, and we select in both time steps the unique perfect matching
165 of the path connecting them; if $q_j \notin Q'$, then neither c_j, d_j is matched to $A \cup B$ in either
166 time step, so we select the perfect matching on the path from c_j to d_j , including these two
167 vertices. Observe that the cost of all edges we use is 0, while we only change at most n edges
168 from one time step to the other, hence the total transition cost is at most nM .

169 Suppose that the original instance does not have a solution and consider any multistage
170 matching in the new instance. We will show that it must make at least $n^{\frac{4}{\epsilon}}$ changes from one
171 time step to the other. We will say that $q_j \in Q$ is selected in time step 1, if in that time step
172 c_j is matched to an element of A . If q_j is selected in time step 1, then d_j is matched to an
173 element of B in that time step, otherwise it would be impossible to have a perfect matching
174 on the path connecting c_j to d_j . If some q_j is selected in time step 1, but not in time step 2,
175 then the solution must change all internal edges on the perfect matching on the path from
176 c_j to d_j , hence it makes at least $n^{\frac{4}{\epsilon}}$ changes, and we are done. What remains therefore to

177 show is that if the solution maintains the set of selected q_j in the two time steps, then we
 178 can construct a solution to the original instance. Indeed, since all of $A \cup B$ is matched, we
 179 have n selected q_j 's. Each element of $C \cup D$ has at most one edge connecting it to $A \cup B$ in
 180 each step, hence if it is selected this edge must be used. But if we select q_{j_1}, q_{j_2} that overlap,
 181 then two selected elements will have a common neighbor in $A \cup B$ and will therefore not be
 182 matched, contradiction.

183 Since the new graph has N vertices with $n^{\frac{1}{\epsilon}} \leq N \leq Cn^{\frac{4}{\epsilon}+4}$ vertices, it is NP-hard to
 184 distinguish if the optimal is at most $nM \leq N^\epsilon M$ or at least $n^{\frac{4}{\epsilon}} M \geq N^{1-\epsilon} M/C$. ◀

185 **3** METRIC-MIN-MPM

186 We consider in this section that c_t obeys the triangle inequality: $c_t(u, v) + c_t(v, w) \geq c_t(u, w)$.
 187 In particular, the graph is complete. As seen before, the problem is hard to approximate
 188 even if there are only 2 time steps with general costs. We show here that while the problem
 189 is APX-hard in the metric case even with only 2 time steps (Section 3.1), it admits a
 190 3-approximation algorithm in this case (2 time steps), see Section 3.2. We then extend this
 191 last result to the case of 3 time steps in Section 3.3.

192 **3.1 APX-hardness for 2 time steps**

193 In the case of 2 time steps the following result is proved.

194 ▶ **Theorem 3.** METRIC-MIN-MPM is APX-hard, even if the input has $T = 2$ time steps.

195 **Proof.** We give a gap-preserving reduction from Max 3DM. We are given an instance of Max
 196 3DM which consists of three sets X, Y, Z , with $|X| = |Y| = |Z| = n$, a set Q of elements
 197 of $X \times Y \times Z$, with $|Q| = m$, and an integer k . We are asked if there exists a subset of k
 198 pair-wise disjoint elements of Q . We assume that n, m and k are even (if not simply make two
 199 independent copies of the initial instance). This problem is APX-hard even if the occurrence
 200 of each element is bounded above by a constant $C = 3$ [12]. Note that in this case the
 201 optimum value is at least $m/7$ (greedy algorithm; at most 6 incompatible triplets are removed
 202 when a triplet is chosen). So m, n and k are linearly related ($3n \geq m \geq k \geq m/7 \geq n/21$).

203 We construct an instance of METRIC-MIN-MPM as follows: first, we create five sets
 204 of vertices X, Y, Z, G, D with $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$, $Z = \{z_1, \dots, z_n\}$, $G =$
 205 $\{g_1, \dots, g_m\}$ and $D = \{d_1, \dots, d_m\}$.

