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Choquet integral with stochastic entries

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Abstract. We analyse the probabilistic features of the Choquet integral $C_\mu(X)$ with respect to a capacity μ over $\{1, \dots, n\}$, where the n entries $X = (X_1, \dots, X_n)$ are random variables. Few papers deal with this issue excepted in [8]. We give two different formula for the density function of $C_\mu(X)$ when X_1, \dots, X_n are i.i.d. random variables. We also calculate the first moment of the Choquet integral and we compare our results with the ones obtained in [8] which mainly concern the cases where the common distribution of X_i are either uniform or exponential.

Key words and phrases : capacity, Choquet integral, density function, moments, exponential distribution, uniform distribution.

AMS 2000 subject classifications : 60E99, 60K10

1 Introduction

Since almost 3 decades, the Choquet integral has become a major "actor" in the field of aggregation function and decision making [6, 7]. Thanks to theoretical works, many applications have been developed in finance [3], sustainable development [9, 1], image processing [12], recommender system [4], clustering [10], deep-learning [2], to only cite few in recent years.

Despite the amount of studies, the question of uncertainty in the Choquet integral remains under-considered. Such a question is of first importance in many applications and uses. Few works deal with uncertainty considering the entries of the Choquet integral as random variables. Grabisch and Raufaste have proposed in [5] Monte-Carlo simulations to compute in particular cases the empirical mean and standard deviation of the Choquet integral. They draw as conclusion that the exact distribution seems to be hard to obtain. The Choquet integral requires a linear ordering of the entries which generates difficulties when they are random variables. Note that

ordering a family of probability distributions is usually done using the notion of stochastic dominance [15]. This new notion of order has permitted to Yager in [14] to give an approximation of the mean of the Choquet integral based on the Shapley indices [11]. To circumvent the initial issue, Yager has introduced in [13] a surrogate that allows to compute "the mean-like aggregated value". In both studies, no clear relation is done between the density function of the entries and the density function of the Choquet integral.

As far as we have investigated the literature, the only one contribution which deals with the probability distribution of the Choquet integral has been done by Kojadinovic and Marichal in [8]. Indeed, they have obtained theoretical formulas concerning the moments of the Choquet integral for any form of input distributions. Explicit expressions of the density function of the Choquet integral has also been given but only in the case where the entries are either exponential or uniform.

Let μ be a capacity on a finite set $S := \{1, \dots, n\}$ and $X = (X_1, \dots, X_n)$ be the stochastic entries. In our setting, X_1, \dots, X_n are independent random variables. The aim of the paper is to study the Choquet integral $C_\mu(X)$ as a random variable. We are able to determine the density of $C_\mu(X)$ in Proposition 3.2. The formula is rather complicated but can be simplified (cf Proposition 3.5) if we suppose furthermore that X_1, \dots, X_n have the same density, i.e. X_1, \dots, X_n are independent and identically distributed (i.i.d.). We have shown a new form of the density of $C_\mu(X)$ given in Proposition 3.5 using auxiliary exponential random variables. We provide a general formulation of the law of the Choquet integral which generalizes the result of Kojadinovic and Marichal. These authors has given a non-explicit formula for the moments of order n of the Choquet integral. In the case of the expectation (i.e. $n = 1$), we obtain an explicit formulation which is in consistence with Proposition 4 in [8] but our proof uses a different way.

Let us briefly explain the structure of the paper. In Section 2, we recall definition of a capacity and the related Choquet integral. In order that the reader has a synthetic overview of our results, the third and fourth sections provides straightforwardly the results for the density function and resp. for the first order moment. Finally, the proofs of the propositions are postponed in Section 5.

2 Few reminders related to Choquet integral

We briefly recall the definitions of a capacity and the associated Choquet integral over a finite set $S := \{1, 2, \dots, n\}$.

Definition 2.1 *A capacity, also called a fuzzy measure, μ over S is a function defined over the family $\mathcal{P}(S)$ of sets included in S , valued in $[0, 1]$ which is non-decreasing:*

$$\mu(A) \leq \mu(B), \quad \forall A, B, \quad A \subset B \subset S \quad (2.1)$$

and satisfying:

$$\mu(\emptyset) = 0, \quad \mu(S) = 1.$$

Notation 2.2 1. \mathcal{S}_n stands for the group of permutations of S .

2. Let $x := (x_1, \dots, x_n) \in \mathbb{R}^n$. There exists $\sigma \in \mathcal{S}_n$ such that:

$$x_{\sigma(1)} \leq \dots \leq x_{\sigma(n)}. \quad (2.2)$$

σ is unique if $x_i \neq x_j$ for all $i \neq j$. Obviously σ depends on x .

3. It is convenient to adopt the notation: $x_{\sigma(i)} = x_{(i)}$ for any $i \in S$. Therefore (2.2) takes the form:

$$x_{(1)} \leq \dots \leq x_{(n)}. \quad (2.3)$$

4. If $\sigma : S \rightarrow S$, we set:

$$\sigma(a : b) := \{\sigma(i), a \leq i \leq b\}, \quad 1 \leq a \leq b \leq n.$$

We convince that $\sigma(a : b) = \emptyset$ if $b > a$.

Definition 2.3 Let μ be a capacity over S and $x \in \mathbb{R}^n$. The Choquet integral of x with respect to μ is the real number:

$$C_\mu(x) := \sum_{i=1}^n x_{(i)} [\mu(\sigma(i : n)) - \mu(\sigma(i + 1 : n))] \quad (2.4)$$

where σ is the permutation defined by (2.3).

