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# A Dialectica-Like Interpretation of a Linear MSO on Infinite Words<sup>\*</sup>

Pierre Pradic<sup>1,2</sup> and Colin Riba<sup>1</sup>

<sup>1</sup> ENS de Lyon, Université de Lyon, LIP<sup>\*\*</sup>

<sup>2</sup> University of Warsaw, Faculty of Mathematics, Informatics and Mechanics

**Abstract.** We devise a variant of Dialectica interpretation of intuitionistic linear logic for LMSO, a linear logic-based version MSO over infinite words. LMSO was known to be correct and complete w.r.t. Church's synthesis, thanks to an automata-based realizability model. Invoking Büchi-Landweber Theorem and building on a complete axiomatization of MSO on infinite words, our interpretation provides us with a syntactic approach, without any further construction of automata on infinite words. Via Dialectica, as linear negation directly corresponds to switching players in games, we furthermore obtain a complete logic: either a closed formula or its linear negation is provable. This completely axiomatizes the theory the realizability model of LMSO. Besides, this shows that in principle, one can solve Church's synthesis for a given  $\forall\exists$ -formula by only looking for proofs of either that formula or its linear negation.

**Keywords:** Linear Logic · Dialectica Interpretation · MSO on Infinite Words

## 1 Introduction

Monadic Second-Order Logic (MSO) over  $\omega$ -words is a simple yet expressive language for reasoning on non-terminating systems which subsumes non-trivial logics used in verification such as LTL (see e.g. [27,2]). MSO on  $\omega$ -words is decidable by Büchi's Theorem [6] (see e.g. [26,22]), and can be completely axiomatized as a subsystem of second-order Peano's arithmetic [25]. While MSO admits an effective translation to finite-state (Büchi) automata, it is a non-constructive logic, in the sense that it has true (*i.e.* provable)  $\forall\exists$ -statements which can be witnessed by no continuous stream function.

On the other hand, Church's synthesis [8] can be seen as a decision problem for a strong form of constructivity in MSO. More precisely (see e.g. [29,12]), Church's synthesis takes as input a  $\forall\exists$ -formula of MSO and asks whether it can be realized by a finite-state causal stream transducer. Church's synthesis is known to be decidable since Büchi-Landweber Theorem [7], which gives an effective solution to  $\omega$ -regular games on finite graphs generated by  $\forall\exists$ -formulae. In

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<sup>\*\*</sup> UMR 5668 CNRS ENS Lyon UCBL INRIA

traditional (theoretical) solutions to Church’s synthesis, the game graphs are induced from deterministic (say parity) automata obtained by McNaughton’s Theorem [17]. Despite its long history, Church’s synthesis has not yet been amenable to tractable solutions for the full language of **MSO** (see e.g. [12]).

In recent works [23,24], the authors suggested a Curry-Howard approach to Church’s synthesis based on intuitionistic and linear variants of **MSO**. In particular, [24] proposed a system **LMSO** based on (intuitionistic) linear logic [13], in which via a translation  $(-)^L : \mathbf{MSO} \rightarrow \mathbf{LMSO}$ , the provable  $\forall\exists(-)^L$ -statements exactly correspond to the realizable instances of Church’s synthesis. Realizer extraction for **LMSO** is done via an external realizability model based on alternating automata, which amounts to see every formula  $\varphi(a)$  as a formula of the form  $(\exists u)(\forall x)\varphi_D(u, x, a)$ , where  $\varphi_D$  represents a deterministic automaton.

In this paper, we use a variant of Gödel’s “Dialectica” functional interpretation as a syntactic formulation of the automata-based realizability model of [24]. Dialectica associates to  $\varphi(a)$  a formula  $\varphi^D(a)$  of the form  $(\exists u)(\forall x)\varphi_D(u, x, a)$ . In usual versions formulated in higher-types arithmetic (see e.g. [1,15]), the formula  $\varphi_D$  is quantifier-free, so that  $\varphi^D$  is a prenex form of  $\varphi$ . This prenex form is constructive, and a constructive proof of  $\varphi$  can be turned to a proof of  $\varphi^D$  with an explicit witness for  $\exists u$ . Even if Dialectica originally interprets intuitionistic arithmetic, it is structurally linear, and linear versions of Dialectica were formulated at the very beginning of linear logic [19,20,21].

We show that the automata-based realizability model of [24] can be obtained by a suitable modification of the usual linear Dialectica interpretation, in which the formula  $\varphi_D$  essentially represents a deterministic automaton on  $\omega$ -words and is in general not quantifier-free, and whose realizers are exactly the finite-state accepting strategies in the model of [24]. In addition to provide a syntactic extraction procedure with internalized and automata-free correctness proof, this reformulation has a striking consequence, namely that an extension **LMSO**( $\mathfrak{C}$ ) of **LMSO** is complete in the sense that for each closed formula  $\varphi$ , it either proves  $\varphi$  or its linear negation  $\varphi \multimap \perp$ . Since **LMSO**( $\mathfrak{C}$ ) has realizers for all provable  $\forall\exists(-)^L$ -statements, its completeness contrasts with the classical setting, in which due to provable non-constructive statements, one can not decide Church’s synthesis by only looking for proofs of  $\forall\exists$ -statements or their negations. Besides, **LMSO**( $\mathfrak{C}$ ) has a linear choice axiom which is realizable in the sense of both  $(-)^D$  and [24], but whose naive **MSO** counterpart is false.

The paper is organized as follows. We present our basic setting in §2, with a particular emphasis on particularities of (finite-state) causal functions to model strategies and realizers. Our variant of Dialectica and the corresponding linear system are discussed in §3, while §4 defines the systems **LMSO** and **LMSO**( $\mathfrak{C}$ ) and shows the completeness of **LMSO**( $\mathfrak{C}$ ).

## 2 Preliminaries

Alphabets (denoted  $\Sigma, \Gamma$ , etc) are finite non-empty sets of the form  $\mathbf{2}^p$  for some  $p \in \mathbb{N}$ . We let  $\mathbf{1} := \mathbf{2}^0$ . Note that alphabets are closed under Cartesian products

and set-theoretic function spaces. It follows that taking  $\llbracket o \rrbracket := \mathbf{2}$ , we have an alphabet  $\llbracket \tau \rrbracket$  for each simple type  $\tau \in \text{ST}$ , where

$$\sigma, \tau \in \text{ST} \quad ::= \quad \mathbf{1} \quad | \quad o \quad | \quad \sigma \times \tau \quad | \quad \sigma \rightarrow \tau$$

We often write  $(\tau)\sigma$  for the type  $\sigma \rightarrow \tau$ . Given an  $\omega$ -word (or stream)  $B \in \Sigma^\omega$  and  $n \in \mathbb{N}$ , we write  $B \upharpoonright n$  for the finite word  $B(0) \cdots B(n-1) \in \Sigma^*$ .

**Church's Synthesis and Causal Functions.** Church's synthesis consists in the automatic extraction of stream functions from input-output specifications (see e.g. [28,12]). These specifications are in general asked to be  $\omega$ -regular, or equivalently definable in MSO over  $\omega$ -words. In practice, proper subsets of MSO (and even of LTL) are assumed (see e.g. [5,11,12]). As an example, the relation

$$(\exists^\infty k)B(k) \Rightarrow (\exists^\infty k)C(k) \quad \text{resp.} \quad (\forall^\infty k)B(k) \Rightarrow (\exists^\infty k)C(k) \quad (1)$$

with input  $B \in \mathbf{2}^\omega$  and output  $C \in \mathbf{2}^\omega$  specifies functions  $F : \mathbf{2}^\omega \rightarrow \mathbf{2}^\omega$  such that  $F(B) \in \mathbf{2}^\omega \simeq \mathcal{P}(\mathbb{N})$  is infinite whenever  $B \in \mathbf{2}^\omega \simeq \mathcal{P}(\mathbb{N})$  is infinite (resp. the complement of  $B$  is finite). One may also additionally require to respect the transitions of some automaton. For instance, following [28], in addition to either case of (1) one can ask  $C \subseteq B$  and  $C$  not to contain two consecutive positions:

$$(\forall n)(C(n) \Rightarrow B(n)) \quad \text{and} \quad (\forall n)(C(n) \Rightarrow \neg C(n+1)) \quad (2)$$

In any case, the realizers must be (finite-state) causal functions. A stream function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  is causal (notation  $F : \Sigma \rightarrow_{\mathbb{S}} \Gamma$ ) if it can produce a prefix of length  $n$  of its output from a prefix of length  $n$  of its input. Hence  $F$  is causal if it is induced by a map  $f : \Sigma^+ \rightarrow \Gamma$  as follows:

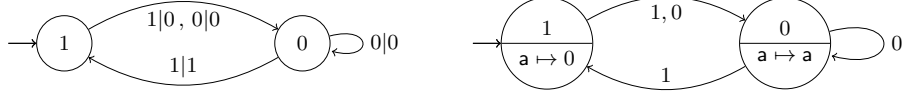
$$F(B)(n) = f(B(0) \cdots B(n)) \quad (\text{for all } B \in \Sigma^\omega \text{ and all } n \in \mathbb{N})$$

The finite-state (f.s.) causal functions are those induced by Mealy machines. A Mealy machine  $\mathcal{M} : \Sigma \rightarrow \Gamma$  is a DFA over input alphabet  $\Sigma$  equipped with an output function  $\lambda : Q_{\mathcal{M}} \times \Sigma \rightarrow \Gamma$  (where  $Q_{\mathcal{M}}$  is the state set of  $\mathcal{M}$ ). Writing  $\partial^* : \Sigma^* \rightarrow Q_{\mathcal{M}}$  for the iteration of the transition function  $\partial$  of  $\mathcal{M}$  from its initial state,  $\mathcal{M}$  induces a causal function via  $(\bar{a}.a \in \Sigma^+) \mapsto (\lambda(\partial^*(\bar{a}), a) \in \Gamma)$ .

Causal and f.s. causal functions form categories with finite products. Let  $\mathbb{S}$  be the category whose objects are alphabets and whose maps from  $\Sigma$  to  $\Gamma$  are causal functions  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ . Let  $\mathbb{M}$  be the wide subcategory of  $\mathbb{S}$  whose maps are finite-state causal functions.

- Example 1.* (a) Usual functions  $\Sigma \rightarrow \Gamma$  lift to (pointwise, one-state) maps  $\Sigma \rightarrow_{\mathbb{M}} \Gamma$ . For instance, the identity  $\Sigma \rightarrow_{\mathbb{M}} \Sigma$  is induced by the Mealy machine with  $\langle \partial, \lambda \rangle : (-, a) \mapsto (-, a)$ .  
(b) Causal functions  $\mathbf{1} \rightarrow_{\mathbb{S}} \Sigma$  correspond exactly to  $\omega$ -words  $B \in \Sigma^\omega$ .  
(c) The conjunction of (2) with either side of (1) is realized by the causal function  $F : \mathbf{2} \rightarrow_{\mathbb{M}} \mathbf{2}$  induced by the machine  $\mathcal{M} : \mathbf{2} \rightarrow \mathbf{2}$  displayed on Fig. 1 (left, where a transition  $a|b$  outputs  $b$  from input  $a$ ), taken from [28].

**Proposition 1.** *The Cartesian product of  $\Sigma_1, \dots, \Sigma_n$  (for  $n \geq 0$ ) in  $\mathbb{S}, \mathbb{M}$  is given by the product of sets  $\Sigma_1 \times \cdots \times \Sigma_n$  (so that  $\mathbf{1}$  is terminal).*



**Fig. 1.** A Mealy machine (left) and an equivalent eager (Moore) machine (right).

**The Logic  $\text{MSO}(\mathbf{M})$ .** Our specification language  $\text{MSO}(\mathbf{M})$  is an extension of  $\text{MSO}$  on  $\omega$ -words with one function symbol for each f.s. causal function. More precisely,  $\text{MSO}(\mathbf{M})$  is a many-sorted first-order logic, with one sort for each simple type  $\tau \in \text{ST}$ , and with one function symbol of arity  $(\sigma_1, \dots, \sigma_n; \tau)$  for each map  $\llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket \rightarrow_{\mathbf{M}} \llbracket \tau \rrbracket$ . A term  $\mathbf{t}$  of sort  $\tau$  (notation  $\mathbf{t}^\tau$ ) with free variables among  $x_1^{\sigma_1}, \dots, x_n^{\sigma_n}$  (we say that  $\mathbf{t}$  is of arity  $(\sigma_1, \dots, \sigma_n; \tau)$ ) thus induces a map  $\llbracket \mathbf{t} \rrbracket : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket \rightarrow_{\mathbf{M}} \llbracket \tau \rrbracket$ . Given a valuation  $x_i \mapsto B_i \in \llbracket \sigma_i \rrbracket^\omega \simeq \mathbb{S}[\mathbf{1}, \llbracket \sigma_i \rrbracket]$  for  $i \in \{1, \dots, n\}$ , we then obtain an  $\omega$ -word

$$\llbracket \mathbf{t} \rrbracket \circ \langle B_1, \dots, B_n \rangle \in \mathbb{S}[\mathbf{1}, \llbracket \tau \rrbracket] \simeq \llbracket \tau \rrbracket^\omega$$

$\text{MSO}(\mathbf{M})$  extends  $\text{MSO}$  with  $\exists x^\tau$  and  $\forall x^\tau$  ranging over  $\mathbb{S}[\mathbf{1}, \llbracket \tau \rrbracket] \simeq \llbracket \tau \rrbracket^\omega$  and with sorted equalities  $\mathbf{t}^\tau \doteq \mathbf{u}^\tau$  interpreted as equality over  $\mathbb{S}[\mathbf{1}, \llbracket \tau \rrbracket] \simeq \llbracket \tau \rrbracket^\omega$ . Write  $\models \varphi$  when  $\varphi$  holds in this model, called the *standard* model. The full definition of  $\text{MSO}(\mathbf{M})$  is deferred to §4.1.

An instance of Church's synthesis problem is given by a closed formula  $(\forall x^\sigma)(\exists u^\tau)\varphi(u, x)$ . A positive solution (or realizer) of this instance is a term  $\mathbf{t}(x)$  of arity  $(\sigma; \tau)$  such that  $(\forall x^\sigma)\varphi(\mathbf{t}(x), x)$  holds.

Proposition 1 implies that  $\text{MSO}(\mathbf{M})$  proves the following equations:

$$\pi_i(\langle \mathbf{t}_1, \dots, \mathbf{t}_n \rangle) \doteq_{\sigma_i} \mathbf{t}_i \quad \text{and} \quad \mathbf{t} \doteq_{\sigma_1 \times \dots \times \sigma_n} \langle \pi_1(\mathbf{t}), \dots, \pi_n(\mathbf{t}) \rangle \quad (3)$$

Hence each formula  $\varphi(a_1^{\sigma_1}, \dots, a_n^{\sigma_n})$  can be seen as a formula  $\varphi(a^{\sigma_1 \times \dots \times \sigma_n})$ .

**Eager Functions.** A causal function  $\Sigma \rightarrow_{\mathbb{S}} \Gamma$  is eager if it can produce a prefix of length  $n+1$  of its output from a prefix of length  $n$  of its input. More precisely, an eager  $F : \Sigma \rightarrow_{\mathbb{S}} \Gamma$  is induced by a map  $f : \Sigma^* \rightarrow \Gamma$  as

$$F(B)(n) = f(B(0) \cdot \dots \cdot B(n-1)) \quad (\text{for all } B \in \Sigma^\omega \text{ and all } n \in \mathbb{N})$$

Finite-state eager functions are those induced by eager (Moore) machines (see also [11]). An eager machine  $\mathcal{E} : \Sigma \rightarrow \Gamma$  is a Mealy machine  $\Sigma \rightarrow \Gamma$  whose output function  $\lambda : Q_{\mathcal{E}} \rightarrow \Gamma$  does not depend on the current input letter. An eager  $\mathcal{E} : \Sigma \rightarrow \Gamma$  induces an eager function via the map  $(\bar{\mathbf{a}} \in \Sigma^*) \mapsto (\lambda_{\mathcal{E}}(\partial_{\mathcal{E}}^*(\bar{\mathbf{a}})) \in \Gamma)$ .

We write  $F : \Sigma \rightarrow_{\mathbb{E}} \Gamma$  when  $F : \Sigma \rightarrow_{\mathbb{S}} \Gamma$  is eager and  $F : \Sigma \rightarrow_{\text{EM}} \Gamma$  when  $F$  is f.s. eager. All functions  $F : \Sigma \rightarrow_{\mathbf{M}} \mathbf{1}$ , and more generally, constants functions  $F : \Sigma \rightarrow_{\mathbb{S}} \Gamma$  are eager. Note also that if  $F : \Sigma \rightarrow_{\mathbb{S}} \Gamma$  is eager, then  $F : \Sigma \rightarrow_{\text{EM}} \Gamma$ . On the other hand, if  $F : \Sigma \rightarrow_{\text{EM}} \Gamma$  is induced by an eager machine  $\mathcal{E}$  then  $F$

is finite-state causal as being induced by the Mealy machine with same states and transitions as  $\mathcal{E}$ , but with output function  $(q, \mathbf{a}) \mapsto \lambda_{\mathcal{E}}(q)$ .