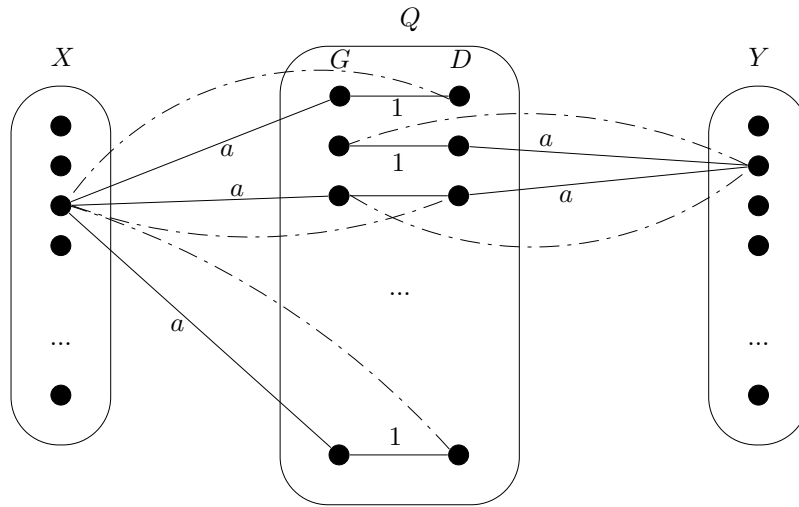
206 The graph is complete, and we set the following costs:

- 207 ■ At time step 1, Z is seen as a single point very far from the rest of the graph: (z_i, z_j) has
 208 cost 0 for $z_i, z_j \in Z$, and (z_i, v) has infinite cost for $z_i \in Z, v \notin Z$.
- 209 ■ The same is done for X at time 2.
- 210 ■ The m edges (g_i, d_i) have cost 1 at both time steps.
- 211 ■ For each triplet $q_i = (x_j, y_p, z_s)$: at time 1 edges (x_j, g_i) and (d_i, y_p) have cost a (a is a
 212 sufficiently large constant, to be specified later), and, for the triangle inequality to hold,
 213 (x_j, d_i) and (g_i, y_p) have cost $a + 1$. Similarly at time 2: (z_s, g_i) and (d_i, y_p) have cost a ,
 214 and, for the triangle inequality to hold, (z_s, d_i) and (g_i, y_p) have cost $a + 1$.

215 All non yet defined costs are equal to $2a$. The transition cost is $M = 1$. Figure 1 gives an
 216 illustration of the construction.

217 Note that the triangle inequality holds in both time steps.

218 We show that (1) if there is a 3DM of size k then there exists a solution of METRIC-MIN-
 219 MPM whose total cost is at most $2m + 4an - k/2$, and (2) conversely from a solution of the



■ **Figure 1** An illustration of the reduction at time $t = 1$, without representing Z - the construction is symmetric for time step $t = 2$. The third element of X is in the first, third and last triplet of Q . The second element of Y is in the second and third triplet. The dashed edges have costs $a + 1$. Not represented edges have cost $2a$.

220 multistage problem of total cost z we can construct a 3DM of size at least $2(2m + 4an - z)$.
 221 This proves APX -hardness since a is a constant, and m , n and k are linearly related.

222 Let us first prove (1), and suppose that we have a 3DM of size k , say (for ease of notation)
 223 q_1, \dots, q_k where $q_i = (x_i, y_i, z_i)$. Then we define a solution S of the multistage matching as
 224 follows:

- 225 ■ We take the $(m - k)$ edges of triplets (g_j, d_j) *not* in the 3DM, at both time steps 1 and 2;
- 226 ■ For $q_i, 1 \leq i \leq k$: we take edges (x_i, g_i) at time 1, (z_i, g_i) at time 2, and (y_i, d_i) at time 1
 227 and 2.
- 228 ■ We match together the $(n - k)$ remaining vertices of Y , choosing the same $\frac{n-k}{2}$ edges at
 229 both time steps.
- 230 ■ We match together the $(n - k)$ remaining vertices of X at time 1. At time 2 we keep
 231 these $\frac{n-k}{2}$ edges and match the remaining k vertices of X together.
- 232 ■ We do the same for Z .

233 We get a solution (M_1, M_2) whose costs are:

- 234 ■ At time 1, the matching cost is $(m - k) + 2ak + 2a\frac{n-k}{2} + 2a\frac{n-k}{2} = m + 2an - k$;
- 235 ■ The matching cost at time 2 is the same.
- 236 ■ The number of modifications is $3k/2$: k edges (x_i, g_i) become (z_i, g_i) , and $k/2$ edges in Z
 237 disappear at time 1 ($k/2$ edges appear in X at time 2).

238 In all, (M_1, M_2) has cost $2m + 4an - k/2$.

239

240 Conversely, suppose that we have a solution (M_1^0, M_2^0) of total cost z for the instance
 241 of METRIC-MIN-MPM. We first structure this solution using local modifications, and then
 242 show how to derive a matching from it.