3 The law of $C_\mu(X)$

Let μ be a capacity over $S = \{1, \dots, n\}$ and $X := (X_1, \dots, X_n)$ be a random vector. ~~In his Phd-thesis (Contributions aux mesures floues k -additives et à l'intégrale de Choquet stochastique. Application aux analyses médico-économiques) Y. Petot has considered the case where only one coordinate of X is random. However,~~ We suppose here that all the components X_1, \dots, X_n are independent random variables and for any $i \in \{1, \dots, n\}$ the random variable X_i has a density function f_i .

~~Let μ be a given capacity over $S = \{1, \dots, n\}$.~~ We focus in this section on the density function of the Choquet integral $C_\mu(X)$. We calculate it in Proposition 3.2. Unfortunately the formula is not completely explicit since

it is the sum of multiple Lebesgue integrals which cannot be simplified in the general case. To get a more tractable result, we suppose moreover that X_1, \dots, X_n have the same density. In that case, we obtain the simpler formula (3.12). In this identity, it is striking (and unexpected) to see that appear elements of Poisson distributions. Introducing additional exponential random variables, independent of X , we can write in Proposition 3.7 the density of $C_\mu(X)$ as a sum of functions which can be expressed as expectation of some random variables. This is a new shape of the density function of $C_\mu(X)$. Along the presentation, some relation with [8] are proposed to show how both are connected. Proof of the proposition are postponed in Section 5.

~~Since the density function of the Choquet integral is complicated, it is natural to wonder if its moments are easier to calculate. We have been only able to (explicitly) calculate the expectation, see Proposition 4.1. We also compare our results to the ones obtained in [8].~~

Notation 3.1 1. Let τ be a permutation of S .

(a) We set

$$i_\tau := \max \{1 \leq j \leq n, \mu(\tau(j : n)) > 0\}.$$

Since $i \mapsto \mu(\tau(i : n))$ is non-increasing

$$\begin{aligned} \mu(\tau(j : n)) &= 0, & j \geq i_\tau + 1 \\ \mu(\tau(j : n)) &> 0, & 1 \leq j \leq i_\tau. \end{aligned}$$

(b) We also introduce:

$$a_\tau := \mu(\tau(i_\tau : n)) - \mu(\tau(i_\tau + 1 : n)) = \mu(\tau(i_\tau : n)). \quad (3.1)$$

Note that $a_\tau > 0$.

(c) Let $\theta_\tau : \mathbb{R} \rightarrow \mathbb{R}$ be the function:

$$\theta_\tau(v) := \int_{\mathbb{R}^{n-i_\tau}} 1_{\{v \leq y_1 \leq \dots \leq y_{n-i_\tau}\}} \prod_{1 \leq j \leq n-i_\tau} f_{\tau(j+i_\tau)}(y_j) \prod_{1 \leq j \leq n-i_\tau} dy_j, \quad v \in \mathbb{R}. \quad (3.2)$$

2. Here, we adhere to the conventions:

$$\int_{\mathbb{R}^0} = 1, \quad \prod_{i=1}^0 = 1, \quad \sum_{i=1}^0 = 0. \quad (3.3)$$

Proposition 3.2 Let ϕ^C be the density function of $C_\mu(X)$. Then

$$\phi^C(u) = \sum_{\tau \in \mathcal{S}_n} \phi_\tau(u) \quad (3.4)$$

where

$$\begin{aligned} \phi_\tau(u) &= \frac{1}{a_\tau} \int_{\mathbb{R}^{i_\tau-1}} f_{\tau(i_\tau)}\left(\frac{u-b_\tau}{a_\tau}\right) \theta_\tau\left(\frac{u-b_\tau}{a_\tau}\right) \prod_{1 \leq j \leq i_\tau-1} f_{\tau(j)}(y_j) \\ &\quad \times \mathbf{1}_{\{y_1 \leq \dots \leq y_{i_\tau-1} \leq \frac{u-b_\tau}{a_\tau}\}} \prod_{1 \leq j \leq i_\tau-1} dy_j \end{aligned} \quad (3.5)$$

and b_τ is the coefficient which depends on $(y_1, \dots, y_{i_\tau-1})$:

$$b_\tau := \sum_{i=1}^{i_\tau-1} y_i [\mu(\tau(i:n)) - \mu(\tau(i+1:n))]. \quad (3.6)$$

For the proof of Proposition 3.2, see Section 5.1. Obviously, Proposition 3.2 is rather complicated. We can explicit the density function ϕ^C when $n = 2$, see **Remark CE N'EST PAS LE BON NUMERO** 3.6 below. Under additional assumptions we obtain a simplified expression, see Proposition 3.5 below.

Remark 3.3 *Let's consider $S = \{1, 2\}$ and $S_2 = \{(1), (2)\}$ where (1) equals the identity and (2) the transposition: $1 \mapsto 2, 2 \mapsto 1$. Thus, the function ϕ^C is the sum of $\phi_{(1)}$ and $\phi_{(2)}$. Let Φ_i be the cumulative distribution function of X_i .*

Case 1.