Eager functions do not form a category since the identity of  $\mathbb{S}$  is not eager. On the other hand, eager functions are closed under composition with causal functions.

**Proposition 2.** *If  $F$  is eager and  $G, H$  are causal then  $H \circ F \circ G$  is eager.*

Isolating eager functions allows a proper treatment of strategies in games and realizers w.r.t. the Dialectica interpretation. Since  $\Sigma^+ \rightarrow \Gamma \simeq \Sigma^* \rightarrow \Gamma^\Sigma$ , maps  $\Sigma \rightarrow_{\mathbb{E}} \Gamma^\Sigma$  are in bijection with maps  $\Sigma \rightarrow_{\mathbb{S}} \Gamma$ . This easily extends to machines. Given a Mealy machine  $\mathcal{M} : \Sigma \rightarrow \Gamma$ , let  $\mathbf{A}(\mathcal{M}) : \Sigma \rightarrow \Gamma^\Sigma$  be the eager machine defined as  $\mathcal{M}$  but with output map taking  $q \in Q_{\mathcal{M}}$  to  $(\mathbf{a} \mapsto \lambda_{\mathcal{M}}(q, \mathbf{a})) \in \Gamma^\Sigma$ .

*Example 2.* Recall the Mealy machine  $\mathcal{M} : \mathbf{2} \rightarrow \mathbf{2}$  of Ex. 1.(c). Then  $\mathbf{A}(\mathcal{M}) : \mathbf{2} \rightarrow \mathbf{2}^{\mathbf{2}}$  is the eager machine displayed in Fig. 1 (right, where the output is indicated within states).

Eager f.s. functions will often be used with the following notations. First, let  $\textcircled{\@}$  be the pointwise lift to  $\mathbb{M}$  of the usual application function  $\Gamma^\Sigma \times \Sigma \rightarrow \Gamma$ . We often write  $(F)G$  for  $\textcircled{\@}(F, G)$ . Consider a Mealy machine  $\mathcal{M} : \Sigma \rightarrow \Gamma$  and the induced eager machine  $\mathbf{A}(\mathcal{M}) : \Sigma \rightarrow \Gamma^\Sigma$ . We have

$$F_{\mathcal{M}}(B) = \textcircled{\@}(F_{\mathbf{A}(\mathcal{M})}(B), B) \quad (\text{for all } B \in \Sigma^\omega)$$

Given  $F : \Gamma \rightarrow_{\mathbb{E}} \Sigma^{\Gamma}$ , we write  $\mathbf{e}(F)$  for the causal  $\textcircled{\@}(F(-), -) : \Gamma \rightarrow_{\mathbb{S}} \Sigma$ . Given  $F : \Gamma \rightarrow_{\mathbb{S}} \Sigma$ , we write  $\mathbf{A}(F)$  for the eager  $\Gamma \rightarrow_{\mathbb{E}} \Sigma^{\Gamma}$  such that  $F = \mathbf{e}(\mathbf{A}(F))$ . We extend these notations to terms.

Eager functions admit fixpoints similar to those of contractive maps in the topos of tree (see e.g. [4, Thm. 2.4]).

**Proposition 3.** *For each  $F : \Sigma \times \Gamma \rightarrow_{\mathbb{E}} \Sigma^{\Gamma}$  there is a  $\text{fix}(F) : \Gamma \rightarrow_{\mathbb{E}} \Sigma^{\Gamma}$  s.t.*

$$\text{fix}(F)(C) = F(\mathbf{e}(\text{fix}(F))(C), C) \quad (\text{for all } C \in \Gamma^\omega)$$

*If  $F$  is induced by the eager machine  $\mathcal{E} : \Sigma \times \Gamma \rightarrow \Sigma^{\Gamma}$ , then  $\text{fix}(F)$  is induced by the eager  $\mathcal{H} : \Gamma \rightarrow \Sigma^{\Gamma}$  defined as  $\mathcal{E}$  but with  $\partial_{\mathcal{H}} : (q, \mathbf{b}) \mapsto \partial_{\mathcal{E}}(q, ((\lambda_{\mathcal{E}}(q))\mathbf{b}, \mathbf{b}))$ .*

**Games.** Traditional solutions to Church's synthesis turn specifications to infinite two-player games with  $\omega$ -regular winning conditions. Consider an  $\text{MSO}(\mathbf{M})$  formula  $\varphi(u^\tau, x^\sigma)$  with no free variable other than  $u, x$ . We see this formula as defining a two-player infinite game  $\mathcal{G}(\varphi)(u^\tau, x^\sigma)$  between the *Proponent*  $\text{P}$  ( $\exists$ loise), playing moves in  $\llbracket \tau \rrbracket$  and the *Opponent*  $\text{O}$  ( $\forall$ bélard), playing moves in  $\llbracket \sigma \rrbracket$ . The Proponent begins, and then the two players alternate, producing an infinite play of the form

$$\chi := u_0 x_0 \cdots u_n x_n \cdots \simeq ((u_k)_k, (x_k)_k) \in \llbracket \tau \rrbracket^\omega \times \llbracket \sigma \rrbracket^\omega$$

The play  $\chi$  is winning for P if  $\varphi((u_k)_k, (x)_k)$  holds. Otherwise  $\chi$  is winning for O. Strategies for P resp. O in this game are functions

$$\llbracket \sigma \rrbracket^* \longrightarrow \llbracket \tau \rrbracket \quad \text{resp.} \quad \llbracket \tau \rrbracket^+ \longrightarrow \llbracket \sigma \rrbracket \simeq \llbracket \tau \rrbracket^* \longrightarrow \llbracket \sigma \rrbracket^{\llbracket \tau \rrbracket}$$

Hence finite-state strategies are represented by f.s. eager functions. In particular, a realizer of  $(\forall x^\sigma)(\exists u^\tau)\varphi(u, x)$  in the sense of Church is a f.s. P-strategy in

$$\mathcal{G}(\varphi((u)x, x))(u^{(\tau)\sigma}, x^\sigma)$$

Most approaches to Church's synthesis reduce to Büchi-Landweber Theorem [7], stating that games with  $\omega$ -regular winning conditions are effectively determined, and that the winner always has a finite-state winning strategy. We will use Büchi-Landweber Theorem in following form. Note that an O-strategy in the game  $\mathcal{G}(\varphi)(u^\tau, x^\sigma)$  is a P-strategy in the game  $\mathcal{G}(\neg\varphi(u, (x)u))(x^{(\sigma)\tau}, u^\tau)$ .

**Theorem 1 ([7]).** *Let  $\varphi(u^\tau, x^\sigma)$  be an MSO( $\mathbf{M}$ )-formula with only  $u, x$  free. Then either there is an eager term  $\mathbf{u}(x)$  of arity  $(\sigma; \tau)$  such that  $\models (\forall x)\varphi(\mathbf{u}(x), x)$  or there is an eager term  $\mathbf{x}(u)$  of arity  $(\tau; (\sigma)\tau)$  such that  $\models (\forall u)\neg\varphi(u, \mathbf{e}(\mathbf{x})(u))$ . It is decidable which case holds and the terms are computable from  $\varphi$ .*

**Curry-Howard Approaches.** Following the complete axiomatization of MSO on  $\omega$ -words of [25] (see also [24]), one can axiomatize MSO( $\mathbf{M}$ ) with a deduction system based on arithmetic (see §4.1). Consider an instance of Church's synthesis  $(\forall x^\sigma)(\exists u^\tau)\varphi(u, x)$ . Then we get from Theorem 1 the alternative

$$\vdash_{\text{MSO}(\mathbf{M})} (\forall x)\varphi(\mathbf{e}(\mathbf{u})(x), x) \quad \text{or} \quad \vdash_{\text{MSO}(\mathbf{M})} (\forall u)\neg\varphi((u)(\mathbf{x}(u)), \mathbf{x}(u)) \quad (4)$$

for an eager term  $\mathbf{u}(x)$  or a causal term  $\mathbf{x}(u)$ . By enumerating proofs and machines, one thus gets a (naïve) syntactic algorithm for Church's synthesis. But it seems however unlikely to obtain a complete classical system in which the provable  $\forall\exists$ -statements do correspond to the realizable instances of Church's synthesis, because MSO( $\mathbf{M}$ ) has true but unrealizable  $\forall\exists$ -statements. Besides, note that

$$\begin{array}{lll} (\forall x^\sigma)\varphi(\mathbf{e}(\mathbf{u})(x), x) & \vdash_{\text{MSO}(\mathbf{M})} & (\forall x^\sigma)(\exists u^\tau)\varphi(u, x) \\ (\forall u^{(\tau)\sigma})\neg\varphi((u)(\mathbf{x}(u)), \mathbf{x}(u)) & \vdash_{\text{MSO}(\mathbf{M})} & (\forall u^{(\tau)\sigma})(\exists x^\sigma)\neg\varphi((u)x, x) \\ \neg(\forall x^\sigma)(\exists u^\tau)\varphi(u, x) & \vdash_{\text{MSO}(\mathbf{M})} & (\forall u^{(\tau)\sigma})(\exists x^\sigma)\neg\varphi((u)x, x) \end{array}$$

while it is possible both for realizable and unrealizable instances to have

$$\vdash_{\text{MSO}(\mathbf{M})} (\forall x^\sigma)(\exists u^\tau)\varphi(u, x) \wedge (\forall u^{(\tau)\sigma})(\exists x^\sigma)\neg\varphi((u)x, x) \quad (5)$$

In previous works [23,24], the authors devised intuitionistic and linear variants of MSO on  $\omega$ -words in which, thanks to automata-based polarity systems, proofs of suitably polarized existential statements correspond exactly to realizers for Church's synthesis. In particular, [24] proposed a system LMSO based on (intuitionistic) linear logic [13], such that via a translation  $(-)^L : \text{MSO} \rightarrow$

LMSO, provable  $\forall\exists(-)^L$ -statements exactly correspond to realizable instances of Church’s synthesis, while (4) exactly corresponds to alternatives of the form

$$\vdash_{\text{LMSO}} (\forall x^\sigma)(\exists u^\tau) [\varphi((u)x, x)]^L \text{ or } \vdash_{\text{LMSO}} (\forall u^{(\tau)\sigma})(\exists x^\sigma) [\neg\varphi((u)x, x)]^L \quad (6)$$

This paper goes further. We show that the automata-based realizability model of [24] can be obtained in a syntactic way, thanks to a (linear) Dialectica-like interpretation of a variant of LMSO, which turns a formula  $\varphi$  to a formula  $\varphi^D$  of the form  $(\exists u)(\forall x)\varphi_D(u, x)$ , where  $\varphi_D(u, x)$  essentially represents a deterministic automaton. While the correctness of the extraction procedure of [23,24] relied on automata-theoretic techniques, we show here that it can be performed syntactically. Second, by extending LMSO with realizable axioms, we obtain a system  $\text{LMSO}(\mathfrak{C})$  in which, using an adaptation of the usual *Characterization Theorem* for Dialectica stating that  $\varphi \multimap \varphi^D$  (see e.g. [15]), alternatives of the form (6) imply that for a closed  $\varphi$ ,

$$\vdash_{\text{LMSO}(\mathfrak{C})} \varphi \quad \text{or} \quad \vdash_{\text{LMSO}(\mathfrak{C})} \varphi \multimap \perp$$

where  $(-) \multimap \perp$  is a *linear* negation. We thus get a complete *linear* system with extraction of suitably polarized  $\forall\exists$ -statements. Such a system can of course not have a standard semantics, and indeed,  $\text{LMSO}(\mathfrak{C})$  has a functional choice axiom

$$(\forall x^\sigma)(\exists y^\tau)\varphi(x, y) \quad \multimap \quad (\exists f^{(\tau)\sigma})(\forall x^\sigma)\varphi(x, (f)x) \quad (\text{LAC})$$

which is realizable in the sense of both  $(-)^D$  and [24], but whose translation to  $\text{MSO}(\mathbf{M})$  (which precludes (5)) is false in the standard model.

### 3 A Monadic Linear Dialectica-like Interpretation

Gödel’s “Dialectica” functional interpretation associates to  $\varphi(a)$  a formula  $\varphi^D(a)$  of the form  $(\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x, a)$ . In usual versions formulated in higher-types arithmetic (see e.g. [1,15]), the formula  $\varphi_D$  is quantifier-free, so that  $\varphi^D$  is a prenex form of  $\varphi$ . This prenex form is constructive, and a constructive proof of  $\varphi$  can be turned to a proof of  $\varphi^D$  with an explicit (closed) witness for  $\exists u$ . We call such witnesses *realizers* of  $\varphi$ . Even if Dialectica originally interprets intuitionistic arithmetic, it is structurally linear: in general, realizers of contraction

$$\varphi(a) \quad \longrightarrow \quad \varphi(a) \wedge \varphi(a)$$

only exist when the term language can decide  $\varphi_D(u, x, a)$ , which is possible in arithmetic but not in all settings. Besides, linear versions of Dialectica were formulated at the very beginning of linear logic [19,20,21].

In this paper, we use a variant of Dialectica as a syntactic formulation of the automata-based realizability model of [24]. The formula  $\varphi_D$  essentially represents a deterministic automaton on  $\omega$ -words and is in general not quantifier-free. Moreover, we extract f.s. causal functions, while the category  $\mathbb{M}$  is not closed.



$$\begin{array}{c}
\frac{}{\varphi \vdash \varphi} \quad \frac{\overline{\varphi} \vdash \gamma, \overline{\varphi'} \quad \overline{\psi}, \gamma \vdash \overline{\psi'}}{\overline{\varphi}, \overline{\psi} \vdash \overline{\varphi'}, \overline{\psi'}} \quad \frac{\overline{\varphi}, \varphi, \psi, \overline{\psi} \vdash \overline{\varphi'}}{\overline{\varphi}, \psi, \varphi, \overline{\psi} \vdash \overline{\varphi'}} \quad \frac{\overline{\varphi} \vdash \overline{\varphi'}, \varphi, \psi, \overline{\psi'}}{\overline{\varphi} \vdash \overline{\varphi'}, \psi, \varphi, \overline{\psi'}} \\
\frac{\overline{\varphi} \vdash \overline{\psi}}{\overline{\varphi}, \mathbf{I} \vdash \overline{\psi}} \quad \frac{}{\vdash \mathbf{I}} \quad \frac{\overline{\varphi}, \varphi_0, \varphi_1 \vdash \overline{\varphi'}}{\overline{\varphi}, \varphi_0 \otimes \varphi_1 \vdash \overline{\varphi'}} \quad \frac{\overline{\varphi} \vdash \varphi, \overline{\varphi'} \quad \overline{\psi} \vdash \psi, \overline{\psi'}}{\overline{\varphi}, \overline{\psi} \vdash \varphi \otimes \psi, \overline{\varphi'}, \overline{\psi'}} \quad \frac{\overline{\varphi}, \varphi \vdash \psi}{\overline{\varphi} \vdash \varphi \multimap \psi} \\
\frac{}{\perp \vdash} \quad \frac{\overline{\varphi} \vdash \overline{\psi}}{\overline{\varphi} \vdash \perp, \overline{\psi}} \quad \frac{\overline{\varphi}, \varphi \vdash \overline{\varphi'} \quad \overline{\psi}, \psi \vdash \overline{\psi'}}{\overline{\varphi}, \overline{\psi}, \varphi \wp \psi \vdash \overline{\varphi'}, \overline{\psi'}} \quad \frac{\overline{\varphi} \vdash \varphi_0, \varphi_1, \overline{\varphi'}}{\overline{\varphi} \vdash \varphi_0 \wp \varphi_1, \overline{\varphi'}} \quad \frac{\overline{\varphi} \vdash \varphi, \overline{\varphi'} \quad \overline{\psi}, \psi \vdash \overline{\psi'}}{\overline{\varphi}, \overline{\psi}, \varphi \multimap \psi \vdash \overline{\varphi'}, \overline{\psi'}} \\
\frac{\overline{\varphi}, \varphi \vdash \overline{\varphi'}}{\overline{\varphi}, (\exists z^\tau) \varphi \vdash \overline{\varphi'}} \quad \frac{\overline{\varphi} \vdash \varphi[\mathbf{t}^\tau/x^\tau], \overline{\varphi'}}{\overline{\varphi} \vdash (\exists x^\tau) \varphi, \overline{\varphi'}} \quad \frac{\overline{\varphi}, \varphi[\mathbf{t}^\tau/x^\tau] \vdash \overline{\varphi'}}{\overline{\varphi}, (\forall x^\tau) \varphi \vdash \overline{\varphi'}} \quad \frac{\overline{\varphi} \vdash \varphi}{\overline{\varphi} \vdash (\forall z^\tau) \varphi}
\end{array}$$

**Fig. 2.** Deduction for MF (where  $z^\tau$  is fresh).

As a result, a realizer of  $\varphi$  is an *open* (eager) term  $\mathbf{u}(x)$  of arity  $(\sigma; \tau)$  satisfying  $\varphi_D(\mathbf{u}(x), x)$ . Thanks to Büchi-Landweber Theorem, contractions on closed  $\varphi$  have realizers which are correct in the standard model. But this is generally not the case for open  $\varphi(a)$ . We thus work in a linear system, in which we obtain witnesses for  $\forall\exists(-)^L$ -statements (and thus for realizable instances of Church's synthesis), but not for all  $\forall\exists$ -statements.