- 243 ■ *Replacement 1.* First, suppose that M_1^0 takes (at time 1) an edge (x_j, g_i) of cost $2a$ - so
 244 x_j is not in the i -th triplet q_i of Q . Then d_i is matched with a vertex v with an edge
 245 of cost at least a . By replacing (at time 1) (x_j, g_i) and (d_i, v) by (x_j, v) and (g_i, d_i) we
 246 get a matching cost for these two edges at most $2a + 1$ instead of (at least) $3a$. Even

247 considering that the transition cost may have increased by two, this replacement does
 248 not increase the cost of the solution for $a \geq 3$. The same argument applies for an edge
 249 (x_j, d_i) (time step 1), an edge (y_j, d_i) or (y_j, g_i) (time step 1 or 2) and for an edge (z_j, g_i)
 250 or (z_j, d_i) in M_2^0 .

251 ■ *Replacement 2.* Now, suppose that M_1^0 takes an edge of cost $2a$ in $G \cup D$, say (g_i, g_j)
 252 with $i \neq j$ (the very same argument works for the 2 other cases (g_i, d_j) and (d_i, d_j)).
 253 Let v and w be the neighbors of d_i and d_j in M_1^0 . By replacing the three edges (g_i, g_j) ,
 254 (d_i, v) and (d_j, w) by (g_i, d_i) , (g_j, d_j) and (v, w) , we get a matching cost at most $(2a + 2)$
 255 instead of (at least) $4a$. Even considering that the transition cost may have increased by
 256 three, this replacement does not increase the cost of the solution for $a \geq 5/2$. The same
 257 holds for M_2^0 .

258 ■ *Replacement 3.* Last, suppose that edges (y_j, g_i) and (y_s, d_i) are both taken at time 1
 259 and 2. This costs $2(a + a + 1) = 4a + 2$. Then we can take instead edges (g_i, d_i) and
 260 (y_j, y_s) at both time steps, with the same cost $2 + 2(2a) = 4a + 2$.

261 In this way, we transform (M_1^0, M_2^0) into a solution (M_1, M_2) of cost at most z such that:

- 262 ■ No g_i (and no d_i) is matched using an edge of cost $2a$ (replacements 1 and 2).
- 263 ■ g_i and d_i cannot be both matched to the same vertices at time 1 and 2, unless they are
 264 matched together (replacement 3).

265 We now show how to find a 3DM from this solution (M_1, M_2) . Let:

266 ■ N_x and N_z be respectively the number of edges in $X \times (G \cup D)$ at time 1 and in $Z \times (G \cup D)$
 267 at time 2.

268 ■ N_y^1 and N_y^2 be respectively the number of edges in $Y \times (G \cup D)$ at time 1 and time 2,
 269 among which λ_1 (resp., λ_2) are of cost $a + 1$.

270 ■ N_y be the number of edges in $Y \times (G \cup D)$ that are taken at both times 1 and 2.

271 At time 1, besides these $N_x + N_y^1$ edges and the $n/2$ edges of cost 0 (vertices of Z), the
 272 other edges of (M_1, M_2) have cost either 1 (edges (g_i, d_i)) or $2a$. Since $N_x + N_y^1$ vertices in
 273 $G \cup D$ are already matched at time 1, there are at most $\frac{2m - N_x - N_y^1}{2}$ edges of cost 1 at time 1.

274 Similarly, there are at most $\frac{2m - N_z - N_y^2}{2}$ edges of cost 1 at time 2.

275 Then, computing the matching cost of (M_1, M_2) we have

$$\begin{aligned}
 276 \quad c(M_1, M_2) &\geq a(N_x + N_z + N_y^1 + N_y^2) + \lambda_1 + \lambda_2 + \frac{4m - N_x - N_z - N_y^1 - N_y^2}{2} \\
 277 &\quad + 2a \left(\frac{n - N_x + n - N_y^1 + n - N_z + n - N_y^2}{2} \right) \\
 278 &\geq 2m + 4na + \lambda_1 + \lambda_2 - \frac{N_x + N_z + N_y^1 + N_y^2}{2}.
 \end{aligned}$$

279 Now, note that at time 1 at least $N_x + N_y^1 - N_y + \frac{N_z}{2}$ edges disappear, so $D(M_1, M_2) \geq$
 280 $N_x + N_y^1 - N_y + \frac{N_z}{2}$. Similarly, at least $N_z + N_y^2 - N_y + \frac{N_x}{2}$ edges appear at time 2. So
 281 $D(M_1, M_2) \geq N_z + N_y^2 - N_y + \frac{N_x}{2}$. Then,

$$D(M_1, M_2) \geq \frac{N_x + N_z + N_y^1 + N_y^2}{2} - N_y + \frac{N_x + N_z}{4}.$$