If $\mu(2) = 0$, then $i_\tau = 1$, $a_\tau = 1$, $b_\tau = 0$ and

$$\phi_{(1)}(u) = f_1(u) \left(\int_u^{+\infty} f_2(y_1) dy_1 \right) = f_1(u) (1 - \Phi_2(u)).$$

If $\mu(\{2\}) > 0$, then $i_\tau = 2$, $a_\tau = \mu(2)$, $b_\tau = y_1(1 - \mu(2))$ and

$$\phi_{(1)}(u) = \frac{1}{\mu(2)} \left(\int_{-\infty}^u f_2\left(\frac{u - y_1(1 - \mu(2))}{\mu(2)}\right) f_1(y_1) dy_1 \right).$$

Case 2.

If $\mu(1) = 0$, then $i_\tau = 1$, $a_\tau = 1$, $b_\tau = 0$ and

$$\phi_{(2)}(u) = f_2(u) \left(\int_u^{+\infty} f_1(y_1) dy_1 \right) = f_2(u) (1 - \Phi_1(u)).$$

If $\mu(1) > 0$, then $i_\tau = 2$, $a_\tau = \mu(1)$, $b_\tau = y_1(1 - \mu(1))$ and

$$\phi_{(1)}(u) = \frac{1}{\mu(1)} \left(\int_{-\infty}^u f_1\left(\frac{u - y_1(1 - \mu(1))}{\mu(1)}\right) f_2(y_1) dy_1 \right).$$

We can go further than Proposition 3.2, in the i.i.d. case. Let f (resp. F) be the common probability density function (resp. cumulative distributive function) of the r.v.'s X_i . Before stating our result we fix few notations.

Notation 3.4 Let τ be a fixed element in \mathcal{S}_n .

1. We set:

$$\alpha_i := \mu(\tau(i : n)) - \mu(\tau(i + 1 : n)), \quad 1 \leq i \leq n. \quad (3.7)$$

2. Let l be number of elements of $\{0 \leq j \leq n, \alpha_j \neq 0\}$. Note that $\alpha_{i_\tau} \neq 0$ and therefore $l \geq 1$. We set:

$$\{1 \leq j \leq n, \alpha_j \neq 0\} = \{j_1, \dots, j_l\} \quad (3.8)$$

where $1 \leq j_1 < \dots < j_l \leq n$ and

$$\delta_i := j_i - j_{i-1} - 1 \quad 2 \leq i \leq l, \quad \delta_1 = j_1 - 1. \quad (3.9)$$

Note that

$$i_\tau = j_l, \quad a_\tau = \alpha_{j_l}. \quad (3.10)$$

3. Let C_τ be the constant:

$$C_\tau := \frac{1}{a_\tau \delta_1! \times \dots \times \delta_l! (n - j_l)!}. \quad (3.11)$$

Proposition 3.5 We suppose that the rv's X_1, \dots, X_n are i.i.d. Then the density function ϕ^C of $C_\mu(X)$ is given by (3.4), where

$$\begin{aligned} \phi_\tau(u) &= C_\tau \int_{\mathbb{R}^{l-1}} f(x_1) \times \dots \times f(x_{l-1}) f\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right) \left(1 - F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right)\right)^{n-j_l} \\ &\quad F(x_1)^{\delta_1} \prod_{r=2}^{l-1} (F(x_r) - F(x_{r-1}))^{\delta_r} \times \left(F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right) - F(x_{l-1})\right)^{\delta_l} \\ &\quad \mathbb{1}_{\{x_1 < \dots < x_{l-1} < \frac{u - \widehat{b}_{l-1}}{a_\tau}\}} dx_1 \otimes \dots \otimes dx_{l-1}. \end{aligned} \quad (3.12)$$

$$\text{and } \widehat{b}_{l-1} := \sum_{r=1}^{l-1} \alpha_{j_r} x_r.$$

Remark 3.6 1. Let $\rho_1, \dots, \rho_{k+1}$ in $[0, +\infty[$. Using the fact that the multiple integral

$$\begin{aligned} &\int_{\mathbb{R}^k} f(x_1) \times \dots \times f(x_k) F(x_1)^{\rho_1} \prod_{r=2}^k (F(x_r) - F(x_{r-1}))^{\rho_r} \\ &\quad \times (1 - F(x_k))^{\rho_{k+1}} dx_1 \otimes \dots \otimes dx_k \\ &\text{equals } \frac{\rho_1! \times \dots \times \rho_{k+1}!}{(k + \rho_1 + \dots + \rho_{k+1})!} \text{ then we can prove that } \int_{\mathbb{R}} \phi_\tau(u) du = \frac{1}{n!}. \end{aligned}$$

2. In the uniform case (i.e. when the distribution of all the random variables is uniform over $[0, 1]$), the authors in [8] have shown that $\phi^C(u) = \sum_{\tau \in \mathcal{S}_n} \phi_\tau^*(u)$, where $\phi_\tau^*(u)$ can be expressed either as an operator iterated $(n - 1)$ times or an integral over $[0, 1]^{n-1}$. Since the function ϕ_τ defined by (3.12) is an integral over $[0, 1]^l$, then $\phi_\tau^*(u)$ and $\phi_\tau(u)$ are a priori different. In the case:

$$\mu(A) < \mu(B), \quad \forall A \subsetneq B \subset S \quad (3.13)$$

Kojodinovic and Marichal proved:

$$\phi^C(u) = \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} \sum_{i=0}^n \frac{(\mu_i^\tau - u)_+^{n-1}}{\prod_{j \neq i} (\mu_i^\tau - \mu_j^\tau)} \quad (3.14)$$

where a_+ stands for the positive part of a and $\mu_i^\tau := \mu(\tau(1, i))$. It is actually possible to recover (3.14) using (3.12) since (3.13) implies $l = n$, $j_1 = \dots = j_n = 1$, $\delta_1 = \dots = \delta_n = 0$.