Fix a set of atomic formulae  $\text{At}$  containing all  $(\mathbf{t}^\tau \doteq \mathbf{u}^\tau)$ .

### 3.1 The Multiplicative Fragment

Our linear system is based on *full intuitionistic linear logic* (see [14]). The formulae of the multiplicative fragment MF are given by the grammar:

$$\varphi, \psi ::= \mathbf{I} \mid \perp \mid \alpha \mid \varphi \multimap \psi \mid \varphi \otimes \psi \mid \varphi \wp \psi \mid (\exists x^\tau) \varphi \mid (\forall x^\tau) \varphi$$

(where  $\alpha \in \text{At}$ ). Deduction is given by the rules of Fig. 2 and the axioms

$$\frac{}{\vdash \mathbf{t}^\tau \doteq \mathbf{t}^\tau} \quad \frac{}{\mathbf{t}^\tau \doteq \mathbf{u}^\tau, \varphi[\mathbf{t}^\tau/x^\tau] \vdash \varphi[\mathbf{u}^\tau/x^\tau]} \quad \frac{\llbracket \mathbf{t}^\tau \rrbracket = \llbracket \mathbf{u}^\tau \rrbracket}{\vdash \mathbf{t}^\tau \doteq \mathbf{u}^\tau} \quad (7)$$

Each formula  $\varphi$  of MF can be mapped to an  $\text{MSO}(\mathbf{M})$ -formula  $[\varphi]$  (where  $\mathbf{I}, \multimap, \otimes, \wp$  are replaced resp. by  $\top, \rightarrow, \wedge, \vee$ ). Hence  $[\varphi]$  holds whenever  $\vdash \varphi$ .

The Dialectica interpretation of MF is the usual one rewritten with the connectives of MF, but for the disjunction  $\wp$  that we treat similarly as  $\otimes$ . To each formula  $\varphi(a)$  with only  $a$  free, we associate a formula  $\varphi^D(a)$  with only  $a$  free, as well as a formula  $\varphi_D$  with possibly other free variables. For atomic formulae we let  $\varphi^D(a) := \varphi_D(a) := \varphi(a)$ . The inductive cases are given on Fig. 3, where  $\varphi^D(a) = (\exists u)(\forall x)\varphi_D(u, x, a)$  and  $\psi^D(a) = (\exists v)(\forall y)\psi_D(v, y, a)$ .

Dialectica is such that  $\varphi^D$  is equivalent to  $\varphi$  via possibly non-intuitionistic but constructive principles. The tricky connectives are implication and universal quantification. Similarly as in the intuitionistic case (see e.g. [15,1,30]),  $(\varphi \multimap \psi)^D$  is prenex a form of  $\varphi^D \multimap \psi^D$  obtained using (LAC) together with linear variants of the *Markov* and *Independence of premises* principles. In our

$$\begin{aligned}
(\varphi \otimes \psi)^D(a) &:= \exists \langle u, v \rangle \forall \langle x, y \rangle. (\varphi \otimes \psi)_D(\langle u, v \rangle, \langle x, y \rangle, a) &:= \\
& & \exists \langle u, v \rangle \forall \langle x, y \rangle. \varphi_D(u, x, a) \otimes \psi_D(v, y, a) \\
(\varphi \wp \psi)^D(a) &:= \exists \langle u, v \rangle \forall \langle x, y \rangle. (\varphi \wp \psi)_D(\langle u, v \rangle, \langle x, y \rangle, a) &:= \\
& & \exists \langle u, v \rangle \forall \langle x, y \rangle. \varphi_D(u, x, a) \wp \psi_D(v, y, a) \\
(\varphi \multimap \psi)^D(a) &:= \exists \langle f, F \rangle \forall \langle u, y \rangle. (\varphi \multimap \psi)_D(\langle f, F \rangle, \langle u, y \rangle, a) &:= \\
& & \exists \langle f, F \rangle \forall \langle u, y \rangle. \varphi_D(u, (F)uy, a) \multimap \psi_D((f)u, y, a) \\
(\exists w. \varphi)^D(a) &:= \exists \langle u, w \rangle \forall x. (\exists w. \varphi)_D(\langle u, w \rangle, x, a) &:= \exists \langle u, w \rangle \forall x. \varphi_D(u, x, \langle a, w \rangle) \\
(\forall w. \varphi)^D(a) &:= \exists f \forall \langle x, w \rangle. (\forall w. \varphi)_D(f, \langle x, w \rangle, a) &:= \exists f \forall \langle x, w \rangle. \varphi_D((f)w, x, \langle a, w \rangle)
\end{aligned}$$


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**Fig. 3.** The Dialectica Interpretation of MF (where types are leaved implicit).

case, the equivalence  $\varphi \multimap \varphi^D$  also requires additional axioms for  $\otimes$  and  $\wp$ . We give details for the full system in §3.3.

The soundness of  $(-)^D$  goes as usual, excepted that we extract *open eager* terms: from a proof of  $\varphi(a^\kappa)$  we extract a realizer of  $(\forall a)\varphi(a)$ , that is an open eager term  $\mathbf{u}(x, a)$  s.t.  $\vdash \varphi_D(\mathbf{u}(x, a), a, x, a)$ . Composition of realizers (in part. required for the cut rule) is given by the fixpoints of Prop. 3. Note that a realizer of a closed  $\varphi$  is a finite-state winning P-strategy in  $\mathcal{G}(\lfloor \varphi_D \rfloor)(u, x)$ .

### 3.2 Polarized Exponentials

It is well-known that the structure of Dialectica is linear, as it makes problematic the interpretation of contraction:

$$\varphi(a) \multimap \varphi(a) \otimes \varphi(a) \quad \text{and} \quad \varphi(a) \wp \varphi(a) \multimap \varphi(a)$$

In our case, the Büchi-Landweber Theorem implies that all closed instances of contraction have realizers which are correct in the standard model. But this is in general not true for open instances.

*Example 3.* Realizers of  $\varphi \multimap \varphi \otimes \varphi$  for a closed  $\varphi$  are given by eager terms  $\mathbf{U}_1(u, x_1, x_2)$ ,  $\mathbf{U}_2(u, x_1, x_2)$ ,  $\mathbf{X}(u, x_1, x_2)$  which must represent P-strategies in the game  $\mathcal{G}(\Phi)(\langle U_1, U_2, X \rangle, \langle u, x_1, x_2 \rangle)$ , where  $\Phi$  is

$$\lfloor \varphi_D(u, (X)ux_1x_2) \rfloor \longrightarrow \lfloor \varphi_D((U_1)u, x_1) \rfloor \wedge \lfloor \varphi_D((U_2)u, x_2) \rfloor$$

By the Büchi-Landweber Theorem 1, either there is an eager term  $\mathbf{U}(x)$  such that  $\lfloor \varphi_D(\mathbf{U}(x), x) \rfloor$  holds, so that

$$\lfloor \varphi_D(u, x_1) \rfloor \longrightarrow \lfloor \varphi_D(\mathbf{e}(\mathbf{U})(x_1), x_1) \rfloor \wedge \lfloor \varphi_D(\mathbf{e}(\mathbf{U})(x_2), x_2) \rfloor$$

or there is an eager term  $\mathbf{X}(u)$  such that  $\neg \lfloor \varphi_D(u, \mathbf{e}(\mathbf{X})(u)) \rfloor$  holds, so that

$$\lfloor \varphi_D(u, \mathbf{e}(\mathbf{X})(u)) \rfloor \longrightarrow \lfloor \varphi_D(u, x_1) \rfloor \wedge \lfloor \varphi_D(u, x_2) \rfloor$$

*Example 4.* Consider the open formula  $\varphi(a^\circ) := (\forall x^\circ)(\mathfrak{t}(x, a) \doteq 0^\omega)$  where  $\llbracket \mathfrak{t} \rrbracket(B, C) = 0^{n+1}1^\omega$  for the first  $n \in \mathbb{N}$  with  $C(n+1) = B(0)$  if such  $n$  exists, and such that  $\llbracket \mathfrak{t} \rrbracket(B, C) = 0^\omega$  otherwise. The game induced by  $((\forall a)(\varphi \multimap \varphi \otimes \varphi))_D$  is  $\mathcal{G}(\Phi)(X, \langle x_1, x_1, a \rangle)$ , where  $\Phi$  is

$$\mathfrak{t}((X)x_1x_2a, a) \doteq 0^\omega \quad \longrightarrow \quad \mathfrak{t}(x_1, a) \doteq 0^\omega \quad \wedge \quad \mathfrak{t}(x_2, a) \doteq 0^\omega$$

In this game, P begins by playing a function  $\mathbf{2}^3 \rightarrow \mathbf{2}$ , O replies in  $\mathbf{2}^3$ , and then P and O keep on alternatively playing moves of the expected type. A finite-state winning strategy for O is easy to find. Let P begin with the function X. Fix some  $\mathbf{a} \in \mathbf{2}$  and let  $i := X(0, 1, \mathbf{a})$ . O replies  $(0, 1, \mathbf{a})$  to X. The further moves of P are irrelevant, and O keeps on playing  $(-, -, 1 - i)$  (the values of  $x_1$  and  $x_2$  are irrelevant after the first round). This strategy ensures

$$\mathfrak{t}((X)x_1x_2a, a) \doteq 0^\omega \quad \wedge \quad \neg(\mathfrak{t}(x_1, a) \doteq 0^\omega \quad \wedge \quad \mathfrak{t}(x_2, a) \doteq 0^\omega)$$

Hence we can not realize contraction while remaining correct w.r.t. the standard model. On the other hand, Dialectica induces polarities generalizing the usual polarities of linear logic. Say that  $\varphi(a)$  is *positive* (resp. *negative*) if  $\varphi^D(a)$  is of the form  $\varphi^D(a) = (\exists u^\tau)\varphi_D(u, -, a)$  (resp.  $\varphi^D(a) = (\forall x^\sigma)\varphi_D(-, x, a)$ ). Quantifier-free formulae are thus both positive and negative.

*Example 5.* Polarized contraction

$$\varphi^+ \multimap \varphi^+ \otimes \varphi^+ \quad \text{and} \quad \psi^- \wp \psi^- \multimap \psi^- \quad (\varphi^+ \text{ positive, } \psi^- \text{ negative})$$

gives realizers of all instances of itself. Indeed, with say  $\varphi^D(a) = (\exists u)\varphi_D(u, -, a)$  and  $\psi^D(a) = (\forall y)\psi_D(-, y, a)$ ,  $\mathbf{A}(\pi_1)$  (for  $\pi_1$  a  $\mathbb{M}$ -projection on suitable types) gives eager terms  $\mathbf{U}(u, a)$  and  $\mathbf{Y}(y, a)$  such that

$$\begin{aligned} & \varphi_D(u, -, a) \multimap \left( \varphi_D(\mathbf{e}(\mathbf{U})(u, a), -, a) \otimes \varphi_D(\mathbf{e}(\mathbf{U})(u, a), -, a) \right) \\ \text{and} \quad & \left( \psi_D(-, \mathbf{e}(\mathbf{Y})(y, a), a) \wp \psi_D(-, \mathbf{e}(\mathbf{Y})(y, a), a) \right) \multimap \psi_D(-, y, a) \end{aligned}$$

We only have exponentials for polarized formulae. First, following the usual polarities of linear logic, we can let

$$\begin{aligned} (!\varphi^+)^D(a) & := (\exists u)(!\varphi^+)_D(u, -, a) & := (\exists u)!\varphi_D(u, -, a) \\ (?(\psi^-))^D(a) & := (\forall y)(?\psi^-)_D(-, y, a) & := (\forall x)?\psi_D(-, y, a) \end{aligned} \quad (8)$$

Hence  $!\varphi$  is positive for a positive  $\varphi$  and  $?\psi$  is negative for a negative  $\psi$ . The following exponential contraction axioms are then interpreted by themselves:

$$!(\varphi^+) \multimap !(\varphi^+) \otimes !(\varphi^+) \quad \text{and} \quad ?(\psi^-) \wp ?(\psi^-) \multimap ?(\psi^-)$$

Second, we can have exponentials  $!(\psi^-)$  and  $?( \varphi^+)$  with the automata-based reading of [24]. Positive formulae are seen as non-deterministic automata, and

$$\begin{array}{ccccc}
\frac{\overline{\psi} \vdash \overline{\psi}'}{\overline{\psi}, !\varphi \vdash \overline{\psi}'} & \frac{\overline{\psi}, !\varphi, !\varphi \vdash \overline{\psi}'}{\overline{\psi}, !\varphi \vdash \overline{\psi}'} & \frac{\overline{\varphi}, \varphi \vdash \overline{\varphi}'}{\overline{\varphi}, !\varphi \vdash \overline{\varphi}'} & \frac{!\overline{\varphi} \vdash \varphi, ?\overline{\psi}}{!\overline{\varphi} \vdash !\varphi, ?\overline{\psi}} & \frac{\overline{\varphi}, !\varphi \vdash \psi, ?\overline{\psi}}{\overline{\varphi} \vdash !\varphi \multimap \psi, ?\overline{\psi}} \\
\frac{\overline{\psi} \vdash \overline{\psi}'}{\overline{\psi} \vdash ?\varphi, \overline{\psi}'} & \frac{\overline{\psi} \vdash ?\varphi, ?\varphi, \overline{\psi}'}{\overline{\psi} \vdash ?\varphi, \overline{\psi}'} & \frac{\overline{\varphi} \vdash \varphi, \overline{\psi}}{\overline{\varphi} \vdash ?\varphi, \overline{\psi}} & \frac{!\overline{\varphi}, \varphi \vdash ?\overline{\psi}}{!\overline{\varphi}, ?\varphi \vdash ?\overline{\psi}} & \frac{\overline{\varphi} \vdash \varphi, ?\overline{\psi}}{\overline{\varphi} \vdash (\forall z)\varphi, ?\overline{\psi}}
\end{array}$$

**Fig. 4.** Exponential rules of PF.

? on positive formulae is determinization on  $\omega$ -words (McNaughton's Theorem [17]). Negative formulae are seen as universal automata, and ! on negative formulae is to co-determinization (an instance of the *Simulation Theorem* [10,18]). Formulae which are both positive and negative (notation  $(-)^{\pm}$ ) correspond to deterministic automata, and are called *deterministic*. We let

$$\begin{aligned}
(!(\psi^-))^D(a) &:= (!(\psi^-))_D(-, -, a) &:= !(\forall x)\psi_D(-, x, a) \\
?((\varphi^+))^D(a) &:= ?(\varphi^+)_D(-, -, a) &:= ?(\exists u)\varphi_D(u, -, a)
\end{aligned} \tag{9}$$

So  $!(\psi^-)$  and  $?((\varphi^+))$  are always deterministic. The corresponding exponential contraction axioms are interpreted by themselves. This leads to the following polarized fragment PF (the deduction rules for exponentials are given on Fig. 4):

$$\begin{aligned}
\varphi^{\pm}, \psi^{\pm} &::= \mathbf{I} \mid \perp \mid \alpha \mid !(\varphi^-) \mid ?(\varphi^+) \mid \varphi^{\pm} \otimes \psi^{\pm} \mid \varphi^{\pm} \wp \psi^{\pm} \mid \varphi^{\pm} \multimap \psi^{\pm} \\
\varphi^+, \psi^+ &::= \varphi^{\pm} \mid !(\varphi^+) \mid (\exists x^{\sigma})\varphi^+ \mid \varphi^+ \otimes \psi^+ \mid \varphi^+ \wp \psi^+ \mid \varphi^- \multimap \psi^+ \\
\varphi^-, \psi^- &::= \varphi^{\pm} \mid ?(\varphi^-) \mid (\forall x^{\sigma})\varphi^- \mid \varphi^- \otimes \psi^- \mid \varphi^- \wp \psi^- \mid \varphi^+ \multimap \psi^-
\end{aligned}$$

### 3.3 The Full System

The formulae of the full system FS are given by the following grammar:

$$\varphi, \psi ::= \varphi^+ \mid \varphi^- \mid \varphi \multimap \psi \mid \varphi \otimes \psi \mid \varphi \wp \psi \mid (\exists x^{\tau})\varphi \mid (\forall x^{\tau})\varphi$$

Deduction in FS is given by Fig. 2, Fig. 4 and (7). We extend  $[-]$  to FS with  $![\varphi] := [?\varphi] := [\varphi]$ . Hence  $[\varphi]$  holds when  $\vdash \varphi$  is derivable. The Dialectica interpretation of FS is given by Fig. 3 and (8), (9) (still taking  $\varphi^D(a) := \varphi_D(a) := \varphi(a)$  for atoms). Note that  $(-)^D$  preserves and reflects polarities.