282 This gives:

$$283 \quad z \geq c(M_1, M_2) + D(M_1, M_2) \geq 2m + 4na + \lambda_1 + \lambda_2 - N_y + \frac{N_x + N_z}{4}.$$

284 Now, consider the set of indices i such that edge (y_j, d_i) is taken at both time steps, or
 285 edge (y_j, g_i) is taken at both time steps. Since, thanks to the preprocessing, for a given i
 286 this cannot concern both d_i or g_i , we know that there are exactly N_y such indices (edges).
 287 Since there are $\lambda_1 + \lambda_2$ edges of cost $a + 1$ between Y and $G \cup D$, among these N_y indices at
 288 least $N_y - (\lambda_1 + \lambda_2)$ are such that: (1) edge (d_i, y_j) is used at both time steps (2) an edge
 289 (x_s, g_i) of cost a is used at time 1 (since no edge of cost $2a$ is used for vertices in G) and (3)
 290 an edge (z_p, g_i) of cost a is used at time 2.

In other words these at least $N_y - (\lambda_1 + \lambda_2)$ indices correspond to triplets of a 3DM. So
 we have a 3DM of size (at least) $k = N_y - (\lambda_1 + \lambda_2)$. Then, $N_x \geq N_y - (\lambda_1 + \lambda_2) = k$ and
 similarly $N_z \geq k$, so $\frac{N_x + N_z}{4} \geq \frac{k}{2}$. All together, we get

$$z \geq 2m + 4an - k + \frac{k}{2} = 2m + 4an - \frac{k}{2}.$$

291

292 3.2 A 3-approximation algorithm for 2 time steps

293 We now devise an approximation algorithm. Informally, this algorithm first guesses the
 294 number k of edges that an optimal solution keeps between steps 1 and 2. Then it computes
 295 a set of k edges with low matching cost that it maintains between time 1 and 2. Finally,
 296 it completes this set of k edges into two perfect matchings, in such a way that, using the
 297 triangle inequality, the matching cost does not increase too much.

298 Formally, the algorithm **Metric2** runs the following procedure for k from 0 to $n/2$.

- 299 1. Let G_{1+2} be the graph where the edge costs are $c(u, v) = c_1(u, v) + c_2(u, v)$. Compute a
 300 minimum cost matching M^k of size exactly k in G_{1+2} .
- 301 2. Compute a minimum cost perfect matching M_1 in G_1 , and a minimum cost perfect
 302 matching M_2 in G_2 .
- 303 3. Consider the symmetric difference of the two matchings M^k and M_1 in G_1 . This is a
 304 (vertex disjoint) set of paths P_1, \dots, P_p and cycles. Define M_1^k as M^k plus the p edges
 305 linking the first vertex and last vertex of each path P_j .
- 306 4. Do the same to get M_2^k .
- 307 5. Consider $S^k = (M_1^k, M_2^k)$.

308 **Metric2** outputs the best solution S^k .

309

310 ► **Theorem 4.** *Metric2* is a (polytime) 3-approximation algorithm for METRIC-MIN-MPM
 311 when $T = 2$.

312 **Proof.** We first prove that S^k is a feasible solution, i.e., M_i^k is a perfect matching of G_i .
 313 Since M_i is a perfect matching, in all paths P_j the first and last edges belong to M_i . Hence
 314 the first and last vertices are not covered by M^k , so M_1^k is a matching. Every other vertex is
 315 covered by M^k , so the matching is perfect.

316 Now, let us prove the claimed approximation ratio. Let us denote $S^* = (M_1^*, M_2^*)$ be an
 317 optimal solution, and consider S^k where $k = |M_1^* \cap M_2^*|$.

