Heights on squares of modular curves
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Heights on squares of modular curves

Pierre Parent

with an appendix by Pascal Autissier

January 15, 2019

Abstract

We develop a strategy for bounding from above the height of rational points of modular curves with values in number fields, by functions which are polynomial in the curve’s level. Our main technical tools come from effective Arakelov descriptions of modular curves and jacobians. We then fulfill this program in the following particular case:

If $p$ is a not-too-small prime number, let $X_0(p)$ be the classical modular curve of level $p$ over $\mathbb{Q}$. Assume Brumer’s conjecture on the dimension of winding quotients of $J_0(p)$. We prove that there is a function $b(p) = O(p^5 \log p)$ (depending only on $p$) such that, for any quadratic number field $K$, the $j$-height of points in $X_0(p)(K)$ which are not lifts of elements of $X_0^+(p)(\mathbb{Q})$, is less or equal to $b(p)$.

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1 Introduction

Let $N$ be an integer, $\Gamma_N$ a level-$N$ congruence subgroup of $\text{GL}_2(\mathbb{Z})$, and $X_{\Gamma_N}$ the associated modular curve over some subfield of $\mathbb{Q}(\mu_N)$ which, to simplify the discussion, we assume from now on to be $\mathbb{Q}$. The genus $g_N$ of $X_{\Gamma_N}$ grows roughly as a polynomial function of $N$. So if $N$ is not too small, $X_{\Gamma_N}$ has only a finite number of rational points with values in any given number field, by Mordell-Faltings. If one is interested in explicitly determining the set of rational points however, finiteness is of course not sufficient; a much more desirable control would be provided by upper bounds, for some handy height, on those points. Proving such an “effective Mordell” is known to be an extremely hard problem for arbitrary algebraic curves on number fields.
In the case of modular curves, however, the situation is much better. Indeed, whereas the jacobian of a random algebraic curve should be a somewhat equally random simple abelian variety, it is well-known that the jacobian $J_{\Gamma_N}$ of $X_{\Gamma_N}$ decomposes up to isogeny into a product of quotient abelian varieties defined by Galois orbits of newforms for $\Gamma_N$. Moreover, in many cases, a nontrivial part of those factors happen to have rank zero over $\mathbb{Q}$. Our rustic starting observation is therefore the following: if $J_{\Gamma_N,e}$ is the “winding quotient” of $J_{\Gamma_N}$, that is the largest quotient $J_{\Gamma_N,e}$ with trivial $\mathbb{Q}$-rank, and

$$X_{\Gamma_N} \hookrightarrow J_{\Gamma_N} \xrightarrow{\pi_e} J_{\Gamma_N,e}$$

is some Albanese map from the curve to its jacobian followed by the projection to $J_{\Gamma_N,e}$, then any rational point on $X_{\Gamma_N}$ has an image which is a torsion point (because rational) on $J_{\Gamma_N,e}$, hence has 0 normalized height. The pull-back of some invertible sheaf defining the (say) theta height on $J_{\Gamma_N,e}$ therefore defines a height on $X_{\Gamma_N}$ which is trivial on rational points. That height in turn necessarily compares to any other natural one, for instance the modular $j$-height. Therefore the $j$-height of any rational point on $X_{\Gamma_N}$ is also zero “up to error terms”. Making those error terms explicit would give us the desired upper bound for the height of rational points on $X_{\Gamma_N}$.

That approach can in principle be generalized to degree-$d$ number fields, by considering rational points on symmetric powers $X_{\Gamma_N}^{(d)}$ of $X_{\Gamma_N}$ (at least if dim $J_{\Gamma_N,e} \geq d$). To be a little bit more precise in the present case of symmetric squares, let us associate to a quadratic point $P$ in $X_{\Gamma_N}(p)$ the $\mathbb{Q}$-point $Q := (P, \sigma P)$ of $X_{\Gamma_N}(p)^{(2)}$. Its image $i(Q)$ via some appropriate Albanese embedding in $J_0(p)$ lies above a torsion point $a$ in $J_{\Gamma_N}^e$: assume for simplicity $a = 0$. We therefore know $i(Q)$ belongs to the intersection of $i(X_0(p)^{(2)})$ with the kernel $\tilde{J}_e^\perp$ of the projection

$$\pi_e: J_0(p) \twoheadrightarrow J_{\Gamma_N}^e.$$  

To improve the situation we can further remark that $i(Q)$ actually lies at the intersection of $i(X_0(p)^{(2)})$ with the “projection”, in some appropriate sense, of the latter surface on $\tilde{J}_e^\perp$. Then one can show that this intersection is 0-dimensional (but here we need to assume Brumer’s conjecture, see below) so that its theta height is controlled, via some arithmetic Bézout theorem, in terms of the degree and height of the two surfaces we intersect. Using an appropriate version of Mumford’s repulsion principle one derives a bound for the height of $i(P)$ too (and not only for its sum $i(Q)$ with its Galois conjugate). Then one makes the translation again from theta height to $j$-height on $X_0(p)$.

Nontrivial technical work is of course necessary to give sense to the straightforward strategy sketched above. The aim of this article is thus to show the possibility of that approach, by making it work in what we feel to be the simplest non-trivial case: that of quadratic points of the classical modular curve $X_0(p)$ as above (or $X_0(p^2)$, for technical reasons), for $p$ a prime number\footnote{Larson and Vaintrob have proven, under the Generalized Riemann Hypothesis, the asymptotic triviality of rational points on $X_0(p)$ with values in any given number field which does not contain the Hilbert class field of some quadratic imaginary field (see \cite{Iw-Sar-Kow-Mic-Van}, Corollary 6.5). Independently of any conjecture, Momose had already proven the same result in the case where $K$ is a given quadratic number field (\cite{Mom}). Our method however provides bounds which do not depend on the field, and should generalize to some other congruence subgroups.}. In the course of the proof we are led to assume the already mentioned conjecture of Brumer, which asserts that the winding quotient of $J_0(p) := J_{\Gamma_N}(p)$ has dimension roughly half that of $J_0(p)$. That hypothesis is actually used in only one, technical, but crucial place, where we prove that a morphism between two curves is a generic isomorphism (see last point of Lemma 7.2). Note that a lower bound of $1/4$ (instead of the desired $1/2$) for the asymptotic ratio $\dim J_{\Gamma_N}^e/\dim J_0(p)$ has been proven by Iwaniec-Sarnak and Kowalski-Michel-VanderKam. (Actually, $(1/3 + \varepsilon)$ would be sufficient for us, see Lemma 7.2 and the proof of Theorem 7.5 below.) In any case we cannot at the moment get rid of this assumption - note it can in principle be numerically checked in all specific cases. In this setting, our main result is the following (see Theorem 7.5).

**Theorem 1.1** For $w_p$ the Fricke involution, set $X_0^+(p) = X_0(p)/w_p$. Assume Brumer’s conjecture (see Section 2, (21)$^2$). Then the quadratic points of $X_0(p)$, which are not lifts of elements of $X_0^+(p)(\mathbb{Q})$, have $j$-height bounded from above by $O(p^3 \log p)$.

\footnote{The weak version of that conjecture we actually need is stated in (22).}
The same holds true for quadratic points of $X_0(p^2)$, without the restriction about $X_0^+(p)$.

Needless to say, this result cries for both sharpening and generalization. Yet it should be possible to immediately use avatars of Theorem 1.1 to prove that rational points are only cusps and CM points, for some specific modular curves of arithmetic interest. If combined with lower bounds for heights furnished by isogeny theorems as in [5], the above theorem already has consequences on rational points (see Corollary 7.6).

Regarding past works about rational points on modular curves, one can notice that most of them use, at least in parts, some variants of Mazur’s method, which can very roughly be divided into two steps: first, map modular curves to winding quotients as described above; then prove some quite delicate properties about completions of that map to $J_0$ (formal immersion criteria). The second step is probably the most difficult to carry over to great generality. The method we here propose therefore allows one to use only the first and crucial fact - the mere existence of nontrivial winding quotients. In many cases, the existence of such quotients is known to be a deep result of Kolyvagin-Logachev-Kato, à la Birch-Swinnerton-Dyer Conjecture which, again, seems to reflect, from the arithmetic point of view, the quite special properties of the image locus (in the moduli space of principally polarized abelian varieties) of modular curves, among all algebraic curves, under Torelli’s map.

The methods used in this paper are mainly explicit Arakelov techniques for modular curves and abelian varieties. Such techniques and results have been pioneered, as far as we know, by Abbes, Michel and Ullmo at the end of the 1990s (see in particular [2], [43] and [62], whose results we here eagerly use). They have subsequently been revisited and extended in the work developed by Edixhoven and his school, as mainly (but not exhaustively) presented in the orange book [13]. That work was motivated by algorithmic Galois-representations issues, but its tools are well suited to our rational points questions, as we wish to show here. We similarly hope that the effective Arakelov results about modular curves and jacobians we work out in the present article shall prove useful in other contexts.

The layout of this article is as follows. In Section 2 we start gathering classical instrumental facts on quotients of modular jacobians and regular models of $X_0(p)$ over rings of algebraic integers. In Section 3 we make a precise description of the arithmetic Chow group of $X_0(p)$. Section 4 provides an explicit comparison theorem between $j$-heights and pull-back of normalized theta height on the jacobian. Section 5 computes the degree and Faltings height of the image of symmetric products within modular jacobians. In Section 6 we prove our arithmetic Bézout theorem (in the sense of [8]) for cycles in $J_0(p)$, relative to cubist metrics (instead of the more usual Fubini-Study metrics). This seems more natural, and has the advantage of being quantitatively much more efficient; that constitutes the technical heart of the present paper. Then we apply that arithmetic Bézout to our modular jacobian after technical computations on metric comparisons. Section 7 concludes the computations of the height bounds for quadratic rational points on $X_0(p)$ by making various intersections, projections and manipulations for which to refer to loc. cit.

Convention. In order to avoid numerical troubles, we safely assume in all what follows that primes are by definition strictly larger than 17.

## 2 Curves, jacobians, their quotients and subvarieties

### 2.1 Abelian varieties

#### 2.1.1 Decompositions

Let $K$ be a field, $J$ an abelian variety of dimension $g$ over $K$ and $\mathcal{L}$ an ample invertible sheaf defining a polarization of $J$. Assume $J$ is $K$-isogenous to a product of two (nonzero) subvarieties,

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3For recent investigations related to more general questions of effective bounds of algebraic points on curves, one can check [11].

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3
that is, there are abelian subvarieties

\[ \iota_A: A \hookrightarrow J, \quad \iota_B: B \hookrightarrow J \]

endowed with the polarization \( \iota_A^*(\mathcal{L}) \) and \( \iota_B^*(\mathcal{B}) \) respectively, such that \( \iota_A + \iota_B: A \times B \to J \) is an isogeny. (Recall that by convention here, all abelian (sub)varieties are assumed to be connected.) Then \( \pi_A: J \to A': := J \bmod B \), and similarly \( \pi_B: J \to B' \), are called **optimal quotients** of \( J \).

To simplify things we also assume from now on that \( \text{End}(A, B) = \{0\} \). The product isogeny \( \pi := \pi_A \times \pi_B: J \to A' \times B' \) induces isogenies \( A \to A' \) and \( B \to B' \). We write

\[ \Phi: A \times B \to J \to A' \times B' \]

for the obvious composition. Taking for instance dual isogenies of \( A \to A' \) and \( B \to B' \), we also define an endomorphism

\[ \Psi: J \to A' \times B' \to A \times B \to J. \]

When \( K = \mathbb{C} \), the above constructions are transparent. There is a \( \mathbb{Z} \)-lattice \( \Lambda \) in \( C^g \), endowed with a symplectic pairing, such that \( J(\mathbb{C}) \simeq C^g / \Lambda \) and one can find a direct sum decomposition \( C^g = C^{gA} \oplus C^{gB} \) such that \( \Lambda_A = \Lambda \cap C^{gA} \) and \( \Lambda_B = \Lambda \cap C^{gB} \), then

\[ A(\mathbb{C}) \simeq C^{gA} / \Lambda_A \quad \text{and} \quad B(\mathbb{C}) \simeq C^{gB} / \Lambda_B. \]

If \( p_A: C^g \to C^{gA} \) and \( p_B: C^g \to C^{gB} \) are the \( \mathbb{C} \)-linear projections relative to that decomposition, the analytic description of \( \pi_A, \pi_C: J(\mathbb{C}) \to A'(\mathbb{C}) \) is then

\[ z \bmod \Lambda \mapsto z \bmod (\Lambda_A + \Lambda_B) = p_A(z) \bmod (p_A(\Lambda)). \]

Summing-up we have lattice inclusions: \( \Lambda_A \subseteq p_A(\Lambda), \quad \Lambda_B \subseteq p_B(\Lambda), \) with finite indices, in \( C^g \) such that our isogenies are induced by

\[ \Lambda_A \oplus \Lambda_B \subseteq \Lambda \subseteq p_A(\Lambda) \oplus p_B(\Lambda). \]

The isogeny \( I'_A: A \to A' \) deduced from the inclusion \( \Lambda_A \subseteq p_A(\Lambda) \) has degree \( \text{card}(p_A(\Lambda) / \Lambda_A) \).

If \( N_A \) is a multiple of the exponent of the quotient \( p_A(\Lambda) / \Lambda_A \), there is an isogeny \( I_{A,N_A}: A' \to A \) such that \( I_{A,N_A} \circ I'_A \) and \( I'_A \circ I_{A,N_A} \) both are multiplication by \( N_A \). The analytic descriptions of the above clearly are:

\[ \left\{ \begin{array}{c} A(\mathbb{C}) \simeq C^{gA} / \Lambda_A \quad \overset{I'_A}{\to} \quad A'(\mathbb{C}) \simeq C^{gA} / p_A(\Lambda) \\ z \mapsto \quad z \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{c} C^{gA} / p_A(\Lambda) \quad \overset{I_{A,N_A}}{\to} \quad C^{gA} / N_A \Lambda_A \\ z \mapsto \quad N_A z \end{array} \right. \]

**Remark 2.1** Instead of considering two immersions as in (1), suppose only \( A \hookrightarrow J \) is given, and \( K \) is a number field. One might apply [21], Théorème 1.3, to deduce the existence of an abelian variety \( B \) over \( K \) such that, with our previous notations, the degree of \( A \times B \to J \):

\[ |A \cap B| = |A / \Lambda_A \oplus \Lambda_B| \]

is bounded from above by an explicit function \( \kappa(J) \) of the stable Faltings’ height \( h_F(J) \):

\[ \kappa(J) = \left( (14g)^{649^3} [K : \mathbb{Q}] \max(h_F(J), \log(K : \mathbb{Q}), 1)^2 \right)^{2^{10}g^3} \]

and this does not depend on the choice of the embedding \( K \hookrightarrow \mathbb{C} \). Note that when \( A \) and \( (J \text{ mod } A) \) are not isogenous (which will be the case for us), then there is actually no choice for that \( B \hookrightarrow J \): it has to be the Poincaré complement to \( A \). The isogeny \( J \to A' \times B' \) given by the two projections has degree \( |p_A(\Lambda) \oplus p_B(\Lambda) / \Lambda| \), which also is \( |A \cap B| := N \). One can therefore take the \( N_A \) appearing in (3) as equal to \( N \), and

\[ N \leq \kappa(J). \]

Making the same for \( B' \to B \), the above morphism \( \Psi \) (see (2)) is then simply the multiplication \( J \overset{[N]}{\longrightarrow} J \) by the integer \( N \). Although we will not need numerical estimates for those quantities in what follows, it is straightforward, using [62], to make them explicit in our setting of modular curves and jacobians.
2.1.2 Polarizations and heights

Keeping the above notations and hypothesis, consider in addition now an ample sheaf \( \Theta \) on \( J \), and let \( I_A := I_{A,N} : A' \to A \) (respectively, \( I_{B,N} \)) be as in (3). We pull-back \( \Theta \) along the composed morphism:

\[
\varphi_A : J \xrightarrow{\pi_A} A' \xrightarrow{I_A} A \xrightarrow{\iota_A} J
\]

so that the immersion \( \iota_A : A \hookrightarrow J \) defines a polarization \( \Theta_A := \iota_A^*(\Theta) \) on \( A \), whence a polarization \( \Theta_{A'} := I_A^*(\Theta_A) \) on \( A' \), and finally an invertible sheaf \( \Theta_{J,A} := \pi_A^*(\Theta_A) \) on \( J \). Composing the morphisms:

\[
J \xrightarrow{\pi_A \times \pi_B} A' \times B \xrightarrow{I_A \times I_B} A \times B \xrightarrow{\iota_A \times \iota_B} J
\]

gives the multiplication-by-\( N \) map: \( J \xrightarrow{[N]} J \). Assuming for simplicity \( \Theta \) is symmetric one therefore has

\[
[N]^* \Theta \simeq \Theta \otimes \Theta^N \simeq \Theta_{J,A} \otimes_{\mathcal{O}_J} \Theta_{J,B}.
\]

If \( K \) is a number field, the Néron-Tate normalization process associates with \( \Theta \) a system of compatible Euclidean norms \( h_{\Theta} = \| \cdot \|_{\Theta}^2 \) on the finite-dimensional \( \mathbb{Q} \)-vector spaces \( J(F) \otimes \mathbb{Z} \mathbb{Q} \), for \( F/K \) running through the number field extensions of \( K \), and similarly Euclidean norms \( h_{\Theta_A} := \| \cdot \|_{\Theta_A}^2 \) on \( A(F) \otimes \mathbb{Z} \mathbb{Q} \) and \( h_{\Theta_B} := \| \cdot \|_{\Theta_B}^2 \) on \( B(F) \otimes \mathbb{Z} \mathbb{Q} \) such that, under the isomorphisms \( J(F) \otimes \mathbb{Z} \mathbb{Q} \simeq (A(F) \otimes \mathbb{Z} \mathbb{Q}) \oplus (B(F) \otimes \mathbb{Z} \mathbb{Q}) \), one has

\[
h_{\Theta} = h_{\Theta_A} + h_{\Theta_B}.
\]

Recall from (3) the definition of \( N_A \), that of the maps \( A' \xrightarrow{I_A,N_A} A \) and \( A \xrightarrow{\iota_A} J \). Denote by \([N_A]_A\) the multiplication by \( N_A \) restricted to \( A \). If \( V \) is a closed algebraic subvariety of \( J \), define

\[
P_A(V) := ([N_A]_A^{-1} I_{A,N_A} \pi_A)(V)
\]

as the reduced closed subscheme with relevant support. That map \( P_A \) would simply be the projection of \( V \) on \( A \) if \( J \) were isomorphic to the product \( A \times B \) of subvarieties, and is the best approximation to that projection in our case when \( J \) is only isogenous to \( A \times B \).

Note that \( P_A(V) \) is a priori highly non-connected. All its irreducible geometric components are however obtained from each other by translation by a \( N_A \)-torsion point of \( A(\mathbb{Q}) \). For our later purposes (see proof of Theorem 7.5), we will have the possibility to replace \( P_A(V) \) by one of its components containing a specific point, say \( P_0 \); we shall denote that component by \( P_A(V)_{P_0} \), and refer to it as the “pseudo-projection” of \( V \) on \( A \) containing \( P_0 \).

Suppose now \( J \sim A \times B \) as above is the jacobian of an algebraic curve \( X \) on \( K \) with positive genus \( g \). For \( P_0 \) a point of \( X(K) \) (or more generally a \( K \)-divisor of degree 1 on \( X \)) let

\[
\iota_{P_0} : \begin{cases} \begin{array}{ccc} X & \hookrightarrow & J \\ P & \mapsto & (P) - (P_0) \end{array} \end{cases}
\]

be the Albanese embedding associated with \( P_0 \). We define the classical Theta divisor \( \Theta \) on \( J \) which is the image of \( \iota_{P_0}^{-1} : X^{g-1} \to J \) and its symmetric version

\[
\Theta := (\Theta \otimes \Theta, [-1]^* \Theta) \oplus \frac{1}{2}
\]

(which is a translate of \( \Theta \) obtained as \( \iota_{P_0}^{-1}(X^{g-1}) \), where \( \iota_{\kappa_0} = t_{\kappa_0} \iota_{P_0} \) for \( \kappa_0 \) the translation by some \( \kappa_0 \) with \( (2g-2)\kappa_0 = \kappa \): the canonical divisor on \( X \). Of course \( \Theta \) does not need to be defined over \( K \). Our first aim will be to compare the height functions \( \| \iota_{P_0}(\cdot) \|_{\Theta_A}^{\frac{1}{2}} \) on \( X(F) \), when \( X \) is a modular curve, with another natural height given by the modular \( j \)-function.
We will discuss in Section 3 an Arakelov description of Néron-Tate height. We conclude this paragraph by a few remarks as a preparation. Let $B_2 := \{\omega_1, \ldots, \omega_g\}$ be a basis of $H^0(X(\mathbb{C}), \Omega^1_{X/\mathbb{C}}) \simeq H^0(J(\mathbb{C}), \Omega^1_{J/\mathbb{C}})$, which is orthogonal with respect to the norm
\[
||\omega||^2 = \frac{i}{2} \int_{X(\mathbb{C})} \omega \wedge \overline{\omega}.
\]
The transcendent writing-up of the Abel-Jacobi map $\iota_{P_0} : P \mapsto (\int_{P_0}^P \omega_i)_{1 \leq i \leq g}$ shows that the pullback to $X(\mathbb{C})$ of the translation-invariant measure on $J(\mathbb{C})$, normalized to have total mass 1, is
\[
\mu_0 = \frac{i}{2g} \sum_{B_2} \frac{\omega \wedge \overline{\omega}}{||\omega||^2}.
\]

More generally, $\pi_A \circ \iota_{P_0}$ is, over $\mathbb{C}$, the map $P \mapsto (\int_{P_0}^P \omega)_{\omega \in B_2^A}$, where $B_2^A$ is some orthogonal basis of $H^0(A(\mathbb{C}), \Omega^1_{A/\mathbb{C}}) \simeq H^0(J(\mathbb{C}), \pi^*_A(\Omega^1_{J/\mathbb{C}})) \subseteq H^0(J(\mathbb{C}), \Omega^1_{J/\mathbb{C}})$. Therefore, writing $g_A := \dim(A') = \dim(A)$ (we assume $A \neq 0$), the pull-back to $X(\mathbb{C})$ of the translation-invariant measure on $A'(\mathbb{C})$ (normalized so to have total mass 1 on the curve again) is
\[
\mu_A = \frac{i}{2g_A} \sum_{B_2^A} \frac{\omega \wedge \overline{\omega}}{||\omega||^2}. \tag{11}
\]

2.2 Modular curves

Here we recall a few classical facts on the minimal regular model of the modular curve $X_0(p)$, for $p$ a prime number, over a ring of algebraic integers. The first general reference on this topic is [14]; see also [13] or [40], [41].

2.2.1 The $j$-height

The quotient of the completed Poincaré upper half-plane $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ by the classical congruence subgroup $\Gamma_0(p)$ defines a Riemann surface $X_0(p)(\mathbb{C})$ which is known to have a geometrically connected smooth and proper model over $\mathbb{Q}$. All through this paper, we denote its genus by $g$.

The first technical theme of this article is the explicit comparison of various heights on $X_0(p)(\mathbb{C})$. When $V$ is an algebraic variety over a number field $K$, any finite $K$-map $\varphi : V \to \mathbb{P}^N_K$ to some projective space defines a naive Weil height on $V(\mathbb{K})$. This applies in particular when $V$ is a curve and $\varphi$ is the finite morphism defined by an element of the function field of $V$, and in the case of a modular curve $X_\Gamma$ associated with some congruence subgroup $\Gamma$, say, a natural height to choose on $X_\Gamma(\mathbb{C})$ is precisely Weil’s height $h(P) = h(j(P))$ relative to the classical $j$-function.

The degree of the associated map $X_\Gamma \to X(1) \simeq \mathbb{P}^1$ is $|\text{PSL}_2(\mathbb{Z}) : \Gamma|$, so that number is the class of our Weil height in the Néron-Severi group $\text{NS}(X_\Gamma)$ identified with $\mathbb{Z}$. More explicitly if $X = X_\Gamma$ is defined over the number field $K$, say, the $j$-morphism is
\[
\left\{ \begin{array}{l}
\quad X \xrightarrow{j} \mathbb{P}^1_K = \text{Proj}(K[X_0, X_1]) \leftrightarrow K_1^1 \leftarrow \text{Spec}(K[X_1/X_0]) \\
\quad P \mapsto (1, j(P)) = (1/j(P), 1) \leftarrow j(P) = \frac{\Delta}{X_1^2}(P),
\end{array} \right.
\]
and the Weil height of a point $P \in X(K)$ is therefore the naive height of its $j$-invariant as an algebraic number:
\[
h(P) = h(j(P)) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log(\max(1, |j(P)|_v))
\]
which is also Weil’s projective height $h(j(P))$ with respect to the above basis $(X_0, X_1 = X_0 j)$ of global sections of $\mathcal{O}_{\mathbb{P}^1_K}(1)$. Our Weil height on $X$ is associated with the linear equivalence classes.
of divisors $D$ corresponding to $j^*(\mathcal{O}_{\mathcal{X}_K}(1))$, so that

$$D \sim (\text{poles of } j \text{ on } X)(\sim (\text{zeroes of } j)) \sim \sum_{c \in \{\text{cusps of } X\}} e_c c$$

where each $e_c$ is the ramification index of $c$ via $j$.

Those considerations lead to explicit comparisons with other heights. Indeed, a more intrinsic way to define heights on algebraic varieties is provided by Arakelov theory. Defining this properly in the case of our modular curves demands a precise description of regular models for them, which we now recall.