**Theorem 2 (Soundness).** *Let  $\varphi$  be closed with  $\varphi^D = (\exists u^{\tau})(\forall x^{\sigma})\varphi_D(u, x)$ . From a proof of  $\varphi$  in FS one can extract an eager term  $u(x)$  such that FS proves  $(\forall x^{\sigma})\varphi_D(u(x), x)$ .*

As usual, proving  $\varphi \multimap \varphi^D$  requires extra axioms. Besides (LAC), we use the following (*linear*) *semi-intuitionistic principles* (LSIP), with polarities as shown:

$$\begin{aligned}
(\forall a)(\varphi^-(a) \otimes \psi^-) &\multimap (\forall a)\varphi^-(a) \otimes \psi^- \\
(\forall a)(\varphi^-(a) \wp \psi^-) &\multimap (\forall a)\varphi^-(a) \wp \psi^- \\
(\exists a)\varphi^-(a) \wp \psi &\multimap (\exists a)(\varphi^-(a) \wp \psi) \\
(\psi^- \multimap (\exists a)\varphi^-(a)) &\multimap (\exists a)(\psi^- \multimap \varphi^-(a)) \\
((\forall a)\varphi^{\pm}(a) \multimap \psi^{\pm}) &\multimap (\exists a)(\varphi^{\pm}(a) \multimap \psi^{\pm})
\end{aligned} \tag{LSIP}$$

as well as the following *deterministic exponential* axioms (DEXP):

$$\delta \multimap !\delta \quad \text{and} \quad ?\delta \multimap \delta \quad (\delta \text{ deterministic})$$

All these axioms but (LAC) are true in the standard model (via  $[-]$ ). Moreover:

**Proposition 4.** *The axioms (LAC) and (LSIP) are realized in FS. The axioms (DEXP) are realized in FS + (DEXP).*

**Theorem 3 (Characterization).** *We have*

$$\begin{array}{ll} \vdash_{\text{FS}+(\text{LAC})+(\text{LSIP})+(\text{DEXP})} \varphi(a) \multimap \varphi^D(a) & (\varphi \text{ FS-formula}) \\ \vdash_{\text{FS}+(\text{LSIP})+(\text{DEXP})} \varphi(a) \multimap \varphi^D(a) & (\varphi \text{ PF-formula}) \end{array}$$

**Corollary 1 (Extraction).** *Consider a closed formula  $\varphi := (\forall x^\sigma)(\exists u^\tau)\delta(u, x)$  with  $\delta$  deterministic. From a proof of  $\varphi$  in FS + (LAC) + (LSIP) + (DEXP) one can extract a term  $\mathfrak{t}(x)$  such that  $\models (\forall x^\sigma)[\delta(\mathfrak{t}(x), x)]$ .*

Note that FS + (DEXP) proves  $\delta \wp (\delta \multimap \perp)$  for all deterministic  $\delta$ .

### 3.4 Translations of Classical Logic

There are many translations from classical to linear logic. Two canonical possibilities are the  $(-)^T$  and  $(-)^Q$ -translation of [9] (see also [16]) targeting resp. negative and positive formulae. Both take classical sequents to linear sequents of the form  $!(-) \vdash ?(-)$ , which are provable in FS in part. thanks to the PF rules

$$\frac{\overline{\varphi}, !\varphi \vdash \psi, ?\overline{\psi}}{\overline{\varphi} \vdash !\varphi \multimap \psi, ?\overline{\psi}} \quad \frac{\overline{\varphi} \vdash \varphi, ?\overline{\psi}}{\overline{\varphi} \vdash (\forall z)\varphi, ?\overline{\psi}}$$

For the completeness of LMSO( $\mathfrak{C}$ ) (Thm. 6, §4), we shall actually require a translation  $(-)^L$  such that the linear equivalences (with polarities as displayed)

$$?\varphi^+ \multimap [\varphi^+]^L \quad \delta^\pm \multimap [\delta^\pm]^L \quad !\psi^- \multimap [\psi^-]^L \quad (10)$$

are provable possibly with extra axioms that we require to realize themselves. In part., (10) implies (DEXP), and  $(-)^L$  should give deterministic formulae. While  $(-)^T$  and  $(-)^Q$  can be adapted accordingly, (10) induces axioms which make the resulting translations equivalent to the deterministic  $(-)^L$ -translation of [24]:

$$\begin{array}{lll} \perp^L := \perp & \top^L := \mathbf{I} & \alpha^L := \alpha \\ (\varphi \vee \psi)^L := \varphi^L \wp \psi^L & (\exists x^\sigma.\varphi)^L := ?(\exists x^\sigma)\varphi^L & \\ (\varphi \rightarrow \psi)^L := \varphi^L \multimap \psi^L & (\varphi \wedge \psi)^L := \varphi^L \otimes \psi^L & (\forall x^\sigma.\varphi)^L := !(\forall x^\sigma)\varphi^L \end{array}$$

**Proposition 5.** *The scheme (10) is equivalent in FS to (DEXP)+(PEXP), where (PEXP) are the following polarized exponential axioms, with polarities as shown:*

$$\begin{array}{ll} ?(\varphi^+) \multimap ?!(\varphi^+) & !?(\psi^-) \multimap !(\psi^-) \\ !(\varphi^-) \multimap ?(\psi^+) \multimap ?(\varphi^- \multimap \psi^+) & ?(\varphi^+) \multimap !(\psi^-) \multimap !(\varphi^+ \multimap \psi^-) \\ ?(\varphi^+) \otimes ?(\psi^+) \multimap ?(\varphi^+ \otimes \psi^+) & !(\varphi^- \otimes \psi^-) \multimap !(\varphi^-) \otimes !(\psi^-) \\ ?(\varphi^+) \wp ?(\psi^+) \multimap ?(\varphi^+ \wp \psi^+) & !(\varphi^- \wp \psi^-) \multimap !(\varphi^-) \wp !(\psi^-) \end{array}$$

**Proposition 6.** *If  $\varphi$  is provable in many-sorted classical logic with equality then  $\text{FS} + (\text{DEXP})$  proves  $\varphi^L$ .*

**Proposition 7.** *The axioms (PEXP) are realized in  $\text{FS} + (\text{LSIP}) + (\text{DEXP}) + (\text{PEXP})$ . Corollary 1 thus extends to  $\text{FS} + (\text{LAC}) + (\text{LSIP}) + (\text{DEXP}) + (\text{PEXP})$ .*

Note that  $\varphi^L$  is deterministic and that  $\lfloor \varphi^L \rfloor = \varphi$ .

## 4 Completeness

In §3 we devised a Dialectica-like  $(-)^D$  providing a syntactic extraction procedure for  $\forall\exists(-)^L$ -statements. In this Section, building on an axiomatic treatment of  $\text{MSO}(\mathbf{M})$ , we show that  $\text{LMSO}$ , an arithmetic extension of  $\text{FS} + (\text{LSIP}) + (\text{DEXP}) + (\text{PEXP})$  adapted from [24], is correct and complete w.r.t. Church's synthesis, in the sense that the provable  $\forall\exists(-)^L$ -statements are exactly the realizable ones. We then turn to the main result of this paper, namely the completeness of  $\text{LMSO}(\mathfrak{C}) := \text{LMSO} + (\text{LAC})$ . We fix the set of atomic formulae

$$\alpha \in \text{At} ::= \mathfrak{t}^\tau \doteq \mathfrak{u}^\tau \mid \mathfrak{t}^o \dot{\subseteq} \mathfrak{u}^o \mid \text{E}(\mathfrak{t}^o) \mid \text{N}(\mathfrak{t}^o) \mid \text{S}(\mathfrak{t}^o, \mathfrak{u}^o) \mid \mathbf{0}(\mathfrak{t}^o) \mid \mathfrak{t}^o \dot{\leq} \mathfrak{u}^o$$

### 4.1 The Logic $\text{MSO}(\mathbf{M})$

$\text{MSO}(\mathbf{M})$  is many-sorted first-order logic with atomic formulae  $\alpha \in \text{At}$ . Its sorts and terms are those given in §2, and standard interpretation extends that of §2 as follows:  $\dot{\subseteq}$  is set inclusion,  $\text{E}$  holds on  $B$  iff  $B$  is empty,  $\text{N}$  (resp.  $\mathbf{0}$ ) holds on  $B$  iff  $B$  is a singleton  $\{n\}$  (resp. the singleton  $\{0\}$ ), and  $\text{S}(B, C)$  (resp.  $B \dot{\leq} C$ ) holds iff  $B = \{n\}$  and  $C = \{n+1\}$  for some  $n \in \mathbb{N}$  (resp.  $B = \{n\}$  and  $C = \{m\}$  for some  $n \leq m$ ). We write  $x^t$  for variables  $x^o$  relativized to  $\mathbf{N}$ , so that  $(\exists x^t)\varphi$  and  $(\forall x^t)\varphi$  stand resp. for  $(\exists x^o)(\text{N}(x) \wedge \varphi)$  and  $(\forall x^o)(\text{N}(x) \rightarrow \varphi)$ . Moreover,  $x^t \dot{\in} \mathfrak{t}$  stands for  $x^t \dot{\subseteq} \mathfrak{t}$ , so that  $\mathfrak{t}^o \dot{\subseteq} \mathfrak{u}^o$  is equivalent to  $(\forall x^t)(x \dot{\in} \mathfrak{t} \rightarrow x \dot{\in} \mathfrak{u})$ .

The logic  $\text{MSO}^+$  [24] is  $\text{MSO}(\mathbf{M})$  restricted to the type  $o$ , hence with only terms for Mealy machines of sort  $(\mathbf{2}, \dots, \mathbf{2}; \mathbf{2})$ . The  $\text{MSO}$  of [24] is the purely relational (term-free) restriction of  $\text{MSO}^+$ . Recall from [24, Prop. 2.6], that for each Mealy machine  $\mathcal{M} : \mathbf{2}^p \rightarrow \mathbf{2}$ , there is an  $\text{MSO}$ -formula  $\delta_{\mathcal{M}}(\overline{X}, x)$  such that for all  $n \in \mathbb{N}$  and all  $\overline{B} \in (\mathbf{2}^\omega)^p$ , we have  $F_{\mathcal{M}}(\overline{B})(n) = 1$  iff  $\delta_{\mathcal{M}}(\{n\}, \overline{B})$  holds.

The axioms of  $\text{MSO}(\mathbf{M})$  are the arithmetic rules of Fig. 5, the axioms (7) and the following, where  $\mathcal{M} : \mathbf{2}^p \rightarrow \mathbf{2}$  and  $y, z, X$  are fresh.

$$\frac{}{\vdash (\forall \overline{X}^o)(\forall x^t)(x \dot{\in} \mathfrak{f}_{\mathcal{M}}(\overline{X}) \leftrightarrow \delta_{\mathcal{M}}(x, \overline{X}))} \quad \frac{}{\vdash (\exists X^o)(\forall x^t)(x \dot{\in} X \leftrightarrow \varphi)}$$

$$\frac{\overline{\varphi}, \mathbf{0}(z) \vdash \varphi[z/x], \overline{\varphi}' \quad \overline{\varphi}, \text{S}(y, z), \varphi[y/x] \vdash \varphi[z/x], \overline{\varphi}'}{\overline{\varphi} \vdash (\forall x^t)\varphi, \overline{\varphi}'}$$

The axioms of  $\text{MSO}(\mathbf{M})$  completely axiomatize its standard model. This extends [24, Thm. 2.11 (via [25])].

**Theorem 4 (Completeness of  $\text{MSO}(\mathbf{M})$ ).** *For closed  $\text{MSO}(\mathbf{M})$ -formulae  $\varphi$ , we have  $\models \varphi$  if and only if  $\vdash_{\text{MSO}(\mathbf{M})} \varphi$ .*

$$\begin{array}{c}
\frac{}{E(\mathbf{t}) \vdash \mathbf{t} \dot{\subseteq} \mathbf{u}} \quad \frac{\overline{\varphi} \vdash \mathbf{t} \dot{\subseteq} z, \overline{\varphi}'}{\overline{\varphi} \vdash E(\mathbf{t}), \overline{\varphi}'} \quad \frac{\overline{\varphi}, z \dot{\subseteq} \mathbf{t} \vdash E(z), z \dot{=} \mathbf{t}, \overline{\varphi}'}{\overline{\varphi} \vdash N(\mathbf{t}), E(\mathbf{t}), \overline{\varphi}'} \quad \frac{\overline{\varphi}, N(z), z \dot{\subseteq} \mathbf{t} \vdash z \dot{\subseteq} \mathbf{u}, \overline{\varphi}'}{\overline{\varphi} \vdash \mathbf{t} \dot{\subseteq} \mathbf{u}, \overline{\varphi}'} \quad \frac{}{N(\mathbf{t}), E(\mathbf{t}) \vdash} \\
\frac{}{\vdash \mathbf{t} \dot{\subseteq} \mathbf{t}} \quad \frac{}{\mathbf{t} \dot{\subseteq} \mathbf{u}, \mathbf{u} \dot{\subseteq} \mathbf{v} \vdash \mathbf{t} \dot{\subseteq} \mathbf{v}} \quad \frac{}{\mathbf{t} \dot{\subseteq} \mathbf{u}, \mathbf{u} \dot{\subseteq} \mathbf{t} \vdash \mathbf{t} \dot{=} \mathbf{u}} \quad \frac{}{N(\mathbf{t}), \mathbf{u} \dot{\subseteq} \mathbf{t} \vdash E(\mathbf{u}), \mathbf{u} \dot{=} \mathbf{t}} \quad \frac{}{S(\mathbf{t}, \mathbf{u}), 0(\mathbf{u}) \vdash} \\
\frac{}{N(\mathbf{t}) \vdash \mathbf{t} \dot{\leq} \mathbf{t}} \quad \frac{}{\mathbf{t} \dot{\leq} \mathbf{u}, \mathbf{u} \dot{\leq} \mathbf{v} \vdash \mathbf{t} \dot{\leq} \mathbf{v}} \quad \frac{}{\mathbf{t} \dot{\leq} \mathbf{u}, \mathbf{u} \dot{\leq} \mathbf{t} \vdash \mathbf{t} \dot{=} \mathbf{u}} \quad \frac{}{S(\mathbf{t}, \mathbf{u}) \vdash \mathbf{t} \dot{\leq} \mathbf{u}} \quad \frac{}{0(\mathbf{t}) \vdash N(\mathbf{t})} \\
\frac{\overline{\varphi}, 0(z) \vdash \overline{\varphi}'}{\overline{\varphi} \vdash \overline{\varphi}'} \quad \frac{}{S(\mathbf{u}, \mathbf{v}), \mathbf{t} \dot{\leq} \mathbf{v} \vdash \mathbf{t} \dot{\leq} \mathbf{u}} \quad \frac{}{\mathbf{t} \dot{\leq} \mathbf{u} \vdash N(\mathbf{t})} \quad \frac{}{\mathbf{t} \dot{\leq} \mathbf{u} \vdash N(\mathbf{u})} \quad \frac{}{S(\mathbf{t}, \mathbf{u}) \vdash N(\mathbf{t})} \\
\frac{\overline{\varphi}, S(\mathbf{t}, z) \vdash \overline{\varphi}'}{\overline{\varphi} \vdash \overline{\varphi}'} \quad \frac{}{0(\mathbf{t}), 0(\mathbf{u}) \vdash \mathbf{t} \dot{=} \mathbf{u}} \quad \frac{}{S(\mathbf{t}, \mathbf{u}), S(\mathbf{t}, \mathbf{v}) \vdash \mathbf{u} \dot{=} \mathbf{v}} \quad \frac{}{S(\mathbf{u}, \mathbf{t}), S(\mathbf{v}, \mathbf{t}) \vdash \mathbf{u} \dot{=} \mathbf{v}} \quad \frac{}{S(\mathbf{t}, \mathbf{u}) \vdash N(\mathbf{u})}
\end{array}$$

**Fig. 5.** The Arithmetic Rules of  $\text{MSO}(\mathbf{M})$  and  $\text{LMSO}$  (with terms of sort  $o$  and  $z$  fresh).