318 Since at least M^k is common between M_1^k and M_2^k , at least k edges are maintained
 319 between time 1 and 2 in S^k , as in S^* . So:

$$320 \quad D(S^k) \leq D(S^*). \tag{1}$$

321 Now, let us prove that:

$$322 \quad c_1(M_1^k) + c_2(M_2^k) \leq 3c_1(M_1^*) + 3c_2(M_2^*). \quad (2)$$

323 Thanks to the triangle inequality, in a path $P = (v_0, v_1, \dots, v_t)$, $c_i(v_0, v_t) \leq \sum_j c_i(v_j, v_{j+1})$:
 324 when adding edges (v_0, v_t) we add in total at most the total length of the paths, hence at
 325 most $c_i(M_i) + c_i(M^k)$. So $c_i(M_i^k) \leq c_i(M_i) + 2c_i(M^k)$. Using that $c_i(M_i) \leq c_i(M_i^*)$, we get:

$$c_1(M_1^k) + c_2(M_2^k) \leq c_1(M_1^*) + c_2(M_2^*) + 2(c_1(M^k) + c_2(M^k)).$$

326 By optimality of M^k and since S^* has k common edges between times 1 and 2, these k
 327 common edges induce a cost in S^* at least $c_1(M^k) + c_2(M^k)$. Then:

$$c_1(M_1^k) + c_2(M_2^k) \leq c_1(M_1^*) + c_2(M_2^*) + 2(c_1(M_1^*) + c_2(M_2^*))$$

and Equation 2 follows. From Equations 1 and 2 we derive:

$$c(S) + D(S) \leq 3c(S^*) + D(S^*).$$

328 The result immediately follows. ◀

329 3.3 A 3-approximation algorithm for 3 time steps

330 We now extend the previous result to the case of $T = 3$. As previously, if an optimal solution
 331 preserves in total k edges (operates in total $n - k$ modifications between time steps 1 and 2,
 332 and 2 and 3) we would like to first compute a set of k ‘preserved’ edges inducing a low cost,
 333 and then to complete this set as perfect matchings in each of the time steps. Now things get
 334 more complex since an edge can be preserved between steps 1 and 2, between steps 2 and 3,
 335 or during the whole process. It seems hard to mimic an optimal solution on these 3 types of
 336 edges (while inducing a low matching cost), but this difficulty can be overcome as follows.

337 Let G be the graph with edge cost $w = \min\{c_1 + c_2 + c_3, c_1 + c_2 + M, c_2 + c_3 + M\}$. If
 338 the minimum is $c_1 + c_2 + c_3$ (resp., $c_1 + c_2 + M$, $c_2 + c_3 + M$) we say that the edge is of type
 339 1 (resp., 2, 3). Intuitively, edges of type 1 will be taken in steps 1, 2 and 3, edges of type
 340 2 (resp., 3) will be taken in steps 1 and 2 (resp., 2 and 3). We present a 3-approximation
 341 algorithm **Metric3**. It runs the following procedure for k from 0 to $n/2$.
 342

- 343 1. Compute a minimum cost matching M^k of size exactly k in G . Denote M_1^k the set of
 344 edges of M^k of type 1 or 2, $M_2^k = M^k$ and M_3^k the set of edges of M^k of type 1 or 3.
- 345 2. Compute a minimum cost perfect matching M_i in G_i , $i = 1, 2, 3$.
- 346 3. Consider the symmetric difference of the two matchings M_i^k and M_i in G_i . This is a
 347 (vertex disjoint) set of paths P_1, \dots, P_p and cycles. Define $M_i'^k$ as the set of p edges
 348 linking the first vertex and last vertex of each path P_j .
- 349 4. Consider $S^k = (M_1^k \cup M_1'^k, M_2^k \cup M_2'^k, M_3^k \cup M_3'^k)$.

350 Then **Metric3** outputs the best solution S^k .

351

352 ▶ **Theorem 5.** *Metric3* is a (polytime) 3-approximation algorithm for METRIC-MIN-MPM
 353 when $T = 3$.

7:10 Multistage Matchings

354 **Proof.** We first note that, as in the case for $T = 2$ time steps, $M_i^k \cup M_i'^k$ is a perfect matching
 355 of G_i , so S^k is a feasible solution.

356 Now let us deal with the approximation ratio. Let $S^* = (M_1^*, M_2^*, M_3^*)$ be an optimal
 357 solution. Let us consider the set $H = (M_1^* \cap M_2^*) \cup (M_2^* \cap M_3^*)$ of edges in S^* that are in (at
 358 least) two consecutive steps. Note that H is a matching (it is included in M_2^*). Consider S^k
 359 where $k = |H|$. We now prove the following result:

360 ► **Lemma 6.** $D(S^k) + \sum_i c_i(M_i^k) \leq D(S^*) + c(S^*)$.