### 2.2.2 Regular models

The normalization of the $j$-map $X_0(p) \to X(1)/\mathbb{Z} \simeq \mathbb{P}^1/\mathbb{Z}$ over $\mathbb{Z}$ defines a model for $X_0(p)$, that we call the modular model, it is smooth over $\mathbb{Z}[1/p]$.

We fix a number field $K$, write $\mathcal{O}_K$ for its ring of integers, and deduce by base change a model for $X_0(p)$ over $\mathcal{O}_K$. We know its only singularities are normal crossing, so after a few blow-ups if necessary we obtain a regular model of $X_0(p)$ over $\mathcal{O}_K$: see Theorem 1.1.d) of the Appendix of [39]. We denote it from now on by $X_0(p)/\mathcal{O}_K$, or simply $X_0(p)$ if the context prevents confusion.

We stress here that for $F/K$ a field extension, $X_0(p)/\mathcal{O}_F$ is not the base change to $\mathcal{O}_F$ of $X_0(p)/\mathcal{O}_K$ if $F/K$ ramifies above $p$. Let $v$ be a place of $\mathcal{O}_K$ above $p$, with residue field $k(v)$. The dual graph of $X_0(p)$ at $v$ is made of two extremal vertices, which we label $C_0$ and $C_\infty$, containing the cusps 0 and $\infty$ respectively (see Figure 1). Those two vertices, which correspond to irreducible components of genus 0, are linked by

$$s := g + 1$$

branches. Each branch corresponds to a singular point $S$ in $X_0(p)(\mathbb{F}_p^\times)$, which in turn parameterizes an isomorphism class of supersingular elliptic curve $E_S$ in characteristic $p$.

The Fricke involution $w$ acts on the dual graph as the continuous isomorphism which exchanges $C_0$ and $C_\infty$ and acts on the branches as a generator of $\text{Gal}(\mathbb{F}_p^\times/\mathbb{F}_p)$.

We list the supersingular points as $S(1), \ldots S(s)$ and for each one define

$$w_n := \#\text{Aut}(S(n))/\langle\pm 1\rangle := \#\text{Aut}_{\mathbb{F}_p^\times}(E_{S(n)})/\langle\pm 1\rangle$$

which is equal to 1 except in the (at most two) cases when the underlying supersingular elliptic curve has $j$-invariant 1728 or 0, where it is equal to 2 or 3 respectively. Now each path, or branch, on our dual graph at $v$ passes through $(w_n e^{-1})$ vertices (for $e$ the ramification index of $K$ at $v$), that is, again, equal to $e^{-1}$ except for at most two branches: one of length $2e - 1$ (obtained by blowing-up the supersingular point of moduli $j = 1728 \text{ mod } v$, if it exists), and a path of length $3e - 1$ (obtained by blowing-up, if needed, at the supersingular point of moduli $j = 0 \text{ mod } v$). We enumerate the vertices $\{C_{n,m}\}_{1 \leq m \leq w_n e^{-1}}$ in the $n^\text{th}$ path. We also denote by $w(E)$ the familiar quantity $\sum_{1 \leq n \leq s} \frac{1}{w_n}$, the sum being taken over the set of all supersingular points of $X_0(p)/\mathcal{O}_{K,s}$. The well-known Eichler mass formula says that

$$w(E) = \sum_{1 \leq n \leq s} \frac{1}{w_n} = \frac{p - 1}{12}$$

(see for instance [24], p. 117). Recall this implies that the genus $g$ of $X_0(p)$ is asymptotically equivalent to $p/12$ (the exact formula depending on the residue class of $p \text{ mod } 12$) and in any case:

$$\frac{p - 13}{12} \leq g \leq \frac{p + 1}{12}$$

(see for instance p. 117 of [24] again).

Abusing a bit notations, $C_\infty$ will sometimes be also denoted as $C_{n,0}$, and similarly $C_0$ might be written as $C_{n,w,e}$. We choose as a basis for $\oplus_{C \mathbb{Z}} \cdot C$ the ordered set

$$\mathcal{B} = (C_\infty, (C_{1,1}, C_{1,2}, \cdots, C_{1,e-1}), (C_{2,1}, \cdots, C_{2,e-1}), \ldots, (C_{s,1}, \cdots, C_{s,w,e-1}), C_0)$$
(that is, we enumerate the vertices by running through each branch successively, and put the possible branches of length twice or thrice the generic length at the end). At bad places $v$ the intersection matrix restricted to each submodule $\oplus_{m=1}^{w_n e^{-1}} \mathcal{C}_{n,m}$ (for some fixed branch of index $n$) is then $(\log(#k(v)) \cdot \mathcal{M}_0$, where

$$
\mathcal{M}_0 = \begin{pmatrix}
-2 & 1 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & -2
\end{pmatrix},
$$

(17)

whose only dependence on $n$ is that its type is $(w_n e^{-1}) \times (w_n e^{-1})$. That matrix has determinant $(-1)^{w_n e^{-1}} w_n e$. Define the row vectors:

$L := (1 \ 0 \ 0 \ \cdots \ 0), \ L' := (0 \ 0 \ 0 \ \cdots \ 1)$

(with length implicitly defined by the next lines) and the transpose column vectors:

$V := L^t, \ V' := L'^t.$

The intersection matrix on the whole space $\mathbb{Z}^B$ is finally $(\log(#k(v)) \cdot \mathcal{M})$ for

$$
\mathcal{M} = \begin{pmatrix}
-s & L & L & \cdots & L & 0 \\
V & \mathcal{M}_0 & 0 & \cdots & 0 & V' \\
V & 0 & \mathcal{M}_0 & \cdots & 0 & V' \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & L' & L' & \cdots & L' & -s
\end{pmatrix}.
$$

(18)

(This has to be modified in the obvious way when $e_v = 1$.)

2.2.3 Winding quotients, their dimension

We denote as usual the jacobian of $X_0(p) \mathbb{Q}$ by $J_0(p)$. As follows from section 2.2.2, $X_0(p)$ is semistable over $\mathbb{Z}$, and the neutral component of the Néron model $\mathcal{J}_0(p)$ of $J_0(p)$ is a semi-abelian scheme over $\mathbb{Z}$ (and an abelian scheme over $\mathbb{Z}[1/p]$). Its neutral component represents the neutral component $\mathcal{P}_{\mathbb{Z}}(X_0(p))$ of the relative Picard functor of $X_0(p)$ over $\mathbb{Z}$.

We know from Shimura’s theory that the natural decomposition of cotangent spaces into Hecke eigenspaces induces a corresponding decomposition over $\mathbb{Q}$ of abelian varieties up to isogenies:

$$
J_0(p) \sim \prod_{f \in B_2/\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} J_f
$$

(19)
Iwaniec, Luo and Sarnak prove one can improve Corollary 13 and [34]). Breaking that \( w_\phi \) are true without restriction on Fourier transform of to be all for the moment.

\[ \prod \text{associated with} \]

whose Fourier transform is supported on the interval \([-2, 2]\). Assuming the Generalized Riemann Hypothesis for \( L \) functions of eigenforms \( f \), that is, \( J_0(p) \) for large enough \( p \).

\[ \text{Identity (3.2).} \]

\[ \text{Equivalently, it is conjectured that dim} \]

\[ \text{density conjecture of [28], p. 56 et seq., see also Remark F on p. 65.} \]

\[ \text{Actually, what we eventually need in this article (see Section 7) is a weaker form of (21), which is:} \]

\[ \dim J_e > \frac{\dim J_0(p)}{3} + \frac{2}{3} \]

for large enough \( p \). An important theorem of Iwaniec-Sarnak and Kowalski, Michel and Vanderkam asserts something nearly as good, that is:

\[ \left( \frac{1}{4} - o(1) \right) \dim J_0(p) \leq \dim J_e \leq \left( \frac{1}{2} + o(1) \right) \dim J_0(p) \]

as \( p \) goes to infinity (so that \( \left( \frac{1}{4} - o(1) \right) \dim J_0(p) \leq \dim J_e \leq \left( \frac{1}{2} + o(1) \right) \dim J_0(p) \), see [29], Corollary 13 and [34]). Breaking that \( \frac{1}{4} \) is known to be closely linked to the Landau-Siegel zero problem. Assuming the Generalized Riemann Hypothesis for \( L \)-functions of modular forms, Iwaniec, Luo and Sarnak prove one can improve \( \frac{1}{4} \) to \( \frac{3}{12} \) ([28], Corollary 1.6, (1.54))... That seems to be all for the moment.

The central object of this paper will eventually be the maps

\[ X_0(p)^{(d)} \rightarrow J_0(p) \rightarrow J_e \]

from symmetric products of \( X_0(p) \) (mainly the curve itself and its square) to the winding quotient.

3 Arithmetic Chow group of modular curves

We now give a description of the Arakelov geometry of \( X_0(p) \), relying on the work of many people: that topic has been pioneered by Abbes, Ullmo and Michel ([2], [43], [62]) and notably developed

Quoting Olga Balkanova (private communication), “Theorem 1.1 in [28] is proved for the test function \( \phi \), whose Fourier transform is supported on the interval \([-2, 2]\). The density conjecture claims that the same results are true without restriction on Fourier transform of \( \phi \), see formula 1.9 [of loc. cit.].”
by Edixhoven, Couveignes and their coauthors (see [13]). We shall also use the work of Bruin ([9]), Jorgenson-Kramer ([32]) and Menares ([40], [41]) among others. We refer to those articles and their bibliography for general facts on Arakelov theory (see [12], [16]).

Let \( \mathcal{X} \) be any regular and proper arithmetic surface over the integer ring \( \mathcal{O}_K \) of a number field \( K \). Fixing in general smooth hermitian metrics \( \mu \) on the base changes of \( \mathcal{X} \) to \( \mathcal{C} \), it follows from the basics of Arakelov theory that for any horizontal divisor \( D \) on \( \mathcal{X} \) over \( \mathcal{O}_K \) there are Green functions \( g_{\mu,D} \) on each Archimedean completion \( \mathcal{X}(\mathcal{C}) \) satisfying the differential equation

\[
\Delta g_{\mu,D} = -\delta_D + \deg(D)\mu
\]

for \( \Delta = \frac{1}{i\pi} \partial \bar{\partial} \) the Laplace operator and \( \delta_D \) the Dirac distribution relative to \( DC \) on \( \mathcal{X}(\mathcal{C}) \). The function \( g_{\mu,D} \) is integrable on the compact Riemann surface \( \mathcal{X}(\mathcal{C}) \) endowed with its measure \( \mu \), and uniquely determined up to an additive constant which is often fixed by imposing the normalizing condition that

\[
\int_{\mathcal{X}(\mathcal{C})} g_{\mu,D} \, d\mu = 0. \tag{24}
\]

When the horizontal divisor \( D \) is a section \( P_0 \) in \( \mathcal{X}(\mathcal{O}_K) \), one will sometimes also use the notation \( g_{\mu}(P_0, z) \) for \( g_{\mu,P_0}(z) \). The Green functions relative to fixed smooth \((1,1)\)-forms \( \mu \) allow one to define an Arakelov intersection product relative to the \( \mu \), which will be denoted by \( [\cdot, \cdot]_\mu \), or \([\cdot, \cdot]\) if there is no ambiguity about the implicit form. In particular the index will often be dropped for divisors intersections of which one at least is vertical, where the choice of \( \mu \) does not intervene.

We shall denote by \( \mu_0 \) the canonical Arakelov \((1,1)\)-form on the Riemann surface \( \mathcal{X}(\mathcal{C}) \) (assumed to have positive genus), inducing the “flat metric”. It corresponds to the pullback, by any Albanese morphism \( \mathcal{X}(\mathcal{C}) \to \text{Jac}(\mathcal{X}_K)(\mathcal{C}) \), of the “cubist” metric in the sense of Moret-Bailly ([46], more about this shortly) on the jacobian \( \text{Jac}(\mathcal{X}_K) \), associated with the Néron-Tate normalized height \( h_0 \).

We now specialize to the case of \( \mathcal{X}_0(p) \) as in Section 2.2. If \( f \) is a modular form of weight 2 for \( \Gamma_0(p) \), let \( \|f\|^2 \) be its Petersson norm. Because newforms are orthogonal in prime level we have, as in (11):

\[
\mu_0 := \frac{i}{2 \dim(J_0(p))} \sum_{f \in B_2} \frac{\int dq \wedge \int \overline{dq}}{\|f\|^2}. \tag{25}
\]

We shall also need to consider Néron-Tate heights \( h_A \) for subabelian varieties \( A \hookrightarrow J_0(p) \) as in section 2.1.2 (recall \( A \neq 0 \)). The associated \((1,1)\)-form \( \mu_A \) is given by (12). More specifically, we focus on \( h_\Theta \) on \( J_c \) (as in (7) and around, for \( A' = J_c \) which induces a height \( h_\Theta \circ \iota_c.P_0 \) on \( X_0(p) \) via the map \( \iota_c.P_0 : X_0(p) \to J \to J_c \). The curvature form of the hermitian sheaf on \( X_0(p) \) defining the Arakelov height associated with \( h_\Theta \circ \iota_c.P_0 \) is

\[
\mu_c := \frac{i}{2 \dim(J_c)} \sum_{f \in B_2} \frac{\int dq \wedge \int \overline{dq}}{\|f\|^2}. \tag{26}
\]

where \( B_2[J_c] \) stands for the set of newforms killed by the ideal \( J_c \) defining \( J_c \) as in (20).

**Remark 3.1** Notice that both \( \mu_0 \) and \( \mu_c \), or any \( \mu_A \) above, are invariant by pull-back \( w_p^* \) by the Fricke involution. In particular the Arakelov intersection products \([\cdot, \cdot]_{\mu_0}\) and \([\cdot, \cdot]_{\mu_c}\), relative to \( \mu_0 \) and \( \mu_c \) respectively, are \( w_p \)-invariant. The latter was clear already from the fact that, more generally, \( w_p \) is an orthogonal symmetry on \( J_0(p) \) endowed with its quadratic form \( h_\Theta \), which respects the orthogonal decomposition \( \prod_f J_f \) of (19).

One can now specialize the Hodge index theorem to our modular setting (see [41], Theorem 4.16, [40], Theorem 3.26, or more generally [46], p. 85 et seq.):
Theorem 3.2 Let $K$ be a number field, $\mu$ be a smooth non-zero $(1,1)$-form on $X_0(p)(\mathbb{C})$ as given in (12), and $\hat{CH}(p)_{num}^\mu$ be the arithmetic Chow group with real coefficients up to numerical equivalence of $X_0(p)$ over $\hat{O}_K$, relative to $\mu$. Denote by $\infty$ the horizontal divisor defined by the $\infty$-cusp on $X_0(p)$ over $\mathbb{Z}$ (which is the Zariski closure of the $\mathbb{Q}$-point $\infty$ in $X_0(p)(\mathbb{Q})$), compactified with the normalizing condition (24). Write $\mathbb{R} \cdot X_\infty$ for the line of divisors with real coefficients supported on some fixed full vertical fiber $X_\infty$. Define, for all $v \in \text{Spec}(O_K)$ above $p$, the $\mathbb{R}$-vector space:

$$G_v := \bigoplus_{C \neq C_\infty} \mathbb{R} \cdot C$$

where the sum runs through all the irreducible components of $X_0(p) \times_{O_K} k(v)$ except $C_\infty$ (the one containing $\infty(k(v))$). Identify finally $J_0(p)(K)/\text{torsion}$ with the subgroup of divisor classes $D_0$ which are compactified under the normalizing condition $g_{D_0}(\infty) = 0$ (which is therefore different from (24)). One has a decomposition:

$$\hat{CH}(p)_{num}^\mu = (\mathbb{R} \cdot \infty \oplus \mathbb{R} \cdot X_\infty) \oplus \mu_1^{\perp} G_v \oplus \mu_1 \oplus (J_0(p)(K) \otimes \mathbb{R})$$

(27)

where the “$\oplus \perp$” mean that the direct factors are mutually orthogonal with respect to the Arakelov intersection product. Moreover, the restriction of the self-intersection product to $J_0(p)(K) \otimes \mathbb{R}$ coincides with twice the opposite of the Néron-Tate pairing.

Proof The proof can be immediately adapted from that of [41], Theorem 4.16, for $L^1$-admissible measures (a setting allowing to define convenient actions of the Hecke algebra on the Chow group). For further computational use we recall how one decomposes divisors in practice. Take $D$ in $\hat{CH}(p)_{num}^\mu$, with degree $d$ on the generic fiber. There is a vertical divisor $\Phi_D$, with support in fibers above places of bad reduction (that is, of characteristic $p$), such that $(D - d\infty - \Phi_D)$ has a real multiple which belongs to the neutral component $\text{Pic}^0(J_0(p))/O_K$. That $\Phi_D$ is well-defined up to multiple of full vertical fibres, so we can assume $\Phi_D$ belongs to $\mu^\perp \oplus G_v$ (and is then unambiguously defined). One associates to $(D - d\infty - \Phi_D) \in \mathbb{R} \cdot J_0(p)(O_K)$ an element $\delta$ in $\hat{CH}(p)_{num}^\mu$ by imposing a compactification such that $[\infty, \delta]_\mu = 0$. The general Hodge index theorem (see for instance [46]) then finally asserts that $(D - d\infty - \Phi_D - \delta)$ can be written as an element in $\mathbb{R} \cdot X_\infty$.

In order to later on interpret the Néron-Tate height (associated with some given (symmetric) invertible sheaf) as an Arakelov intersection in a suitable sense (see [1] paragraph 3, or [47]), we will need to compute explicitly, given $P \in X_0(p)(K)$, the vertical divisor $\Phi_P = \oplus \mu \Phi_{P,v}$ such that

$$[C, P - \infty - \Phi_P] = 0$$

(28)

for any irreducible component of any fiber of $X_0(p) \to \text{Spec}(O_K)$, as in the proof of Theorem 3.2.

Lemma 3.3 Consider a bad fiber $X_0(p)_{k(v)}$, with $e_v$ the absolute ramification index of $v$, and write $\#k(v) = p^{e_v}$. Let $P \in X_0(p)(K)$ and let $C_{P,v}$ be the irreducible component of $X_0(p)_{k(v)}$ which contains $P(k(v))$. As $X_0(p)$ is assumed to be regular, the section $P$ hits each fiber on its smooth locus, so that the component $P$ belongs to is unambiguously defined in each bad fiber. Write

$$\Phi_{P,v} = \sum_{n,m} a_{n,m} [C_{n,m}]$$

with notations as in (16). Recall that, by our convention, $a_{C,\infty} = a_{s,0} = 0$.

(a) If $C_{P,v} = C_0$ then for all $n$ and $m$,

$$a_{n,m} = \frac{-12}{(p - 1) \cdot w_n} \cdot m.$$
For further use we henceforth write \( \Phi_{C_0} \) for the above vector \( \Phi_{P,v} \in \mathbb{Z}^B \).

(b) If \( C_{P,v} = C_{n_0,m_0} \neq C_0, C_\infty \) then

- for \( n = n_0 \) and \( m \in \{0, m_0\} \), one has \( a_{n,m} = \left( \frac{m_0}{w_{n_0}v} (1 - \frac{12}{(p-1)w_{n_0}}) - 1 \right) \cdot m; \)
- for \( n = n_0 \) and \( m \in \{m_0, w_{n_0}e_v\} \), one has \( a_{n,m} = \left( \frac{m_0}{w_{n_0}v} (1 - \frac{12}{(p-1)w_{n_0}}) \right) \cdot m - m_0; \)
- for \( n \neq n_0 \) and \( m \in \{0, w_{n_0}e_v\} \), one has \( a_{n,m} = \frac{-12m_0}{(p-1)w_{n_0}} \cdot \frac{m}{w_{n_0}}. \)

(c) (Of course if \( C_{P,v} = C_\infty \) then \( \Phi_{P,v} = 0. \))

**Remark 3.4** We have distinguished different cases above because the proof naturally leads to doing so, and it will be of interest below to have the simpler case (a) explicitly displayed. Note however that all outputs are actually covered by the formulae of case (b). Notice also that, in case (a), all coefficients of \( \Phi_{P,v} \) satisfy

\[
0 \geq a_{n,m} \geq a_0 := a_{C_0} = a_{n,w_{n_0}m} = -12e_v/(p-1).
\]

As for case (b), all coefficients of \( \Phi_{P,v} \) satisfy

\[
0 \geq a_{n,m} \geq a_{n,m_0} = \left( \frac{m_0}{w_{n_0}e_v} (1 - \frac{12}{(p-1)w_{n_0}}) - 1 \right) \cdot m_0
\]

(remember \( 0 \leq m \leq w_{n_0}e_v \) for all \( m \)). Computing the minimum of the above right-hand as a polynomial in \( m_0 \) gives

\[
0 \geq a_{n,m} \geq \frac{-e_vw_{n_0}}{4(1 - \frac{12}{(p-1)w_{n_0}})} \geq \frac{-e_vw_{n_0}}{4 - \frac{3}{w_{n_0}}} \geq -3e_v
\]

(recalling we always assume \( p \geq 17 \)).

**Proof** Given the intersection matrix (18) and condition (28): \([C, P = \infty - \Phi_{P,v}] = 0\) for all \( C \) in the fiber at \( v \) gives the matrix equation:

\[
\log(\#k(v))M \cdot \Phi_{P,v} = \log(\#k(v))(-1, 0, \ldots, 1, 0, \ldots, 0)^t
\]

where the coefficient 1 (respectively, −1) in the right-hand column vector is at the place corresponding to \( C_{P,v} = C_{n,m} \) (respectively, \( C_\infty = C_{n,0} \)) in the ordering of our component basis (16). That is however more easily solved by running through the dual graph of \( X_0(p)k(e) \) “branch by branch” as follows. Suppose first that \( C_{P,v} = C_0 \), and recall \( a_{C_\infty} = 0 \) by convention. Equation (28) translates into:

- \((-1 - \sum_{n=1}^s a_{n,1} = 0) \text{ for } C = C_\infty; \)
- \((1 + sa_0 - \sum_{n=1}^s a_{n,w_{n_0}e_v-1} = 0) \text{ for } C = C_0; \)
- \((a_{n,m-1} - 2a_{n,m} + a_{n,m+1} = 0) \text{ for all others } C = C_{n,m}. \)

The equations of the third line in turn define, for each branch (that is, for fixed \( n \)), a sequence defined by linear double induction with solution \( a_{n,m} = m\alpha_n \) for some \( \alpha_n \) which is easily computed to be \( \frac{12}{w(s+1)e_v} w_{n_0} = \frac{12}{(p-1)w_{n_0}} \) (see (14)). (Note this is true even for \( e_v = 1 \).)

For case (b), the intersection equations become:
Lemma 3.5

Hence, because of the first equation $(-1 - \sum_{n=1}^s a_n,1 = 0)$,

$$0 = -1 - \beta_n - \sum_{1 \leq n, s \neq n_0 \leq s} \beta_w - \beta w(Eis) - \frac{m_0}{w_{n_0}w_v}.$$ 

so that

$$\beta = -\frac{m_0}{w(Eis)w_{n_0}w_v} = -\frac{12m_0}{(p-1)w_{n_0}w_v}.$$ 

□

Lemma 3.5 Let $\mu$ be some $(1, 1)$-form on $X_0(p)(\mathbb{C})$ as in Theorem 3.2.

(a) The class in $\overline{CH}(p)_{\mathbb{Q}, \mu}^{\text{num}}$ of the cuspidal divisor $(0) - (\infty)$ satisfies

$$(0) - (\infty) \equiv \Phi^0_{C_0} := \Phi_{C_0} + \sum_{v \mid p} \frac{6e_v}{p-1} \left(\sum_C [C]\right) = \sum_{v \mid p} \sum_{n,m} \frac{6}{(p-1)} (e_v - \frac{2m}{w_n}) [C_{n,m}]$$

with notations as in Lemma 3.3 (a). This is an eigenvector of the Fricke $\mathbb{Z}$-automorphism $w_p$ with eigenvalue $-1$.

(b) One has $[\infty, \infty]_{\mu} = [0, 0]_{\mu} - \frac{6\log p}{p-1}$. If $\mu$ is the Green-Arakelov measure $\mu_0$ then $0 \geq [\infty, \infty]_{\mu_0} = O(\log p/p)$ and similarly $[0, \infty]_{\mu_0} = O(\log p/p)$ with $[0, \infty]_{\mu_0}$ non-positive too, at least for large enough $p$. If $\mu = \mu_c$ (see (26)) - or more generally any sub-measure of $\mu_0$ - then $[0, \infty]_{\mu_c} = O(p \log p)$.

Proof

By the Manin-Drinfeld theorem, $(0) - (\infty)$ is torsion as a divisor in the generic fiber $X_0(p) \times_{\mathbb{Z}} \mathbb{Q}$. One therefore has

$$(0) - (\infty) \equiv \Phi + eX_\infty$$

in the decomposition (27) of $\overline{CH}(p)_{\mathbb{Q}, \mu}^{\text{num}}$, for $\Phi$ some vertical divisor with support in the fibers above $p$. This divisor is determined by the same equations (28) as $\Phi_{C_0}$ in Lemma 3.3 (a). For each $v \mid p$ the full $v$-fiber $\sum_C [C]$ is numerically equivalent to some real multiple of the archimedean fiber $X_\infty$; there is therefore a real number $a$ such that

$$\Phi^0_{C_0} := \Phi_{C_0} + \sum_{v \mid p} \frac{6e_v}{p-1} \left(\sum_C [C]\right) \equiv \Phi_{C_0} + aX_\infty.$$ 

Now $w_p$ switches the cusps $0$ and $\infty$ so the divisor $(0) - (\infty)$ is anti-symmetric for $w_p$:

$$w_p^*((0) - (\infty)) = -(0) - (\infty)$$

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and clearly $w_p^*(\Phi_{C_0}) = -\Phi_{C_0}$. The fact that $w_p$ preserves the archimedean fiber concludes the proof of (a).