## 4.2 The Logic $\text{LMSO}$

The system  $\text{LMSO}$  is  $\text{FS} + (\text{LSIP}) + (\text{DEXP}) + (\text{PEXP})$  extended with Fig. 5 and

$$\begin{array}{c}
\frac{}{\vdash (\forall \overline{X}^o)(\forall x^t) (x \dot{\in} \mathbf{f}_{\mathcal{M}}(\overline{X}) \multimap \delta_{\mathcal{M}}^L(x, \overline{X}))} \quad \frac{}{\vdash ?(\exists X^o)!(\forall x^t) (x \dot{\in} X \multimap \delta^\pm)} \\
\frac{\overline{\varphi}, 0(z) \vdash \varphi^-[z/x], ?\overline{\varphi}' \quad !\overline{\varphi}, S(y, z), !\varphi^-[y/x] \vdash \varphi^-[z/x], ?\overline{\varphi}'}{!\overline{\varphi} \vdash (\forall x^t)\varphi^-, ?\overline{\varphi}'}
\end{array}$$

Let  $\text{LMSO}(\mathfrak{C}) := \text{LMSO} + (\text{LAC})$ . Note that  $\vdash_{\text{MSO}(\mathbf{M})} [\varphi]$  whenever  $\vdash_{\text{LMSO}} \varphi$ . Proposition 6 extends so that similarly as in [24] we have

**Proposition 8.** *If  $\vdash_{\text{MSO}(\mathbf{M})} \varphi$  then  $\vdash_{\text{LMSO}} \varphi^L$ . In part., for a realizable instance of Church's synthesis  $(\forall x^\sigma)(\exists u^\tau)\varphi(u, x)$ , we have  $\vdash_{\text{LMSO}} (\forall x^\sigma)(\exists u^\tau)\varphi^L(u, x)$ .*

Moreover, the soundness of  $(-)^D$  extends to  $\text{LMSO}$ . It follows that  $\text{LMSO}(\mathfrak{C})$  is coherent and proves exactly the realizable  $\forall\exists(-)^L$ -statements.

**Theorem 5 (Soundness).** *Let  $\varphi$  be closed with  $\varphi^D = (\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x)$ . From a proof of  $\varphi$  in  $\text{LMSO}(\mathfrak{C})$  one can extract an eager term  $\mathbf{u}(x)$  such that  $\text{LMSO}$  proves  $(\forall x^\sigma)\varphi_D(\mathbf{u}(x), x)$ .*

**Corollary 2 (Extraction).** *Consider a closed formula  $\varphi := (\forall x^\sigma)(\exists u^\tau)\delta(u, x)$  with  $\delta$  deterministic. From a proof of  $\varphi$  in  $\text{LMSO}(\mathfrak{C})$  one can extract a term  $\mathbf{t}(x)$  such that  $\models (\forall x^\sigma)[\delta(\mathbf{t}(x), x)]$ .*

## 4.3 Completeness of $\text{LMSO}(\mathfrak{C})$

The completeness of  $\text{LMSO}(\mathfrak{C})$  follows from a couple of important facts. First,  $\text{LMSO}(\mathfrak{C})$  proves the elimination of linear double negation, using (via Thm. 3) the same trick as in [24].

**Lemma 1.** *For all  $\text{LMSO}$ -formula  $\varphi$ , we have  $(\varphi \multimap \perp) \multimap \perp \vdash_{\text{LMSO}(\mathfrak{C})} \varphi$ .*

Combining Lemma 1 with (LAC) gives classical linear choice.

**Corollary 3.**  $(\forall f)(\exists x)\varphi(x, (f)x) \vdash_{\text{LMSO}(\mathfrak{C})} (\exists x)(\forall y)\varphi(x, y)$ .

The key to the completeness of  $\text{LMSO}(\mathfrak{C})$  is the following quantifier inversion.

**Lemma 2.**  $(\forall x^\sigma)\varphi(\mathfrak{t}^\tau(x), x) \vdash_{\text{LMSO}(\mathfrak{C})} (\exists u^\tau)(\forall x^\sigma)\varphi(u, x)$ , where  $\mathfrak{t}(x)$  is eager.

Lemma 2 follows (via Cor. 3) from the fixpoints on eager machines (Prop. 3). Fix an eager  $\mathfrak{t}^\tau(x^\sigma)$ . Taking the fixpoint of  $\llbracket (f)\mathfrak{t}(x) \rrbracket : \llbracket \sigma \rrbracket \times \llbracket (\sigma)\tau \rrbracket \rightarrow_{\text{EM}} \llbracket \sigma \rrbracket^{\llbracket (\sigma)\tau \rrbracket}$  gives a term  $\mathfrak{v}^\sigma(f^{(\sigma)\tau})$  such that  $\mathfrak{v}(f) \doteq \text{@}(f, \mathfrak{t}(\mathfrak{v}(f)))$ . Then conclude with

$$\begin{array}{l} (\forall x^\sigma)\varphi(\mathfrak{t}(x), x) \vdash_{\text{LMSO}} \varphi(\mathfrak{t}(\mathfrak{v}(f)), \mathfrak{v}(f)) \\ \vdash_{\text{LMSO}} \varphi(\mathfrak{t}(\mathfrak{v}(f)), \text{@}(f, \mathfrak{t}(\mathfrak{v}(f)))) \\ \vdash_{\text{LMSO}} (\exists u^\tau)\varphi(u, (f)u) \\ \vdash_{\text{LMSO}} (\forall f^{(\sigma)\tau})(\exists u^\tau)\varphi(u, (f)u) \\ \vdash_{\text{LMSO}(\mathfrak{C})} (\exists u^\tau)(\forall x^\sigma)\varphi(u, x) \end{array}$$

Completeness of  $\text{LMSO}(\mathfrak{C})$  then follows via  $(-)^D$ , Proposition 5, completeness of  $\text{MSO}(\mathbf{M})$  and Büchi-Landweber Theorem 1. The idea is to lift a f.s. winning P-strat. in  $\mathcal{G}(\llbracket \varphi_D(u, x) \rrbracket)(u, x)$  to a realizer of  $\varphi^D = (\exists u)(\forall x)\varphi_D(u, x)$  in  $\text{LMSO}(\mathfrak{C})$ .

**Theorem 6 (Completeness of  $\text{LMSO}(\mathfrak{C})$ ).** *For each closed formula  $\varphi$ , either  $\vdash_{\text{LMSO}(\mathfrak{C})} \varphi$  or  $\vdash_{\text{LMSO}(\mathfrak{C})} \varphi \multimap \perp$ .*

## 5 Conclusion

We provided a linear Dialectica-like interpretation of  $\text{LMSO}(\mathfrak{C})$ , a linear variant of  $\text{MSO}$  on  $\omega$ -words based on [24]. Our interpretation is correct and complete w.r.t. Church's synthesis, in the sense that it proves exactly the realizable  $\forall\exists(-)^L$ -statements. We thus obtain a syntactic extraction procedure with correctness proof internalized in  $\text{LMSO}(\mathfrak{C})$ . The system  $\text{LMSO}(\mathfrak{C})$  is moreover complete in the sense that for every closed formula  $\varphi$ , it proves either  $\varphi$  or its linear negation. While completeness for a linear logic necessarily collapse some linear structure, the corresponding axioms (DEXP) and (PEXP) do respect the structural constraints allowing for realizer extraction from proofs. The completeness of  $\text{LMSO}(\mathfrak{C})$  contrasts with that of the classical system  $\text{MSO}(\mathbf{M})$ , since the latter has provable unrealizable  $\forall\exists$ -statements. In particular, proof search in  $\text{LMSO}(\mathfrak{C})$  for  $\forall\exists(-)^L$ -formulae and their negation is correct and complete w.r.t. Church's synthesis. The design of the Dialectica interpretation also clarified the linear structure of  $\text{LMSO}$ , as it allowed us to decompose it starting from a system based on usual intuitionistic linear logic (see e.g. [3] for recent references on the subject).

An outcome of witness extraction for  $\text{LMSO}(\mathfrak{C})$  is the realization of a simple version of the fan rule (in the usual sense of e.g. [15]). We plan to investigate monotone variants of Dialectica for our setting. Thanks to the compactness of  $\Sigma^\omega$ , we expect this to allow extraction of uniform bounds, possibly with translations to stronger constructive logics than  $\text{LMSO}$ .



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## A Proofs of §2 (Preliminaries)

We decompose Prop. 2 as follows.

**Proposition 9 (Prop. 2).**

- (1) If  $F : \Sigma \rightarrow_{\mathbb{E}} \Gamma$  is eager and  $G : \Gamma \rightarrow_{\mathbb{S}} \Delta$  is causal then  $G \circ F : \Sigma \rightarrow_{\mathbb{E}} \Delta$  is eager.
- (2) If  $F : \Sigma \rightarrow_{\mathbb{E}} \Gamma$  is eager and  $G : \Delta \rightarrow_{\mathbb{S}} \Sigma$  is causal then  $F \circ G : \Delta \rightarrow_{\mathbb{E}} \Gamma$  is eager.

*Proof.* Assume that  $F : \Sigma \rightarrow_{\mathbb{E}} \Gamma$  is induced by  $f : \Sigma^* \rightarrow \Gamma$ .

- (a) Assume that  $G : \Gamma \rightarrow_{\mathbb{S}} \Delta$  is induced by  $g : \Gamma^+ \rightarrow \Delta$ . Then

$$(F \circ G)(B)(n) = F(G(B))(n) = f(G(B)(0) \cdot \dots \cdot G(B)(n-1)) = f(g(B(0)) \cdot \dots \cdot g(B \upharpoonright n))$$

So  $(F \circ G)(n)$  only depends on  $B(0), \dots, B(n-1)$ .

- (b) Assume that  $G : \Delta \rightarrow_{\mathbb{S}} \Sigma$  is induced by  $g : \Delta^* \rightarrow \Gamma$ . Then

$$(G \circ F)(B)(n) = G(F(B))(n) = g(F(B)(0) \cdot \dots \cdot F(B)(n)) = g(f(\varepsilon) \cdot \dots \cdot f(B \upharpoonright n))$$

So  $(G \circ F)(n)$  only depends on  $B(0), \dots, B(n-1)$ .

### A.1 Proof of Proposition 3 (Fixpoints for Eager Functions)

We split Proposition 3 into two statements.

**Proposition 10.** For each  $F : \Sigma \times \Gamma \rightarrow_{\mathbb{E}} \Sigma^{\Gamma}$  there is a  $\text{fix}(F) : \Gamma \rightarrow_{\mathbb{E}} \Sigma^{\Gamma}$  such that

$$\text{fix}(F)(C) = F(\mathbf{e}(\text{fix}(F))(C), C) \quad (\text{for all } C \in \Gamma^{\omega})$$

Consider an eager function

$$F : \Sigma \times \Gamma \longrightarrow_{\mathbb{E}} \Sigma^{\Gamma}$$

We are going to define a causal function

$$\text{fix}(F) : \Gamma \longrightarrow_{\mathbb{E}} \Sigma^{\Gamma}$$

such that for all  $C \in \Gamma^{\omega}$ ,

$$\text{fix}(F)(C) = F(\mathbf{e}(\text{fix}(F))(C), C) = F(@(\text{fix}(F)(C), C), C) \in (\Sigma^{\Gamma})^{\omega}$$

Intuitively,  $\text{fix}(F)(C)$  is the fixpoint of the map  $(B \mapsto F(B, C))$ . Assume first that  $F$  is induced by

$$\tilde{f} : \bigcup_{n \in \mathbb{N}} (\Sigma^n \times \Gamma^{n+1}) \longrightarrow \Sigma \simeq (\Sigma \times \Gamma)^* \longrightarrow \Sigma^{\Gamma}$$

and let us look at how

$$\text{fix}(F) : \Gamma \longrightarrow_{\mathbb{M}} \Sigma \simeq \Gamma \longrightarrow_{\mathbb{E}} \Sigma^{\Gamma}$$

can be induced induced by a function  $\tilde{h} : \Gamma^+ \rightarrow \Sigma$  defined from  $\tilde{f}$ . We should have

$$\begin{aligned} \tilde{h}(\mathbf{b}_1) &= \tilde{f}(\varepsilon, \mathbf{b}_1) = \mathbf{a}_1 \\ \tilde{h}(\mathbf{b}_1\mathbf{b}_2) &= \tilde{f}(\tilde{f}(\mathbf{b}_1, \varepsilon), \mathbf{b}_1\mathbf{b}_2) = \tilde{f}(\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2) = \mathbf{a}_2 \\ \tilde{h}(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) &= \tilde{f}(\tilde{f}(\mathbf{b}_1, \varepsilon) \cdot \tilde{f}(\tilde{f}(\varepsilon, \mathbf{b}_1), \mathbf{b}_1\mathbf{b}_2), \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = \tilde{f}(\mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = \mathbf{a}_3 \\ &\vdots \\ \tilde{h}(\mathbf{b}_1 \dots \mathbf{b}_n \mathbf{b}_{n+1}) &= \tilde{f}(\mathbf{a}_1 \dots \mathbf{a}_n, \mathbf{b}_1 \dots \mathbf{b}_n \mathbf{b}_{n+1}) = \mathbf{a}_{n+1} \end{aligned}$$

That is, for  $n > 0$ ,  $\tilde{h}(\mathbf{b}_1 \dots \mathbf{b}_n) = \mathbf{a}_n$ , where the  $\mathbf{a}_k$ 's are given by the recurrence:

$$\mathbf{a}_1 := \tilde{f}(\varepsilon, \mathbf{b}_1) \quad \text{and} \quad \mathbf{a}_{k+1} := \tilde{f}(\mathbf{a}_1 \dots \mathbf{a}_k, \mathbf{b}_1 \dots \mathbf{b}_k \mathbf{b}_{k+1})$$

In terms of

$$f : (\Sigma \times \Gamma)^* \longrightarrow \Sigma^{\Gamma} \quad \text{such that} \quad F(B, C)(n) = f(B \upharpoonright n, C \upharpoonright n)$$

this amounts to define

$$h : \Gamma^* \longrightarrow \Sigma^{\Gamma} \quad \text{such that} \quad \text{fix}(F)(C)(n) = h(C \upharpoonright n)$$

as follows:

$$\begin{aligned} h(\varepsilon) &:= f(\varepsilon) = \mathbf{g}_1 \in \Sigma^{\Gamma} \\ h(\mathbf{b}_1) &:= f(\mathbf{a}_1, \mathbf{b}_1) = \mathbf{g}_2 \in \Sigma^{\Gamma} \\ h(\mathbf{b}_1\mathbf{b}_2) &:= f(\mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2) = \mathbf{g}_3 \in \Sigma^{\Gamma} \\ &\vdots \\ h(\mathbf{b}_1 \dots \mathbf{b}_n) &:= f(\mathbf{a}_1 \dots \mathbf{a}_n, \mathbf{b}_1 \dots \mathbf{b}_n) = \mathbf{g}_{n+1} \in \Sigma^{\Gamma} \end{aligned}$$

where  $\mathbf{a}_k = \mathbf{g}_k(\mathbf{b}_k)$ , that is

$$\mathbf{a}_1 := @ (f(\varepsilon), \mathbf{b}_1) \quad \text{and} \quad \mathbf{a}_{k+1} := @ (f(\mathbf{a}_1 \dots \mathbf{a}_k, \mathbf{b}_1 \dots \mathbf{b}_k), \mathbf{b}_{k+1})$$

This easily gives an eager machine for  $\text{fix}(F)$  given an eager machine for  $F$ .

**Lemma 3.** *Consider a f.s. eager function*

$$F : \Sigma \times \Gamma \longrightarrow_{\text{EM}} \Sigma^{\Gamma}$$

*induced by an eager machine*

$$(\mathcal{E} : \Sigma \times \Gamma \rightarrow \Sigma^{\Gamma}) = (Q, q^i, \partial, \lambda)$$

*Then*

$$\text{fix}(F) : \Gamma \longrightarrow_{\text{EM}} \Sigma^{\Gamma}$$

is induced by the eager machine

$$(\mathcal{H} : \Gamma \rightarrow \Sigma^\Gamma) = (Q, q^i, \partial_{\mathcal{H}}, \lambda)$$

where

$$\begin{aligned} \partial_{\mathcal{H}} : Q \times \Gamma &\longrightarrow Q \\ (q, \mathbf{b}) &\longmapsto \partial(q, (@(\lambda(q), \mathbf{b}), \mathbf{b})) \end{aligned}$$

Proposition 10 will mostly be used in the following context. Consider a Mealy term  $\mathbf{u}(z, y)$  of sort  $(\varsigma, \sigma; \tau)$  and an eager term  $\mathbf{t}(x, y)$  of sort  $(\tau, \sigma; \varsigma)$ . Intuitively, the term  $\mathbf{u}(\mathbf{t}(x, y), y)$  is “eager in  $x$ ”, in the sense that it can be seen as an eager Moore function

$$[[\tau]] \times [[\sigma]] \longrightarrow_{\text{EM}} [[\tau]]^{[[\sigma]]}$$

**Lemma 4.** *Given*

$$\begin{aligned} F &: \Sigma \times \Gamma \longrightarrow_{\mathbb{E}} \Theta \\ \text{and } G &: \Theta \times \Sigma \times \Gamma \longrightarrow_{\mathbb{E}} (\Delta^\Gamma)^\Theta \end{aligned}$$

there is an eager  $H : \Sigma \times \Gamma \rightarrow_{\mathbb{E}} \Delta^\Gamma$  such that for all  $B \in \Sigma^\omega$  and all  $C \in \Gamma^\omega$ , we have

$$@\!(H(B, C), C) = \overline{\text{@}}\!(G(F(B, C), B, C), F(B, C), C)$$

Moreover  $H$  is finite-state whenever  $F$  and  $G$  are finite-state.