361 **Proof.** To prove this, let $k_1 = |M_1^* \cap M_2^* \cap M_3^*|$ be the number of edges in S^* that are taken
 362 at each of the 3 time steps. Hence, $k - k_1$ edges are taken at (only) 2 consecutive time steps.
 363 So there are $(n/2 + n/2 - 2k_1 - (k - k_1))$ modifications in total, and:

$$364 \quad D(S^k) = M(n - k - k_1). \quad (3)$$

365 Recall that in G , $w = \min\{c_1 + c_2 + c_3, c_1 + c_2 + M, c_2 + c_3 + M\}$. k_1 edges of H are present
 366 on the 3 time steps (matching cost $c_1 + c_2 + c_3$), while $k - k_1$ are present in two consecutive
 367 time steps (matching cost $c_1 + c_2$ or $c_2 + c_3$).

$$368 \quad w(H) \leq c(S^*) + M(k - k_1). \quad (4)$$

369 Similarly, let λ_1 be the number of edges of type 1 in M^k . There are $(k - \lambda_1)$ edges of
 370 type 2 or 3, hence

$$371 \quad w(M^k) = \sum_i c_i(M_i^k) + M(k - \lambda_1). \quad (5)$$

372 Indeed, in G cost c_1 applies to edges of type 1 and 2 ($c_1(M_1^k)$), cost c_2 applies to all edges of
 373 M^k ($c_2(M_2^k)$), cost c_3 applies to edges of type 1 and 3 ($c_3(M_3^k)$), and cost M to the $(k - \lambda_1)$
 374 edges of type 2 and 3.

375 Also, the number of preserved edges in S^k is at least $k + \lambda_1$, so:

$$376 \quad D(S^k) \leq M(n - k - \lambda_1). \quad (6)$$

377 Since H is a matching, in G we have $w(H) \geq w(M^k)$. This gives using Equations 4 and 5:

$$\sum_i c_i(M_i^k) + M(k - \lambda_1) \leq c(S^*) + M(k - k_1)$$

378 so $\sum_i c_i(M_i^k) \leq c(S^*) + M(\lambda_1 - k_1)$. Then using Equations 3 and 6 we get:

$$\begin{aligned} \sum_i c_i(M_i^k) + D(S^k) &\leq c(S^*) + M(\lambda_1 - k_1) + M(n - k - \lambda_1) = c(S^*) + M(n - k - k_1) \\ &= c(S^*) + D(S^*) \end{aligned}$$

379 which concludes the proof of Lemma 6. ◀

380 Now, by triangle inequality, and the fact that $c_i(M_i) \leq c_i(M_i^*)$, we know that:

$$381 \quad c_i(M_i'^k) \leq c_i(M_i^*) + c_i(M_i^k). \quad (7)$$

382 Then, from Lemma 6 and Equation 7 we get:

$$\begin{aligned}
 383 \quad c(S^k) + D(S^k) &= \sum_i (c_i(M_i^k) + c_i(M_i'^k)) + D(S^k) \leq \sum_i (2c_i(M_i^k) + c_i(M_i^*)) + D(S^k) \\
 384 \quad &\leq c(S^*) + 2 \left(\sum_i c_i(M_i^k) + D(S^k) \right) \leq 3c(S^*) + 2D(S^*).
 \end{aligned}$$

385 The result follows. ◀

386 **4** MAX-MPM

387 In the maximization version, we consider that $c_t(e)$ is the *profit* obtained by taking edge e (at
 388 time t). Then a solution sequence S has a matching profit $c(S) = \sum_t c_t(M_t)$. We define the
 389 transition profit $D(S)$ as $D(S) = \sum_{t \leq T-1} D_t$ where $D_t = M \cdot |M_{t+1} \cap M_t|$ is proportional
 390 to the number of edges that *remain* between time t and $t+1$. The goal now is to maximize
 391 $c(S) + D(S)$. Recall that in the maximization version we may no longer assume that the
 392 graphs are complete.

393 **4.1 APX-hardness for 2 time steps**

394 We first show that MAX-MPM, even in the case of 2 time steps is APX-hard.

395 ▶ **Theorem 7.** MAX-MPM is APX-hard even if $T = 2$.

396 **Proof.** As previously, we consider the maximum 3DM problem in the case where the
 397 occurrence of each element is bounded by 3, hence the optimal value, the number of triplets
 398 and the size of the ground sets are linearly related.

399 Given three sets X, Y, Z each of size n , and m triplets q_i of $X \times Y \times Z$, we build two
 400 graphs G_1 and G_2 with $n' = 2m + 4n$ vertices:

- 401 ■ 4 sets D, E, F, G of size n ;
- 402 ■ 2 sets $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_m\}$ of size m .