3. In the exponential case, a nice formula has been given in [8]:

$$\phi^C(u) = \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} \sum_{i=1}^n \frac{(\mu_i^\tau/i)^{n-2}}{\prod_{j \neq i} (\mu_i^\tau/i - \mu_j^\tau/j)} \exp \left\{ -\frac{u}{(\mu_i^\tau/i)} \right\} \quad (3.15)$$

under the additional assumption

$$\frac{\mu(A)}{|A|} \neq \frac{\mu(B)}{|B|}, \quad \forall A \neq B \quad (3.16)$$

where $|C|$ is the number of elements in C .

Proposition 3.5 is a consequence of Proposition 3.2 and its proof is postponed in Section 5.2. Recall that if Z is a random variable which is Poisson distributed with parameter λ , then

$$\mathbb{P}(Z = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \in \mathbb{N}.$$

It is striking to observe that in the right hand-side of (3.12), appears a product of such quantities. Using a Poisson process, we can write (3.12) as the expectation of a random variable.

Let $(\xi_k)_{k \geq 1}$ a sequence of i.i.d. random variables exponentially distributed with parameter 1, independent of X_1, \dots, X_n and

$$T_0 := 0, \quad T_n := \xi_1 + \dots + \xi_n \quad \forall n \geq 1. \quad (3.17)$$

We keep the assumptions given in Proposition 3.5.

Proposition 3.7 *Let $\tau \in \mathcal{S}_n$ and $u \in \mathbb{R}$. Then,*

$$\phi_\tau(u) = \frac{e}{a_\tau} \mathbb{E} \left[f \left(\frac{u - B^*}{a_\tau} \right) 1_{A_1 \cap A_2(u)} \right] \quad (3.18)$$

where

$$A_1 := \{T_{n-l} \leq 1 < T_{n-l+1}\} \bigcap_{r=1}^{l-1} \{T_{j_{r-r}} \leq F(X_r) < T_{j_{r-r+1}}\}$$

$$A_2(u) := \{X_1 < \dots < X_{l-1} < \frac{u - B^*}{a_\tau}\} \bigcap \{T_{j_{l-l}} \leq F \left(\frac{u - B^*}{a_\tau} \right) < T_{j_{l-l+1}}\}$$

$$\text{and } B^* =: \sum_{r=1}^{l-1} \alpha_{j_r} X_r.$$

The proof of Proposition 3.7 is given in Section 5.3.

4 Expectation

Since the density function of the Choquet integral is complicated, it is natural to wonder if its moments are easier to calculate. We have been only able to (explicitly) calculate the expectation, see Proposition 4.1. We also compare our results to the ones obtained in [8].

~~We are interested in the calculation of the expectation and the variance of $C_\mu(X)$ where X is a random vector and μ is a given capacity. We set:~~

$$\bar{\mu}(k) := \sum_{A \subset S, |A|=k} \mu(A), \quad 0 \leq k \leq n. \quad (4.1)$$

where $|A|$ is the number of elements of A .

Proposition 4.1 *The first moment of $C_\mu(X)$ is*

$$\mathbb{E}(C_\mu(X)) = \sum_{i=1}^n \bar{\mu}(i) \mathbb{E} \left[F(X_1)^{n-i-1} (1 - F(X_1))^{i-1} X_1 (nF(X_1) + i - n) \right]. \quad (4.2)$$

The proof of Proposition 4.1 is given in Subsection 5.4.

Remark 4.2 *1. In (Proposition 4, [8]), for any integer r , an expression of the moment of order r of $C_\mu(X)$ has been given but it involves expectations of random variables. In the case $r = 1$, the formula coincides with (4.2) The calculation of the second moment of $C_\mu(X)$ is possible but is very complicated and do not lead to a simple formula.*

2. In the particular case of uniform distribution, Kojadinovic and Marichal have given an explicit formula for $\mathbb{E}[C_\mu(X)]$ which takes the form

$$\mathbb{E}(C_\mu(X)) = \frac{1}{(n+1)!} \sum_{i=1}^n i! (n-i)! \bar{\mu}(i)$$

when $r = 1$. It is easy to recover the previous identity from (4.2).

5 Proofs

5.1 Proof of Proposition 3.2

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a test function. Note that if $\tau \in \mathcal{S}_n$, then $\{\sigma = \tau\} = \{X_{\tau(1)} < \dots < X_{\tau(n)}\}$. Therefore,

$$\mathbb{E}\left[g(C_\mu(X))\right] = \sum_{\tau \in \mathcal{S}_n} \mathbb{E}\left[g(C_\mu(X)) \mathbf{1}_{\{X_{\tau(1)} \leq \dots \leq X_{\tau(n)}\}}\right].$$

Since the random variables X_1, \dots, X_n are independent and X_i admits a density function f_i , we can write the above expectation as:

$$\mathbb{E}\left[g(C_\mu(X))\right] = \sum_{\tau \in \mathcal{S}_n} \int_{\mathbb{R}^n} g(C_\mu(x)) \mathbf{1}_{\{x_{\tau(1)} \leq \dots \leq x_{\tau(n)}\}} f_1(x_1) \times \dots \times f_n(x_n) dx_1 \otimes \dots \otimes dx_n. \quad (5.1)$$