To prove (b) we compute

$$0 = \left[0 - \infty - \Phi_{C_0}, \infty\right]_\mu = \left[0, \infty\right]_\mu - \left[\infty, \infty\right]_\mu - \frac{6}{p-1} \log p$$

and

$$0 = \left[0 - \infty - \Phi_{C_0}, 0\right]_\mu = \left[0, 0\right]_\mu - \left[0, \infty\right]_\mu + \frac{6}{p-1} \log p$$

so that $[\infty, \infty]_\mu = [0, 0]_\mu = [0, \infty]_\mu - \frac{6 \log p}{p-1}$. The cusps 0 and $\infty$ are known not to intersect on $X_0(p)_{/\mathbb{Z}}$, so that $[0, \infty]_\mu = -g_\mu(0, \infty)$. When $\mu = \mu_0$, this special value of the Arakelov-Green function has been computed by Michel and Ullmo: it satisfies

$$g_{\mu_0}(0, \infty) = \frac{1}{2g} \log p(1 + O(\frac{\log \log p}{\log p})) = O(\frac{\log p}{p})$$

by [43], formula (12) on p. 650. Finally, using [9], Theorem 7.1 (c) and paragraph 8, and plugging into Bruin’s method the estimates of [43] regarding the comparison function $F(z) = O((\log p)/p)$ between Green-Arakelov and Poincaré measures, we obtain a bound of shape $O(p \log p)$ for $|g_{\mu_0}(0, \infty)|$ (see also Remark 4.5). This completes the proof of (b). $\square$

Instrumental in the sequel will be the explicit decomposition of the relative dualizing sheaf $\omega$ in the arithmetic Chow group.

**Proposition 3.6** The relative dualizing sheaf $\omega$ of the minimal regular model $X_0(p) \to \mathcal{O}_K$ can be written, in the decomposition (27) of $\overline{CH}(p)_{\mathbb{R},\mu_0}^{\text{num}}$ relative to the canonical Green-Arakelov (1,1)-form $\mu_0$, as:

$$\omega = (2g - 2)\infty + \sum_{v\mid p} \Phi_{\omega,v} + \omega^0 + [K : \mathbb{Q}]c_\omega X_{\infty}$$

(32)

where the above components satisfy the following properties.

- The number $c_\omega$ is equal to $\frac{(1-2g)}{[K : \mathbb{Q}]_\mu} [\infty, \infty]_{\mu_0}$, so that $0 \leq c_\omega \leq O(\log p)$.

- Set

$$H_4 := \frac{1}{2} \sum_{p \in \mathcal{H}_4} (P - \frac{1}{2}(0 + \infty)), \quad H_3 := \frac{2}{3} \sum_{p \in \mathcal{H}_3} (P - \frac{1}{2}(0 + \infty))$$

where the sums run over the sets $\mathcal{H}_4$ and $\mathcal{H}_3$, whose number of elements can be 0 or 2, of Heegner points of $X_0(p)$ with $j$-invariant 1728 and 0 respectively. Define

$$H_4^0 := H_4 + [K : \mathbb{Q}]c_4 X_{\infty} \quad \text{and} \quad H_3^0 := H_3 + [K : \mathbb{Q}]c_3 X_{\infty}$$

for two numbers $c_4$ and $c_3$ with $c_4 = O(\log p)$, and the same for $c_3$. (Recall this means the $H_4$ are compactified with the normalizing condition (24), whereas the $H_3$ are the orthogonal projections on $(J_0(p)(K) \otimes \mathbb{R}) \subseteq \overline{CH}(p)_{\mathbb{R},\mu_0}^{\text{num}}$ of the $H_4$, so that $[\infty, H_4^0]_{\mu_0} = 0$, for $* = 3$ or 4.) One sets $\omega^0 := -H_3^0 - H_3^0$, which can be chosen in $J_0(p)^0(\overline{\mathbb{Q}})$.

- Finally, the component $\Phi_{\omega,v}$ in each $G_v$ for $v | p$ is

$$\Phi_{\omega,v} = -12 \frac{(g-1)}{p-1} \sum_{n,m} m C_{n,m}$$

(33)

with notations as in (16). We therefore have $\Phi_{\omega,v} = (g-1)C_{C_0}$ using notations of Lemma 3.3. In particular, recalling $e_v$ is the ramification index of $K/\mathbb{Q}$ at $v$, the coefficients $\omega_{n,m}$ of $\Phi_{\omega,v}$ in (33) satisfy

$$0 \geq \omega_{n,m} \geq -e_v.$$  

(34)
Proof. Many parts of those statements are deduced from [43], Section 6, and results of Edixhoven et al. from [18]. See also [41], Section 4.4.

We start by estimating $c_\omega$. By Arakelov’s adjunction formula,

$$-[\infty, \infty]_{\mu_0} = [\infty, \omega]_{\mu_0} = (2g - 2)[\infty, \infty]_{\mu_0} + [K:Q]c_\omega$$

because of the orthogonality of the decomposition (27). Lemma 3.5 therefore implies

$$0 \leq c_\omega = \frac{1 - 2g}{[K:Q]}[\infty, \infty]_{\mu_0} = O(\log p).$$

The computations of the $J_0(p)$-part $\omega_\delta := -(H_0^4 + H_0^4)$ follows from the Hurwitz formula, as explained in [43], paragraph 6, p. 670. One indeed checks that, on the generic fiber $X_0(p)/\mathbb{Q} = X_0(p) \times_{\mathbb{Z}} Q$, the canonical divisor is linearly equivalent to

$$(2g - 2)\infty - \left( \frac{1}{2} \sum_{(P)_{\omega,e^{1/2}}} (P - \infty) + \frac{2}{3} \sum_{(P)_{\omega,e^{2/3}}} (P - \infty) \right)$$

where the sums $\sum'$ are here restricted to points $P$ at which $X_0(p) \to X(1)$ is unramified (these are the Heegner points alluded to in our statement). It follows from the modular interpretation that in each of those sums there are two Heegner points (if any), which are then ordinary at $p$ (recall we assume $p > 13 > 3$). This proves that the $J_0(p)(K) \otimes \mathbb{R}$-part of $\omega$ is indeed $-(H_0^4 + H_0^4)$ with $H_0^4 = H_4 + [K:Q]c_4 X_\infty$ and $H_0^4 = H_3 + [K:Q]c_3 X_\infty$ for some real numbers $c_3$ and $c_4$.

(Note that, as Heegner points are preserved by the Atkin-Lehner involution ([23], paragraph 5, p. 90), their specializations above $p$ share themselves between the two components $C_0$ and $C_\infty$ of $X_0(p)/\mathbb{F}_p$, so that $2H_0^4 = \sum_{(P)_{\omega,e^{1/2}}} (P - \infty)$ and $\frac{2}{3}H_0^4 = \sum_{(P)_{\omega,e^{2/3}}} (P - \infty)$ belong to the neutral component $J_0(p)^0(\mathcal{O}_K)$.) The estimates on $c_3$ and $c_4$ will be justified at the end of the proof.

The bad fibers divisors $\Phi_{\omega,v} := \sum_{n,m} \omega_{n,m}[C_{n,m}]$ can be computed with the “vertical” adjunction formula ([37] Chapter 9, Theorem 1.37) as in [41], Lemma 4.22. Indeed, for each irreducible component $C$ in the $v$-fiber having genus 0, one has

$$[C, C + \omega] = -2 \log(\#k(v)).$$

If $\mathcal{M}$ is the intersection matrix displayed in (18), and $\delta_{+, +}$ is Kronecker’s symbol, we therefore have

$$C \cdot \mathcal{M} \cdot \Phi_{\omega,v} = -2 - \frac{1}{\log(\#k(v))}[C, C] - (2g - 2)\delta_{C, C_\infty} = \begin{cases} 0 & \text{if } C \neq C_\infty, C_0 \\ s - 2 & \text{if } C = C_\infty \\ s - 2 & \text{if } C = C_0 \end{cases}$$

that is, as $s = g + 1$:

$$\mathcal{M} \cdot \Phi_{\omega,v} = (g - 1)(-1, 0, \cdots, 0, 1)^t.$$

That equation is (30) (up to a multiplicative scalar), which has been solved in the first case of Lemma 3.3. Therefore

$$\Phi_{\omega,v} = (g - 1)\Phi_{C_0},$$

that is: $\omega_{n,m} = \frac{12(1 - g)}{(p - 1)} \cdot \frac{m}{w_n}.$

As noted in Remark 3.4 and using (15), this implies the coefficients $\omega_{n,m}$ of $\Phi_{\omega,v}$ satisfy

$$0 \geq \omega_{n,m} \geq \frac{12(1 - g)}{p - 1} c_v > -c_v.$$
We finally estimate the intersection products

\[ c_3 = \frac{-1}{[K : \mathbb{Q}]}[\infty, H_3]_{\mu_0} \quad \text{and} \quad c_4 = \frac{-1}{[K : \mathbb{Q}]}[\infty, H_4]_{\mu_0}. \]

By the adjunction formula and Hriljac-Faltings’ theorem ([12], Theorem 5.1 (ii)) we compute that for any \( P \in \mathcal{X}_0(p)(K) \),

\[
-2[K : \mathbb{Q}]h_\Theta(P - \frac{1}{2g-2}\omega) = [P - \frac{1}{2g-2}\omega - \Phi_\omega(P), P - \frac{1}{2g-2}\omega - \Phi_\omega(P)]_{\mu_0}
\]

\[
= \frac{1}{(2g-2)^2} [\omega, \omega]_{\mu_0} + \frac{g}{g-1} [P, P]_{\mu_0} - \Phi_\omega(P)^2
\]

where here \( \Phi_\omega(P) \) is a vertical divisor supported at bad fibers such that

\[
[C, P - \frac{1}{2g-2}\omega - \Phi_\omega(P)] = 0
\]

(37) for any irreducible component \( C \) of any bad fiber of \( \mathcal{X}_0(p)/\mathcal{O}_K \). Hence

\[
\frac{1}{(2g-2)^2} \omega^2 + \frac{g}{g-1} [P, P]_{\mu_0} - \Phi_\omega(P)^2 = -2[K : \mathbb{Q}]h_\Theta((P - \infty) + \frac{1}{2g-2}(H_3 + H_4)).
\]

(38)

We specialize to the case when \( P = P^*_v \) (where the upper star is 1 or 2 and the lower star is 4 or 3) is one of the Heegner points occurring in \( H_4 \) or \( H_3 \), respectively. We replace for now the base field \( K \) by \( F := \mathbb{Q}(P^*_v) = \mathbb{Q}(\sqrt{-1}) \) (respectively, \( \mathbb{Q}(\sqrt{-3}) \)). The right-hand of (38), if non-zero, is then

\[
-8 \log(p)(1 + o(1)) \quad \text{or} \quad -12 \log(p)(1 + o(1)) \quad \text{respectively},
\]

(39)

by [43], p. 673. If those Heegner points occur we know that \( p \) splits in \( F \), so there are two bad primes \( v \) and \( v' \) on \( \mathcal{O}_F \) (therefore two bad fibers on \( \mathcal{X}_0(p)/\mathcal{O}_F \) and two \( G_v, G_{v'} \)) to take into account. We compute \( \Phi_\omega(P^*_v) \) and \( \Phi_\omega(P^*_v)^2 \). As mentioned at the beginning of the proof, \( P^*_v \) specializes to the component \( C_0 \) at a place, say \( v \), of \( F \) above \( p \), and to \( C_\infty \) at the conjugate place \( v' \). Conditions (37) therefore give that, for any irreducible component \( C \) of the fiber at \( v \),

\[
0 = [C, P^*_v - \frac{1}{2g-2}\omega - \Phi_\omega(P^*_v)] = [C, 0 - \infty - \frac{1}{2g-2}\Phi_\omega, v - \Phi_\omega(P^*_v)]
\]

and using Lemma 3.3, Lemma 3.5 and (36) one obtains

\[
\Phi_\omega(P^*_v) = -\frac{1}{2g-2}\Phi_\omega, v + \Phi_{C_0, v} = \frac{1}{2}\Phi_{C_0, v}
\]

whereas, at \( v' \):

\[
\Phi_\omega(P^*_v) = -\frac{1}{2g-2}\Phi_\omega, v' = -\frac{1}{2}\Phi_{C_0, v'}.
\]

Using Lemmas 3.3 and 3.5 again we therefore have

\[
\Phi_\omega(P^*_v)^2 = \sum_{w|p} \Phi_{C_0, w}^2 = \sum_{w|p} \Phi_{C_0, w, 0 - \infty} = \frac{1}{2} a_0 \log p = -\frac{6 \log(p)}{p - 1}.
\]

(40)

As for the self-intersection of \( \omega \) one knows from [62], Introduction, that

\[
\omega^2 \mathcal{X}_0(p)_{//} = 3g \log(p)(1 + o(1)).
\]
As the quantity \( \frac{1}{|P_K|}[\omega]^2 \) is known to be independent from the number field extension \( F/K \), the dualizing sheaf \( \omega_{\mathcal{X}_0(p)/C_v} \) of \( \mathcal{X}_0(p) \) over \( O_F \) (instead of \( \mathbb{Z} \)) satisfies \( \omega^2 = 6g \log(p)(1 + o(1)) \). Summing-up, equation (38) implies that

\[
[P^*_v, P^*_v]_{\mu_0} = O(\log(p))
\]

(41)

for each Heegner point \( P^*_v \). Now, on the other hand, the vertical divisor \( \Phi_{P^*_v} \) in the sense of (28) and Lemma 3.3 is \( \Phi_{P^*_v} = \Phi_{C_n,v} \) for the place \( v \) of \( F \) where \( P^*_v \) specializes on \( C_0 \) and not \( C_\infty \). Therefore

\[
-4h_\Theta(P^*_v - \infty) = [P^*_v - \infty - \Phi_{P^*_v}, P^*_v - \infty - \Phi_{P^*_v}]_{\mu_0} = -2[P^*_v, \infty]_{\mu_0} + [P^*_v, P^*_v]_{\mu_0} + [\infty, \infty]_{\mu_0} - (\Phi_{P^*_v})^2
\]

(42)

whence, using (39), (40), (41) and Lemma 3.5(b):

\[
[P^*_v, \infty]_{\mu_0} = \frac{1}{2} \left( [P^*_v, P^*_v]_{\mu_0} + [\infty, \infty]_{\mu_0} - (\Phi_{C_n,v})^2 + 4h_\Theta(P^*_v - \infty) \right) = O(\log(p)).
\]

Putting everything together and using Lemma 3.5 once more we conclude that

\[
c_4 = -\frac{1}{[K : \mathbb{Q}]}[\infty, H_4]_{\mu_0} = \frac{1}{2[K : \mathbb{Q}]} \left( -[\infty, P^2 + P^2]_{\mu_0} + [\infty, 0 + \infty]_{\mu_0} \right) = O(\log(p))
\]

(43)

and similarly for \( c_3 \). (Note that the Arakelov intersection products, in the computations around (42), were performed over \( F = \mathbb{Q}(P^*_v) \) and not \( K \), although we did not indicate this in the notations in order to keep it from becoming too heavy. We however want quantities over \( K \) for the statement of the theorem, so we need considering Arakelov products over \( K \) in (43) above.)

\[\Box\]

Remark 3.7 It may be convenient to write, with notations as in (32), a more symmetric \( \omega \) as

\[
\omega = (g - 1)(\infty + 0) + (-H_0^0 - H_0^0) + [K : \mathbb{Q}]c_\omega \mathcal{X}_\infty
\]

(44)

which yields an element with no vertical component at bad fibers.

4 \( j \)-height and \( \Theta \)-height

In this section we compare two natural heights on \( \mathcal{X}_0(p)(\overline{\mathbb{Q}}) \), namely the \( j \)-height and the one induced from the Néron-Tate \( \Theta \)-height on \( \mathcal{X}_0(p)(\overline{\mathbb{Q}}) \). We start with an explicit description of the latter, for which it is actually convenient to use a bit of Zhang’s language about “adelic metrics” (see [64]) which, in our modular setting, has a very concrete form.

Using notations and results from Section 2.2.2 we therefore consider the limit, as \( e_v \) goes to \( \infty \), of the dual graph of the special fiber of \( \mathcal{X}_0(p) \) at a place \( v \) of a \( p \)-adic local field with ramification index \( e_v \) at \( p \) (see Figure 2.2.2). Here we normalize the length of the \( s = g + 1 \) edges from \( C_\infty \) to \( C_0 \) to be 1, so that the vertex \( C_{n,m} \) corresponds to the point of the \( n^{th} \) edge with distance \( \frac{m}{e_v} \) from the origin \( C_\infty \). Now associate to any edge \( n \in \{1, \ldots, s\} \) the quadratic polynomial function

\[
g_n(x) : [0, 1] \to \mathbb{R}, \ x \mapsto \frac{1}{2} \left( \frac{w_n - 12}{p - 1} \right)x - w_n - 12 \frac{(g - 1)}{(p - 1)}.
\]

(45)

For \( K \) any number field, \( P \) in \( \mathcal{X}_0(p)(K) \), and \( v \) a place of \( K \) whose ramification degree and residual degree are still denoted by \( e_v \) and \( f_v \) respectively, let

\[
G(P(K_v)) = e_v f_v \log(p) \cdot g_n(C_{P(K_v)})
\]

(46)

where \( C_{P(K_v)} \) is the component to which the specialization of \( P \) belongs at \( v \), identified to a point of the \( n^{th} \) edge where it lives.
Theorem 4.1 For any number field $K$, there is an element
\[
\tilde{\omega}_{\Theta,K} = (g \cdot \infty + \Phi_{\Theta,K} + c_{\Theta,K}X_{\infty})
\] (47)
of $\tilde{CH}(p)_{\text{num}}$ such that, for any $P \in X_0(p)(K)$ one has, with notations as in Proposition 3.6,
\[
h_{\Theta}(P - \infty + \frac{1}{2} \omega^0) = \frac{1}{[K : \mathbb{Q}]} [P, \tilde{\omega}_{\Theta,K}]_{\mu_0}
\] (48)
and the terms of (47) satisfy:
\[
0 \geq [P, \Phi_{\Theta,K}] \geq -2[K : \mathbb{Q}] \log(p) \quad \text{and} \quad c_{\Theta,K} = [K : \mathbb{Q}]O(\log p). \tag{49}
\]
Passing to the limit on all number fields, the height induced on $X_0(p)(\overline{\mathbb{Q}})$ by pulling-back Néron-Tate’s $\Theta$-height on $J_0(p)(\overline{\mathbb{Q}})$ via the embedding $P \mapsto P - \infty + \frac{1}{2} \omega^0$ can be written as:
\[
h_{\Theta}(P - \infty + \frac{1}{2} \omega^0) = \frac{1}{[K : \mathbb{Q}]} \left( g[P, \infty]_{\mu_0} + \sum_{v \in M_K, v | p} G(P(K_v)) + c_{\Theta,K} \right) \tag{50}
\]
where Zhang’s Green function $G$ at bad fibers is defined in (45) and (46).

In any case one has that the height satisfies
\[
h_{\Theta}(P - \infty + \frac{\omega^0}{2}) = \frac{1}{[K : \mathbb{Q}]} [P, g \cdot \infty]_{\mu_0} + O(\log p). \tag{51}
\]

Proof We prove (48) and (49); from there reformulation (50) and (51) are straightforward.

Recall $X_0(p)$ denotes the minimal regular model of $X_0(p)$ on $\text{Spec}(\mathcal{O}_K)$, that $J_0(p)$ is the Néron model of $J_0(p)$ on the same base, and $J_0(p)^0$ stands for its neutral component. Let $\delta$ be an element of $J_0(p)(K)$, seen as a degree 0 divisor on $X_0(p)$. Up to making a base extension we can assume $\delta$ is linearly equivalent to a sum of points in $X_0(p)(K)$. We shall denote by $\delta = \delta + \Phi_\delta$ (for $\Phi_\delta$ some vertical divisor on $X_0(p)$, with multiplicity 0 on the component containing $\infty$, following our running conventions) the associated element of the neutral component $J_0(p)^0(\mathcal{O}_K)$ (that is, the one whose associated divisor has degree zero on each irreducible component, in any fiber, of $X_0(p)$, and therefore defines a point of $J_0(p)(\mathcal{O}_K)$). For any point $P$ in $X_0(p)(K) \hookrightarrow X_0(p)(\mathcal{O}_K)$ let similarly $\Phi_P$ be the vertical divisor on $X_0(p)$, with support on the bad fibers, such that $(P - \infty - \Phi_P)$ has divisor class belonging to the neutral component $J_0(p)^0(\mathcal{O}_K)$ and, again, $\Phi_P$ has everywhere trivial $\infty$-component, see (28). Recall we can compute $\Phi_P$ explicitly by Lemma 3.3. We write $\Phi_P = \sum_{v \in M_K, v | p} \sum_{C_v} a_{C_v} [C_v]$ where the sum is taken on irreducible components $C_v$ of vertical bad fibers of $X_0(p)$. Using notations of Lemma 3.3 (b) we also define the following new vertical divisor at bad fibers:
\[
\Phi_{\Theta,K} := \sum_{v \in M_K, v | p} \sum_{Q_v} a_{C_{Q_v}} C_{Q_v} = \sum_{v | p} \sum_{(n_0, m_0)} a_{n_0, m_0}^w C_{n_0, m_0} \tag{52}
\]
so that
\[
a_{n_0, m_0}^w = \left( \frac{m_0}{\omega_{n_0} c_v} \left( 1 - \frac{12}{(p-1)\omega_{n_0}} \right) - 1 \right) \cdot m_0.
\]
Our very definitions imply
\[
\Phi_P^2 = [P, \Phi_P] = [P, \Phi_{\Theta,K}] \tag{53}
\]
for any $P \in X_0(p)(K)$. Using Faltings’ Hodge index theorem we can write the Néron-Tate
height \( h_\Theta(P - \infty + \delta) \) as:

\[
\begin{align*}
\frac{1}{|K : \mathbb{Q}|} h_\Theta(P - \infty + \delta) &= \frac{-1}{2[K : \mathbb{Q}]} \left[ P - \infty + \tilde{\delta} - \Phi_P, P - \infty + \tilde{\delta} - \Phi_P \right]_{\mu_0} \\
&= \frac{1}{2[K : \mathbb{Q}]} \left( [P, \omega + 2\infty - 2\tilde{\delta}]_{\mu_0} + 2[P, \Phi_P]_{\mu_0} - [\Phi_P, \Phi_P]_{\mu_0} + [\tilde{\delta}, 2\infty - \tilde{\delta}]_{\mu_0} - [\infty, \infty]_{\mu_0} \right) \\
&= \frac{1}{2[K : \mathbb{Q}]} \left( [P, \omega + 2\infty - 2\tilde{\delta} + \Phi_{\vartheta,K}]_{\mu_0} + [\tilde{\delta}, 2\infty - \tilde{\delta}]_{\mu_0} - [\infty, \infty]_{\mu_0} \right) \\
&= \frac{1}{|K : \mathbb{Q}|} [P, \tilde{\omega}_\delta]_{\mu_0}
\end{align*}
\]

with

\[
\tilde{\omega}_\delta := \left( \frac{1}{2}(\omega + \Phi_{\vartheta,K}) + \infty - \tilde{\delta} \right) + c_\delta X_\infty
\]

for \( X_\infty \) some fixed archimedean fiber of \( X_0(p) \) and \( c_\delta \) is the real number

\[
c_\delta = \frac{1}{2} \left( -\infty, \infty \right)_{\mu_0} + [\tilde{\delta}, 2\infty - \tilde{\delta}]_{\mu_0}
\]

(56)

Note that \( \tilde{\omega}_\delta \) does not depend on \( P \) (as \( \Phi_{\vartheta,K} \) was introduced to that aim).

Let us now take \( \delta = \omega^0/2 = -(H_3 + H_4)/2 \in \frac{1}{2} \cdot J_0(p)^0(\mathbb{Q}) \), as defined in Proposition 3.6. (This is Riemann’s characteristic (the “\( \kappa \)” of [26], p. 138 for instance, that is the generic fiber of the \( J_0(p)(\mathbb{Q}) \) part of \( \omega \) in the decomposition (32).) Set \( \Phi_{\Theta,K} := \frac{1}{2}(\Phi_\omega + \Phi_{\vartheta,K}) \). Then

\[
\tilde{\omega}_\Theta := \tilde{\omega}_\delta = (g \cdot \infty + \Phi_{\Theta,K} + c_{\Theta,K} X_\infty)
\]

(57)

for \( c_{\Theta,K} \) which, still using notations of Proposition 3.6 and its proof, is explicitly given by:

\[
\begin{align*}
\frac{1}{|K : \mathbb{Q}|} c_{\Theta,K} &= \frac{1}{2} \left( c_\omega - c_4 - c_3 + \frac{1}{2} h_\Theta(H_3 + H_4) - \frac{1}{|K : \mathbb{Q}|} \left( [\infty]^2_{\mu_0} + [\infty, H_3 + H_4]_{\mu_0} \right) \right) \\
&= \frac{1}{2} \left( c_\omega - \frac{1}{2} \left[ K : \mathbb{Q} \right] [\infty]^2_{\mu_0} - \frac{1}{2} h_\Theta(H_3 + H_4) \right).
\end{align*}
\]

As in the proof of Proposition 3.6 we invoke p. 673 of [43] to assert \( h_\Theta(H_3 + H_4) = O(\log(p)) \).