403 Vertices of D will represent elements of X , vertices of E and F elements of Y (twice), vertices
 404 of G elements of Z . Each triplet q_i is represented by one edge (a_i, b_i) in both graphs. It has
 405 cost 0.

406 If a triplet q_i is (x_j, y_k, z_l) then:

- 407 ■ In G_1 we put edges (d_j, a_i) and (b_i, e_k) , both with cost M' ;
- 408 ■ In G_2 we put edges (f_k, a_i) and (b_i, z_l) , both with cost M' .

409 Note that vertices in F, G have degree 0 in G_1 , vertices in D, E have degree 0 in G_2 .

410 We fix $M' = \frac{M+1}{4}$, and $M \geq 3$.

411 Let us show that there is a 3DM of size (at least) k if and only if there is a solution of
 412 profit at least $Mm + k$.

413 Suppose first that there is a set S of k independent triplets. Then we build matchings
 414 (M_1, M_2) as follows:

- 415 ■ if q_i is not in S , we take (a_i, b_i) both in M_1 and M_2 . This gives transition profit $M(m-k)$.
- 416 ■ if $q_i = (x_j, y_k, z_l)$ is in S , then we take in M_1 the two edges (d_j, a_i) and (b_i, e_k) , and in
 417 M_2 the two edges (f_k, a_i) and (b_i, z_l) . This gives a matching profit $4kM'$.

418 Note that since any element of X, Y, Z is in at most one triplet of S , vertices in D, E, F, G
 419 are adjacent to at most one chosen edge. In other words M_1 and M_2 are matchings.

420 The profit of the solution is $4kM' + M(m-k) = k(M+1) + M(m-k) = Mm + k$.

421 Suppose now that there is a solution (M_1, M_2) of profit at least $Mm + k$. Suppose first
 422 that there is an edge (a_i, b_i) which is in M_1 but not in M_2 . Then we get no transition profit
 423 for this edge. In M_2 we have taken at most one edge incident to a_i , and one edge incident
 424 to b_i , with matching profit at most $2M'$. Since these edges are not in G_1 they cannot give
 425 transition profit. So we can put in M_2 the edge (a_i, b_i) and remove the edges incident to a_i
 426 and b_i (if any). The profit increases by $M - 2M' = M/2 - 1/2 \geq 0$.

427 So we can assume that M_1 and M_2 have the same set of edges between A and B .
 428 Suppose now that there are two edges (a_i, b_i) and (a_s, b_s) both not in M_1 (equiv. not in M_2)
 429 corresponding to two intersecting triplets. Suppose for instance that x_j is in both triplets.
 430 This means that in M_1 we cannot take both edges (c_j, a_i) and (c_j, a_s) , for instance (c_j, a_s) is
 431 not in M_1 . Then we can add (a_s, b_s) to M_1 and M_2 , and remove the (at most) 3 incident
 432 edges. This increases profit by $M - 3M' \geq 0$.

433 So, the set of edges (a_i, b_i) not in M_1 (or not in M_2) corresponds to a set of independent
 434 triplets. Let t the number of such edges. Since M_1 is a matching, besides these edges between
 435 A and B , there is at most two edges for each (a_i, b_i) not in M_1 . Similarly, there is at most
 436 two edges in M_2 for each (a_i, b_i) not in M_2 . So the matching profit is at most $4tM'$, and
 437 the transition profit is $M(m - t)$. The profit is $M(m - t) + 4tM' = Mm + t \geq Mm + k$. So
 438 $t \geq k$. ◀

439 4.2 Constant factor approximation algorithms

440 ► **Theorem 8.** MAX-MPM is $1/2$ -approximable. If $T = 2$ it is $2/3$ -approximable, if $T = 3$
 441 it is $3/5$ -approximable.

442 **Proof.** Note that if the graphs are assumed to be complete (bipartite complete) then the
 443 ratio $1/2$ is easily achievable. Indeed, consider two solutions:

- 444 ■ The first one S_1 consisting of the same perfect matching M_0 at all time steps;
- 445 ■ The second one S_2 consisting of a matching \hat{M}_t of maximum profit on G_t for each t .

446 Output the best one.

447 Let $S^* = (M_1^*, \dots, M_T^*)$ be an optimal solution. Clearly the profit of S_1 is at least the
 448 transition profit $D(S^*)$ of S^* . Also, $c(M_t^*) \leq c(\hat{M}_t)$ so the matching profit of S^* is at most
 449 the one of S_2 . The ratio $1/2$ follows.