Using the definition of i_τ , we have

$$\begin{aligned} C_\mu(x) &= \sum_{i=1}^n x_{\tau(i)} [\mu(\tau(i:n)) - \mu(\tau(i+1:n))] \\ &= \sum_{i=1}^{i_\tau-1} x_{\tau(i)} [\mu(\tau(i:n)) - \mu(\tau(i+1:n))] + x_{\tau(i_\tau)} \mu(\tau(i_\tau:n)). \end{aligned}$$

In each integral of (5.1), we change the variable $x_{\tau(i_\tau)}$, setting

$$\begin{aligned} u &= \sum_{i=1}^{i_\tau-1} x_{\tau(i)} [\mu(\tau(i:n)) - \mu(\tau(i+1:n))] + x_{\tau(i_\tau)} \mu(\tau(i_\tau:n)) \\ &= b_\tau + a_\tau x_{\tau(i_\tau)} \end{aligned}$$

where

$$\begin{aligned} a_\tau &= \mu(\tau(i_\tau:n)) \\ b_\tau &= \sum_{i=1}^{i_\tau-1} x_{\tau(i)} [\mu(\tau(i:n)) - \mu(\tau(i+1:n))]. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\left[g(C_\mu(X))\right] &= \sum_{\tau \in \mathcal{S}_n} \int_{\mathbb{R}} g(u) \left(\frac{1}{a_\tau} \int_{\mathbb{R}^{n-1}} f_{\tau(i_\tau)}\left(\frac{u-b_\tau}{a_\tau}\right) \prod_{j \neq i_\tau} f_{\tau(j)}(x_{\tau(j)}) \right. \\ &\quad \left. \times \mathbf{1}_{\{x_{\tau(1)} \leq \dots \leq x_{\tau(i_\tau-1)} \leq \frac{u-b_\tau}{a_\tau} \leq x_{\tau(i_\tau+1)} \leq \dots \leq x_{\tau(n)}\}} \prod_{j \neq i_\tau} dx_{\tau(j)} \right) du \end{aligned}$$

We can integrate with respect to $dx_{\tau(i_\tau+1)} \otimes \dots \otimes dx_{\tau(n)}$

$$\begin{aligned} \mathbb{E}\left[g(C_\mu(X))\right] &= \sum_{\tau \in \mathcal{S}_n} \int_{\mathbb{R}} g(u) \left(\frac{1}{a_\tau} \int_{\mathbb{R}^{i_\tau-1}} f_{\tau(i_\tau)}\left(\frac{u-b_\tau}{a_\tau}\right) \theta_\tau\left(\frac{u-b_\tau}{a_\tau}\right) \right. \\ &\quad \left. \times \mathbf{1}_{\{x_{\tau(1)} \leq \dots \leq x_{\tau(i_\tau-1)} \leq \frac{u-b_\tau}{a_\tau}\}} \prod_{1 \leq j \leq i_\tau-1} f_{\tau(j)}(x_{\tau(j)}) \right. \\ &\quad \left. \times \prod_{1 \leq j \leq i_\tau-1} dx_{\tau(j)} \right) du \end{aligned}$$

with

$$\theta_\tau(v) := \int_{\mathbb{R}^{n-i_\tau}} \mathbf{1}_{\{v \leq x_{\tau(i_\tau+1)} \leq \dots \leq x_{\tau(n)}\}} \prod_{i_\tau+1 \leq j \leq n} f_{\tau(j)}(x_{\tau(j)}) \prod_{i_\tau+1 \leq j \leq n} dx_{\tau(j)}.$$

This proves (3.5).

5.2 Proof of Proposition 3.5

Recall that $\theta_\tau(v)$ has been defined by (3.2). This quantity can be calculated explicitly.

Lemma 5.1 *For any v :*

$$\theta_\tau(v) = \frac{1}{(n-i_\tau)!} (1-F(v))^{n-i_\tau}. \quad (5.2)$$

Proof. Since the r.v.'s X_1, \dots, X_n are i.i.d., they have the same probability density f . Therefore $\theta_\tau(v)$ takes the simpler form:

$$\theta_\tau(v) = \int_{\mathbb{R}^m} f(y_1) \times \dots \times f(y_m) \mathbf{1}_{\{v < y_1 < \dots < y_m\}} dy_1 \otimes \dots \otimes dy_m$$

where $m := n - i_\tau$.

Let g be a symmetric function of m variables, i.e. $g(y_1, \dots, y_m) = g(y_{\rho(1)}, \dots, y_{\rho(m)})$ for any permutation ρ of $\{1, \dots, m\}$ and $(y_1, \dots, y_m) \in \mathbb{R}^m$. Then

$$\int_{\mathbb{R}^m} g(y_1, \dots, y_m) dy_1 \otimes \dots \otimes dy_m = m! \int_{\mathbb{R}^m} g(y_1, \dots, y_m) \mathbf{1}_{\{y_1 < \dots < y_m\}} dy_1 \otimes \dots \otimes dy_m.$$

(5.3)

Applying this identity with $g(y_1, \dots, y_m) = f(y_1)1_{\{v < y_1\}} \times \dots \times f(y_m)1_{\{v < y_m\}}$ we get:

$$\begin{aligned}\theta_\tau(v) &= \frac{1}{m!} \int_{\mathbb{R}^m} f(y_1)1_{\{v < y_1\}} \times \dots \times f(y_m)1_{\{v < y_m\}} dy_1 \otimes \dots \otimes dy_m \\ &= \frac{1}{m!} \left(\int_{\mathbb{R}} f(y)1_{\{v < y\}} \right)^m.\end{aligned}$$