*Proof.* Assume that  $F$  and  $G$  are induced respectively by

$$\begin{aligned} f &: (\Sigma \times \Gamma)^* \longrightarrow \Theta \\ g &: (\Theta \times \Sigma \times \Gamma)^* \longrightarrow (\Delta^\Gamma)^\Theta \end{aligned}$$

Then for all  $B \in \Sigma^\omega$ , all  $C \in \Gamma^\omega$  and all  $n \in \mathbb{N}$  we have

$$\begin{aligned} \overline{\text{@}}\!(G(F(B, C), B, C), F(B, C), C)(n) &= \overline{\text{@}}\!(G(F(B, C)(n), B(n), C(n))) \\ &= \overline{\text{@}}\!(g(F(B, C)\upharpoonright n, B\upharpoonright n, C\upharpoonright n), f(B\upharpoonright n, C\upharpoonright n), C(n)) \end{aligned}$$

Let  $h : (\Sigma \times \Gamma)^* \rightarrow \Delta^\Gamma$  take  $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in (\Sigma \times \Gamma)^*$  to

$$\left( \mathbf{b} \longmapsto \overline{\text{@}}\!(g(f(\varepsilon) \cdots f(\bar{\mathbf{a}}, \bar{\mathbf{b}}), \bar{\mathbf{a}}, \bar{\mathbf{b}}), f(\bar{\mathbf{a}}, \bar{\mathbf{b}}), \mathbf{b}) \right) \in \Delta^\Gamma$$

and let  $H : \Sigma \times \Gamma \rightarrow_{\mathbb{E}} \Delta^\Gamma$  be the eager function induced by  $h$ . Then we are done since

$$@\!(H(B, C), C)(n) = @\!(h(B\upharpoonright n, C\upharpoonright n), C(n))$$

□

In particular, given  $F$  as in Lemma 4 and given  $G : \Theta \times \Gamma \rightarrow_{\mathbb{S}} \Delta$ , since  $\mathbf{AG} : \Theta \times \Gamma \rightarrow_{\mathbb{E}} \Delta^{\Gamma^\Theta}$  we obtain  $H$  such that

$$@\!(H(B, C), C) = \overline{\text{@}}\!(\mathbf{AG}(F(B, C), C), F(B, C), C) = G(F(B, C), C)$$

**Corollary 4.** *If  $G : \Theta \times \Gamma \rightarrow_{\mathbb{S}} \Delta$  is causal and  $F : \Sigma \times \Gamma \rightarrow_{\mathbb{E}} \Theta$  is eager, then there is an eager  $H : \Sigma \times \Gamma \rightarrow_{\mathbb{E}} \Delta^{\Gamma}$  such that for all  $B \in \Sigma^{\omega}$  and all  $C \in \Gamma^{\omega}$ , we have*

$$\textcircled{\text{@}}(H(B, C), C) = G(F(B, C), C)$$

Moreover,  $H$  is finite-state whenever so are  $F$  and  $G$ .

Returning to  $\mathbf{u}(z, y)$  and  $\mathbf{t}(x, y)$ , consider  $H$  defined from  $G := \llbracket \mathbf{u} \rrbracket$  and  $F := \llbracket \mathbf{t} \rrbracket$  as in Corollary 4. Note that  $H$  is finite state since so are  $\llbracket \mathbf{u} \rrbracket$  and  $\llbracket \mathbf{t} \rrbracket$ . By applying Proposition 10 to

$$H : \Sigma \times \Gamma \rightarrow_{\text{EM}} \Sigma^{\Gamma}$$

we thus obtain a fixpoint of  $(x \mapsto \mathbf{u}(\mathbf{t}(x, y), y))$  since

$$\text{fix}(H) : \Gamma \rightarrow_{\text{EM}} \Sigma^{\Gamma}$$

is such that

$$\begin{aligned} \llbracket \mathbf{u} \rrbracket \left( \llbracket \mathbf{t} \rrbracket (\mathbf{e}(\text{fix}(H))(C), C), C \right) &= \mathbf{e}(H)(\mathbf{e}(\text{fix}(H))(C), C) \\ &= \mathbf{e}(\text{fix}(H))(C) \end{aligned}$$

## B Proofs of §3 (A Monadic Linear Dialectica-like Interpretation)

### B.1 Soundness of Dialectica (Theorem 2)

**Notation 7.** *Consider formulae  $\varphi(a)$  and  $\psi(a)$  with free variable  $a^v$  and let*

$$\varphi^D(a) = (\exists u^{\tau})(\forall x^{\sigma})\varphi_D(u, x, a) \quad \text{and} \quad \psi^D(a) = (\exists v^{\kappa})(\forall y^{\varsigma})\psi_D(v, y, a)$$

We write

$$\varphi(a) \dashv\vdash_{\text{Ax}} \psi(a)$$

if there are eager terms

$$\begin{aligned} &\mathbf{v}(u, y, a) \quad \text{of sort} \quad (\tau, \varsigma, v; (\kappa)v\tau) \\ \text{and} \quad &\mathbf{x}(u, y, a) \quad \text{of sort} \quad (\tau, \varsigma, v; (\sigma)v\varsigma\tau) \end{aligned}$$

such that

$$\varphi_D(u, \textcircled{\text{@}}(\mathbf{x}(u, y, a), u, y, a), a) \vdash_{\text{FS+Ax}} \psi_D(\textcircled{\text{@}}(\mathbf{v}(u, y, a), u, a), y, a)$$

In particular,  $\varphi(a) \dashv\vdash \psi(a)$  stands for  $\dashv\vdash$  w.r.t. the system FS without further axioms.

We will prove Theorem 2 with the following inductive invariant. Let

$$\textcircled{\text{@}}_{1 \leq i \leq 0} \varphi_i(a) := \mathbf{I} \quad \text{and} \quad \textcircled{\text{@}}_{1 \leq j \leq 0} \psi_j(a) := \perp$$

**Theorem 8.** Assume given  $\overline{\varphi}(a) = \varphi_1(a), \dots, \varphi_n(a)$  and  $\overline{\psi}(a) = \psi_1(a), \dots, \psi_m(a)$  with free variables among  $a$ . Then we have

$$\varphi_1(a), \dots, \varphi_n(a) \vdash_{\text{FS}+\text{Ax}} \psi_1(a), \dots, \psi_m(a) \implies (\otimes_i \varphi_i(a)) \dashv\vdash_{\text{Ax}} (\wp_j \psi_j(a))$$

*Remark 1.* Note that

$$\varphi(a) \dashv\vdash_{\text{Ax}} \psi(a) \iff \varphi'(a) \dashv\vdash_{\text{Ax}} \psi'(a)$$

whenever

$$\left( (\forall a)(\varphi(a) \multimap \psi(a)) \right)^D = \left( (\forall a)(\varphi'(a) \multimap \psi'(a)) \right)^D$$

We first list some (expected) structure of  $\dashv\vdash_{\text{Ax}}$  and then give the proof of Theorem 8.

### Basic Structure.

**Lemma 5.**  $\varphi(a) \dashv\vdash_{\text{Ax}} \varphi(a)$ .

*Proof.* We have

$$\varphi^D(a^v) = (\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x, a)$$

and we have to provide eager terms

$$\begin{array}{ll} \mathbf{u}(u, x, a) & \text{of sort } (\tau, \sigma, v; (\tau)v\tau) \\ \text{and } \mathbf{x}(u, y, a) & \text{of sort } (\tau, \sigma, v; (\sigma)v\sigma\tau) \end{array}$$

such that

$$\varphi_D(u, \overline{\mathbb{Q}}(\mathbf{x}(u, x, a), u, x, a), a) \vdash_{\text{FS}+\text{Ax}} \varphi_D(\overline{\mathbb{Q}}(\mathbf{u}(u, x, a), u, a), x, a)$$

For  $\mathbf{x}(u, x, a)$ , consider the  $\mathbb{M}$ -projection

$$[\pi] : [\tau] \times [\sigma] \times [v] \longrightarrow_{\mathbb{M}} [\sigma]$$

We obtain  $\mathbf{x}(u, x, a)$  by composing the eager term for

$$\mathbf{A}([\pi]) : [\tau] \times [\sigma] \times [v] \longrightarrow_{\text{EM}} [\sigma]^{[\tau] \times [\sigma] \times [v]}$$

with the  $\mathbb{M}$ -iso

$$[\sigma]^{[\tau] \times [\sigma] \times [v]} \simeq (([\sigma]^{[v]})^{[\sigma]})^{[\tau]}$$

For  $\mathbf{u}(u, x, a)$ , we take the eager term obtained by composing the eager function

$$[\tau] \times [v] \xrightarrow{\mathbf{A}([\pi])}_{\text{EM}} [\sigma]^{[\tau] \times [v]} \xrightarrow{\simeq}_{\mathbb{M}} (([\sigma]^{[v]})^{[\tau]})$$

(where  $\pi$  is a suitable projection) with the Mealy projection

$$[\tau] \times [\sigma] \times [v] \longrightarrow_{\mathbb{M}} [\tau] \times [v]$$

Then we are done since FS proves

$$\vdash \overline{\mathbb{Q}}(\mathbf{u}(u, x, a), u, a) \doteq_{\sigma} u \quad \text{and} \quad \vdash \overline{\mathbb{Q}}(\mathbf{x}(u, x, a), u, x, a) \doteq_{\sigma} x$$

□

**Proposition 11.**

$$\varphi_0(a) \dashrightarrow_{\text{Ax}} \varphi_1(a) \implies \varphi_1(a) \dashrightarrow_{\text{Ax}} \varphi_2(a) \implies \varphi_0(a) \dashrightarrow_{\text{Ax}} \varphi_2(a)$$

The proof of Proposition 11 relies on the fixpoints of (finite-state) eager functions given by Proposition 3 (see also §A.1). For legibility reasons, we may refrain from writing free variables of terms explicitly and manipulate explicit substitutions. In such a case, given a term  $\mathfrak{t}(\dots, x, \dots)$  with free variable  $x$  and a term  $\mathfrak{u}$  of the appropriate sort, we write  $\mathfrak{t}[\mathfrak{u}/x]$  for the substitution of  $x$  by  $\mathfrak{u}$  in  $\mathfrak{t}$ . Let

$$\begin{aligned} \varphi_0^D(a^v) &= (\exists u_0^{\tau_0})(\forall x_0^{\sigma_0})(\varphi_0)_D(u_0, x_0, a) \\ \varphi_1^D(a^v) &= (\exists u_1^{\tau_1})(\forall x_1^{\sigma_1})(\varphi_1)_D(u_1, x_1, a) \\ \varphi_2^D(a^v) &= (\exists u_2^{\tau_2})(\forall x_2^{\sigma_2})(\varphi_2)_D(u_2, x_2, a) \end{aligned}$$

By assumption, there are eager terms

$$\begin{array}{ll} \mathfrak{u}_1(u_0, x_1, a) & \text{of sort } (\tau_0, \sigma_1, v; (\tau_1)\nu\tau_0) \\ \mathfrak{x}_0(u_0, x_1, a) & \text{of sort } (\tau_0, \sigma_1, v; (\sigma_0)\nu\sigma_1\tau_0) \\ \mathfrak{u}_2(u_1, x_2, a) & \text{of sort } (\tau_1, \sigma_2, v; (\tau_2)\nu\tau_1) \\ \text{and } \mathfrak{x}_1(u_1, x_2, a) & \text{of sort } (\tau_1, \sigma_2, v; (\sigma_1)\nu\sigma_2\tau_1) \end{array}$$

such that

$$\begin{array}{ll} (\varphi_0)_D(u_0, (\mathfrak{x}_0)u_0x_1a, a) & \vdash_{\text{FS+Ax}} (\varphi_1)_D((\mathfrak{u}_1)u_0a, x_1, a) \\ \text{and } (\varphi_1)_D(u_1, (\mathfrak{x}_1)u_1x_2a, a) & \vdash_{\text{FS+Ax}} (\varphi_2)_D((\mathfrak{u}_2)u_1a, x_2, a) \end{array}$$

From this data, our goal is to produce eager terms

$$\begin{array}{ll} \mathfrak{v}(u_0, x_2, a) & \text{of sort } (\tau_0, \sigma_2, v; (\tau_2)\nu\tau_0) \\ \text{and } \mathfrak{y}(u_0, x_2, a) & \text{of sort } (\tau_0, \sigma_2, v; (\sigma_0)\nu\sigma_2\tau_0) \end{array}$$

such that

$$(\varphi_0)_D(u_0, (\mathfrak{y})u_0x_2a, a) \vdash_{\text{FS+Ax}} (\varphi_2)_D((\mathfrak{v})u_0a, x_2, a)$$

We would like  $\mathfrak{v}$  and  $\mathfrak{y}$  to satisfy the following equations in FS:

$$\begin{array}{ll} (\mathfrak{y})u_0x_2a \doteq (\mathfrak{x}_0)u_0x_1a & \text{where } u_1 := (\mathfrak{u}_1)u_0a \\ (\mathfrak{v})u_0a \doteq (\mathfrak{u}_2)u_1a & \quad \quad \quad x_1 := (\mathfrak{x}_1)u_1x_2a \end{array}$$

But the variables  $x_1^{\sigma_1}$  and  $u_1^{\tau_1}$ , which are free in  $\mathfrak{u}_1$  and  $\mathfrak{x}_1$ , should not occur in the terms  $\mathfrak{y}, \mathfrak{v}$ . We are thus lead to solve the following equations in FS

$$\begin{array}{ll} (\mathfrak{y})u_0x_2a \doteq (\mathfrak{x}_0[\mathfrak{y}_1/x_1])u_0\mathfrak{y}_1a & \text{where } \mathfrak{v}_1 \doteq (\mathfrak{u}_1[\mathfrak{y}_1/x_1])u_0a \\ (\mathfrak{v})u_0a \doteq (\mathfrak{u}_2[\mathfrak{v}_1/u_1])\mathfrak{v}_1a & \quad \quad \quad \mathfrak{y}_1 \doteq (\mathfrak{x}_1[\mathfrak{v}_1/u_1])\mathfrak{v}_1x_2a \end{array} \quad (11)$$

with terms

$$\begin{array}{ll} \mathfrak{y}_1(u_0, x_2, a) & \text{of sort } (\tau_0, \sigma_2, v; \sigma_1) \\ \mathfrak{v}_1(u_0, x_2, a) & \text{of sort } (\tau_0, \sigma_2, v; \tau_1) \end{array}$$



Assuming (11) satisfied, we are done since

$$\begin{array}{lcl}
(\varphi_0)_D(u_0, (\mathbf{y})u_0x_2a, a) & \vdash_{\text{FS}} & (\varphi_0)_D(u_0, (\mathbf{x}_0[\mathbf{y}_1/x_1])u_0\mathbf{y}_1a, a) \\
& \vdash_{\text{FS+Ax}} & (\varphi_1)_D(\mathbf{u}_1[\mathbf{y}_1/x_1]u_0a, \mathbf{y}_1, a) \\
& \vdash_{\text{FS}} & (\varphi_1)_D(\mathbf{v}_1, (\mathbf{x}_1[\mathbf{v}_1/u_1])\mathbf{v}_1x_2a, a) \\
& \vdash_{\text{FS+Ax}} & (\varphi_2)_D(\mathbf{u}_2\mathbf{v}_1a, x_2, a) \\
& \vdash_{\text{FS}} & (\varphi_2)_D(\mathbf{v})u_0a, x_2, a)
\end{array}$$

We now turn to the resolution of (11). We first discuss the construction of  $\mathbf{y}_1$  and  $\mathbf{v}_1$  and then turn to  $\mathbf{y}$  and  $\mathbf{v}$ .