450 If the graphs are not assumed to be complete things get harder since one cannot trivially
 451 optimize the transition profit by keeping a perfect matching along the multistage process.

452 Let us consider three consecutive time steps $t - 1, t, t + 1$. Let us consider the graph G'_t
 453 which is the same as G_t up to the profit on edges, which is now $c'_t(e)$ where:

- 454 1. $c'_t(e) = c_t(e) + 2M$ if e is in G_{t-1} and G_{t+1} ;
- 455 2. otherwise, $c'_t(e) = c_t(e) + M$ if e is in G_{t-1} or G_{t+1} ;
- 456 3. otherwise $c'_t(e) = c_t(e)$.

457 Let us consider a matching M'_t of maximum profit in G'_t .

458 ► **Lemma 9.** $c'_t(M'_t) \geq D_{t-1}(S^*) + c_t(M_t^*) + D_t(S^*)$.

459 **Proof.** Let us consider the profit of M'_t on G'_t . Since the set of edges preserved from time
 460 $t - 1$ to time t is included in M_t^* , the profit $D_{t-1}(S^*)$ appears in the profit of M'_t on G'_t
 461 ($+M$ on each common edges between the two consecutive graphs). This is also the case
 462 for $D_t(S^*)$, for the same reason. Of course, the profit $c_t(e)$ appears as well. Since M'_t is of
 463 maximum profit, the Lemma follows. ◀

464 Because of Lemma 9, choosing the matching M'_t at time steps $t - 1$, t and $t + 1$ in a
 465 solution generates a profit at least $D_{t-1}(S^*) + c_t(M'_t) + D_t(S^*)$.

466 Note that, with similar arguments, if two times steps $t, t + 1$ are involved, we can compute
 467 a matching H_i that we take at time steps $t, t + 1$ generating a profit at least $c_t(M'_t) + D_t(S^*)$.
 468 Symmetrically, we can compute a matching H'_i that we take at time steps $t, t + 1$ generating
 469 a profit at least $c_{t+1}(M'_t) + D_t(S^*)$.

470

471 Now we consider the following 2 solutions:

472 ■ S_1 consists of choosing H_1 at steps 1, 2, H_3 at step 3, 4, If T is even then we are
 473 done, otherwise we take an optimal matching \hat{M}_T at step T .

474 ■ S_2 consisting of choosing an optimal matching \hat{M}_1 at step 1, then H_2 at steps 2, 3, H_4 at
 475 steps 4, 5, If T is even we take an optimal matching \hat{M}_T at step T .

476 Output the best of these two solutions. Then: S_1 covers the transition profit of an optimal
 477 solution D_t for t odd, plus the matching profits for t odd. S_2 covers the transition profit
 478 of an optimal solution D_t for t even, plus the matching profits for t even. The ratio $1/2$ follows.
 479

480 *Improvement for $T = 3$.* The previous solutions S_1 and S_2 have profit (respectively) at
 481 least $c_1(S^*) + D_1(S^*) + c_3(S^*)$ and $c_1(S^*) + D_2(S^*) + c_2(S^*)$. S_3 takes \hat{M}_1 at step 1 and H'_2
 482 at time steps 2 and 3, with profit at least $c_1(S^*) + D_2(S^*) + c_3(S^*)$; S_4 takes H'_1 at steps 1
 483 and 2, and \hat{M}_3 at step 3, with profit at least $D_1(S^*) + c_2(S^*) + c_3(S^*)$. S_5 uses M'_2 at the 3
 484 steps with profit at least $D_1(S^*) + c_2(S^*) + D_2(S^*)$ (thanks to Lemma 9). Take the best of
 485 these 5 solutions, and the ratio follows.

486 *Improvement for $T = 2$.* Simply take 3 solutions: S_1 is defined as previously, with profit
 487 at least $c_1(S^*) + D_1(S^*)$. S_2 takes H'_1 at both steps with profit at least $D_1(S^*) + c_2(S^*)$. S_3
 488 consists of one optimal matching at step 1, and an optimal matching at step 2, with profit at
 489 least $c_1(S^*) + c_2(S^*)$. The ratio $2/3$ follows.

490

491 5 Concluding remarks

492 Following the results of Section 3, we leave as an open question the existence of a constant
 493 factor approximation algorithm for the metric case for a number of time steps bigger than 3.
 494 Also, we considered here an off-line version of the problem where the whole set of instances
 495 is known in advance. It would be worth investigating the on-line case where data are not
 496 known in advance.

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