The result follows since $F(v) = \int_{-\infty}^v f(y)dy$. ■

We are now able to prove Proposition 3.5. By Proposition 3.2, Lemma 5.1 and (3.10), we have:

$$\begin{aligned}\phi_\tau(u) &= \frac{1}{a_\tau(n-j_l)!} \int_{\mathbb{R}^{j_l-1}} f\left(\frac{u-b_\tau}{a_\tau}\right) \left(1 - F\left(\frac{u-b_\tau}{a_\tau}\right)\right)^{n-j_l} \\ &\quad \times f(y_1) \times \dots \times f(y_{j_l-1}) 1_{\{y_1 \leq \dots \leq y_{j_l-1} \leq \frac{u-b_\tau}{a_\tau}\}} dy_1 \otimes \dots \otimes dy_{j_l-1}.\end{aligned}$$

By (3.7), (3.8) and (3.10), the coefficient b_τ defined by (3.6) equals

$$b_\tau = \sum_{i=1}^{j_l-1} \alpha_i y_i = \sum_{i=1}^{l-1} \alpha_{j_i} y_{j_i}$$

and therefore only depends on the variables $y_{j_1} \dots y_{j_{l-1}}$. Our strategy is to fix $y_{j_1} \dots y_{j_{l-1}}$ and to integrate with respect to the other variables. We integrate first with respect to $dy_{j_{l-1}+1} \otimes \dots \otimes dy_{j_l-1}$ the function $f(y_{j_{l-1}+1}) \times \dots \times f(y_{j_l-1})$ over $\left[y_{j_{l-1}}, \frac{u-b_\tau}{a_\tau}\right]$ with the constrain $y_{j_{l-1}+1} < \dots < y_{j_l-1}$. Using (5.3), we get :

$$\frac{1}{\delta_l!} \left(F\left(\frac{u-b_\tau}{a_\tau}\right) - F(y_{j_{l-1}}) \right)^{\delta_l}$$

which is fixed since $y_{j_1}, \dots, y_{j_{l-1}}$ are supposed to be given. We continue, integrating with respect to $dy_{j_{l-2}+1} \otimes \dots \otimes dy_{j_{l-1}-1}$ the function $f(y_{j_{l-2}+1}) \times \dots \times f(y_{j_{l-1}-1})$ over $[y_{j_{l-2}}, y_{j_{l-1}}]$ with the constrain $y_{j_{l-2}+1} < \dots < y_{j_{l-1}-1}$ leads to

$$\frac{1}{\delta_{l-1}!} \left(F(y_{j_{l-1}}) - F(y_{j_{l-2}}) \right)^{\delta_{l-1}}.$$

And so on. The last integration is related to $dy_1 \otimes \dots \otimes dy_{j_1-1}$ and gives

$$\frac{1}{(j_1-1)!} F(y_{j_1})^{j_1-1} = \frac{1}{\delta_1!} F(y_{j_1})^{\delta_1}.$$

Finally, setting $x_1 = y_{j_1}, \dots, x_{l-1} = y_{j_{l-1}}$, we get :

$$\begin{aligned} \phi_\tau(u) &= C_\tau \int_{\mathbb{R}^{l-1}} f(x_1) \times \dots \times f(x_{l-1}) f\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right) \left(1 - F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right)\right)^{n-j_l} \\ &\quad \times F(x_1)^{\delta_1} \left(\prod_{r=2}^{l-1} (F(x_r) - F(x_{r-1}))^{\delta_r}\right) \left(F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right) - F(x_{l-1})\right)^{\delta_l} \\ &\quad \times \mathbf{1}_{\{x_1 \leq \dots \leq x_{l-1} \leq \frac{u - \widehat{b}_{l-1}}{a_\tau}\}} dx_1 \otimes \dots \otimes dx_{l-1} \end{aligned}$$

where:

$$C_\tau = \frac{1}{a_\tau(n - j_l)!} \frac{1}{\delta_1! \times \dots \times \delta_l!}$$

and $\widehat{b}_{l-1} := \alpha_{j_1} x_1 + \dots + \alpha_{j_{l-1}} x_{l-1}$.

5.3 Proof of Proposition 3.7

According to Proposition 3.5,

$$\phi_\tau(u) = \frac{1}{a_\tau} \int_{\mathbb{R}^{l-1}} f(x_1) \times \dots \times f(x_{l-1}) \psi_1(x_1, \dots, x_{l-1}) dx_1 \otimes \dots \otimes dx_{l-1} \quad (5.4)$$

where

$$\psi_1(x_1, \dots, x_{l-1}) := f\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right) \psi_2(x_1, \dots, x_{l-1}) \mathbf{1}_{\{x_1 < \dots < x_{l-1} < \frac{u - \widehat{b}_{l-1}}{a_\tau}\}}$$

and

$$\begin{aligned} \psi_2(x_1, \dots, x_{l-1}) &= \frac{F(x_1)^{\delta_1}}{\delta_1!} \prod_{r=2}^{l-1} \frac{(F(x_r) - F(x_{r-1}))^{\delta_r}}{\delta_r!} \\ &\quad \times \frac{\left(F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right) - F(x_{l-1})\right)^{\delta_l}}{\delta_l!} \frac{\left(1 - F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right)\right)^{n-j_l}}{(n-j_l)!} \end{aligned}$$