We moreover know from the same Proposition and from Lemma 3.5 that both \( |c_\omega| = O(\log(p)) \) and \( [\infty, \infty]_{\mu_0} = [K : \mathbb{Q}]O(\log(p)) \), so that

\[
c_{\Theta,K} = [K : \mathbb{Q}]O(\log(p)).
\]

(58)

The contribution of \( \Phi_{\Theta,K} \) is controlled by Lemma 3.3 and Remark 3.4: on one hand,

\[
0 \geq [P, \Phi_{\Theta,K}] = [P, \Phi_P] = \sum_{v \in M_K,v|p} a_{C,v} \log(\#k_v) \geq \sum_{v \in M_K,v|p} -3c_v \log(p^f_v) \geq -3|K : \mathbb{Q}| \log(p)
\]

(59)

On the other hand, by (34), the coefficients of the vertical components \( \Phi_{\omega,v} \) satisfy \( 0 \geq \omega_{n,m} \geq -c_v \), so writing \( \omega_{n,m,v} \) for the coefficient in \( \Phi_{\omega,v} \) of the component containing \( P(k(v)) \) we have:

\[
0 \geq [P, \Phi_\omega] = \sum_{v|p} \omega_{n,m,v} \log(\#k(v)) \geq \sum_{v|p} -c_v \log(p^f_v) = -|K : \mathbb{Q}| \log(p).
\]

(60)

Putting (58), (59) and (60) together completes the proof of (48) and (49) and the proof. □
Remark 4.2 Estimates on the Green-Zhang function on $X_0(p)$ as in the above theorem will be extended below to the Néron model over $\mathbb{Z}$ of the whole jacobian $J_0(p)$, see Proposition 5.8.

Remark 4.3 As already noticed, the involution $w_p$ acts as an isometry (actually, an orthogonal symmetry) with respect to the quadratic form $h_\Theta$ on $J_0(p)(K) \otimes \mathbb{R}$. Indeed $w_p$ acts as multiplication by $\pm 1$ on each factor of Shimura’s decomposition up to isogeny:

$$J_0(p) \sim \prod_{f \in G_0 \cdot S_2(G_0(p))^{\text{new}}} J_f$$

whose factors are $h_\Theta$-orthogonal subspaces. (See also [40], Corollaire 4.3, or [41], Theorem 4.5 (3).) As $w_p(\omega^0) = \omega^0$ (see the proof of Proposition 3.6) this implies

$$h_\Theta(P - \infty + \frac{1}{2} \omega^0) = h_\Theta(w_p(P - \infty + \frac{1}{2} \omega^0)) = h_\Theta(w_p(P) - 0 + \frac{1}{2} \omega^0) = h_\Theta(w_p(P) - \infty + \frac{1}{2} \omega^0)$$

using once more that $(0) - (\infty)$ is torsion, so that

$$[P, \tilde{\omega}_\Theta]_{\mu_0} = [w_p(P), \tilde{\omega}_\Theta]_{\mu_0} = [P, w_p^*(\tilde{\omega}_\Theta)]_{w_p^*(\mu_0)} = [P, w_p^*(\tilde{\omega}_\Theta)]_{\mu_0}$$

(see Remark 3.1). This suggests it could sometimes be convenient to write $\tilde{\omega}_\Theta$ in a $w_p$-eigenbasis of $\tilde{CH}(p)^{\text{num}}_{\mathbb{R}, \mu}$ instead of that of Theorem 3.2, for instance

$$\tilde{CH}(p)^{\text{num}}_{\mathbb{R}, \mu_0} = \mathbb{R} \cdot \frac{1}{2} (0 + \infty) \oplus \mathbb{R} \cdot X_\infty \oplus \chi_{[p]} \Gamma_v \oplus (J_0(p)(K) \otimes \mathbb{R})$$

(62)

where now the $\Gamma_v$ decompose as the direct sum of eigenspaces $\Gamma_{w_p=-1}^{v}$ and $\Gamma_{w_p=+1}^{v}$, with bases:

$$\{C_{n,m}^+ := C_{n,m} - w_p(C_{n,m}) \}_{1 \leq m \leq u_{n/2}} \text{ and } \{C_{n,m}^- := C_{n,m} + w_p(C_{n,m}) - C_0 - C_\infty \}_{1 \leq m \leq u_{n/2}}$$

(63)

respectively. Using Lemma 3.5 and Proposition 3.6, a lengthy but easy computation allows one to check that

$$\tilde{\omega}_\Theta = g \cdot \frac{1}{2} (0 + \infty) + \Phi_\Theta^+ + \gamma_\Theta X_\infty$$

where $\Phi_\Theta^+$ is an explicit vertical divisor above $p$ with $w_p^*(\Phi_\Theta^+) = \Phi_\Theta^+$, so that indeed

$$w_p^*(\tilde{\omega}_\Theta) = \tilde{\omega}_\Theta$$

thus recovering (61).

Consider for instance the case of $X_0(p)$ over $\mathbb{Z}$, for $p \equiv 1 \mod 12$ (that is, $X_0(p)/\mathbb{Z}$ is regular, so that there is no need to blow-up singular points of width larger than 1). Here $\Gamma_v = \Gamma_v^- = \mathbb{R} \cdot C_0^- = \mathbb{R} \cdot ([C_\infty] - [C_0])$ and one readily checks that

$$\tilde{\omega}_\Theta = \frac{g}{2} (0 + \infty) + \gamma_\Theta X_\infty$$

(64)

that is, there is no $\Gamma_v$-component at all in that case. Evaluating $h_\Theta(\frac{1}{2} \omega^0)$ as in the proof of Proposition 3.6 and using Lemma 3.5,

$$\gamma_\Theta = -\frac{g}{2} [\infty, 0 + \infty]_{\mu_0} + h_\Theta(\frac{1}{2} \omega^0) = gO(\log p/p) + O(\log p) = O(\log p).$$

We then turn to the $j$-height, first making a comparison of $h_j$ with the “degree component” (in the sense of Theorem 3.2) of the hermitian sheaf $\omega$. 

20
Proposition 4.4 Let $h_j$ be Weil’s $j$-height on $X_0(p)$ as defined in in Section 2.2, and let $\mu_0$ and $\mu_e$ be the $(1,1)$-forms defined in (25) and (26). Recall $\sup_{X_0(p)(\mathbb{C})} g_\mu$ stands for the upper bound for all Green functions $g_{\mu,a}$ relative to some point $a$ of $X_0(p)(\mathbb{C})$ and to the measure $\mu$.

If $p$ is a prime number, $K$ is a number field, and $P$ belongs to $X_0(p)(K)$, then

$$h_j(P) \leq (p + 1) \left( \frac{1}{[K : \mathbb{Q}]} [P, \infty]_{\mu_0} + \sup_{X_0(p)(\mathbb{C})} g_{\mu_0} + O(1) \right)$$

and similarly

$$h_j(P) \leq (p + 1) \left( \frac{1}{[K : \mathbb{Q}]} [P, \infty]_{\mu_e} + \sup_{X_0(p)(\mathbb{C})} g_{\mu_e} + O(1) \right)$$

(65)

Remark 4.5 As explained in the proof below, the function $O(p^2 \log p)$ of (65) comes from [63], Corollary 1.5, together with [62], Corollaire 1.3 for the estimate of Faltings’ $\delta$ invariant for $X_0(p)$, which imply the suprema of our functions verify:

$$\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} \leq O(p \log p).$$

(67)

The function $O(p^3)$ of (66) in turns follows from the main result of [9]. Indeed this states explicitly that $\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} \leq 0.088 \cdot p^2 + 7.7 \cdot p + 1.6 \cdot 10^4$, see [9], Theorem 1.2. It follows from measures comparison (see (74) below) and the method of P. Bruin that this holds for $\sup_{X_0(p)(\mathbb{C})} g_{\mu_e}$ too, so that

$$\sup_{X_0(p)(\mathbb{C})} g_{\mu_e} \leq O(p^2).$$

(68)

It seems that, at least in the case of $X_0(p)$, if we plug into Bruin’s method the estimates of [43] regarding the comparison function $F(z)$ between Green-Arakelov and Poincaré measures, we recover bounds of shape $O(p \log p)$ instead of $O(p^2)$ (see [9], p. 263, and Paragraph 8 (Theorem 7.1 in particular)), and the same again holds true for the Green function $g_{\mu_e}$. One should therefore be able to obtain the same error term $O(p^2 \log p)$ for (66) as for (65).

Note that the main theorems of [32] and [3] might even yield that the above functions $O(p^2)$ or $O(p \log p)$ could be replaced by a uniform bound $O(1)$.

Proof This is essentially a question of measure comparisons on $X_0(p)(\mathbb{C})$, between $j^*(\mu_{FS})$ on one hand (where $\mu_{FS}$ is the Fubini-Study $(1,1)$-form on $X(1)(\mathbb{C}) \simeq \mathbb{P}^1(\mathbb{C})$) and the Green-Arakelov form $\mu_0$ (respectively, $\mu_e$) on the other hand. We adapt the main result of [17].

We define first a somewhat canonical Arakelov intersection product $[\cdot, \cdot]_{\mu_{FS}}$ on the projective line using $\mu_{FS}$. Write $\mathbb{P}^1_{/\mathcal{O}_K} = \text{Proj}(\mathcal{O}_K[x_0, x_1]) = \text{Spec}^\text{Zad}(\mathcal{O}_K[j])$ (with $j = x_1/x_0$), so that the horizontal divisor $\infty(\mathcal{O}_K)$ is $V(x_0)$ and, for any $P = [x_0 : x_1]$, let the associated Green function be

$$g_{\mu_{FS, \infty}}(P) = g_{\mu_{FS, \infty}}(j(P)) = \frac{1}{2} \log \left( \frac{|x_0|^2}{|x_0|^2 + |x_1|^2} \right) = \frac{1}{2} \log(1 + |j(P)|^2)$$

at any point different from $\infty = [0 : 1]$. (We note in passing this ad hoc Green function does not need to fulfill the normalization condition (24).) Then for any $P$ in $X(1)(K)$ one easily checks that

$$\left| h_j(P) - \frac{1}{[K : \mathbb{Q}]} [j(P), \infty]_{\mu_{FS}} \right| \leq \frac{1}{2} \log(2).$$

(69)
Applying [17], Theorem 9.1.3 and its proof to the setting described above gives, for any $P$ in $X_0(p)(K)$,

$$[j(P), \infty]_{\mu_F} \leq [P, j^*(\infty)]_{\mu_0} + (p + 1) \sum \sup_{X_0(p)_\sigma} g_{\mu_0} + \frac{1}{2} \sum_{\sigma} \int_{X_0(p)_\sigma} \log(|j|^2 + 1)\mu_0$$

(70)

where $\sigma$ runs through the infinite places of $K$ and $X_0(p)_\sigma := X_0(p) \times_{G_K, \sigma} \mathbb{C}$.

We estimate the right-hand terms of (70). As for the last integrals we recall that, on the union of disks of ray $|q| < r$ around the cusps (that is, on the image in $X_0(p)(\mathbb{C})$ of the open subset $D_r := \{ z \in \mathcal{H}, \Re(z) > - (\log r)/2\pi \}$ in Poincaré upper-half plane $\mathcal{H}$) for some fixed $r$ in $[0, 1[$, one has

$$\left| \frac{f(q)}{q} \right| \leq \frac{2}{(1 - r)^2}$$

for any newform $f$ in $S_2(\Gamma_0(p))$. (See for instance [18], Lemma 11.3.7 and its proof.) We also know that the Petersson norm of such an $f$ satisfies $\|f\|^2 \geq \pi e^{-4\pi}$ ([18], Lemma 11.1.2). Choose $r = 1/2$ to fix ideas. On $D_{1/2}$, we have (see (25)):

$$\mu_0 = \frac{1}{2} \dim(J) \sum_{f \in B_2} \frac{f dq}{\|f\|^2} \leq \frac{64e^{4\pi}}{\pi} \int dq \wedge \overline{dq}.$$

(Sharper bounds should be achievable, but the one above is good enough for our present purpose.) It follows that there exists some real $A$ such that, in the decomposition

$$\int_{X_0(p)(\mathbb{C})} \log(|j|^2 + 1)\mu_0 = \int_{X_0(p)(\mathbb{C}) \cap D_{1/2}} \log(|j|^2 + 1)\mu_0 + \int_{X_0(p)(\mathbb{C}) \setminus D_{1/2}} \log(|j|^2 + 1)\mu_0$$

(71)

the first term of the right-hand side satisfies

$$\int_{X_0(p)(\mathbb{C}) \cap D_{1/2}} \log(|j|^2 + 1)\mu_0 \leq \frac{64e^{4\pi}}{\pi} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(p)] \int_{X(1)(\mathbb{C}) \cap D_{1/2}} \log(|j|^2 + 1) \frac{i}{2} dq \wedge \overline{dq} \leq (p + 1)A.$$

As for the second term, remembering that $\mu_0$ has total mass 1 on $X_0(p)(\mathbb{C})$ we check that

$$\int_{X_0(p)(\mathbb{C}) \setminus D_{1/2}} \log(|j|^2 + 1)\mu_0 \leq M_{1/2} := \max_{X(1)(\mathbb{C}) \setminus D_{1/2}} \log(|j|^2 + 1)$$

whence the existence of some absolute real number $A_0$ such that

$$\int_{X_0(p)(\mathbb{C})} \log(|j|^2 + 1)\mu_0 \leq (p + 1)A_0.$$

(72)

Putting this together with (70) we obtain a constant $C$ for which (69) reads

$$h_j(P) \leq \frac{1}{[K : \mathbb{Q}]} [P, j^*(\infty)]_{\mu_0} + (p + 1)\left( \sup_{X_0(p)(\mathbb{C})} g_{\mu_0} + A_0 \right).$$

With notations of Lemma 3.5, one further has

$$j^*(\infty) = p(0) + (\infty) \equiv (p + 1)\infty + p \cdot \Phi_{C_0}^0$$

(73)

as elements of $\overline{CH}(p)_{\text{num}}^{\mu_0}$. Using Lemma 3.5 (a) we get

$$\| [P, \Phi_{C_0}^0] \| \leq [K : \mathbb{Q}] \frac{6 \log p}{p - 1}.$$
so that, with (67),
\[
 h_j(P) \leq \frac{1}{[K : \mathbb{Q}]} \left[ P, (p+1) \mathcal{X}_0 \right]_{\mu_0} + (p+1) \left( \sup_{X_0(p)(\mathbb{C})} g_{\mu_0} + A_0 \right) + O(\log p)
\]
\[
 \leq \frac{1}{[K : \mathbb{Q}]} \left[ P, (p+1) \mathcal{X}_0 \right]_{\mu_0} + C_0 \cdot p^2 \log p
\]
which is (65).

The proof of (66) proceeds along the same lines, with one more ingredient. Applying Theorem 9.1.3 of [17] with the measure \( \mu_e \) instead of \( \mu_0 \) gives the corresponding version of (70). To obtain an upper bound for \( \sup_{X_0(p)(\mathbb{C})} g_{\mu_e} \) we recall that the theorem of Kowalski, Michel and Vanderkam asserts that \( \dim(J_e) \geq \dim(J_0(p))/5 \) for large enough \( p \). Our measure \( \mu_e := \frac{1}{\dim(J_e)} \sum_{\mathbb{C}} \frac{f_{\mathcal{H}_e}(q)}{g_{J_e}^2} \) (see (26)) therefore satisfies
\[
 0 \leq \mu_e \leq \frac{g}{\dim(J_e)} \mu_0 \leq 5\mu_0.
\]
This shows that as in (68), Bruin’s theorem ([9], Theorem 7.1) provides a universal \( c_e \) such that
\[
 \sup_{X_0(p)(\mathbb{C})} g_{\mu_e} \leq c_e p^2.
\]
Using (72) we obtain:
\[
 \int_{X_0(p)(\mathbb{C})} \log(|j|^2 + 1) \mu_e \leq (p + 1) A_e.
\]
Finally, equivalence (73) remains naturally true in the Chow group \( \hat{\mathcal{H}}(p)_{\text{num}} \) relative to the measure \( \mu_e \) instead of \( \mu_0 \), as remarked in Lemma 3.5 (a). This completes the proof of (66).

We can finally relate \( h_j \) and the Néron-Tate height \( h_0 \) relative to the \( \Theta \)-divisor (see (10)):

**Theorem 4.6** There are real numbers \( \gamma, \gamma_1 \) such that the following holds. Let \( K \) be a number field and \( p \) a prime number. Let \( \omega^0 := -(H_1 + H_3) \) be the 0-component of the canonical sheaf \( \omega \) on \( X_0(p) \) over \( K \) (as in Proposition 3.6 and Theorem 4.1). If \( P \) is a point of \( X_0(p)(K) \) then
\[
 h_j(P) \leq (12 + o(1)) \cdot h_0(P - \infty + \frac{1}{2} \omega^0) + \gamma \cdot p^2 \log p
\]
and
\[
 h_j(P) \leq (24 + o(1)) \cdot h_0(P - \infty) + \gamma_1 \cdot p^2 \log p.
\]

**Remark 4.7** Theorem 4.6 offers only one direction of inequality between \( j \)-height and \( \Theta \)-height: with our method of proof, it is harder to give an effective form to the reverse inequality, because of the metrics comparisons we use (see below).

Notice also that going through the above proofs using the estimate \( \sup_{X_0(p)(\mathbb{C})} g_{\mu_{00}} = O(1) \) of [32] and [3] (see Remark 4.5) would even give an error term of shape \( O(p) \) instead of \( O(p^2 \log p) \) in (78).

Those results are in some sense (hopefully sharp) special cases of the main results of [54], after rewriting the \( j \)-function in terms of classical \( \Theta \).

**Proof** Using Theorem 4.1, (51), Proposition 4.4 and (15) we obtain
\[
 h_j(P) \leq 12 \frac{p + 1}{p - 13} h_0(P - \infty + \frac{1}{2} \omega^0) + O(p^2 \log p).
\]
The last estimate (78) of the theorem comes from the fact that \( h_0 \) is a quadratic form and that
\[
 h_0(\omega^0) = O(\log p)
\]
by the results of [43] now many times mentioned.
5 Height of modular curves and the various $W_d$

We prove in this section a certain number of technical results about heights of cycles in the modular jacobian, which will be useful in the sequel. For applications of the explicit arithmetic Bézout theorem displayed in next section (Proposition 6.1), we indeed first need estimates for the degree and height of the image of $X_0(p)$, together with its various $d^{th}$-symmetric products (usually called “$W_d^n$”), within either $J_0(p)$ or its quotient $J$, relative to the $\Theta$-polarization. (For more general considerations on this topic, we also refer to [30].) We estimate those heights both in the normalized Néron-Tate sense and for some good (“Moret-Bailly”) projective models, to be defined shortly.

Let us first define the height of cycles relative to some hermitian bundle. For further details on this we refer to [65], or to [1], Section 2 for a more informal introduction.

**Definition 5.1** Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathcal{X}$ be an arithmetic scheme over $\mathcal{O}_K$, that is an integral scheme which is projective and flat over $\mathcal{O}_K$, having smooth generic fiber $X$ over $K$. Let $\mathcal{F}$ be a generically ample and relatively semiample hermitian sheaf with smooth metric, see [65], Section 5. We denote by $c_1(\mathcal{F})$ the first arithmetic Chern class of $\mathcal{F}$, and similarly by $c_1(F)$ the first Chern class of $F$.

Such a pair $(\mathcal{X}, \mathcal{F})$ will be called a model, in the sense of Zhang, of its pull-back $(X, F) = (\mathcal{X}_K, F_K)$ to the generic fiber.

Consider a model $(\mathcal{X}, \mathcal{F})$ as in Definition 5.1, and let $Y$ be a $d$-dimensional subvariety of $X$. The degree of $Y$ with respect to $F$ is as usual the non-negative integer given by the $d^{th}$-power self-intersection of $c_1(F)$ with $Y$, that is

$$\deg_F(Y) = (c_1(F)^d|Y).$$

We shall sometimes also write that quantity as $\deg_F(Y)$.

Now let $Y \to X$ be some “generic resolution of singularities” of $Y$ (that is, some good integral model for some desingularization of $Y$, see Section 1 of [65]). The height of $Y$ with respect to $\mathcal{F}$ will similarly be the real number obtained by taking the $(\dim Y)^{th}$-power self-intersection of $c_1(\mathcal{F})$ with $Y$, divided by the degree of $Y$ and normalized so that:

$$h_\mathcal{F}(Y) = \frac{(c_1(\mathcal{F})^{d+1}|Y)}{[K: \mathbb{Q}](d+1)\deg_F(Y)}. \tag{80}$$

One can check that definition\(^5\) does not depend on the desingularization $Y \to X$.

Instrumental to us will here be Zhang’s control of heights in terms of essential minima. Recall that the (first) essential minimum $\mu^{\text{ess}}_\mathcal{F}(Y)$ of $Y$ is the minimum of the set of real numbers $\mu$ such that there is a sequence of points $(x_n)$ in $Y(\overline{\mathbb{Q}})$ which is Zariski dense in $Y$ and $h_\mathcal{F}(x_n) \leq \mu$ for all $n$. Zhang’s Theorem (5.2) of [65] then asserts that

$$h_\mathcal{F}(Y) \leq \mu^{\text{ess}}_\mathcal{F}(Y). \tag{81}$$

Note that if $h_\mathcal{F} \geq 0$ on $Y(\overline{\mathbb{Q}})$ one also knows from [65], Theorem 5.2 the reverse inequality

$$h_\mathcal{F}(Y) \geq \frac{\mu^{\text{ess}}_\mathcal{F}(Y)}{d+1}. \tag{82}$$

If $(\mathcal{X}, \mathcal{F})$ is a model over $\mathcal{O}_K$, in the sense of Definition 5.1, of a polarized abelian variety $(X, F)$ over $K = \text{Frac}(\mathcal{O}_K)$, and $Y$ again is a $d$-dimensional subvariety of the generic fiber $X$, we still define its normalized Néron-Tate height relative to $F$ as the limit

$$h_F(Y) := \lim_{n \to \infty} \frac{1}{N^{2n}} h_\mathcal{F}([N^n]Y)$$

\(^5\)It could have been simpler to systematically use the definition of height of [8], Section 3.1, which does not demand desingularization, as we do in the proof of Proposition 6.1 at the end of Section 6. We could not find references however for Zhang’s inequality (see (81)) in that setting, so we stick to the above definitions.
where $N$ is any fixed integer larger than 1 and $[N^n]Y$ is the image of $Y$ under multiplication by $N^n$ in $X$. This normalized height, which is a direct generalization of the classical notion of Néron-Tate height for points, is known not to depend neither on the model $X$ of $X$, nor the extension $F$ of $F$, nor its hermitian structure (and not on $N$), so that the notation $h_F(\cdot)$ is finally unambiguous. We refer to [1], Proposition-Définition 3.2 of Section 3 for more details. We will actually use the extension of the two inequalities (81) and (82) to the case where the heights and essential minima are those given by the limit process defining Néron-Tate height (which is known to be non-negative on points) that is, with obvious notations

$$\frac{\mu_F^{\text{ess}}(Y)}{d+1} \leq h_F(Y) \leq \mu_F^{\text{ess}}(Y) \quad (83)$$

see Théorème 3.4 of [1]. As we will see in Section 5.3 and below, Moret-Bailly theory allows, under certain conditions, to interpret Néron-Tate heights as Arakelov projective heights (that is, without going through limit process).

### 5.1 Néron-Tate heights

We shall apply the above to cycles in modular abelian varieties endowed with their symmetric theta divisor: the notation $h_0$ will always stand for normalized Néron-Tate height of cycles.

**Proposition 5.2** Let $X$ be the image via $\pi_A \circ \iota_\infty: X_0(p) \rightarrow A$ of the modular curve $X_0(p)$ mapped to a non-zero quotient $\pi_A: J_0(p) \rightarrow A$ of its jacobian, endowed with the polarization $\Theta_A$ induced by the $\Theta$-divisor (see (4), (9) and around). The degree and normalized Néron-Tate height of $X$ satisfy:

$$\deg_{\Theta_A}(X) = \dim(A) = O(p)$$

and

$$h_{\Theta_A}(X) = O(\log p).$$

**Proof** If $(A, \Theta_A) = (\text{Jac}(X_0(p)), \Theta)$, it is well-known that the $\Theta$-degree of $X_0(p)$ (or in fact any curve) embedded in its jacobian via some Albanese embedding, equals its genus. That can be seen in many ways, among which one can invoke Wirtinger’s theorem ([22], p. 171), which yields in fact the desired result for any quotient $(A, \Theta_A)$: using the notation before (12) we have

$$\deg_{\Theta_A}(X) = \int_{X_0(p)} \sum_{f \in B_2} \frac{1}{2} \frac{f dq}{\|f\|^2} = \dim A \leq g(X_0(p)).$$

We then apply once more the fact (15) that the genus $g(X_0(p))$ is roughly $p/12$. (We could also have more simply say that the degree is decreasing by projection, as in the argument below.)

As for the height, the main result of [43] gives that the essential minimum of the normalized Néron-Tate height, $\mu^{\text{ess}}_{\Theta_A}(X_0(p))$ is $O(\log p)$. As the height of points decreases by projection (see Section 2.1.2, and in particular (7)) the same is true for $\mu^{\text{ess}}_{\Theta_A}(X)$ and we conclude with Zhang’s (83). □

Now for the Néron-Tate normalized height of symmetric squares and variants:

**Proposition 5.3** Assume $X := X_0(p)$ has gonality strictly larger than 2 (which is true as soon as $p > 71$, see [52]). Let $\iota := \iota_\infty: X_0(p) \hookrightarrow J_0(p)$ be the Albanese embedding as in Proposition 5.2.