*Definition of  $\mathbf{y}_1(u_0, x_2, a)$  and  $\mathbf{v}_1(u_0, x_2, a)$ .* Note that if  $\mathbf{y}_1$  satisfies

$$\vdash_{\text{FS}} \mathbf{y}_1 \doteq (\mathbf{x}_1[(\mathbf{u}_1[\mathbf{y}_1/x_1])u_0a/u_1])((\mathbf{u}_1[\mathbf{y}_1/x_1])u_0a)x_2a \quad (12)$$

then  $\mathbf{a}$  by taking  $\mathbf{v}_1 := (\mathbf{u}_1[\mathbf{y}_1/x_1])u_0a$  one obtains a term satisfying the corresponding equation in (11). The Mealy term

$$\mathfrak{t}(u, u_0, x_2, a) := (\mathbf{x}_1[(u)u_0a/u_1])((u)u_0a)x_2a$$

induces a finite-state causal function

$$G : \llbracket (\tau_1)v\tau_0 \rrbracket \times (\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket) \longrightarrow_{\mathbb{E}} \llbracket \sigma_1 \rrbracket$$

while the eager term  $\mathbf{u}_1(u_0, x_1, a)$  induces (via Prop. 9) a f.s. eager function

$$F : \llbracket \sigma_1 \rrbracket \times (\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket) \longrightarrow_{\text{EM}} \llbracket (\tau_1)v\tau_0 \rrbracket$$

By Corollary 4, there is a (f.s.) eager function

$$H : \llbracket \sigma_1 \rrbracket \times (\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket) \longrightarrow_{\text{EM}} \llbracket \sigma_1 \rrbracket^{\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket}$$

such that, for all  $X_1 \in \llbracket \sigma_1 \rrbracket^\omega$ ,  $U_0 \in \llbracket \tau_0 \rrbracket^\omega$ ,  $X_2 \in \llbracket \sigma_2 \rrbracket^\omega$  and  $B \in \llbracket v \rrbracket^\omega$ ,

$$\mathbf{e}(H)(X_1, U_0, X_2, B) = G(F(X_1, U_0, X_2, B), U_0, X_2, B)$$

Take the fixpoint

$$\text{fix}(H) : \llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket \longrightarrow_{\text{EM}} \llbracket \sigma_1 \rrbracket^{\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket}$$

of  $H$  given by Proposition 10. We thus get

$$\text{fix}(H)(U_0, X_2, B) = H(\mathbf{e}(\text{fix}(H))(U_0, X_2, B), U_0, X_2, B)$$

Note that  $\text{fix}(H)$  is finite-sate by Lemma 3. Let  $\mathbf{y}_1$  be the Mealy term of sort  $(\tau_0, \sigma_2, v; \sigma_1)$  for  $\mathbf{e}(\text{fix}(H))(-, -, -)$ . It follows that  $\mathbf{y}_1$  satisfies (12) since

$$\vdash_{\text{FS}} \mathbf{y}_1(u_0, x_2, a) \doteq \mathfrak{t}(\mathbf{u}_1(u_0, \mathbf{y}_1(u_0, x_2, a), a), u_0, x_2, a)$$

*Definition of  $y(u_0, x_2, a)$ .* We have to provide a term

$$y(u_0, x_2, a) \quad (\tau_0, \sigma_2, v; (\sigma_0)v\sigma_2\tau_0) \quad \text{of sort}$$

First, since  $x_0$  is eager it follows from Prop. 9 that the term

$$x_0(u_0, y_1(u_0, x_2, a), a)$$

is eager. Let

$$F_0 := \llbracket x_0(u_0, y_1(u_0, x_2, a), a) \rrbracket : \llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket \longrightarrow_{\text{EM}} \llbracket (\sigma_0)v\sigma_2\tau_0 \rrbracket$$

and

$$G_0 := \llbracket (f)u_0y_1a \rrbracket : \llbracket (\sigma_0)v\sigma_2\tau_0 \rrbracket \times (\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket) \longrightarrow_{\text{M}} \llbracket \sigma_0 \rrbracket$$

By Corollary 4 there is a finite-state eager function

$$H_0 : \llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket \longrightarrow_{\text{EM}} \llbracket \sigma_0 \rrbracket^{\llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket}$$

such that for  $U_0 \in \llbracket \tau_0 \rrbracket^\omega$ ,  $X_2 \in \llbracket \sigma_2 \rrbracket^\omega$  and  $B \in \llbracket v \rrbracket^\omega$ ,

$$e(H_0)(U_0, X_2, B) = G_0(F_0(U_0, X_2, B), U_0, X_2, B)$$

Letting  $y(u_0, x_2, a)$  be the term for  $e(H_0)$ , we thus get

$$\vdash_{\text{FS}} (y)u_0x_2a \doteq (x_0[y_1/x_1])u_0y_1a$$

*Definition of  $v(u_0, x_2, a)$ .* Our goal is to find a finite-state eager

$$V : \llbracket \tau_0 \rrbracket \times \llbracket \sigma_2 \rrbracket \times \llbracket v \rrbracket \longrightarrow_{\text{EM}} (\llbracket \tau_2 \rrbracket^{\llbracket v \rrbracket})^{\llbracket \tau_0 \rrbracket}$$

with term  $v(u_0, x_2, a)$  such that

$$\overline{\text{Q}}(v(u_0, x_2, a), u_0, a) = \overline{\text{Q}}\left(\mathbf{u}_2(v_1(u_0, x_2, a), x_2, a), v_1(u_0, x_2, a), a\right)$$

Let  $\Sigma := \llbracket \sigma_2 \rrbracket$ ,  $\Gamma := \llbracket \tau_0 \rrbracket \times \llbracket v \rrbracket$  and  $\Theta := \llbracket \tau_1 \rrbracket$ . We apply Lemma 4 to

$$\begin{aligned} F &:= \llbracket v_1(u_0, x_2, a) \rrbracket : \Sigma \times \Gamma \longrightarrow_{\text{EM}} \Theta \\ \text{and } G &:= \llbracket \mathbf{u}_2(u_1, x_2, a) \rrbracket : \Theta \times \Sigma \times \Gamma \longrightarrow_{\text{EM}} (\Delta^\Gamma)^\Theta \end{aligned}$$

Then we are done since we obtain obtain a finite-state eager

$$V : \Sigma \times \Gamma \longrightarrow_{\text{EM}} \Delta^\Gamma$$

with term  $v(u_0, x_2, a)$  such that

$$\vdash_{\text{FS}} \overline{\text{Q}}(v(u_0, x_2, a), u_0, a) \doteq \overline{\text{Q}}\left(\mathbf{u}_2(v_1(u_0, x_2, a), x_2, a), v_1(u_0, x_2, a), a\right)$$

This concludes the proof of Proposition 11.  $\square$

## Monoidal Structure.

### Lemma 6.

- (1) If  $\varphi(a) \multimap_{\mathbf{Ax}} \varphi'(a)$  and  $\psi(a) \multimap_{\mathbf{Ax}} \psi'(a)$ , then  $\varphi(a) \otimes \psi(a) \multimap_{\mathbf{Ax}} \varphi'(a) \otimes \psi'(a)$ .
- (2) If  $\varphi(a) \multimap_{\mathbf{Ax}} \varphi'(a)$  and  $\psi(a) \multimap_{\mathbf{Ax}} \psi'(a)$ , then  $\varphi(a) \wp \psi(a) \multimap_{\mathbf{Ax}} \varphi'(a) \wp \psi'(a)$ .
- (3)  $\varphi(a) \otimes \psi(a) \multimap_{\mathbf{Ax}} \psi(a) \otimes \varphi(a)$ .
- (4)  $\varphi(a) \wp \psi(a) \multimap_{\mathbf{Ax}} \psi(a) \wp \varphi(a)$ .
- (5)  $\varphi(a) \otimes \mathbf{I} \multimap_{\mathbf{Ax}} \varphi(a)$ .
- (6)  $\varphi(a) \multimap_{\mathbf{Ax}} \perp \wp \varphi(a)$ .
- (7)  $(\varphi(a) \wp \psi(a)) \otimes \gamma(a) \multimap_{\mathbf{Ax}} \varphi(a) \wp (\psi(a) \otimes \gamma(a))$

## Exponential Structure.

### Lemma 7.

- (1)  $!\varphi(a) \multimap_{\mathbf{Ax}} \varphi(a)$ .
- (2)  $!\varphi(a) \multimap_{\mathbf{Ax}} \mathbf{I}$ .
- (3)  $!\varphi(a) \multimap_{\mathbf{Ax}} !\varphi(a) \otimes !\varphi(a)$ .
- (4)  $\varphi(a) \multimap_{\mathbf{Ax}} ?\varphi(a)$ .
- (5)  $\perp \multimap_{\mathbf{Ax}} ?\varphi(a)$ .
- (6)  $?\varphi(a) \wp ?\varphi(a) \multimap_{\mathbf{Ax}} ?\varphi(a)$ .

*Proof.* Let

$$\varphi^D(a^v) = (\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x, a)$$

The cases of  $!(\varphi^-)$  and  $?( \varphi^+)$  are easy. The cases of  $!(\varphi^+)$  and  $?( \varphi^-)$  amount to the corresponding rules in PF and follow by taking terms similar to those of Lemma 5. We only detail some cases. We first consider cases of  $!(\varphi^-)$  and  $?( \varphi^+)$ .

(1) We have

$$\begin{aligned} \left( (\forall a)(!\varphi^-(a) \multimap \varphi^-(a)) \right)^D &= \\ & (\forall x_1)(\forall a)(!(\forall x_0)\varphi_D(-, x_0, a) \multimap \varphi_D(-, x_1, a)) \end{aligned}$$

and the result follows from

$$\frac{\frac{\varphi_D(-, x_1, a) \vdash \varphi_D(-, x_1, a)}{(\forall x_0)\varphi_D(-, x_0, a) \vdash \varphi_D(-, x_1, a)}}{!(\forall x_0)\varphi_D(-, x_0, a) \vdash \varphi_D(-, x_1, a)}$$

(2) We have

$$\left( (\forall a)(!\varphi^-(a) \multimap \mathbf{I}) \right)^D = (\forall a)(!(\forall x)\varphi_D(-, x, a) \multimap \mathbf{I})$$

and the result follows from the that FS proves  $!\psi \multimap \mathbf{I}$  for all  $\psi$ .

(3) We have

$$\begin{aligned} \left( (\forall a)(! \varphi^-(a) \multimap ! \varphi^-(a) \otimes ! \varphi^-(a)) \right)^D &= \\ (\forall a)(!(\forall x)\varphi_D(-, x, a) \multimap !(\forall x)\varphi_D(-, x, a) \otimes !(\forall x)\varphi_D(-, x, a)) & \end{aligned}$$

and the result follows from the that FS proves  $! \psi \multimap ! \psi \otimes ! \psi$  for all  $\psi$ .

We now turn to cases of  $!(\varphi^+)$  and  $?( \varphi^-)$ .

(1) We have

$$\begin{aligned} \left( (\forall a^v)(! \varphi^+(a) \multimap \varphi^+(a)) \right)^D &= \\ (\exists u_1^{(\tau)v\tau})(\forall u_0^\tau)(\forall a^v)(! \varphi_D(u_0, -, a) \multimap \varphi_D((u_1)u_0a, -, a)) & \end{aligned}$$

and we conclude as in Lemma 5, using that FS proves  $! \psi \multimap \psi$  for all  $\psi$ .

(2) We have

$$\left( (\forall a)(! \varphi^+(a) \multimap \mathbf{I}) \right)^D = (\forall u)(\forall a)(! \varphi_D(u, -, a) \multimap \mathbf{I})$$

and we conclude from the fact that FS proves  $! \psi \multimap \mathbf{I}$  for all  $\psi$ .

(3) We have

$$\begin{aligned} \left( (\forall a)(! \varphi^+(a) \multimap ! \varphi^+(a) \otimes ! \varphi^+(a)) \right)^D &= \\ (\exists u_1)(\exists u_2)(\forall u_0)(\forall a)(! \varphi_D(u_0, -, a) \multimap ! \varphi_D((u_1)u_0a, -, a) \otimes ! \varphi_D((u_2)u_0a, -, a)) & \end{aligned}$$

and we conclude as in Lemma 5, using that FS proves  $! \psi \multimap ! \psi \otimes ! \psi$  for all  $\psi$ .  $\square$

**Lemma 8.**

(1) If  $! \gamma(a) \multimap_{\text{Ax}} \varphi(a) \wp ? \psi(a)$  then  $! \gamma(a) \multimap_{\text{Ax}} ! \varphi(a) \wp ? \psi(a)$ .

(2) If  $! \gamma(a) \otimes \varphi(a) \multimap_{\text{Ax}} ? \psi(a)$  then  $! \gamma(a) \otimes ? \varphi(a) \multimap_{\text{Ax}} ? \psi(a)$ .

*Proof.* First, note that  $! \gamma(a)$  is positive and that  $? \psi(a)$  is negative. We thus have

$$(! \gamma)^D(a) = (\exists w)(! \gamma)_D(w, -, a) \quad \text{and} \quad (? \psi)^D(a) = (\forall y)(? \psi)_D(-, y, a)$$

where  $(! \gamma)_D$  (resp.  $(? \psi)_D$ ) is an  $!$ -formula (resp. a  $?$ -formula).

(1) Consider first the case of  $\varphi$  negative. We have

$$(! \varphi)^D(a) = (! \varphi)_D(-, -, a) = !(\forall x)\varphi_D(-, x, a)$$

By assumption, FS + Ax proves

$$(! \gamma)_D(w, -, a) \vdash \varphi_D(-, x, a) \wp (? \psi)_D(-, y, a)$$

and the result follows from

$$\frac{\frac{(!\gamma)_D(w, -, a) \vdash \varphi_D(-, x, a), (?\psi)_D(-, y, a)}{(!\gamma)_D(w, -, a) \vdash (\forall x)\varphi_D(-, x, a), (?\psi)_D(-, y, a)}}{(!\gamma)_D(w, -, a) \vdash !(\forall x)\varphi_D(-, x, a), (?\psi)_D(-, y, a)}$$

Consider now the case of  $\varphi$  positive. We have

$$(!\varphi)^D(a) = (\exists u)(!\varphi)_D(u, -, a) = (\exists u)!\varphi_D(u, -, a)$$

By assumption there is an eager term  $u(w, y, a)$  such that

$$(!\gamma)_D(w, -, a) \vdash_{\text{FS}+\text{Ax}} \varphi_D(\overline{\text{@}}(u(w, y, a), w, a), -, a) \wp (?\psi)_D(-, y, a)$$

and the result follows from

$$\frac{(!\gamma)_D(w, -, a) \vdash \varphi_D(\overline{\text{@}}(u(w, y, a), w, a), -, a), (?\psi)_D(-, y, a)}{(!\gamma)_D(w, -, a) \vdash !\varphi_D(\overline{\text{@}}(u(w, y, a), w, a), -, a), (?\psi)_D(-, y, a)}$$

(2) Consider first the case of  $\varphi$  positive. We have

$$(? \varphi)^D(a) = (? \varphi)_D(-, -, a) = ?(\exists u)\varphi_D(u, -, a)$$

By assumption,  $\text{FS} + \text{Ax}$  proves

$$(!\gamma)_D(w, -, a) \otimes \varphi_D(u, -, a) \vdash (? \psi)_D(-, y, a)$$

and the result follows from

$$\frac{\frac{(!\gamma)_D(w, -, a), \varphi_D(u, -, a) \vdash (? \psi)_D(-, y, a)}{(!\gamma)_D(w, -, a), (\exists u)\varphi_D(u, -, a) \vdash (? \psi)_D(-, y, a)}}{(!\gamma)_D(w, -, a), ?(\exists u)\varphi_D(u, -, a) \vdash (? \psi)_D(-, y, a)}$$

Consider now the case of  $\varphi$  negative. We have

$$(? \varphi)^D(a) = (\forall x)(? \varphi)_D(-, x, a) = (\forall x)? \varphi_D(-, x, a)$$

By assumption there is an eager term  $u(w, y, a)$  such that

$$(!\gamma)_D(w, -, a) \otimes \varphi_D(-, \overline{\text{@}}(\mathbf{x}(w, y, a), w, y, a), a) \vdash_{\text{FS}+\text{Ax}} (? \psi)_D(-, y, a)$$

and the result follows from

$$\frac{(!\gamma)_D(w, -, a), \varphi_D(-, \overline{\text{@}}(\mathbf{x}(w, y, a), w, y, a), a) \vdash (? \psi)_D(-, y, a)}{(!\gamma)_D(w, -, a), ? \varphi_D(-, \overline{\text{@}}(\mathbf{x}(w, y, a), w, y, a), a) \vdash (? \psi)_D(-, y, a)}$$

□

**Closed Structure.**

**Lemma 9.**

- (1) We have  $\gamma(a) \otimes \varphi(a) \dashv\vdash_{\mathbf{Ax}} \psi(a)$  if and only if  $\gamma(a) \dashv\vdash_{\mathbf{Ax}} \varphi(a) \multimap \psi(a)$ .  
(2) If  $\gamma(a) \otimes !\varphi(a) \dashv\vdash_{\mathbf{Ax}} \psi(a) \wp ?\gamma'(a)$  then  $\gamma(a) \dashv\vdash_{\mathbf{Ax}} (!\varphi(a) \multimap \psi(a)) \wp ?\gamma'(a)$ .