It is very convenient to introduce:

$$y_0 := 0, \quad y_i := F(x_i) \quad \forall 1 \leq i \leq l-1, \quad y_l := F\left(\frac{u - \widehat{b}_{l-1}}{a_\tau}\right), \quad y_{l+1} := 1.$$

Since $x_1 < \dots < x_{l-1} < \frac{u - \widehat{b}_{l-1}}{a_\tau}$, then $y_0 \leq \dots \leq y_{l+1}$ and

$$\psi_2(x_1, \dots, x_{l-1}) = e \prod_{r=1}^{l+1} \frac{(y_r - y_{r-1})^{\delta_r}}{\delta_r!} e^{-(y_r - y_{r-1})}.$$

where $\delta_{l+1} := n - j_l$.

Let $(N_t)_{t \geq 0}$ be a Poisson process with parameter 1 and independent of X_1, \dots, X_n . Then

1. The random variables $N_{y_1}, N_{y_2} - N_{y_1}, \dots, N_{y_{l+1}} - N_{y_l}$ are independent,
2. for any $r \in \{1, \dots, l+1\}$, the random variable $N_{y_r} - N_{y_{r-1}}$ is Poisson distributed with parameter $y_r - y_{r-1}$.

We are now able to interpret $\psi_2(x_1, \dots, x_{l-1})$:

$$\begin{aligned}
\psi_2(x_1, \dots, x_{l-1}) &= e \prod_{r=1}^{l+1} \mathbb{P}(N_{y_r} - N_{y_{r-1}} = \delta_r) \\
&= e \mathbb{P}\left(\bigcap_{r=1}^{l+1} \{N_{y_r} - N_{y_{r-1}} = \delta_r\}\right) \\
&= e \mathbb{P}\left(\bigcap_{r=1}^{l+1} \{N_{y_r} = \delta_1 + \dots + \delta_r\}\right)
\end{aligned}$$

Let $(T_n)_{n \geq 1}$ be the increasing sequence of the jump times of $(N_t)_{t \geq 0}$:

$$N_t = \sum_{n \geq 1} 1_{\{T_n \leq t\}}, \quad t \geq 0.$$

Consequently:

$$\{N_t = k\} = \{T_k \leq t < T_{k+1}\}, \quad t \geq 0, k \in \mathbb{N}$$

and therefore

$$\psi_2(x_1, \dots, x_{l-1}) = e \mathbb{P}\left(\bigcap_{r=1}^{l+1} \{T_{j_r - r} \leq y_r < T_{j_r - r + 1}\}\right)$$

because relation (3.9) implies:

$$\delta_1 + \dots + \delta_r = j_r - r, \quad 1 \leq r \leq l+1$$

with the convention $j_{l+1} = n+1$.

It is well-know that $(T_n)_{n \geq 1}$ is distributed as $(\xi_1 + \dots + \xi_n)_{n \geq 1}$ where $(\xi_k)_{k \geq 1}$ is a sequence of i.i.d. random variables exponentially distributed with parameter 1. Finally relation (5.4) and the independency of $(\xi_k)_{k \geq 1}$ and (X_1, \dots, X_n) implies (3.18).

5.4 Proof of Proposition 4.1

Recall that the quantity $\bar{\mu}(k)$ has been defined by (4.1).

Lemma 5.2 *Let $1 \leq i \leq n$. Then*

$$\sum_{\tau \in \mathcal{S}_n} \mu(\tau(i : n)) = (i-1)! (n-i+1)! \bar{\mu}(n-i+1). \quad (5.5)$$

Proof. We have:

$$\begin{aligned}
\sum_{\tau \in \mathcal{S}_n} \mu(\tau(i:n)) &= \sum_{A \subset S, |A|=n-i+1} \left(\sum_{\tau \in \mathcal{S}_n, A=\tau(i:n)} \mu(\tau(i:n)) \right) \\
&= \sum_{A \subset S, |A|=n-i+1} \mu(A) \left(\sum_{\tau \in \mathcal{S}_n, A=\tau(i:n)} 1 \right) \\
&= (i-1)!(n-i+1)! \sum_{A \subset S, |A|=n-i+1} \mu(A)
\end{aligned}$$

since if $A \subset S$ is given and $A = \tau(i:n)$ then the image of $\{1, \dots, i-1\}$ (resp. $\{i, \dots, n\}$) is A^c (resp. A) and the number of possibilities is $(i-1)! \times (n-i+1)!$. Then (5.5) is a direct consequence of (4.1). \blacksquare

Lemma 5.3 For any $1 \leq i \leq n$,

$$\mathbb{E}(X_i 1_{\{X_1 < \dots < X_n\}}) = \frac{1}{(i-1)!(n-i)!} \mathbb{E}\left(F(X_1)^{i-1} (1-F(X_1))^{n-i} X_1\right). \quad (5.6)$$

Proof. Let $1 \leq i \leq n$, then

$$\mathbb{E}(X_i 1_{\{X_1 < \dots < X_n\}}) = \mathbb{E}(1_{\{X_1 < \dots < X_{i-1} < X_i\}} X_i 1_{\{X_i < X_{i+1} < \dots < X_n\}}).$$