Let $X^{(2)}$ be the symmetric square of $X_0(p)$ embedded in $J_0(p)$ via $(P_1, P_2) \mapsto \iota(P_1) + \iota(P_2)$, and similarly let $X^{(2)}_{\perp}$ be the image of $(P_1, P_2) \mapsto \iota(P_1) - \iota(P_2)$. Let $X^{(2)}_{\perp}$ and $X^{(2)}_{\perp}$ be the projections of $X^{(2)}$ and $X^{(2)}$, respectively, to $J^\perp_1$ (the “orthogonal complement” to the winding quotient $J_1$, see paragraph 2.2.3). Then with notations as in Proposition 5.2 taking $A = J_0(p)$ and $A = J^\perp_1$ respectively one has

$$\deg_{\Theta}(X^{(2)}) = O(p^2) = \deg_{\Theta}(X^{(2)}_{\perp}),$$

$$h_\Theta(X^{(2)}) = O(\log p) = h_\Theta(X^{(2)}_{\perp}).$$
and the same holds for the quotient objects:

\[ \deg_{\Theta^\perp}(X^{(2)}_{e^\perp}) = O(p^2) = \deg_{\Theta^\perp}(X^{(2),-}_{e^\perp}) ; \quad \h_{\Theta^\perp}(X^{(2)}_{e^\perp}) = O(\log p) = \h_{\Theta^\perp}(X^{(2),-}_{e^\perp}). \]

**Proof** Denoting by \( p_1 \) and \( p_2 \) the obvious projections below we factor in the common way (see [49], paragraph 3, Proposition 1 on p. 320) our maps over \( \mathbb{Q} \) as follows:

\[
X_0(p) \times X_0(p) \xrightarrow{\pi_A \times \pi_A} A \times A \xrightarrow{M} A \times A \xrightarrow{p_2} A
\]

so \( X^{(2)} = p_1 \circ M \circ (\pi_A \times \pi_A)(X_0(p) \times X_0(p)) \) and \( X^{(2),-} = p_2 \circ M \circ (\pi_A \times \pi_A)(X_0(p) \times X_0(p)) \) when \( A = J_0(p) \), and the same with \( X^{(2)}_e \) and \( X^{(2),-} \) with \( A = J_0^e \). We endow \( A \times A \) with the hermitian sheaf \( \Theta^e_A := \pi_1^* \Theta_A \otimes \pi_2^* \Theta_A \). Then \( \mathcal{M}^e(\Theta^e_A) \simeq (\Theta^e_A)_{\otimes 2} \) ([49], p. 320). Therefore, writing \( X \) for \( \pi_A(X_0(p)) \) in short and using Proposition 5.2,

\[ \deg_{\Theta^e_A}(M(X \times X)) = 4 \deg_{\Theta^e_A}(X \times X) = 8(\deg_{\Theta_A}(X))^2 = O(g^2). \]

As degree decreases by our projections and \( O(g^2) = O(p^2) \), \( \deg_{\Theta_A}(X^{(2)}) \) and \( \deg_{\Theta_A}(X^{(2),-}) \) are \( O(p^2) \).

By definition of essential minima,

\[ \mu_{\Theta^e_A}(X \times X) \leq 2 \mu_{\Theta^e_A}(X). \]

This implies that \( \mu_{\Theta^e_A}(M(X \times X)) \leq 4 \mu_{\Theta^e_A}(X) \). Invoking (83) again and Proposition 5.2 together with the fact that the height of points also decreases by projection,

\[ \mu_{\Theta^e_A}(X^{(2)}) \leq \mu_{\Theta^e_A}(M(X \times X)) \leq 4 \mu_{\Theta^e_A}(X) \leq 8 \h_{\Theta_A}(X) \leq O(\log p). \]

Therefore

\[ \h_{\Theta_A}(X^{(2)}) = O(\log p). \]

\[ \square \]

Note that this proof applies more generally to any sub-quotient of \( J_0(p) \).

### 5.2 Moret-Bailly models and associated projective heights

To build-up the projective models of the jacobian (over \( \mathbb{Z} \), or finite extensions), and associated heights, that we shall need for our arithmetic Bézout, we use Moret-Bailly theory, in the sense of [47], as follows. For more about similar constructions in the general setting of abelian varieties we refer to [7], 2.4 and 4.3; see also [54].

Let therefore \( (J, \mathcal{L}(\Theta)) \) stand for the principally polarized abelian variety \( J_0(p) \) endowed with the invertible sheaf associated with its symmetric theta divisor, defined over some small extension of \( \mathbb{Q} \) (see (89) below and around for more details). Endow the complex base-changes of the associated invertible sheaf \( L(\Theta) \) with its cubist hermitian metric. If \( N_{J,\mathcal{O}_{K}} \) is the Néron model of \( J \) over the ring of integers \( \mathcal{O}_{K} \) of a number field \( K \), we know it is a semi-stable scheme over \( \mathcal{O}_{K} \), whose only non-proper fibers are above primes \( \mathfrak{P} \) of characteristic \( p \), where it then is purely toric. At any such \( \mathfrak{P} \), with ramification index \( e_{\mathfrak{P}} \), the group scheme \( N_{J,\mathcal{O}_{K}} \) has components group

\[ \Phi_{\mathfrak{P}} \simeq (\mathbb{Z}/N_0e_{\mathfrak{P}}\mathbb{Z}) \times (\mathbb{Z}/e_{\mathfrak{P}}\mathbb{Z})^{g-1} \]

for \( g := \dim J \) and \( N_0 := \text{num}(\frac{p-1}{12}) \) (see e.g. [36], Proposition 2.11).

We choose and fix an integer \( N > 0 \) and a number field \( K \supseteq \mathbb{Q}(J[2N]) \), for all this paragraph, so that all the \( 2N \)-torsion points in \( J \) have values in \( K \). One then observes from (85) that \( 2N \)
divides all the ramification indices \( e_P \), and Proposition II.1.2.2 on p. 45 of [47] asserts that \( L(\Theta) \) has a cubist extension, let us denote it by \( L(\Theta) \), to the open subgroup scheme \( \mathcal{N}_{J,N} \) of the Néron model \( \mathcal{N}_{J,O_K} \) over \( O_K \) whose fibers have component group killed by \( N \).

Such an extension \( L(\Theta) \) is actually symmetric ([47], Remarque II.1.2.6.2) and unique (see Théorème II.1.1.i) on p. 40 of loc. cit.). Moreover \( L(\Theta) \) is ample on \( \mathcal{N}_{J,N} \) ([47], Proposition VI.2.1 on p. 134). Its powers \( L(\Theta)^{\otimes r} \) are even very ample on \( \mathcal{N}_{J,N} \otimes_{O_K} O_K[1/2p] \) as soon as \( r \geq 3 \), as follows from the general theory of theta functions. Provided \( N > 1 \), the sheaf \( L(\Theta)^{\otimes N} \) is spanned by its global sections on the whole of \( \mathcal{N}_{J,N} \) ([47], Proposition VI.2.2), although we shall not use that last fact as such.

Picking up a basis of generic global sections in \( H^0(J_0(p)K, L(\Theta)^{\otimes N}) \), with \( N \geq 3 \), we thus define a map \( J_0(p)_K \to \mathbb{P}^n_K, \) for \( n = N^g - 1 \). Assume our generic global sections extend to a set \( S \) in \( H^0(\mathcal{N}_{J,N}, L(\Theta)^{\otimes N}) \). Let \( J \to \mathbb{P}^n_{O_K} \) be the schematic closure in \( \mathbb{P}^n_{O_K} \) of the generic fiber \( (\mathcal{N}_{J,N})_K = J_K \) via the associated composed embedding \( J_K \hookrightarrow \mathbb{P}^n_K \hookrightarrow \mathbb{P}^n_{O_K} \). Define \( \mathcal{M} = J \cdot O_{\mathcal{P}^n_{O_K}}(1) \) on \( J \). Let on the other hand \( \mathcal{M}_{J,N,S} := (\sum s \in S O_K \cdot s) \) be the subsheaf of \( L(\Theta)^{\otimes N} \) on \( \mathcal{N}_{J,N} \) spanned by \( S \). Write \( \nu : \mathcal{N}_{J,N} \to \mathcal{N}_{J,N,S} \) for the blowup at base points for \( \mathcal{M}_{J,N,S} \) on \( \mathcal{N}_{J,N} \), that is, the blowup along the closed subscheme of \( \mathcal{N}_{J,N} \) defined by the sheaf \( L(\Theta)^{\otimes N}/\mathcal{M}_{J,N,S} \). We have a commutative diagram

\[
\begin{array}{ccc}
J_K & \xrightarrow{\iota_N} & \mathcal{N}_{J,N} \\
\downarrow \iota_N & & \downarrow \nu \\
\mathbb{P}^n_{O_K} & \xleftarrow{\sim} & \mathcal{M}_{J,N,S}
\end{array}
\]  

where the only non-trivial map \( J \to \mathbb{P}^n_{O_K} \) (whence \( \iota_N \)) is deduced from the fundamental properties of blowups. Considering the complex base-changes of the generic fiber we note that \( M \) is automatically endowed with a cubist hermitian structure induced by that of \( L(\Theta)_\mathbb{C} \) (see [7], (4.3.3) and following lines).

**Definition 5.4** Given an integer \( N \geq 3 \), and a number field \( K \) containing \( \mathbb{Q}(J_0(p)[2N]) \), we define the “good model” for \( (J_0(p), L(\Theta)^{\otimes N}) \) relative to some finite set \( S \) in \( H^0(\mathcal{N}_{J,N}, L(\Theta)^{\otimes N}) \), which spans \( H^0(J_0(p), L(\Theta)^{\otimes N}) \), as the projective scheme \( J \) over \( \text{Spec}(O_K) \) enhanced with the hermitian sheaf \( \mathcal{M} \) constructed above, and \( h_M \) the associated height.

Outside base points for \( \mathcal{M}_{J,N,S} \) on \( \mathcal{N}_{J,N} \) the blowup \( \nu : \mathcal{N}_{J,N} \to \mathcal{N}_{J,N,S} \) is an isomorphism and on that open locus we have

\[
L(\Theta)^{\otimes N} \simeq \mathcal{M}_{J,N,S} \simeq \iota_N^* \mathcal{M} = j_N^* O_{\mathbb{P}^n_{O_K}}(1)
\]

so we dwell on the fact that the height \( h_M \) of our “good models” for \( (J_0(p), L(\Theta)^{\otimes N}) \) will indeed compute \( N \) times the Néron-Tate height of certain \( \mathbb{Q} \)-points (those whose closure factorizes through \( \mathcal{N}_{J,N} \) deprived from the base points for \( S \)), but definitely not all. For arbitrary points, still, one can deduce from the work of Bost ([7], 4.3) the following inequality.

**Proposition 5.5** For any point \( P \) in \( J_0(p)(\mathbb{Q}) \), the height \( h_M(P) \) of Definition 5.4 satisfies

\[
h_M(P) \leq N h_0(P).
\]

**Proof** We briefly adapt [7], 2.4 and 4.3, using our above notations. Of course this statement has nothing to see with modular Jacobians, and holds for any abelian variety over a number field. Let \( N' \) be some integer such that \( P \) defines a section of \( \mathcal{N}_{J,N'}(O_F) \) for some ring of integers \( O_F \). Up to replacing \( O_F \) by a sufficiently ramified finite extension, we can assume \( L(\Theta)^{\otimes N} \) has a cubist extension \( L(\Theta)^{\otimes N} \) to all of \( \mathcal{N}_{J,N'} \) over \( O_F \) ([47], Proposition II.1.2.2). One has

\[
h_0(P) = \frac{1}{N} \frac{1}{[F : \mathbb{Q}]} \deg(P^*(L(\Theta)^{\otimes N})).
\]
As in (86) however we see that there is no well-defined map from \(N_{\mathcal{J},N^\prime} \) to \(\mathbb{P}^n \) because \(L(\Theta)^{\otimes N} \) needs not be spanned by elements of \(S \) on all of \(N_{\mathcal{J},N^\prime} \) (even though it is, by hypothesis, on the generic fiber). To remedy this we adapt the construction (86).

If \(\pi': N_{\mathcal{J},N^\prime} \to \text{Spec}(O_F) \) is the structural morphism, we define now \(\mathcal{M}'_{N^\prime} := \left( \sum_{s \in S} O_F \cdot s \right) \) as the subsheaf of \(L(\Theta)^{\otimes N} \) on \(N_{\mathcal{J},N^\prime} \) spanned by \(S \), still endowed with the metric induced by that of \(L(\Theta)^{\otimes N} \). One checks (see [7], (4.3.8)) that the projective model \(\overline{\mathcal{J}}_{O_F} \) of \( (N_{\mathcal{J},N^\prime})_F \) is \( J_F \) in \( \mathbb{P}^n \) defined as in (86) yields a sheaf \(\mathcal{M}' \) on \( \mathcal{J}_{O_F} \), whence a height \( h_{\mathcal{M}'} \), which coincides with the height \( h_{\mathcal{M}} \) on the base change of the good model \(\overline{\mathcal{J}}_{O_F} \).

Replacing \(N_{\mathcal{J},N^\prime} \) by its blowup \(\nu': \tilde{N}_{\mathcal{J},N^\prime} \to N_{\mathcal{J},N^\prime} \) base points for \(\mathcal{M}'_{N^\prime} \) in \(L(\Theta)^{\otimes N} \) on \(N_{\mathcal{J},N^\prime} \), we keep on following construction (86) to obtain maps \(\iota^*_N': \tilde{N}_{\mathcal{J},N^\prime} \to \mathcal{J}_{O_F} \) and \(\mathbf{f}'_N: \tilde{N}_{\mathcal{J},N^\prime} \to \mathbb{P}^n \) such that the Zariski closure of \(f'_N(\tilde{N}_{\mathcal{J},N^\prime}) \) identifies with \(\mathcal{J}_{O_F} \). We moreover have

\[
\iota^*_N'(\mathcal{M}') = \nu'^*(L(\Theta)^{\otimes N}) \otimes \mathcal{O}(-E)
\]

where \(E \) is the exceptional divisor of the blowup which is by definition effective. The section \(P \) of \( N_{\mathcal{J},N^\prime}(O_F) \) lifts to some \(\tilde{P} \) of \( \tilde{N}_{\mathcal{J},N^\prime}(O_F) \). Let \(e_P \) be the section of \(\mathcal{J}(O_F) \) defined by the Zariski closure of \(P(F) \) in \(J\). One can finally compute

\[
h_{\mathcal{M}}(P) = h_{\mathcal{M}'}(P) = \frac{1}{[F: \mathbb{Q}]} \deg(e_P(\mathcal{M}')) = \frac{1}{[F: \mathbb{Q}]} \deg(\tilde{P}^*(\iota^*_N'(\mathcal{M}'))) \\
\leq \frac{1}{[F: \mathbb{Q}]} \deg(\tilde{P}^*(\nu'^*(L(\Theta)^{\otimes N}))) = \frac{1}{[F: \mathbb{Q}]} \deg(P^*(L(\Theta)^{\otimes N})) = N h_\Theta(P). \quad \square
\]

The following straightforward generalization to higher dimension will be useful in next section.

**Corollary 5.6** If \(Y\) is a \(d\)-dimensional irreducible subvariety of \(J_0(p)\) then

\[
h_{\mathcal{M}}(Y) \leq (d + 1) N h_\Theta(Y).
\]

**Proof** Combine Zhang’s formulas (81) and (83) with Proposition 5.5. \(\square\)

Recall from (8) that one can define the “pseudo-projection” \(P_{J_{0\pm}}(\iota(0(X_0(p)))\) of the image of \(X_0(p)^{\iota(0)} \subseteq J_0(p)\) on the subabelian variety \(J_{0\pm} \subseteq J_0(p)\). Let \(X_{0\pm}\) be any of its irreducible components. Define similarly \(X_{0\pm}^{(2)}, X_{0\pm}^{(2),-}\), \(X_{0\pm}^{(2)}\) and \(X_{0\pm}^{(2),-}\) as in Proposition 5.3. Note that, by construction, the degree and normalized Néron-Tate height of \(X_{0\pm}\) (and other similar pseudo-projections: \(X_{0\pm}^{(2)}\) etc.), as an irreducible subvariety of \(J_0(p)\) endowed with \(h_\Theta\), are those of \(\pi_{J_{0\pm}}(X_0(p)) = X_{0\pm}^{(2),-}\) relative to the only natural hermitian sheaf of \(J_{0\pm}^{\times}\), that is, the \(\Theta_{0\pm} = \Theta_{J_{0\pm}}\) described in paragraph 2.1.2 and estimated in Proposition 5.2.

**Corollary 5.7** For any fixed integer \(N \geq 3\), and any number field \(K\) containing \(\mathbb{Q}(J_0(p)[2N])\), let \((\mathcal{J}_{\mathcal{M}},K)\) be the good model for \((J_0(p),L(\Theta)^{\otimes N})\), and \(h_{\mathcal{M}}\) the associated projective height, given in Definition 5.4. Let \(X\) be the image of \(X_0(p)^{\iota(0)} \subseteq J_0(p)\), and more generally \(X_{0\pm}^{(2)}, X_{0\pm}^{(2),-}\), \(X_{0\pm}^{(2)}\) and \(X_{0\pm}^{(2),-}\) be the objects \(X^{(2)}, \ldots\) defined in Proposition 5.3 (or their pseudo-projections). Then their \(\mathcal{M}^{\times\pm}\)-heights are bounded from above by similar functions as their Néron-Tate height (Proposition 5.3). Explicitly, \(h_{\mathcal{M}^{\times\pm}}(X_0(p))\) is less than \(O(\log(p))\), and \(h_{\mathcal{M}^{\times\pm}}(X_{0\pm}^{(2)},\ldots)\), are all less than \(O(\log(p))\). Similarly the \(\mathcal{M}^{\times\pm}\)-degree of \(X_0(p)\) is \(O(p)\), and the \(\mathcal{M}^{\times\pm}\)-degrees of \(X_{0\pm}^{(2)},\ldots\), etc., are all \(O(p^2)\).

**Proof** Combine Zhang’s formulas (81) and (83) with Propositions 5.2, 5.3 and 5.5. \(\square\)
5.3 Estimates on Green-Zhang functions for $J_0(p)$

We shall later on need some control on the $p$-adic Néron-Tate metric of $\Theta$ as alluded to in Remark 4.3. (Those statements can probably be best formulated in the setting of Berkovich theory, for which one might check in particular [15], Proposition 2.12, and [61]. A useful point of view is also proposed by that of “tropical Jacobians”, see [44] and [31]. We will content ourselves here with our down-to-earth point of view.) We therefore define

$$\Phi_p := \lim_{\kappa_p \rightarrow \mathbb{Q}_p} \Phi_p$$

as the direct limit, on a tower of totally ramified extensions $K_p/\mathbb{Q}_p$, of the component groups $\Phi_p$ of the Néron models of $J_0(p)$ at $\mathfrak{p}$, see (85). The compatible embeddings

$$Z := (C_0 - C_\infty) \simeq (0) - (\infty) \simeq \mathbb{Z}/N\mathbb{Z} \hookrightarrow \Phi_p$$

for each $\mathfrak{p}$ induce an exact sequence $0 \rightarrow Z \rightarrow \Phi_p \rightarrow \lim_{\kappa_p} (\mathbb{Z}/\kappa_p\mathbb{Z})^g \simeq (\mathbb{Q}/\mathbb{Z})^g \rightarrow 0$. Passing to the real completion yields a presentation:

$$0 \rightarrow Z \simeq \mathbb{Z}/N\mathbb{Z} \rightarrow \Phi_{p,\mathbb{R}} \rightarrow (\mathbb{R}/\mathbb{Z})^g \rightarrow 0$$

(88)

(see [36], Proposition 2.11.(c)).

We then sum-up useful properties about theta divisors and theta functions “over $\mathbb{Z}$”.

As $J_0(p)$ is principally polarized over $\mathbb{Q}$, the complex extension of scalars $J_0(p)(\mathbb{C})$ can be given a classical complex uniformization $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ for some $\tau$ in Siegel’s upper half plane. The associated Riemann theta function:

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp(i\pi^4 m \cdot \tau \cdot m + 2i\pi^4 \cdot \tau \cdot z)$$

defines the tautological global section 1 of a trivialization of $\mathcal{O}_{J_0(p)}(\Theta_\mathbb{C}) = \mathcal{M}_{\mathbb{C}}^{g,1/N}$ for $\Theta_\mathbb{C}$ the image $W_{g-1}$ of some $(g - 1)$st power of $X_0(p)$ in $J_0(p)$. More precisely, Riemann’s classical results (e.g. [22], Theorem on p. 338) assert that $\text{div}(\theta(z)) = \Theta_\mathbb{C}$ is the divisor with support

$$\{\kappa_p \mathfrak{p} + \sum_{i=1}^{g-1} \mathfrak{p}_i(P_i), P_i \in X_0(p)(\mathbb{C})\},$$

where for any $P_0 \in X_0(p)(\mathbb{C})$ we write $\mathfrak{p}_i : X_0(p) \hookrightarrow J_0(p)$ for the Albanese morphism with base point $P_0$, and $\kappa = \kappa_{\mathfrak{p}_0} = \frac{1}{4} \mathfrak{p}_0(K_{X_0(p)})$ for the image of Riemann’s characteristic, which is some pre-image under duplication in $J_0(p)$ of the image of some canonical divisor: $\omega^0 = \mathfrak{p}_0(K_{X_0(p)})$ (see Theorem 4.6 above).

Among the translates $\Theta_D = t_D^* \Theta$, for $D \in J_0(p)(\mathbb{C})$, of the above symmetric $\Theta$, the divisor

$$\Theta_\kappa = t^*_\kappa \Theta = \sum_{i=1}^{g-1} t^*_\kappa(X_0(p)\mathbb{C})$$

defines an invertible sheaf $L(\Theta_\kappa)$ on $J_0(p)$ over $\mathbb{Q}$. If $\mathcal{N}_{1,1}$ denotes the neutral component of the Néron model of $J$ over $\mathbb{Z}$ and $L(\Theta_\kappa)$ is the cubist extension of $L(\Theta_\kappa)$ to $\mathcal{N}_{1,1}$ (compare [47], Proposition II.1.2.2, as in Section 5.2 above), we know that $H^0(\mathcal{N}_{1,1}, L(\Theta_{\kappa}))$ is a (locally...) free $\mathbb{Z}$-module of rank 1, so that the complex base-change $H^0(J_0(p)(\mathbb{C}), L(\Theta_{\kappa}))$ is similarly a complex line. This means that if $s_\kappa$ is a generator of the former space, whose image in the later we denote by $s_\kappa \mathcal{C}$, there is a nonzero complex number $C_\kappa$ such that

$$s_{\kappa,\mathcal{C}}(z) = C_\kappa \cdot \theta(z + \kappa).$$

(90)

Up to making some base-change from $\mathbb{Z}$ to some $\mathcal{O}_K$ we can now forget about $\kappa$ and come back to the symmetric $\Theta$: we define a global section

$$s_{\mathcal{O}} := (t^*_{\kappa})s_\kappa \in H^0(\mathcal{N}_{1,1}, L(\Theta)\mathcal{O}_K)$$

so that $s_{\mathcal{O},\mathcal{C}}(z) = C_\kappa \cdot \theta(z)$. 

(91)
If one replaces \( \mathcal{N}_{J,1} \) by the Néron model, say \( \mathcal{N}_{0,K_1} \), of \( J_0(p) \) over any extension \( K_1 \) of \( K \), then [47], Proposition II.1.2.2 insures that up to making some further field extension \( K_2/K_1 \) the sheaf \( L(\Theta) \mathcal{O}_{K_2} \) has a cusp extension \( L(\Theta) \mathcal{O}_{K_1} \times \mathcal{O}_{K_2} \). Therefore \( s_{J^0} \) extends to a rational section (we shall sometimes write meromorphic section) of \( L(\Theta) \mathcal{O}_{K_1} \times \mathcal{O}_{K_1} \mathcal{O}_{K_2} \). Abusing notations we still denote that extended section by \( s_{J^0} \), and write accordingly \( \Theta \) for its divisor \( \text{div}(s_{J^0}) \) on \( \mathcal{N}_{0,K_1} \times \mathcal{O}_{K_1} \mathcal{O}_{K_2} \). Because \( s_{J^0} \) is well-defined (and non-zero) on the neutral component of the Néron model, its poles on \( \mathcal{N}_{0,K_1} \times \mathcal{O}_{K_1} \mathcal{O}_{K_2} \) can only show-up at places of bad reduction.

**Proposition 5.8** The multiplicity of the \( \Theta \)-divisor at any component of the Néron model of \( J_0(p) \) over \( \mathbb{Z} \), normalized to be 0 along the neutral component, is \( O(p) \).

**Proof** We start by the following observations. Let us write \( s_{J^0,z}(z) = C_0 \cdot \theta(z) \) as in (91). Take \( D \) in \( J_0(p)(\mathbb{C}) \) which can written as the linear equivalence class of some divisor

\[
D = \sum_{i=1}^{g} -(Q_i - \infty)
\]

for points \( Q_i \) in \( X_0(p)(\mathbb{C}) \). We associate to \( D \) the embedding:

\[
\iota_{\kappa+D} : \begin{cases}
X_0(p) & \mapsto J_0(p) \\
P & \mapsto \iota_{\kappa}(P) + D
\end{cases}
\]

where \( \kappa \) is Riemann’s characteristic (see just before (57)). For such a \( D \) whose \( Q_i \) are assumed to belong to \( X_0(p)(\mathbb{Q}) \), we know from the proof of Theorem 4.1 (see (54)) that

\[
\text{h}_\Theta(\iota_{\kappa+D}(P)) = \frac{1}{[K(P,D) : \mathbb{Q}]}[P, \omega_D]_{\mu_0}
\]

(92)

with

\[
\omega_D = \sum_i Q_i + \Phi_D + c_D X_{\infty}
\]

(93)

and \( \Phi_D \) is the explicit vertical divisor

\[
\Phi_D = \frac{1}{2}(\Phi_\omega + \Phi_\theta) - \sum_{i=1}^{g} \Phi_Q_i
\]

(94)

at each bad place, with notations as those of the proof of Theorem 4.1, see (55).