Recall that X_1, \dots, X_n are i.i.d. then

$$\mathbb{E}(X_i 1_{\{X_1 < \dots < X_n\}}) = \mathbb{E}(\theta_{i-1}(X_i) X_i \theta^{n-i}(X_i))$$

where

$$\begin{aligned}
\theta_k(x) &:= \int_{\mathbb{R}^k} f(x_1) \times \dots \times f(x_k) 1_{\{x_1 < \dots < x_k < x\}} dx_1 \otimes \dots \otimes dx_k \\
\theta^k(x) &:= \int_{\mathbb{R}^k} f(x_1) \times \dots \times f(x_k) 1_{\{x < x_1 < \dots < x_k\}} dx_1 \otimes \dots \otimes dx_k.
\end{aligned}$$

Using (5.3), $\theta_k(x)$ can be simplified:

$$\begin{aligned}
\theta_k(x) &= \frac{1}{k!} \int_{\mathbb{R}^k} f(x_1) \times \dots \times f(x_k) 1_{\{x_1 < x, \dots, x_k < x\}} dx_1 \otimes \dots \otimes dx_k \\
&= \frac{F(x)^k}{k!}.
\end{aligned}$$

Similarly,

$$\theta^k(x) = \frac{(1-F(x))^k}{k!}.$$

The proof of (5.6) is now complete since X_i and X_1 have the same distribution. \blacksquare

We are now able to prove Proposition 4.1.
By definition,

$$C_\mu(X) = \sum_{i=1}^n X_{\sigma(i)} \left[\mu(\sigma(i:n)) - \mu(\sigma(i+1:n)) \right]$$

where σ is the random permutation such that $X_{\sigma(1)} < \dots < X_{\sigma(n)}$. We modify the previous sum:

$$\begin{aligned} C_\mu(X) &= \sum_{i=1}^n X_{\sigma(i)} \mu(\sigma(i:n)) - \sum_{i=1}^n X_{\sigma(i)} \mu(\sigma(i+1:n)) \\ &= \sum_{i=1}^n \mu(\sigma(i:n)) (X_{\sigma(i)} - X_{\sigma(i-1)}) \end{aligned}$$

with the conventions: $\sigma(0) = 0$, $X_0 = 0$ and $\mu(\sigma(n+1:n)) = 0$. Therefore

$$\begin{aligned} C_\mu(X) &= \sum_{\tau \in \mathcal{S}_n} \left(\sum_{i=1}^n \mu(\sigma(i:n)) (X_{\sigma(i)} - X_{\sigma(i-1)}) \right) 1_{\{\sigma=\tau\}} \\ &= \sum_{\tau \in \mathcal{S}_n} \left(\sum_{i=1}^n \mu(\tau(i:n)) (X_{\tau(i)} - X_{\tau(i-1)}) \right) 1_{\{X_{\tau(1)} < \dots < X_{\tau(n)}\}} \end{aligned} \tag{5.7}$$

We take the expectation and we use the crucial fact that $(X_{\tau(1)}, \dots, X_{\tau(n)})$ is distributed as (X_1, \dots, X_n) :

$$\begin{aligned} \mathbb{E}(C_\mu(X)) &= \sum_{\tau \in \mathcal{S}_n} \left(\sum_{i=1}^n \mu(\tau(i:n)) \mathbb{E}[(X_{\tau(i)} - X_{\tau(i-1)}) 1_{\{X_{\tau(1)} < \dots < X_{\tau(n)}\}}] \right) \\ &= \sum_{\tau \in \mathcal{S}_n} \left(\sum_{i=1}^n \mu(\tau(i:n)) \mathbb{E}[(X_i - X_{i-1}) 1_{\{X_1 < \dots < X_n\}}] \right) \\ &= \sum_{i=1}^n \left(\sum_{\tau \in \mathcal{S}_n} \mu(\tau(i:n)) \right) \mathbb{E}[(X_i - X_{i-1}) 1_{\{X_1 < \dots < X_n\}}]. \end{aligned}$$

Using Lemma 5.3, we get for $i \geq 2$:

$$\begin{aligned} \mathbb{E}[(X_i - X_{i-1}) 1_{\{X_1 < \dots < X_n\}}] &= \mathbb{E}[X_i 1_{\{X_1 < \dots < X_n\}}] - \mathbb{E}[X_{i-1} 1_{\{X_1 < \dots < X_n\}}] \\ &= \frac{1}{(i-1)!(n-i)!} \mathbb{E} \left(F(X_1)^{i-1} (1-F(X_1))^{n-i} X_1 \right) \\ &\quad - \frac{1}{(i-2)!(n-i+1)!} \mathbb{E} \left(F(X_1)^{i-2} (1-F(X_1))^{n-i+1} \right. \\ &\quad \left. \times X_1 \right). \end{aligned}$$

Consequently

$$\mathbb{E}[(X_i - X_{i-1}) 1_{\{X_1 < \dots < X_n\}}] = \frac{1}{(i-1)!(n-i+1)!} \mathbb{E} \left(F(X_1)^{i-2} (1-F(X_1))^{n-i} X_1 Y \right)$$

where

$$\begin{aligned} Y &:= (n - i + 1)F(X_1) - (i - 1)(1 - F(X_1)) \\ &= nF(X_1) - i + 1. \end{aligned}$$

We use finally Lemma 5.2:

$$\mathbb{E}(C_\mu(X)) = \sum_{i=1}^n \bar{\mu}(n-i+1)\mathbb{E}\left(F(X_1)^{i-2}(1-F(X_1))^{n-i}X_1(nF(X_1)-i+1)\right).$$

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