Moreover, it is well-known that there is a subset of \( J_0(p)(\mathbb{C}) \) which is open for the complex topology, and even the Zariski topology, in which all points \( D = \sum_i -(Q_i - \infty) \) as above are such that

\[
\dim_\mathbb{C} H^0(X_0(p)(\mathbb{C}), L(-D + g \cdot \infty)) = \dim_\mathbb{C} H^0(X_0(p)(\mathbb{C}), \iota_{\kappa+D}^* L(\Theta)) = 1
\]

(95)

so that \( \iota_{\kappa+D}(\Theta_\mathbb{C}) = \sum_i Q_i \mathbb{C} \), the latter being an equality between effective divisors, not just a linear equivalence ([22], pp. 336–340). As the height \( \text{h}_\Theta \), in the Néron model of \( J_0(p) \), can be understood as the Arakelov intersection with \( \Theta = \text{div}(s_{J^0}) \) it follows that, on the curve \( X_0(p)_\mathbb{C} \), \( \text{div}(s_{J^0}) \cap \iota_{\kappa+D}(X_0(p)_\mathbb{C}) = \cup_i \iota_{\kappa+D}(Q_i \mathbb{C}) \), or \( \text{div}(\iota_{\kappa+D}^* s_{J^0}) = \sum_i Q_i \mathbb{C} \). More precisely, extending base to some ring of integers \( \mathcal{O}_K \) so that the \( Q_i \) define sections of the minimal regular model \( X_0(p)_{\mathcal{O}_K} \) of \( X_0(p) \) over \( \mathcal{O}_K \), and making if necessary a further base extension such that \( L(\Theta) \) has a cusp extension on the whole Néron model of \( J_0(p) \) over \( \mathcal{O}_K \) (as after (91)), one sees that \( s_{J^0} \) defines a meromorphic section of \( L(\Theta)_{\mathcal{O}_K} \) and the restriction to the generic fiber \( X_0(p)_K \) of \( \text{div}(\iota_{\kappa+D}^* s_{J^0}) \) has to be equal (and not merely linearly equivalent) to \( \sum_i Q_i \mathbb{C} \).

Now in such a situation, the multiplicity of \( \text{div}(s_{J^0}) \) on a component of the Néron model to
which $X_0(p)_K^{\text{smooth}}$ is mapped via $\iota_{k+D}$, can be read on the multiplicity of $\iota_{k+D}^*(\Theta)$ along that component of $X_0(p)_K^{\text{smooth}}$. In turn, because of decompositions of the arithmetic Chow group similar to that of Theorem 3.2, multiplicities of $\text{div}(\Theta)$ are determined by the $\Phi_D$ of (93), up to constant addition of vertical fibers. The property that $\text{div}(\Theta)$ has multiplicity 0 along the neutral component of the Néron model (see (91)) fixes that last indetermination. Now if $\mathfrak{P}$ is a place of bad reduction for $X_0(p)_{\mathfrak{O}_K}$, and if the $Q_i$ move slightly in the $\mathfrak{P}$-adic topology (without modifying their specialization component at $\mathfrak{P}$), the vertical divisor $\Phi_D$ does not change either at $\mathfrak{P}$, and the above reasoning regarding the components values of $\Theta$ is actually independent from the fact that condition (95) holds true or not (provided, we insist, that the specialization components of the $Q_i$ at $\mathfrak{P}$ do not vary).

We shall gain some flexibility with a last preliminary remark. If $k$ is any integer between 0 and $N_0 - 1$ (recall $N_0$ is the order of the Eisenstein element $(0 - \infty)$), the divisor $\tilde{\omega}_D$ of (93) can still be written as

$$\tilde{\omega}_D = \left( k \cdot 0 + (g - k) \cdot \infty - k \Phi_{C_n} + \frac{1}{2} (\Phi_{\omega} + \Phi_{\eta}) - \tilde{D} \right) + c_D X_{\infty}$$

so that if

$$D = \left( \sum_{i=1}^{g} (Q_i - \infty) \right) + k(0 - \infty) = \sum_{i=1}^{k} -(Q_i - 0) + \sum_{i=k+1}^{g} -(Q_i - \infty)$$

then $\tilde{\omega}_D = \sum_{i=1}^{g} Q_i + \Phi_D + c_D X_{\infty}$ where $\Phi_D$ is still

$$\Phi_D = \frac{1}{2} (\Phi_{\omega} + \Phi_{\eta}) - \sum_{i=1}^{g} \Phi_{Q_i}.$$  \hspace{1cm} (96)

Coming back to the proof of the present Proposition 5.8, and assuming first $D = 0$, it follows from what we have just discussed that the multiplicity of the $\Theta$-divisor on the components of the jacobian to which the components of $X_0(p)_K^{\text{smooth}}$ map under $\iota_k$ is given by the functions $g_n$ and $G$ of (45) and (46), see Theorem 4.1. To obtain the multiplicity of the $\Theta$-divisor on all components of the jacobian we shall shift our Albanese embeddings $\iota_{k+D}$ in order to explore all of $J_0(p)/J_0(p)^0$ with successive translations of $X_0(p)_K^{\text{smooth}}$ inside $J_0(p)$.

To be more explicit, let $\mathfrak{E}$ be an element of the component group $J_0(p)/J_0(p)^0$ at $\mathfrak{P}$, and $D = \sum_{i=1}^{g} (P_i - \infty)$ be a divisor, with all $P_i$ in $X_0(p)(K)$, which reduces to $\mathfrak{E}$ at $\mathfrak{P}$. For all $r$ in $\{1, \ldots, g\}$, set $D_r = \sum_{i=1}^{r} (P_i - \infty)$ and let also $k_r$ in $\{1, \ldots, N_0 - 1\}$ and $Q_{i,r}$ be $g$ associated points on the curve such that one can write both

$$D_r = \sum_{i=1}^{r} (P_i - \infty) \quad \text{and} \quad D_r = \sum_{i=1}^{r} -(Q_{i,r} - \infty) + k_r(0 - \infty).$$

As always in this proof, up to making a finite base-field extension one can assume all points have values in $K$. Recall also from the discussion above that one can move slightly the $Q_i$ in the $\mathfrak{P}$-adic topology, as all that interests us here is the component $\mathfrak{E}_r$, $1 \leq r \leq g$, of $(J_0(p)/J_0(p)^0)_{\mathfrak{P}}$ to which $D_r$ maps. One can therefore assume if one wishes that $\iota_{k+D_r}^*(\Theta_{\mathfrak{E}}) = \sum_{i} Q_{i,r} (\text{equality, not just linear equivalence})$. The presentation of $\Phi_D$ given in (88) and above also shows one can assume that the specialization components at $\mathfrak{P}$ of the $Q_{i,r}$, in $X_0(p)_K^{\text{smooth}}$, which are not $C_{\infty}$, are all different (see Figure 2.2.2).

Taking first $D = 0$, that is, using the map $\iota_k$, we already remarked that (94) implies the value $V_1$ of $\text{div}(\Theta)$ on $\mathfrak{E}_1$ is $V_1 = \left[ \frac{1}{2} (\Phi_{\omega} + \Phi_{\eta}), P_1 \right] = \frac{1}{2} (|\Phi_{\omega}|^2 + |\Phi_{\eta}|^2)$ (see (53)). By Remark 3.4 and (34), $|V_1| \leq 2$.

Going one step further we reach $\mathfrak{E}_2$ by considering the Albanese image $\iota_{k+D_1}(X_0(p)_K^{\text{smooth}})$ and looking at the image of $P_2$. Here we need not to forget that the $\infty$-cusp in $X_0(p)$ now maps to
\(C_1\), so the normalization of components-divisor on the curve \(X_0(p)_{\overline{\mathbb{Q}}}^{\text{smooth}}\) at \(\mathfrak{P}\) cannot be fixed to be \(0\) along the \(\infty\)-component any longer: it needs to take the value \(V_1\) found above, in order to match with the normalization of the theta divisor on the Jacobian. Applying the same reasoning as before with formula (96) gives that the value of \(\Theta\) on \(C_2\) is

\[
V_2 = \left[P_2, \frac{1}{2} (\Phi_\omega + \Phi_\eta) - \sum_{i=1}^g \Phi_{Q_{i,1}} + V_1\right] = \frac{1}{2} (\{\Phi_\omega, P_2\} + \{\Phi_\eta, P_2\}) - \sum_{i=1}^g \{\Phi_{Q_{i,1}}, P_2\} + V_1
\]

so that \(|V_2| \leq 9\) invoking Remark 3.4 again, and recalling the \(Q_{i,1}\) specialize to different branches of Figure 2.2.

From here the inductive process is clear which yields that the value of \(\Theta\) on \(C_r\) has absolute value less or equal to \(7\), whence the proof of Proposition 5.8. \(\square\)

### 5.4 Explicit modular version of Mumford’s repulsion principle

We conclude this section by writing-down, for later use, an explicit version of Mumford’s well-known “repulsion principle” for points, in the case of modular curves.

**Proposition 5.9** For \(P\) and \(Q\) two different points of \(X_0(p)(\overline{\mathbb{Q}})\) one has

\[
h_\Theta(P - Q) \geq \frac{g - 2}{4g} (h_\Theta(P - \infty) + h_\Theta(Q - \infty)) - O(p \log p).
\]  

**Proof** Let \(K\) be a number field such that both \(P\) and \(Q\) have values in \(K\). Using notations of Section 3, the adjunction formula and Hodge index theorem give

\[
2[K : \mathbb{Q}] h_\Theta(P - Q) = -[P - Q - \Phi_P + \Phi_Q, P - Q - \Phi_P + \Phi_Q]_{\mu_0} = [P + Q, \omega]_{\mu_0} + 2[P, Q]_{\mu_0} + [\Phi_P - \Phi_Q]^2 \\
g \geq [K : \mathbb{Q}] h_\Theta(P - \infty + \frac{1}{2} \omega^0) - 2[P, \infty]_{\mu_0} + [\infty]_{\mu_0}^2 - [\Phi_P]^2
\]

In the same way,

\[
[P, \omega]_{\mu_0} = 2[K : \mathbb{Q}] h_\Theta(P - \infty) - 2[P, \infty]_{\mu_0} + [\infty]_{\mu_0}^2 - [\Phi_P]^2
\]

where the last inequality comes from the quadratic nature of \(h_\Theta\), plus the fact that the error term of (97) allows us to assume \(h_\Theta(P - \infty) \geq 12^{-8/\sqrt{2}} h_\Theta(\omega^0) = O(\log p)\) (see (79) and the end of proof of Theorem 4.6). Now by (51),

\[
h_\Theta(P - \infty + \frac{1}{2} \omega^0) = \frac{1}{[K : \mathbb{Q}]} [P, g \cdot \infty]_{\mu_0} + O(\log p)
\]

and using Remark 3.4 and Lemma 3.5 gives

\[
[P, \omega]_{\mu_0} \geq \frac{g - 2}{g} [K : \mathbb{Q}] h_\Theta(P - \infty + \frac{1}{2} \omega^0) + [K : \mathbb{Q}] O(\log p).
\]

As \([\Phi_P, \Phi_Q] = [P, \Phi_Q] = [Q, \Phi_P]\), we have \(\| [\Phi_P, \Phi_Q] \| \leq 3[K : \mathbb{Q}] \log p\) using Remark 3.4 again. Putting everything together with Remark 4.5 about \(sup g_{\mu_0}\) we obtain

\[
h_\Theta(P - Q) \geq \frac{g - 2}{2g} \left( h_\Theta(P - \infty + \frac{1}{2} \omega^0) + h_\Theta(Q - \infty + \frac{1}{2} \omega^0) \right) - O(p \log p)
\]

which, by our previous remarks, can again be written as

\[
h_\Theta(P - Q) \geq \frac{g - 2}{4g} (h_\Theta(P - \infty) + h_\Theta(Q - \infty)) - O(p \log p).
\]

(For large \(p\), the angle between two points of equal large enough height is here therefore at least \(\arccos(3/4) - \varepsilon > \pi/6\). Of course the natural value is \(\pi/2\), to which one tends when sharpening the computations.)
6 Arithmetic Bézout theorem with cubist metric

We display in this section an explicit version of Bézout arithmetic theorem, in the sense of Philippon or Bost-Gillet-Soulé ([56], [8]), for intersections of cycles in our modular abelian varieties over number fields, with the following variants: we use Arakelov heights (as in Section 5 above, see (80)) on higher-dimensional cycles, and we endow the implicit hermitian sheaf for this height with its cubist metric (instead of Fubini-Study).

It indeed seems that one generally uses Fubini-Study metrics for arithmetic Bézout because they are the only natural explicit ones available on a general projective space (a necessary frame for the approach we follow for Bézout-like statements). They moreover have the pleasant feature that the relevant projective embeddings have tautological basis of global sections with sup-norm less than 1 which, for instance, allows for proving that the induced Faltings height is non-negative on effective cycles (see [19], Proposition 2.6). For our present purposes however, we need bounds for the Néron-Tate heights of points, that is, Arakelov heights induced by cubist metrics. One could in principle have tried working with Fubini-Study metrics as in [8] and then directly compare with Néron-Tate heights, but comparison terms tend to be huge. In the case of rational points, for instance (that is, horizontal cycles of relative dimension 0), within jacobians, those error terms are bounded by Manin and Zarhin ([38]) linearly in the ambient projective dimension, that is exponential in the dimension of the abelian variety. In other words, for our modular curves, the error terms would be exponential in the level \( p \). It is therefore much preferable to stick to cubist metrics. This implies we avoid the use of joins as in [8], as those need a sheaf metrization on the whole of the ambient projective spaces, and we instead use plain Segre embeddings. The extra numerical cost essentially consists of the appearance of modest binomial coefficients, which do not significantly alter the quantitative bounds we eventually obtain.

We also need to work with projective models which are “almost” compactifications of relevant Néron models of our jacobians. This we do with the help of Moret-Bailly theory as introduced in Section 5.

Let us also recall that there still is another approach for such arithmetic Bézout theorems which uses Chow forms ([56], [57]). That is however known to amount to working again with Faltings height relative to the Fubini-Study metrics ([56]-I, [60]) that we said we cannot afford.

Finally, regarding generality: it would of course be desirable to have a proof available for arbitrary abelian varieties. Many of the present arguments are however quite particular to our application to \( J_0(p) \). We therefore prefer working in our concrete setting from the beginning, instead of considering a somewhat artificial generality.

**Proposition 6.1 (Arithmetic Bézout theorem for \( J_0(p) \)).** Let \((J_0(p), \Theta)\) be defined over some number field \( K \), endowed with the principal and symmetric polarization \( \Theta \). Let \( V \) and \( W \) be two irreducible \( K \)-subvarieties of \( J_0(p) \), of dimension \( d_V := \dim_K V \) and \( d_W := \dim_K W \) respectively, such that

\[
d_V + d_W \leq g = \dim J_0(p)
\]

and assume \( V \cap W \) has dimension 0.

If \( P \) is an element of \((V \cap W)(K)\) then its Néron-Tate \( \Theta \)-height satisfies

\[
h_\Theta(P) \leq \frac{4^{d_V+d_W}}{2} \frac{(d_V + d_W + 1)!}{d_V!d_W!} \deg_\Theta(V) \deg_\Theta(W) \left[ (d_V + 1)h_\Theta(W) + (d_W + 1)h_\Theta(V) \right] + O(p \log p). \quad (98)
\]

**Remark 6.2** The general aspect of the above release of arithmetic Bézout might look a bit different from the original ones, as can be found in [8]: this is due to the fact that our definition of the height of some cycle \( Y \) (see Section 5, (80)) amounts to dividing its height in the sense of [8] by the product of the degree and absolute dimension of \( Y \).
Let us first sketch the strategy of proof, which occupies the rest of this Section 6. We henceforth fix a prime number \( p \) and some perfect square integer \( N := r^2 \). (We shall eventually take \( r = 2 \).)

We write \((J, \mathcal{M})\) for the Moret-Bailly projective model of \((J_0(p), L(\Theta)^\otimes N)\) given by Definition 6.4, relative to some given set of global sections \( S \) in \( H^0(J, \mathcal{L}(\Theta)^\otimes N) \), of size \( N^2 \), to be described later (Lemma 6.5). That model is defined over some ring of integers \( \mathcal{O}_K \). Consider the morphisms:

\[
\mathcal{J} \xrightarrow{\Delta} \mathcal{J} \times \mathcal{J} \xrightarrow{\mathcal{P}} \mathbb{P}^n_{\mathcal{O}_K} \times \mathbb{P}^n_{\mathcal{O}_K} \xrightarrow{\iota} \mathbb{P}^{n^2+2n}_{\mathcal{O}_K}
\]

(99)

where \( \Delta \) is the diagonal map, \( n = N^2 - 1, \mathcal{P} \) is the product of two \( S \)-embeddings \( \mathcal{J} \hookrightarrow \mathbb{P}^n = \mathbb{P}^n_{\mathcal{O}_K} \) and the application \( \iota: \mathcal{J} \times \mathcal{J} \to \mathbb{P}^{n^2+2n} \) is the composition of the Segre embedding \( \mathcal{S} \) with \( \mathcal{P} \). As sheaves,

\[
S^* (\mathcal{O}_{\mathbb{P}^n} (1)) = \mathcal{O} \otimes \mathcal{O}_{\mathcal{J}} (\mathbb{P}^n (1))
\]

and

\[
\mathcal{P}^* (\mathcal{O}_{\mathbb{P}^n} (1) \otimes \mathcal{O}_{\mathcal{J}} (\mathbb{P}^n (1)) = \mathcal{M} \otimes \mathcal{O}_{\mathcal{J}} =: \mathcal{M}^{\otimes 2}
\]

so that

\[
\iota^* (\mathcal{O}_{\mathbb{P}^{n^2+2n}} (1)) = \mathcal{M}^{\otimes 2}
\]

and

\[
\Delta^* \iota^* (\mathcal{O}_{\mathbb{P}^{n^2+2n}} (1)) = \mathcal{M} \otimes \mathcal{O}_{\mathcal{J}} \mathcal{M} = \mathcal{M}^{\otimes 2}.
\]

(100)

We naturally endow the sheaves \( \mathcal{M}^{\otimes 2}, \mathcal{M}^{\otimes 2} \), and so on, with the hermitian structures induced by the cubist metric on the various \( \mathcal{M}_i \) for \( \sigma: \mathcal{J} \hookrightarrow \mathcal{C} \), denoted by \( \| \cdot \|_{\text{cub}} \).

We then pick two copies \((x_i)_{0 \leq i \leq n} \) and \((y_j)_{0 \leq j \leq n} \) of the canonical basis of global sections for each \( \mathcal{O}_{\mathbb{P}^n} (1) \) on the two factors of \( \mathbb{P}^n_{\mathcal{O}_K} \) of (99), which give our basis \( \mathcal{S} \) by restriction to \( \mathcal{J} \). Then we provide the sheaf \( \mathcal{O}_{\mathbb{P}^{n^2+2n}} (1) \) on \( \mathbb{P}^{n^2+2n} \) with the basis of global sections \((z_{ij})_{0 \leq i \leq n, 0 \leq j \leq n} \), each of which is mapped to \( x_i \otimes \mathcal{O}_{\mathcal{J}} y_j \) under \( S^* \). Define \( \mathcal{D} \) as the diagonal linear subspace of \( \mathbb{P}^{n^2+2n} \) defined by the linear equations \( z_{ij} = z_{ij} \) for all \( i \) and \( j \).

Let \( V, W \subseteq J = \mathcal{J}_K \) be two closed subvarieties over \( K \). The support of \( V \cap W \) is the same as that of \((\iota \circ \Delta)^{-1} (\mathcal{D} \cap \iota(V \times W)) \). To bound from above the height of points in \( V \cap W \) it is therefore sufficient to estimate Faltings’ height of \( \mathcal{D} \cap \iota(V \times W) \), relative to the hermitian line bundle \( \mathcal{O}_{\mathbb{P}^{n^2+2n}} (1) \mid_{\mathcal{J} \times \mathcal{J}} \) endowed with the cubist metric. As \( \mathcal{D} \) is a linear subspace that height is essentially the same as that of \( (V \times W) \), up to an explicit error term which depends on the degree. In turn this error term is a priori linear in the number of (relevant) equations for \( \mathcal{D} \), and this is way too high. But if one knows \( V \cap W \) has dimension 0, it is enough to choose \((\dim V + \dim W) \) equations (up to perhaps increasing a bit the size of the set whose height we estimate), which makes the error term much smaller.

That is the basic strategy of proof for Proposition 6.1. To make it effective however we must control the “error terms” alluded to in the preceding lines, and those crucially depend on the supremum, on the set \( \mathcal{S} \), of values for the cubist metric of global sections defining the projective embedding \( \mathcal{J} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K} \). We shall build that \( \mathcal{S} \) using theta functions as follows.

Recall Riemann’s theta function on \( J_0(p) \) introduced in Section 5.3, see (89). Its usual analytic norm is

\[
||\theta(z)||_{an} := \det(3(\tau))^{1/4} \exp(-\pi y \Im(\tau)^{-1} y) |\theta(z)|
\]

(101)

for \( z = x + iy \in \mathbb{C}^\ast \) (see [48], (3.2.2)). That analytic metric will have to be compared to the cubist one, about which we recall the following basic facts.

Let \( A \) be an abelian variety over a number field \( K \), which extends to a semialbelian scheme \( \mathcal{A} \) over the ring of integers \( \mathcal{O}_K \). We endow \( \mathcal{A} \) with a symmetric ample invertible sheaf \( \mathcal{L} \). Define,
for $I \subseteq \{1, 2, 3\}$, the projection $p_I : \mathbb{A}^3 \to \mathbb{A}$, $p_I(x_1, x_2, x_3) = \sum_{i \in I} x_i$. It is known to follow from the theorem of the cube ([47]) that the sheaf $\mathcal{D}_3(L) := \bigotimes_{I \subseteq \{1, 2, 3\}} p_I^* \mathcal{O}^{\vartriangleleft(1)}$ is trivial on $\mathbb{A}^3$. Let us therefore fix an isomorphism $\phi : \mathcal{O}_{\mathbb{A}^3} \to \mathcal{D}_3(L)$. For every complex place $\sigma$ of $\mathcal{O}_K$ one can endow $L_\sigma$ with some cubist metric $\| \cdot \|_\sigma$ such that one obtains through $\phi$ the trivial metric on $\mathcal{O}_{\mathbb{A}^3}$. Each cubist metric $\| \cdot \|_\sigma$ is determined only up to multiplication by some constant factor so we perform the following rigidification to remove that ambiguity. If $\mathcal{O}_A : \text{Spec}(\mathcal{O}_K) \to \mathbb{A}$ denotes the zero section, we replace $L$ by $L \otimes \mathcal{O}_K (\pi^0 \mathcal{O}^{\vartriangleleft(1)})$ on $\mathbb{A}$. Then

$$0^*_{\mathcal{A}}(L) \simeq \mathcal{O}_K$$

and we demand that the $\| \cdot \|_\sigma$ be adjusted so that the above sheaf isomorphism is an isometry at each $\sigma$, where $\mathcal{O}_K$ is endowed with the trivial metric so that $\|1\| = 1$. This uniquely determines our cubist metrics $\| \cdot \|_\sigma$. Now by construction the hermitian sheaf $L$ on $\mathbb{A}$ defines a height $h$ verifying the expected normalization condition $h(0) = 0$.

Having the same curvature form, the analytic and cubist metrics are known to differ by constant factors, at each complex place, on the Theta sheaf, as we shall use in the proof of Lemma 6.4 below.

Recall we also defined in (91) a “meromorphic theta function $s_{r^0}$ over $\mathbb{F}_r$”, which can be generalized: we have $[r]^* \mathcal{L}(\Theta)|_{N_{I,r}} \simeq \mathcal{L}(\Theta)^{r^2}$ on $N_{I,r}$ ([54], Proposition 5.1) so we define a global section

$$s_M := ([r]^* t_{cub}) s_{r^0} \in H^0(N_{I,r}, [r]^* \mathcal{L}(\Theta)|_{\mathcal{O}_K}). \quad (102)$$

We will shortly show how to control the supremum of $\|s_{r^0}\|_{\text{cub}}$, therefore of $\|s_M\|_{\text{cub}}$, on $J_0(p)(\mathbb{C})$ (see Lemma 6.4). Writing $N = r^2$, we shall moreover fix the morphism $j_M : N_{I,N} \to \mathcal{J} \to \mathbb{F}_r \to \mathcal{O}_K$ of (86) by mapping the canonical coordinates $(x_{0i \leq i \leq n})$ to sections $(s_i)$ which will be translates by $r$-torsion points of a multiple of the above $s_M$ by some constant, as explained in Lemma 6.5 and its proof.

This will allow us to control as well the supremum of those $s_i$, relative to the cubist metrics, on the complex base change of our abelian varieties, as is required by the proof of arithmetic Bézout theorems.

We now start the technical preparation for the proof of Proposition 6.1, for which we need some Lemmas on the behavior of heights and degree under Segre maps, comparison between cubist and analytic metrics on theta functions, and estimates for all.

**Lemma 6.3** There is an infinite sequence $(P_i)_{i \in \mathbb{N}}$ of points in $X_0(p)(\mathbb{Q})$ which are ordinary at all places dividing $p$ and have everywhere integral $j$-invariant. Moreover their normalized theta height satisfies $h_\Theta(P_i - \infty + \frac{1}{2} w^0) = O(p^3)$, with notations of Theorem 4.1.

**Proof** Let $(\zeta_i)_{i \in \mathbb{N}}$ be an infinite sequence of roots of unity. One can assume none are congruent to some supersingular $j$-invariant in characteristic $p$, modulo any place of $\mathbb{Q}$ above $p$. (Indeed, as the supersingular $j$-invariants are quadratic over $\mathbb{F}_p$, it is enough for instance to choose for the $\zeta_i$ some primitive $\ell_i$-roots of unity, with $\ell_i$ running through the set of primes larger than $p^2 - 1$.) Lift each $j$-invariant equal to $\zeta_i$ to some point $P_i$ in $X_0(p)(\mathbb{Q})$. By construction, this makes a sequence of points with $j$-height $h_j(P_i)$ equal to 0. As for their (normalized) theta height one sees from Theorem 4.1 that

$$h_\Theta(P_i - \infty + \frac{1}{2} w^0) = \frac{1}{[K(P_i) : \mathbb{Q}]} [P_i, \tilde{\omega}_\Theta]_{w^0} = -\frac{1}{[K(P_i) : \mathbb{Q}]} \sum_{\sigma : K(P_i) \to \mathbb{C}} g \cdot g_{w_0}(\infty, \sigma(P_i)) + O(\log p)$$

as the contribution at finite places of $[P_i, \infty]$ is 0. It is therefore enough to bound the $|g_{w_0}(\infty, \sigma(P_i))|$. Now $|j(P_i)| = 1$ for all $\sigma : K(P_i) \to \mathbb{C}$, so the corresponding elements $\tau$ in the usual fundamental domain in Poincaré upper half-plane for $X_0(p)$ or $X(p)$ are absolutely bounded, and the same for the absolute values of $q_\tau = e^{2i\pi \tau}$. (For a useless explicit estimate of this bound, one can
Lemma 6.4 Let \( s_\theta \) be the “theta function over \( \mathbb{Z} \), that is, the global section introduced just before (90). One has:

\[
\sup_{J_0(p)(\mathbb{C})} (\log \|s_\theta\|_{\text{cub}}) \leq O(p \log p).
\]

Proof Writing \( s_{\theta, \mathbb{C}}(z) = C_\theta \cdot \theta(z + \kappa) \) as in (90), we shall bound from above both \(|C_\theta|\) and the contribution of the difference between cubic and analytic metrics. Then we will use upper bounds for the analytic norm of the theta function due to P. Autissier and proven in the Appendix of the present paper.

We invoke again some key arguments of the proof of Proposition 5.8. For \( D \) in \( J_0(p)(\mathbb{C}) \), written as the linear equivalence class of some divisor \( \sum_{i=1}^g (P_i - \infty) \) on \( X_0(p)(\mathbb{C}) \), we indeed once more consider the embedding

\[
\theta_{\kappa-D}: \left\{ \begin{array}{l}
X_0(p) \mapsto J_0(p) \\
P \mapsto \text{cl}(P - \infty + \kappa - D)
\end{array} \right.
\]

as in Proposition 5.8. For such a \( D \) whose \( P_i \) are assumed to belong to \( X_0(p)(\overline{\mathbb{Q}}) \), we recall (92) that

\[
h_{\theta}(\theta_{\kappa-D}(P)) = \frac{1}{[K(P, D) : \mathbb{Q}]} [P, \sum_i P_i + \Phi_D + c_D X_\infty]_{\eta_0}.
\]

If the \( P_i \) all have everywhere ordinary reduction, as will be the case in (105) below, the vertical divisor \( \Phi_D \) will contribute at most \( O(\log p) \) to the height of points (see Remark 3.4).

Note that we can fulfill condition (95) considering only points \( P_i \) of same type as occurring in Lemma 6.3 (which, in particular, are integral and have integral \( j \)-invariants), because those \( P_i \) make a Zariski-dense subset of \( X_0(p)(\overline{\mathbb{Q}}) \) (and the onto-ness of the map \( X_0(p)(\mathbb{Q}) \to J_0(p) \)).

We therefore conclude as in the proof of Proposition 5.8 that \( \text{div}(\theta_{\kappa-D}(s_\theta)) \) has indeed to be \((\sum_i P_i + \Phi_D)\) on \( X_0(p)(\mathbb{Q})_{\text{smooth}} \).

On the other hand, for some of those choices of \((P_i)_{1 \leq i \leq g}\), our \( \mathbb{Z} \)-theta function \( s_\theta \) does not vanish at \( \theta_{\kappa-D}(\infty)(\mathbb{C}) \), so \( h_{\theta}(\theta_{\kappa-D}(\infty)) \) can also be computed as the Arakelov degree:

\[
h_{\theta}(\theta_{\kappa-D}(\infty)) = \deg(\theta_{\kappa-D}(\mathcal{L}(\Theta))).
\]

Integrality of the \( P_i \) shows the intersection numbers \([\infty, P_i]\) have trivial non-archimedean contribution. The only finite contribution to our Arakelov degree therefore comes from intersection with vertical components, that is, if \( K_D \) is a sufficiently large field, over which \( D \) is defined, then for a set of elements \((z_\sigma)_{\sigma : K_D \to \mathbb{C}}\) which lift \( \sigma(-D) \) in the complex tangent space of \( J_0(p)(\mathbb{C}) \) to 0 one has:

\[
h_{\theta}(\theta_{\kappa-D}(\infty)) = \frac{1}{[K_D : \mathbb{Q}]} \sum_{K_D \ni \sigma \in \mathbb{C}} \log \|s_\theta(z_\sigma)\|_{\text{cub}} + O(\log p)
\]

\[ \text{[Note: Although we shall not use this, one can check that } h_{\theta}(\theta_{\kappa-D}(\infty)) = \| - (\sum_i P_i - \infty) + \frac{1}{2} \|s_\theta\|_{\text{cub}}^2 = O(p^5) \text{ by Lemma 6.3 and (79).]} \text{[19]}} \]
whence as \( s_{\theta, \mathbb{C}}(z) = C_{\theta} \cdot \theta(z + \kappa) \):

\[
\log |C_{\theta}| = -h_{\theta}(\kappa_{-D}(\infty)) - \frac{1}{|K_D(\kappa) : \mathbb{Q}|} \sum_{\kappa_D(\kappa) \subset \mathbb{C}} \log \|\theta((z + \kappa)_{\tau})\|_{\text{cub}} + O(\log p). \tag{105}
\]

Following [20], paragraph 8, we now write \( J_0(p)(\mathbb{C}) = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g) \) for \( \tau \) in Siegel’s fundamental domain, write \( z \in \mathbb{C}^g \) as \( z = \tau \cdot p + q \) for \( p, q \in \mathbb{R}^g \), and introduce the function \( F : \mathbb{C}^g \to \mathbb{C} \) defined as

\[
F(z) = \det(2\Im(z))^{1/4} \sum_{n \in \mathbb{Z}^g} \exp(i\pi^2(n + p)\tau(n + p) + 2i\pi^4nq).
\]

One then has \( |F(z)| = 2^{g/4} \|\theta(z)\|_{\text{an}} \). Indeed there is a constant \( A \in \mathbb{R}_+^* \) such that \( |F(z)| = A \cdot \|\theta(z)\|_{\text{an}} \) (see the end of proof of Lemma 8.3 of [20]). \( \int_{J_0(p)(\mathbb{C})} |F|^2 d\nu = 1 \) (where \( d\nu \) is the probability Haar measure on \( J_0(p)(\mathbb{C}) \); see [20], Lemma 8.2 (1)), and \( \int_{J_0(p)(\mathbb{C})} \|\theta(z)\|_{\text{an}}^2 d\nu = 2^{-g/2} \) (see e.g [48], (3.2.1) and (3.2.2)). Therefore Lemme 8.3 of [20] gives, using definitions of loc. cit., Théorème 8.1,

\[
-\frac{1}{|K_D(\kappa) : \mathbb{Q}|} \sum_{\kappa_D(\kappa) \subset \mathbb{C}} \left( \log \|\theta((z + \kappa)_{\tau})\|_{\text{an}} + \frac{q}{4} \log 2 \right) \leq h_{\theta}(\kappa_{-D}(\infty)) + \frac{1}{2} h_F(J_0(p)) + \frac{q}{4} \log 2\pi.
\]

Remember Faltings height of \( J_0(p) \) is known to satisfy \( h_F(J_0(p)) = O(p \log p) \) by [62], Théorème 1.2. (We remark that Ullmo’s normalization of Faltings’ height differs from that of Gaudron-Rémond, but the difference term is linear in \( q = O(p) \) so the bound \( O(p \log p) \) remains valid for the above \( h_F(J_0(p)) \)). Writing \( \| \cdot \|_{\text{cub}} = e^{\theta^2} \| \cdot \|_{\text{an}} \) we therefore see that (105) implies

\[
\log |C_{\theta}| + \varphi \leq \frac{1}{2} h_F(J_0(p)) + O(p) \leq O(p \log p).
\]

Given this upper bound for \( e^{\varphi} |C_{\theta}| \) we can now go the other way round to derive an upper bound for \( \|s_{\theta}\|_{\text{cub}} = C_{\theta} \cdot \|\theta(z + \kappa)\|_{\text{cub}} \), by using estimates for analytic theta functions. For any principally polarized complex abelian variety whose complex invariant \( \tau \) is chosen within Siegel’s fundamental domain \( F_2 \), Autissier’s result in the Appendix (Proposition 8.1 below) indeed gives, with notations as in (101), that:

\[
\frac{1}{\det(\Im(\tau))^{1/4}} \|\theta(z)\|_{\text{an}} = \exp(-\pi y \Im(\tau)^{-1}y) \|\theta(z)\| \leq g^{\theta/2}. \tag{106}
\]

We refer to the Appendix for a bound which is even slightly sharper.\(^7\)

As for the factor \( \det(\Im(\tau))^{1/4} \), Lemma 11.2.2 of [18] gives the general result:

\[
\det(\Im(z))^{1/4} \leq \frac{(2g)! V_{2g}}{2^g V_g} \prod_{g+1 \leq i \leq 2g} \lambda_i
\]

where for any \( k \) we write \( V_k \) for the volume of the unit ball in \( \mathbb{R}^k \) endowed with its standard Euclidean structure, and the \( \lambda_i \) are the successive minima, relative to the Riemann form, of the lattice \( \Lambda = \mathbb{Z}^g + \tau \cdot \mathbb{Z}^g \). To bound the \( \lambda_i \) we need to invoke an avatar of loc. cit., Lemma 11.2.3. But the very same proof shows that for any integer \( N \), the group \( \Gamma_0(N) \) has a set of generators having entries of absolute value less or equal to the very same bound \( N^3/4 \). (That term could be improved, but this would have an invisible impact on the final bounds so we here content ourselves with it.) We can therefore rewrite the proof of Lemma 11.2.4 verbatim. This gives that \( A \) is generated by elements having naive hermitian norm \( \|x\|_E^2 \) less or equal to \( g \beta^{4g} \). Finally, in our case the Gram matrix is diagonal (no \( 2 \times 2 \)-blocks, at the difference of Lemma 11.1.4 of loc.

\(^7\) Works of Igusa and Edixhoven-de Jong ([18], pp. 231-232) give \( \frac{1}{\det(\Im(\tau))^{1/4}} \|\theta(z)\|_{\text{an}} \leq 2g^{\theta/2 + 5g} \).
cit.) so Lemma 11.2.5 a fortiori holds: if $\| \cdot \|_p$ denotes the hermitian product on $C^g$ induced by the polarization, $\| \cdot \|_p^2 \leq \frac{e^{4\pi}}{\pi} \| \cdot \|_E^2$. This allows to conclude as in p. 228 of [18]:

$$\left( \prod_{i=g+1}^{2g} \lambda_i \right)^2 \leq \left( \frac{e^{4\pi}}{\pi} gp^{4g}\right)^g$$

so that

$$\log(\det(\Im(\tau))) \leq O(p \log p)$$

and combining with (106),

$$\log \|\theta(z)\|_\an \leq O(p \log p).$$

Putting everything together finally yields:

$$\sup_{z \in J_0(p)(C)} \log \|s_{\theta,C}(z)\|_{\cub} = \sup_{z \in J_0(p)(C)} \log \|C_0 \cdot \theta(z + \kappa)\|_{\cub} = (\log |C_0| + \varphi) + \sup_{z \in J_0(p)(C)} \log \|\theta(z + \kappa)\|_\an \leq O(p \log p). \quad \Box$$

**Lemma 6.5** Assume the same hypothesis and notations as in Definition 5.4. After possibly making some finite base extension one can pick a set $S$ in $H^0(N_{J,4} \times L(\Theta)^{\otimes 4})$ of $4^g$ global sections $(s_i)_{1 \leq i \leq 4^g}$, which span $L(\Theta)^{\otimes 4}$ on $N_{J,4}[1/2p]$, and verify

$$\sup_{J_0(p)} (\log \|s_i\|_{\cub}) \leq O(p \log p). \quad (107)$$

**Proof** We fix $N = r^2 = 4$ for the construction of a good model as in Definition 5.4. Up to making a base extension, we can assume $L(\Theta)^{\otimes 4}$ and $[2]^*L(\Theta)$ have cubist extensions $L(\Theta)^{\otimes 4}$ and $[2]^*L(\Theta)$ on $N_{J,4}$. As $\Theta$ is symmetric one knows there is an isomorphism $[2]^*L(\Theta) \rightarrow L(\Theta)^{\otimes 4}$ which actually is an isometry ([54], Proposition 5.1), by which we identify those two objects from now on. On the other hand, every element $x$ of $J_0(p)[4](\Q)$ is defined in the proof of Lemma 6.4 and Proposition 5.8. which gives estimates on the poles of this object. From here, Lemma 6.4 and Proposition 5.8 give (107).
By the theory of theta functions ([54], Proposition 2.5 and its proof, [49] and [47], Chapitre VI) the $s_i$ make a generic basis of global sections, which span $L(\Theta)_{\otimes^4}$ over Spec($O_K[1/2p]$).

\textbf{Lemma 6.6} Let $V$ and $W$ be two closed $K$-subvarieties, with dimension $d_V$ and $d_W$ respectively, of a smooth projective variety $A$ over a number field $K$, endowed with an ample sheaf $M$. Assume the flat projective scheme $(A, M)$ over Spec($O_K$), with $M$ an hermitian sheaf on $A$, is a model for $(A, M)$. Let $\mathcal{V}$ and $\mathcal{W}$ be the Zariski closure in $A$ of $V$ and $W$ respectively. Then, with definitions as in [8], Section 3.1,

\[(c_1(M^{\otimes 2})^{d_V+d_W}|(V \times W)) = \left(\frac{d_V + d_W}{d_V}\right) (c_1(M)^{d_V}|V)(c_1(M)^{d_W}|W) \tag{111}\]

and

\[(\hat{c}_1(M^{\otimes 2})^{d_V+d_W+1}|V \times W) = \left(\frac{d_V + d_W + 1}{d_V}\right) (c_1(M)^{d_V}|V) \left(\hat{c}_1(M)^{d_W+1}|W\right) + \left(\frac{d_V + d_W + 1}{d_W}\right) (\hat{c}_1(M)^{d_V+1}|V)(c_1(M)^{d_W}|W). \tag{112}\]

\textbf{Remark 6.7} Equation (111) can be read as

\[\deg_{M^{\otimes 2}}(V \times W) = \left(\frac{d_V + d_W}{d_V}\right) \deg_M(V) \deg_M(W)\]

Equation (112) in turn fits with Zhang’s interpretation (83) in terms of essential minima, compare the proof of Proposition 6.1 below.

\textbf{Proof (of Lemma 6.6).} For (111), one can realize it is elementary, or refer to Lemme 2.2 of [58], or proceed as follows. Using (2.3.18), (2.3.19), and Proposition 3.2.1, (iii) of [8], and noticing for (111), one can realize it is elementary, or refer to Lemme 2.2 of [58], or proceed as follows. Using (2.3.18), (2.3.19), and Proposition 3.2.1, (iii) of [8], and noticing

\[c_1(M^{\otimes 2}) = c_1(M) \times 1 + 1 \times c_1(M)\]

(and same with $\hat{c}_1(M)$ and $\hat{c}_1(M^{\otimes 2})$ instead) one computes

\[(c_1(M^{\otimes 2})^{d_V+d_W}|(V \times W)) = \left(\sum_{k=0}^{d_V+d_W} \left(\frac{d_V + d_W}{k}\right) c_1(M)^k \times c_1(M)^{d_V+d_W-k}|V \times W\right)\]

\[= \left(\sum_{k=0}^{d_V+d_W} \left(\frac{d_V + d_W}{k}\right) c_1(M)^k \times c_1(M)^{d_V+d_W-k}|V \times W\right)\]

\[= \left(\sum_{k=0}^{d_V+d_W} \left(\frac{d_V + d_W}{k}\right) c_1(M)^k|V)(c_1(M)^{d_V+d_W-k}|W\right)\]

\[= \left(\frac{d_V + d_W + 1}{d_V}\right) (c_1(M)^{d_V}|V)(c_1(M)^{d_W}|W).\]

An analogous computation, using [8], (2.3.19), can be used for the arithmetic degree:

\[(\hat{c}_1(M^{\otimes 2})^{d_V+d_W+1}|V \times W) = \left(\sum_{k=0}^{d_V+d_W+1} \left(\frac{d_V + d_W + 1}{k}\right) (\hat{c}_1(M)^k \times \hat{c}_1(M)^{d_V+d_W+1-k}|V \times W\right)\]

\[= \left(\frac{d_V + d_W + 1}{d_V}\right) (c_1(M)^{d_V}|V) \left(\hat{c}_1(M)^{d_W+1}|W\right) + \left(\frac{d_V + d_W + 1}{d_W}\right) (\hat{c}_1(M)^{d_V+1}|V)(c_1(M)^{d_W}|W).\]
For the rest of this Section we fix the model $(\mathcal{J} , \mathcal{M})$ for $(\mathcal{J}_0(p), \Theta)$ (see (99)) as the one built with the set $\mathcal{S}$ of $N' = 4^g$ sections provided by Lemma 6.5. Before settling the proof of the arithmetic Bézout theorem, we need a last lemma on comparison between the projective height on $(\mathcal{J} , \mathcal{M})$ and its normalized Néron-Tate avatar.

**Lemma 6.8** Up to translation by torsion points, the projective height $h_{\mathcal{M}}$ on points in $\mathcal{J}_0(p)(\overline{\mathbb{Q}})$ (associated with the good model $(\mathcal{J} , \mathcal{M})$) differs from the Néron-Tate theta-height $4h_{\mathcal{M}}$ by an error term of shape $O(p \log p)$.

**Proof** Lemma 6.5 implies that the elements of $\mathcal{S}$ extend as holomorphic sections to any component of the Néron model $\overline{\mathcal{N}}$ of $\mathcal{J}_0(p)$ over $\mathbb{Z}$ (see (109)). As remarked in the proof of Lemma 6.5, Mumford’s algebraic theory of theta-functions implies that the sections in $\mathcal{S}$ do define a projective embedding of $\overline{\mathcal{N}}$ over $\mathbb{Z}[1/2p]$; the only fibers of $\overline{\mathcal{N}}$ over $\mathbb{Z}$ where base points for $S$ can show up are above 2 and $p$. If one seeks to approximate the Néron-Tate height of a given point $P$ in $\mathcal{J}_0(p)(\overline{\mathbb{Q}})$ by the projective height of our good model $(\mathcal{J} , \mathcal{M})$, one needs the section of the Néron model $\overline{\mathcal{N}}$ defined by $P$ to avoid those base points, or at least control their length.

Given $P$ in $\mathcal{J}_0(p)(\overline{\mathbb{Q}})$, we claim one can translate $P$ by some torsion point in $\mathcal{J}_0(p)(\overline{\mathbb{Q}})$ so that the translated new point $P + t$ does avoid base points in characteristic 2. Indeed, choose a Galois extension $F'/\mathbb{Q}$ such that the base locus is defined over Spec$(\mathcal{O}_F \otimes \mathbb{F}_2)$. Summing up, as divisors, all the Galois conjugates of that base locus in each fiber of characteristic 2, one obtains a constant cycle $C_\kappa$, in each fiber at $\kappa$, which is defined over $\mathbb{F}_2$. (In our case one actually could have taken $F = \mathbb{Q}$.) Density of torsion points then shows that one can replace our point $P$ by $P + t$, for some torsion point $t$, such that $P + t$ does not belong to $C_\kappa$ for some $\kappa_0$, then for all $\kappa$ of characteristic 2 because $C_\kappa$ is constant. This proves our claim. Now in characteristic $p$, we know from Proposition 5.8 again that possible base points have length at most $O(p)$, which gives an estimate of size $O(p \log p)$ for the difference error term between projective height on $\mathcal{J}$ and Néron-Tate height ([54], Proposition 4.1).

**Proof of Proposition 6.1.** Before proceeding we will allow ourselves, for this proof only, and in the hope not to weighten too much the computations, to work with heights defined as in [8], Section 3.1. Namely, for $\mathcal{Y}$ a cycle of dimension $(d + 1)$ in a regular arithmetic variety endowed with a hermitian sheaf $\mathcal{F}$, we multiply our definition (80) of its height by degree and absolute dimension and we set:

$$h'_{\mathcal{F}}(\mathcal{Y}) = \frac{(\hat{e}_1(\mathcal{F})^{d+1})[\mathcal{Y}]}{[\mathcal{K} : \mathbb{Q}]}.$$  

Note that $h$ and $h'$ coincide on $K$-rational points, in which case we might use either notation.

Construction (99) gives a $\mathbb{Q}$-embedding $V \times W \hookrightarrow \mathbb{P}^{n^2 + 2n}$ via a Segre map. We set

$$s_{\hat{z}, \hat{z}} := \iota'(\hat{z} \hat{z} - \hat{z} \hat{z})$$

for all $(\hat{z}, \hat{z})$, and denote by $\mathcal{O}_N$ the ambiant line bundle $\iota^*(\mathcal{O}_{\mathbb{P}^{n^2 + 2n}}(1)) = \mathcal{M}^{\otimes 2}$ as before (100). (Recall we will eventually specialize to $N = 4$.) Set also $\mathcal{O}_N := \mathcal{O}_N \otimes \mathbb{Q}$. We intersect $\iota(V \times W)$ with one of the div$(\hat{z} \hat{z} - \hat{z} \hat{z})/\mathbb{Q}$ such that the two cycles meet properly: define

$$J_1 = \text{div}(s_{\hat{z}, \hat{z}})/\mathbb{Q} \cap (V \times W)$$

in the generic fiber $(\mathcal{J}_0(p) \times \mathcal{J}_0(p))_Q$. As div$(\hat{z} \hat{z} - \hat{z} \hat{z})/\mathbb{Q}$ is a projective hyperplane we have by definition

$$\deg_{\mathcal{O}_N}(J_1) = \deg_{\mathcal{O}_N}(V \times W).$$

For the same linearity reason, a similar statement is true for heights. Indeed, let $\mathcal{V}$ and $\mathcal{W}$ denote the schematic closure in $\mathcal{J}$ of $V$ and $W$ respectively, and $\mathcal{J}_1$ the schematic closure of $J_1$ in $\mathcal{J} \times \mathcal{J}$, which satisfies

$$h'_{\mathcal{O}_N}(\mathcal{J}_1) \leq h'_{\mathcal{O}_N}(\text{div}(s_{\hat{z}, \hat{z}}) \cap (V \times W)).$$

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Proposition 3.2.1 (iv) of [8] gives, with notations of loc. cit., that:
\[
h'_\mathcal{O}_\mathcal{N}(\text{div}(\frac{s_{\mathcal{M}}}{\pi}) \cap (\mathcal{V} \times \mathcal{W})) = h'_\mathcal{O}_\mathcal{N}(\mathcal{V} \times \mathcal{W}) + \frac{1}{[K: \mathbb{Q}]} \sum_{\sigma: K \to \mathbb{C}} \int_{(\mathcal{V} \times \mathcal{W})\sigma(\mathbb{C})} \log \|s_{\mathcal{M}}\mathcal{C}\| c_1(\mathcal{O}_\mathcal{N})^{d_V+d_W} \quad (113)
\]
where \(\| \cdot \| = \| \cdot \|_{\text{cub}}\) shall denote the cubist metric, or the metric induced by the cubist metric on products or powers of relevant sheaves. To estimate the last integral we note that at any point of \((\mathcal{V} \times \mathcal{W})\sigma(\mathbb{C})\) and for any \((\frac{z}{i}, \frac{z}{j})\),
\[
\|s_{\mathcal{M}}\| = \|z_{i,j} - z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}} \leq \|z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}} + \|z_{i,j} - z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}} \\
\leq \|z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}} \|y_{j}\|_{\mathcal{M}_{\mathcal{W}}} + \|z_{i,j} - z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}} \leq 2(\sup_{\|z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}}}) \leq \exp(2\log(\sup_{\|z_{i,j}\|_{\mathcal{M}_{\mathcal{W}}}}) + \log 2)
\]
with notations of Lemma 6.5. Setting \(M_{\mathcal{J},\mathcal{M}} = \log(\sup \|s_{i,j}\|_{\text{cub}})\) we obtain
\[
h'_\mathcal{O}_\mathcal{N}(\mathcal{J}_1) \leq h'_\mathcal{O}_\mathcal{N}(\mathcal{V} \times \mathcal{W}) + (2M_{\mathcal{J},\mathcal{M}} + \log 2) \deg\mathcal{O}_\mathcal{N}(\mathcal{V} \times \mathcal{W})
\]
and we can iterate the process with \(I_1\) in place of \(\mathcal{V} \times \mathcal{W}\): we obtain some \(J_2, J_3, I_2, I_3\) such that
\[
h'_\mathcal{O}_\mathcal{N}(I_2) \leq h'_\mathcal{O}_\mathcal{N}(\mathcal{I}_1) + (2(2M_{\mathcal{J},\mathcal{M}} + \log 2)) \deg\mathcal{O}_\mathcal{N}(\mathcal{V} \times \mathcal{W})
\]
(If the only obstruction to this step is if all the \(s_{i,j}\) vanish on \(I_1\), which implies it is contained in the diagonal of \(J_0(p) \times J_0(p)\) - so that \(I_1 = \mathcal{I}(\mathcal{P}(\mathcal{N}))\) by construction and that means we are already done.) Processing, one builds a sequence \((\mathcal{I}_k)\) of integral closed subschemes of \(\mathcal{J} \times \mathcal{J}\), with decreasing dimension, such that the last step gives
\[
h'_\mathcal{O}_\mathcal{N}(\mathcal{I}_{d_V+d_W}) \leq h'_\mathcal{O}_\mathcal{N}(\mathcal{V} \times \mathcal{W}) + (d_V + d_W)(2M_{\mathcal{J},\mathcal{M}} + \log 2) \deg\mathcal{O}_\mathcal{N}(\mathcal{V} \times \mathcal{W})
\]
Now
\[
h'_\mathcal{O}_\mathcal{N}(\mathcal{I}_{d_V+d_W}) \geq h'_\mathcal{O}_\mathcal{N}(\mathcal{P}(\mathcal{N})) = h'_{\mathcal{M}^{\oplus 2}}(P) = h_{\mathcal{M}^{\oplus 2}}(P) = 2N h_\mathcal{O}(P) + O(p \log p),
\]
for \(h_\mathcal{O}(P)\) the Néron-Tate normalized theta height. Indeed the statement of the present Proposition 6.1 is invariant by translation of every object by some fixed torsion point, so that one can apply Lemma 6.8.Using Lemma 6.6 and Corollary 5.6 and writing \(h'_\mathcal{O}(\mathcal{Y}) = (\dim(\mathcal{Y}) + 1) \deg\mathcal{O}(\mathcal{Y})\) we therefore obtain
\[
2Nh_\mathcal{O}(P) \leq N^{d_V+d_W+1} \left( (d_V+1) \left( (d_V + d_W + 1) h'_\mathcal{O}(\mathcal{V}) \deg\mathcal{O}(\mathcal{V}) + \\
+ (d_V + 1) \left( (d_V + d_W + 1) h'_\mathcal{O}(\mathcal{W}) \deg\mathcal{O}(\mathcal{W}) \right) \right) \\
+ N^{d_V+d_W} (d_V + d_W)(2M_{\mathcal{J},\mathcal{M}} + \log 2) \left( (d_V + d_W) h'_\mathcal{O}(\mathcal{V}) \deg\mathcal{O}(\mathcal{V}) + O(p \log p) \right)
\]
From here, fixing \(N = 4\), the bound \(M_{\mathcal{J},\mathcal{M}} \leq O(p \log p)\) (Lemma 6.5) concludes the proof, after expressing quantities \(h'_\mathcal{O}\) back into \(h_\mathcal{O}\). \(\square\)

That arithmetic Bézout theorem will be our principal tool in the sequel.
7 Height bounds for quadratic points on $X_0(p)$

**Proposition 7.1** Let $\iota: X \hookrightarrow J$ be some Albanese map from a curve (of positive genus) over some field $K$ to its jacobian $J$. Let $\pi: J \to A$ be some quotient of $J$, with $\dim(A) > 1$, and $X'$ be the normalization of the image $\pi \circ \iota(X)$ of $X$ in $A$. Then the map $\pi': X \to X'$ induced by $\pi \circ \iota$ verifies

$$\deg(\pi') \leq \frac{\dim(J) - 1}{\dim(A) - 1}.$$ 

**Proof** The map $\pi \circ \iota$ induces an inclusion of function fields which defines the map $\pi': X \to X'$. If $J'$ is the jacobian of $X'$, Albanese functoriality says that $\pi$ factorizes through surjective morphisms $J \to J' \to A$. Hurwitz formula writes:

$$\deg(\pi') = \frac{\dim(J) - 1 - \frac{1}{2} \deg R}{\dim(J') - 1}$$

for $R$ the ramification divisor of $\pi'$, whence the result. □

**Lemma 7.2** For all large enough prime $p$, let $X := X_0(p)$ and $\pi_e: J_0(p) \to J_e$ be the projection. Let

$$\iota_{P_0} : \begin{cases} X_0(p) & \hookrightarrow J_0(p) \\ P & \mapsto \cl(P - P_0) \end{cases}$$

for some $P_0 \in X_0(p)(\overline{\mathbb{Q}})$ such that $w_p(P_0) = P_0$ (there are roughly $\sqrt{p}$ such points, see Proposition 3.1 of [25]), and set $\varphi_e := \pi_e \circ \iota_{P_0}$. Then:

- if $a \in J_e(\mathbb{Q})$ is some (necessarily torsion) point, the equality $\varphi_e(X_0(p)) = a - \varphi_e(X_0(p))$ implies

$$\varphi_e(X_0(p)) = a + \varphi_e(X_0(p)) \quad (114)$$

and $a = 0$;

- If $d$ is the degree of the map $X_0(p) \to \varphi_e(X_0(p))$ to the normalization of $\varphi_e(X_0(p))$, then $d$ is either 1, 3 or 4;

- Assuming moreover Brumer’s conjecture (see (21) and (22)) equality (114) implies $d = 1$ for large enough $p$.

**Proof** Notice first that, by our choice of $P_0$ (whence $\iota$), and because $J_e$ belongs to the $w_p$-minus part of $J_0(p)$, one has:

$$\varphi_e(w_p(P)) = w_p(\varphi_e(P)) = -\varphi_e(P)$$

for all $P \in X_0(p)(\mathbb{C})$, whence equality (114). So let $n$ be the order of $a$, which also is that of the automorphism “translation by $a$ restricted to $\varphi_e(X_0(p))$”. We remark that the degree $d$ cannot be equal to 2, as otherwise the extension of fraction fields $K(X_0(p))/K(\varphi_e(X_0(p)))$ would be Galois and $X_0(p)$ would possess an involution different from $w_p$, which it does not by Ogg’s theorem ([53], or [33]). If $d = 1$, the same reason that $\text{Aut}(X_0(p)) = \langle w_p \rangle$ implies that $n = 1$. Let now $X'$ be the normalization of the quotient of $\varphi_e(X_0(p))$ by the automorphism $P \mapsto P + a$, that is, the image of $\varphi_e(X_0(p))$ by the quotient morphism $J_e \to J_e/\langle a \rangle$. Let $\pi$ be the composed map $J_0(p) \xrightarrow{\pi_{P_0}} J_e \to J_e/\langle a \rangle$. The degree of $X_0(p) \to X'$ is $d \cdot n$ and Proposition 7.1 together with the left part of inequalities (23) implies:

$$d \cdot n \leq \frac{g - 1}{(1 - o(1))g - 1} \leq 4 + o(1)$$

for large enough $p$. This shows that if $d = 3$ or 4 one still has $a = 0$, whence the Proposition’s first two statements. Assuming (22) we have $d \cdot n < 3$, so that $d = 1$ and $a = 0$ by previous arguments. □
Remark 7.3 Replace, in Lemma 7.2, the map $X_0(p) \to J_c$ by $X_0(p) \to \varphi(X_0(p))$ (by which the former factorizes, by the way). The above proof shows that the map $X_0(p) \to \varphi(X_0(p))$ is of generic degree 1 (independently on any conjecture), but of course it needs not be injective on points: a finite number of points can be mapped together to singular points on $\varphi(X_0(p))$. In our case one checks those are among the Heegner points $P$ such that $P = w_p(P)$ (for which we again refer to Proposition 3.1 of [25]). Indeed, the endomorphism of $J_0(p)$ defined by multiplication by $(1 - w_p)$ factorizes through $\varphi$, and $(1 - w_p)$ is the map considered in (4) and what follows, inducing multiplication by 2 on tangent spaces. Therefore, if $P$ maps to a multiple point of $\varphi(X_0(p))$, it also maps to a multiple point of $(1 - w_p) \circ \iota(X_0(p))$. Now assuming $X_0(p)$ has gonality larger than 2 (which is true as soon as $p > 71$, [52], Theorem 2), the equality $cl((1 - w_p)P) = cl((1 - w_p)P')$ in $J_0(p)$, for some $P'$ on $X_0(p)$ different from $P$, implies $P = w_pP$ and $P' = w_pP'$. That is, $P$ and $P'$ are Heegner points.

Lemma 7.4 Suppose $P$ belongs to $X_0(p^2)(K)$ for some quadratic number field $K$, and $P$ is not a complex multiplication point. Then for one of the two natural degeneracy morphisms $\pi$ from $X_0(p^2)$ to $X_0(p)$, the point $Q := \pi(P)$ in $X_0(p)(K)$ does not define a $\mathbb{Q}$-valued point of the quotient curve $X_0^+(p) := X_0(p)/w_p$.

Proof Using the modular interpretation, we write $P = (E, C_{p^2})$ for $E$ an elliptic curve over $K$ and $C_{p^2}$ a cyclic $p$-isogeny of degree $p^2$, from which we obtain the two points $Q_1 := (E, p \cdot C_{p^2})$ and $Q_2 := (E/p, C_{p^2}, C_{p^2} \, mod \, p \cdot C_{p^2})$ in $X_0(p)(K)$. Assume both $Q_1$ and $Q_2$ do define elements of $X_0^+(p)(\mathbb{Q})$. If $\sigma$ denotes a generator of $\text{Gal}(K/\mathbb{Q})$ we then have

$$w_p(Q_1) = (E/p \cdot C_{p^2}, E[p] \, mod \, p \cdot C_{p^2}) \simeq \sigma(Q_1)$$

and

$$w_p(Q_2) = (E/C_{p^2}, E[p] + C_{p^2} \, mod \, C_{p^2}) \simeq \sigma(Q_2).$$

Therefore $E \simeq (E/p \cdot C_{p^2}) \simeq E/C_{p^2}$, which means $E$ has complex multiplication. \(\square\)

We can now conclude with the main result of this paper.

Theorem 7.5 There is an integer $C$ such that the following holds. If $p$ is a prime number such that (22), the weak form of Brumer’s conjecture, holds, and $P$ is a quadratic point of $X_0(p)$ (that is: $P$ is an element of $X_0(p)(K)$ for some quadratic number field $K$) which does not come from $X_0(p)^+(\mathbb{Q})$, then its $j$-height satisfies

$$h_j(P) < C \cdot p^2 \log p.$$  \hspace{1cm} (115)

If $P$ is a quadratic point of $X_0(p^2)$ then the same conclusion holds without further assumption apart from (22).

Proof In the case $P$ is a quadratic point of $X_0(p^2)$, by Lemma 7.4 one can deduce from $P$ a point $P'$ in $X_0(p)(K)$ which does not induce an element of $X_0^+(p)(\mathbb{Q})$, and whose $j$-height, say, is equal to $h_j(P) + O(\log p)$ for an explicit function $O(\log p)$ (see e.g. [55], inequality (51) on p. 240 and [5], Proposition 4.4 (i)). Replace $P$ by $P'$ if necessary. By Theorem 4.6 it is now sufficient to prove that $h_0(P - \infty) = O(p^3 \log p)$.

Keep the notation of Lemma 7.2. By construction, the point:

$$a := \varphi_c(P) + \varphi_c(w_p(P)) = \varphi_c(P) - \varphi_c(w_p(P)) = \varphi_c(P - w_p(P))$$

is torsion. First assume $a = 0$. Set $X^{(2)} := \{i_\infty(x) - i_\infty(y), (x, y) \in X_0(p)^2\}$ as in Proposition 5.3. Recall from Section 2 that $I_{J_c^\perp, N_c^\perp} : J_c^\perp \to J_c$ is the map defined as in (3), that $i_{J_c^\perp, N_c^\perp}$...
is the embedding $\tilde{J}_e \hookrightarrow J_0(p)$, and denote by $[N_{J_e}]_\tilde{J}_e$ the multiplication by $N_{J_e}$ restricted to $\tilde{J}_e$. As in (8) and before Corollary 5.7 we use our pseudo-projections and define

$$\tilde{X}^{(2),-} := \epsilon_{J_e,N_e}[N_{J_e}]_\tilde{J}_e^{-1} \tilde{I}_{J_e,N_e} \pi_{J_e}(X^{(2),-}).$$

Then $P - w_p(\sigma P)$ belongs to $X^{(2),-} \cap \tilde{J}_e$, and even to the intersection of surfaces (in the generic fiber):

$$X^{(2),-} \cap \tilde{X}^{(2),-}.$$  

Recall (see (8)) that $\tilde{X}^{(2),-}$ is a priori highly non-connected, being the inverse image of multiplication by $N_{J_e}$ in $\tilde{J}_e$ of the (irreducible) surface $\tilde{I}_{J_e,N_e} \pi_{J_e}(X^{(2),-})$. However, in what follows we can replace $\tilde{X}^{(2),-}$ by one of its connected components containing $P - w_p(\sigma P)$. Denote that component by $\tilde{X}_P^{(2),-}$.

By construction, the theta degree and height of $\tilde{X}_P^{(2),-}$, as an irreducible subvariety of $J_0(p)$ endowed with $\Theta$, are those of $\pi_{J_e}(X^{(2),-}) = X^{(2),-}$ relative to the only natural hermitian sheaf of $J_e$, that is, the $\Theta ^{J_e} = \Theta _{J_e}$ described in paragraph 2.1.2. One can therefore apply Proposition 5.3 to obtain that all theta degrees are $O(p^2)$, all Néron-Tate theta heights are $O(\log p)$. We claim the dimension of $(X^{(2),-} \cap \tilde{X}_P^{(2),-})$ is zero. That intersection indeed corresponds to pairs of distinct points on $X_0(p)$ having same image (0) under $\varphi_\epsilon$. On the other hand, Brumer’s conjecture implies $X_0(p) \rightarrow \varphi_\epsilon(X_0(p))$ has generic degree one (see Lemma 7.2), so our intersection points correspond to singular points on $\varphi_\epsilon(X_0(p))$, which of course make a finite set.

We therefore are in position to apply our arithmetic Bézout theorem (Proposition 6.1), which yields $h_\Theta(P - w_p(\sigma P)) \leq O(p^5 \log p)$. The two points $(P - \infty)$ and $(w_p(\sigma P) - \infty)$ have same $\Theta$-height (recall $w_p$ is an isometry on $J_0(p)$ for $h_\Theta$, compare the end of Remark 4.3), and are by hypothesis different, so one can apply them Mumford’s repulsion principle (Proposition 5.9) to obtain

$$h_\Theta(P - \infty) \leq O(p^5 \log p). \quad (116)$$

Let us finally deal with the case when the torsion point $a = \varphi_\epsilon(P) + \varphi_\epsilon(\sigma P)$ is nonzero. We adapt the previous argument: pick a lift $\tilde{a} \in J_0(p)(\overline{\mathbb{Q}})$ of $a$ by $\pi^+_e$ which also is torsion, and let $t_{\tilde{a}}$ be the translation by $\tilde{a}$ in $J_0(p)$. Replace $(P - w_p(\sigma P))$ by $t_{\tilde{a}}(P - w_p(\sigma P))$, $X^{(2),-}$ by $t_{\tilde{a}} X^{(2),-}$ and $\tilde{X}^{(2),-}$ by

$$t_{\tilde{a}}^{-1} X^{(2),-} := \epsilon_{J_e,N_e} [N_{J_e}]_{\tilde{J}_e}^{-1} \tilde{I}_{J_e,N_e} \pi_{J_e}(t_{\tilde{a}}^{-1} X^{(2),-}).$$

Now $t_{\tilde{a}} (P - w_p(\sigma P))$ belongs to $(t_{\tilde{a}} X^{(2),-} \cap t_{\tilde{a}}^{-1} \tilde{X}^{(2),-})$. The theta degree and height of $t_{\tilde{a}} X^{(2),-}$ and $t_{\tilde{a}}^{-1} \tilde{X}^{(2),-}$ (or rather, as above, some connected component $t_{\tilde{a}}^{-1} \tilde{X}^{(2),-}$ of it containing $t_{\tilde{a}} (P - w_p(\sigma P))$) are the same as for the former objects in the case $a = 0$. The fact that the intersection

$$t_{\tilde{a}} X^{(2),-} \cap t_{\tilde{a}}^{-1} \tilde{X}^{(2),-}$$

is zero-dimensional comes from the fact that otherwise, we would have $\varphi_\epsilon(X_0(p)) = a - \varphi_\epsilon(X_0(p))$, a contradiction with our present hypothesis $a \neq 0$ by Proposition 7.2. The height bound for $P$ is therefore the same as (116). \[\Box\]

**Corollary 7.6** Under the assumptions of Theorem 7.5, if $p$ is a large enough prime number and $P$ is a quadratic point of $X_0(p^\gamma)$ for some integer $\gamma$, such that $P$ is not a cusp nor a complex multiplication point, then $\gamma \leq 10$. 

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Proof Let $P$ be a point in $X_0(p)(K)$, which is not a cusp nor a CM point, for some quadratic number field $K$. Then the isogeny bounds of [20], Theorem 1.4 imply there is some real $\kappa$ with
\[ p^{\gamma} < \kappa(h_j(P))^2. \]

Now Theorem 7.5 gives that there is some absolute real constant $B$ such that, if $p \geq B$ then $\gamma \leq 10$. □

Remark 7.7 A similar (but technically simpler) approach for the morphism $X_0(p) \to J_e$ over $\mathbb{Q}$ should give (independently of any conjecture) a bound of shape $O(p^3 \log p)$ for the $j$-height of $\mathbb{Q}$-rational (non-cuspidal) points of $X_0(p)$ (which are known not to exist for $p > 163$ by Mazur’s theorem). The same should apply for $\mathbb{Q}$-points of $X_{\text{split}}(p)$ (and here again, we obtain a weak version of known results).

Actually, sharpening results directly coming from Section 4 (that is, avoiding the use of Bézout) might even yield the full strength of the above results about $X_0(p)(\mathbb{Q})$ and $X_{\text{split}}(p)(\mathbb{Q})$, with more straightforward (unconditional) proofs.

8 Appendix: An upper bound for the theta function, by P. Autissier

In this appendix, I give a new upper bound for the norm of the classical theta function on any complex abelian variety. This result, apart from its role in the present paper (see Section 6), has been used by Wilms [63] to bound the Green-Arakelov function on curves.

8.1 Result

Let $g$ be a positive integer. Write $\mathbb{H}_g$ for the Siegel space of symmetric matrices $Z \in \mathbb{M}_g(\mathbb{C})$ such that $\text{Im} Z$ is positive definite. To every $Z \in \mathbb{H}_g$ is associated the theta function defined by
\[ \theta_Z(z) = \sum_{m \in \mathbb{Z}^g} \exp(i\pi t m Z m + 2i\pi t m z), \quad \forall z \in \mathbb{C}^g, \]
and its norm defined by
\[ \|\theta_Z(z)\| = \sqrt{\text{det} Y} \exp(-\pi y Y^{-1} y)|\theta_Z(z)|, \quad \forall z = x + iy \in \mathbb{C}^g, \]
where $Y = \text{Im} Z$.

My contribution here is the following:

Proposition 8.1 Let $Z \in \mathbb{H}_g$ and assume that $Z$ is Siegel-reduced. Put $c_g = \frac{g + 2}{2}$ if $g \leq 3$ and $c_g = \frac{g + 2}{2} \left( \frac{g + 2}{\pi \sqrt{3}} \right)^{g/2}$ if $g \geq 4$. The upper bound $\|\theta_Z(z)\| \leq c_g(\text{det} \text{Im} Z)^{1/4}$ holds for every $z \in \mathbb{C}^g$.

Let us remark that $c_g \leq g^{g/2}$ for every $g \geq 2$. In comparison, Edixhoven and de Jong ([18] page 231) obtained the statement of Proposition 8.1 with $c_g$ replaced by $2^{3g^3 + 5g}$.
8.2 Proof

Fix a positive integer \( g \). Denote by \( S_g \) the set of symmetric matrices \( Y \in M_g(\mathbb{R}) \) that are positive definite. Let us recall a special case of the functional equation for the theta function (see equation (5.6) of [50] page 195): for every \( Y \in S_g \) and every \( z \in \mathbb{C}^g \), one has

\[
\theta_{iY^{-1}}(-iz) = \sqrt{\det Y} \exp(i\pi z Y^{-1} z) \theta_{iY}(z).
\]

Lemma 8.2 Let \( Z \in \mathbb{H}_g \) and \( z \in \mathbb{C}^g \). Putting \( Y = \Im Z \), one has the inequality

\[
\|\theta_Z(z)\| \leq \|\theta_{iY}(0)\| = \theta_{iY}(0)\sqrt{\det Y}.
\]

Proof Put \( y = \Im z \). One has

\[
|\theta_Z(z)| = \left| \sum_{m \in \mathbb{Z}^g} \exp(i\pi mZm + 2i\pi mz) \right| \leq \sum_{m \in \mathbb{Z}^g} \left| \exp(i\pi mZm + 2i\pi mz) \right| = \theta_{iY}(iy),
\]

that is, \( \|\theta_Z(z)\| \leq \|\theta_{iY}(iy)\| \). The functional equation (117) gives \( \|\theta_{iY^{-1}}(Y^{-1}y)\| = \|\theta_{iY}(iy)\| \), and one deduces

\[
\|\theta_Z(z)\| \leq \|\theta_{iY^{-1}}(Y^{-1}y)\|. \tag{118}
\]

Applying again (118) with \( Z \) replaced by \( iY^{-1} \) and \( z \) by \( Y^{-1}y \), one gets

\[
\|\theta_{iY^{-1}}(Y^{-1}y)\| \leq \|\theta_{iY}(0)\|.
\]

Whence the result. \( \square \)

Let \( Y \in S_g \). Define \( \lambda(Y) = \min_{m \in \mathbb{Z}^g - \{0\}} b_m Y m \). For every \( t \in \mathbb{R}^+ \), put

\[
f_Y(t) = \theta_{itY}(0) = \sum_{m \in \mathbb{Z}^g} \exp(-\pi t^2 b_m Y m).
\]

Lemma 8.3 Let \( Y \in S_g \) and put \( \lambda = \lambda(Y) \). The following properties hold.

(a) The function \( \mathbb{R}^+ \rightarrow \mathbb{R} \) that maps \( t \) to \( t^{g/2} f_Y(t) \) is increasing.

(b) One has the estimate \( f_Y \left( \frac{g+2}{2\pi \lambda} \right) \leq \frac{g+2}{2} \).

Proof (a) The functional equation (117) implies \( \sqrt{\det Y} t^{g/2} f_Y(t) = f_{Y^{-1}}(1/t) \) for every \( t \in \mathbb{R}^+ \); conclude by remarking that \( f_{Y^{-1}} \) is decreasing.

(b) Part (a) gives \( \frac{d}{dt} [t^{g/2} f_Y(t)] \geq 0 \), that is, \( \frac{g}{2t} f_Y(t) \geq f'_Y(t) \) for every \( t > 0 \). On the other hand,

\[
-\frac{1}{\pi} f'_Y(t) = \sum_{m \in \mathbb{Z}^g} b_m Y m \exp(-\pi t^2 b_m Y m) \geq \sum_{m \in \mathbb{Z}^g - \{0\}} \lambda \exp(-\pi t^2 b_m Y m) = \lambda[f_Y(t) - 1].
\]

One infers \( \frac{g}{2t} f_Y(t) \geq \pi \lambda[f_Y(t) - 1] \). Choosing \( t = \frac{g+2}{2\pi \lambda} \), one obtains the result. \( \square \)

Proposition 8.4 Let \( Y \in S_g \). Putting \( \lambda = \lambda(Y) \), one has the upper bound

\[
\theta_{iY}(0) \leq \frac{g+2}{2} \max \left( \left( \frac{g+2}{2\pi \lambda} \right)^{g/2}, 1 \right).
\]
Proof  Put \( t = \frac{g + \frac{2}{2\pi \lambda}}{} \). If \( t \geq 1 \), then Lemma 8.3 (a) implies the inequality \( f_Y(1) \leq t^{\frac{g}{2}} f_Y(t) \). If \( t \leq 1 \), then \( f_Y(1) \leq f_Y(t) \) since \( f_Y \) is decreasing. In any case, one obtains

\[ \theta_Y(0) = f_Y(1) \leq \max(t^{\frac{g}{2}}, 1) f_Y(t). \]

Conclude by applying Lemma 8.3 (b). \( \square \)

Now, to prove Proposition 8.1 from Lemma 8.2 and Proposition 8.4, it suffices to observe that if \( Z \in \mathbb{H}_g \) is Siegel-reduced, then \( \lambda(\text{Im} Z) \geq \frac{\sqrt{3}}{2} \) (see lemma 15 of [27] page 195).

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\(^9\)Although, as goes without saying, he bears no responsibility for the mistakes which remain.

\(^{10}\)They have already been used by R. Wilms in [63], see the introduction to Autissier’s Appendix.
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