*Proof.* Let

$$\begin{aligned}\varphi^D(a) &= (\exists u)(\forall x)\varphi_D(u, x, a) \\ \psi^D(a) &= (\exists v)(\forall y)\psi_D(v, y, a) \\ \gamma^D(a) &= (\exists w)(\forall z)\gamma_D(w, z, a)\end{aligned}$$

- (1) Note that

$$(\varphi(a) \multimap \psi(a))^D = (\exists v, x)(\forall u, y)\left(\varphi_D(u, (x)uy, a) \multimap \psi_D((v)u, y, a)\right)$$

First, note that

$$\begin{aligned}&\left((\forall a)(\gamma \otimes \varphi \multimap \psi)\right)^D = \\ &(\exists v, z, x)(\forall w, u, y, a)\left(\gamma_D(w, (z)wuy, a) \otimes \varphi_D(u, (x)wuy, a) \multimap \psi_D((v)wua, y, a)\right)\end{aligned}$$

and

$$\begin{aligned}&\left((\forall a)(\gamma \multimap (\varphi \multimap \psi))\right)^D = \\ &(\exists v, x, z)(\forall w, u, y, a)\left(\gamma_D(w, (z)wuy, a) \multimap (\varphi_D(u, (x)uywa, a) \multimap \psi_D((v)uwa, y, a))\right)\end{aligned}$$

So both formulae have the same realizers modulo  $(\Sigma^{\Gamma})^{\Delta} \simeq_{\mathbb{M}} (\Sigma^{\Delta})^{\Gamma}$ . The result then follows from

$$\frac{\gamma_D, \varphi_D \vdash \psi_D}{\gamma_D \vdash \varphi_D \multimap \psi_D} \quad \text{and} \quad \frac{\frac{\frac{\gamma_D \vdash \varphi_D \multimap \psi_D}{\gamma_D, \varphi_D \vdash \psi_D} \quad \frac{\varphi_D \vdash \varphi_D \quad \psi_D \vdash \psi_D}{\varphi_D \multimap \psi_D, \varphi_D \vdash \psi_D}}{\gamma_D, \varphi_D \vdash \psi_D}}{\gamma_D \otimes \varphi_D \vdash \psi_D}$$

- (2) Note that  $!\varphi$  is positive and that  $?\gamma'$  is negative, so that

$$(!\varphi)^D(a) = (\exists u)(!\varphi)_D(u, -, a) \quad \text{and} \quad (? \gamma')^D(a) = (\forall z')(?\gamma')_D(-, z', a)$$

where  $(!\varphi)_D$  (resp.  $(?\gamma')_D$ ) is an  $!$ -formula (resp. a  $?$ -formula). We have

$$\begin{aligned}(\psi(a) \wp ?\gamma'(a))^D &= (\exists v)(\forall y, z')(\psi_D(v, y, a) \wp (? \gamma')_D(-, z', a)) \\ \text{and } (!\varphi(a) \multimap \psi(a))^D &= (\exists v)(\forall u, y)((!\varphi)_D(u, -, a) \multimap \psi_D((v)u, y, a))\end{aligned}$$

It follows that

$$\gamma(a) \otimes !\varphi(a) \dashv\vdash_{\mathbf{Ax}} \psi(a) \wp ?\gamma'(a) \quad \Longrightarrow \quad \gamma(a) \dashv\vdash_{\mathbf{Ax}} (!\varphi(a) \multimap \psi(a)) \wp ?\gamma'(a)$$

since

$$\begin{aligned} & \left( \gamma(a) \otimes !\varphi(a) \multimap \psi(a) \wp ?\gamma'(a) \right)^D = \\ & (\exists v, z)(\forall w, u, y, z') \left( \gamma_D(w, (z)wuyz', a) \otimes (!\varphi)_D(u, -, a) \multimap \psi_D((v)wu, y, a) \wp (? \gamma')_D(-, z', a) \right) \end{aligned}$$

and

$$\begin{aligned} & \left( \gamma(a) \multimap (!\varphi(a) \multimap \psi(a)) \wp ?\gamma'(a) \right)^D = \\ & (\exists v, z)(\forall w, u, y, z') \left( \gamma_D(w, (z)wuyz', a) \multimap ((!\varphi)_D(u, -, a) \multimap \psi_D((v)wu, y, a)) \wp (? \gamma')_D(-, z', a) \right) \end{aligned}$$

□

### Quantifiers.

#### Lemma 10.

- (1)  $\varphi(a, \mathfrak{t}(a)) \dashv\vdash_{\mathbf{Ax}} (\exists z)\varphi(a, z)$ .
- (2)  $(\forall z)\varphi(a, z) \dashv\vdash_{\mathbf{Ax}} \varphi(a, \mathfrak{t}(a))$ .

*Proof.* Let

$$\varphi^D(a, z) = (\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x, a, z)$$

We thus have

$$(\exists z.\varphi(a, z))^D = (\exists u)(\exists z)(\forall x)\varphi_D(u, x, a, z) \quad \text{and} \quad (\forall z.\varphi(a, z))^D = (\exists u^{(\tau)\kappa})(\forall x)(\forall z)\varphi_D((u)z, x, a, z)$$

In both cases we assume  $a^v$ ,  $z^\kappa$  and  $\mathfrak{t}(a)$  to be of sort  $(v; \kappa)$

- (1) We have to find eager terms  $\mathbf{u}(u, x, a)$ ,  $\mathbf{z}(u, x, a)$  and  $\mathbf{x}(u, x, a)$  such that

$$\varphi_D(u, \overline{\mathbb{Q}}(\mathbf{x}(u, x, a), u, x, a), a, \mathfrak{t}(a)) \multimap \varphi_D(\overline{\mathbb{Q}}(\mathbf{u}(u, x, a), u, a), x, a, \overline{\mathbb{Q}}(\mathbf{z}(u, x, a), a))$$

We let  $\mathbf{u}$  and  $\mathbf{x}$  be given as in Lemma 5. As for  $\mathbf{z}(u, x, a)$ , we take the eager term obtained from the composite

$$[[\tau]] \times [[\sigma]] \times [[v]] \xrightarrow{[\pi]_{\mathbf{M}}} [[v]] \xrightarrow{\mathbf{A}(\mathfrak{t})_{\mathbf{EM}}} [[\kappa]]^{[v]}$$

where  $[\pi]$  is a suitable Mealy projection.

- (2) Note that the variable  $u$  now has type  $(\tau)\kappa$ . We have to find eager terms  $\mathbf{u}(u, x, a)$ ,  $\mathbf{z}(u, x, a)$  and  $\mathbf{x}(u, x, a)$  such that

$$\begin{aligned} & \varphi_D((u)(\overline{\mathbb{Q}}(\mathbf{z}(u, x, a), z)), \overline{\mathbb{Q}}(\mathbf{x}(u, x, a), u, x, a), a, \overline{\mathbb{Q}}(\mathbf{z}(u, x, a), a)) \multimap \\ & \varphi_D(\overline{\mathbb{Q}}(\mathbf{u}(u, x, a), u, x, a), x, a, \mathfrak{t}(a)) \end{aligned}$$

First, for  $\mathbf{z}(u, x, a)$ , we take as above the eager term obtained from the composite

$$[[\tau]]^{[\kappa]} \times [[\sigma]] \times [[v]] \xrightarrow{[\pi]_{\mathbf{M}}} [[v]] \xrightarrow{\mathbf{A}(\mathfrak{t})_{\mathbf{EM}}} [[\kappa]]^{[v]}$$

It thus remains to find  $\mathbf{u}$  and  $\mathbf{x}$  such that

$$\varphi_D(\textcircled{\@}(u, \mathfrak{t}(a)), \overline{\textcircled{\@}}(\mathbf{x}(u, x, a), u, x, a), a, \mathfrak{t}(a)) \text{ ---} \circ \varphi_D(\overline{\textcircled{\@}}(\mathbf{u}(u, x, a), u, x, a), x, a, \mathfrak{t}(a))$$

We then take for  $\mathbf{x}$  the same term as in Lemma 5. It remains to deal with  $\mathbf{u}$ . Consider the Mealy term  $\tilde{\mathbf{u}}$  for

$$\llbracket \tau \rrbracket^{\llbracket \kappa \rrbracket} \times \llbracket \sigma \rrbracket \times \llbracket \nu \rrbracket \xrightarrow{\llbracket \tau \rrbracket}_{\mathbb{M}} \llbracket \tau \rrbracket^{\llbracket \kappa \rrbracket} \times \llbracket \nu \rrbracket \xrightarrow{\textcircled{\@}(u, \mathfrak{t}(a))}_{\mathbb{M}} \llbracket \tau \rrbracket$$

Then we take for  $\mathbf{u}$  the eager term

$$\llbracket \tau \rrbracket^{\llbracket \kappa \rrbracket} \times \llbracket \sigma \rrbracket \times \llbracket \nu \rrbracket \xrightarrow{\Lambda(\tilde{\mathbf{u}})}_{\text{EM}} \llbracket \tau \rrbracket^{\llbracket \tau \rrbracket^{\llbracket \kappa \rrbracket} \times \llbracket \sigma \rrbracket \times \llbracket \nu \rrbracket} \xrightarrow{\sim}_{\mathbb{M}} ((\llbracket \tau \rrbracket^{\llbracket \nu \rrbracket}) \llbracket \sigma \rrbracket) \llbracket \tau \rrbracket^{\llbracket \kappa \rrbracket} \quad \square$$

**Lemma 11.**

- (1) If  $\gamma(a) \otimes \varphi(a, b) \text{ ---} \rightarrow_{\mathbf{Ax}} \psi(a)$ , then  $\gamma(a) \otimes (\exists b)\varphi(a, b) \text{ ---} \rightarrow_{\mathbf{Ax}} \psi(a)$ .  
(2) If  $\gamma(a) \text{ ---} \rightarrow_{\mathbf{Ax}} \varphi(a, b) \text{ } \mathfrak{A} ? \psi(a)$ , then  $\gamma(a) \text{ ---} \rightarrow_{\mathbf{Ax}} (\forall b)\varphi(a, b) \text{ } \mathfrak{A} ? \psi(a)$ .

*Proof.* Let

$$\gamma^D(a) = (\exists w)(\forall z)\gamma_D(w, z, a) \quad \text{and} \quad \varphi^D(a, b) = (\exists u)(\forall x)\varphi_D(u, x, a, b)$$

(1) Let

$$\psi^D(a, b) = (\exists v)(\forall y)\psi_D(v, y, a, b)$$

We have

$$\begin{aligned} (\gamma(a) \otimes \varphi(a, b))^D &= (\exists w, u)(\forall z, x)(\gamma_D(w, z, a) \otimes \varphi_D(u, x, a, b)) \\ \text{and } (\gamma(a) \otimes (\exists b)\varphi(a, b))^D &= (\exists w, u, b)(\forall z, x)(\gamma_D(w, z, a) \otimes \varphi_D(u, x, a, b)) \end{aligned}$$

It follows that

$$\begin{aligned} \left( \gamma(a) \otimes \varphi(a, b) \text{ ---} \circ \psi(a) \right)^D &= \\ (\exists v, z, x)(\forall w, u, y) \left( \gamma_D(w, (z)wuy, a) \otimes \varphi_D(u, (x)wuy, a, b) \text{ ---} \circ \psi_D((v)wu, y, a) \right) \end{aligned}$$

and

$$\begin{aligned} \left( \gamma(a) \otimes (\exists b)\varphi(a, b) \text{ ---} \circ \psi(a) \right)^D &= \\ (\exists v, z, x)(\forall w, u, b, y) \left( \gamma_D(w, (z)wuby, a) \otimes \varphi_D(u, (x)wuby, a, b) \text{ ---} \circ \psi_D((v)wub, y, a) \right) \end{aligned}$$

We thus obtain

$$\gamma(a) \text{ ---} \rightarrow_{\mathbf{Ax}} \varphi(a, b) \text{ } \mathfrak{A} ? \psi(a) \quad \iff \quad \gamma(a) \text{ ---} \rightarrow_{\mathbf{Ax}} (\forall b)\varphi(a, b) \text{ } \mathfrak{A} ? \psi(a)$$



since

$$\begin{aligned} & \left( (\forall a, b) (\gamma(a) \otimes \varphi(a, b) \multimap \psi(a)) \right)^D = \\ & (\exists v, z, x) (\forall w, u, y, b, a) \left( \gamma_D(w, (z)wuyba, a) \otimes \varphi_D(u, (x)wuyba, a, b) \multimap \psi_D((v)wuba, y, a) \right) \end{aligned}$$

and

$$\begin{aligned} & \left( (\forall a) (\gamma(a) \otimes (\exists b) \varphi(a, b) \multimap \psi(a)) \right)^D = \\ & (\exists v, z, x) (\forall w, u, b, y, a) \left( \gamma_D(w, (z)wubya, a) \otimes \varphi_D(u, (x)wubya, a, b) \multimap \psi_D((v)wuba, y, a) \right) \end{aligned}$$

(2) Note that  $?\psi$  is negative. We thus have

$$(?\psi)^D(a) = (\forall y) (?\psi)_D(-, y, a)$$

where  $(?\psi)_D$  is a  $?$ -formula. Moreover,

$$(\forall b. \varphi(a, b))^D = (\exists u) (\forall x, b) \varphi_D((u)b, x, a, b)$$

We have

$$\begin{aligned} & (\varphi(a, b) \wp ?\psi(a))^D = (\exists u) (\forall x, y) (\varphi_D(u, x, a, b) \wp (?\psi)_D(-, y, a)) \\ \text{and} \quad & ((\forall b) \varphi(a, b) \wp ?\psi(a))^D = (\exists u) (\forall x, b, y) (\varphi_D((u)b, x, a, b) \wp (?\psi)_D(-, y, a)) \end{aligned}$$

It follows that

$$\begin{aligned} & \left( \gamma(a) \multimap \varphi(a, b) \wp \psi(a) \right)^D = \\ & (\exists u, z) (\forall w, x, y) \left( \gamma_D(w, (z)wxy, a) \multimap \varphi_D((u)w, x, a, b) \wp \psi_D(-, y, a) \right) \end{aligned}$$

and

$$\begin{aligned} & \left( \gamma(a) \multimap (\forall b) \varphi(a, b) \wp \psi(a) \right)^D = \\ & (\exists u, z) (\forall w, x, b, y) \left( \gamma_D(w, (z)wxb, a) \multimap \varphi_D((u)bw, x, a, b) \wp \psi_D(-, y, a) \right) \end{aligned}$$

We thus obtain

$$\gamma(a) \multimap_{\mathbf{Ax}} \varphi(a, b) \wp ?\psi(a) \iff \gamma(a) \multimap_{\mathbf{Ax}} (\forall b) \varphi(a, b) \wp ?\psi(a)$$

since

$$\begin{aligned} & \left( (\forall a, b) (\gamma(a) \multimap \varphi(a, b) \wp \psi(a)) \right)^D = \\ & (\exists u, z) (\forall w, x, y, b, a) \left( \gamma_D(w, (z)wxyba, a) \multimap \varphi_D((u)wba, x, a, b) \wp \psi_D(-, y, a) \right) \end{aligned}$$

and

$$\begin{aligned} & \left( (\forall a) (\gamma(a) \multimap (\forall b) \varphi(a, b) \wp \psi(a)) \right)^D = \\ & (\exists u, z) (\forall w, x, b, y, a) \left( \gamma_D(w, (z)wxb, a) \multimap \varphi_D((u)b, x, a, b) \wp \psi_D(-, y, a) \right) \end{aligned}$$

□

**The Equality Axioms (7).** Realization of the equality axioms (7), follows from the fact that atomic formulae are interpreted by themselves, so these axioms are interpreted by instances of themselves.

**Proof of Theorem 8.** Assume given  $\bar{\varphi}(a) = \varphi_1(a), \dots, \varphi_n(a)$  and  $\bar{\psi}(a) = \psi_1(a), \dots, \psi_m(a)$  with free variables among  $a$ . We show

$$\varphi_1(a), \dots, \varphi_n(a) \vdash_{\text{FS+Ax}} \psi_1(a), \dots, \psi_m(a) \implies (\otimes_i \varphi_i(a)) \dashv\vdash_{\text{Ax}} (\wp_j \psi_j(a))$$

with, as expected,

$$\otimes_{1 \leq i \leq 0} \varphi_i(a) = \mathbf{I} \quad \text{and} \quad \wp_{1 \leq j \leq 0} \psi_j(a) = \perp$$

We reason by induction on derivations.

– The cases of

$$\frac{}{\varphi \vdash \varphi} \quad \frac{\bar{\varphi}, \varphi, \psi, \bar{\psi} \vdash \bar{\varphi}'}{\bar{\varphi}, \psi, \varphi, \bar{\psi} \vdash \bar{\varphi}'} \quad \frac{\bar{\varphi} \vdash \bar{\varphi}', \varphi, \psi, \bar{\psi}'}{\bar{\varphi} \vdash \bar{\varphi}', \psi, \varphi, \bar{\psi}'}$$

directly follow from Lemmas 5 and 6.

– Case of

$$\frac{\bar{\varphi} \vdash \gamma, \bar{\varphi}' \quad \bar{\psi}, \gamma \vdash \bar{\psi}'}{\bar{\varphi}, \bar{\psi} \vdash \bar{\varphi}', \bar{\psi}'}$$

We show that if

$$\varphi(a) \dashv\vdash_{\text{Ax}} \gamma(a) \wp \varphi'(a) \quad \text{and} \quad \psi(a) \otimes \gamma(a) \dashv\vdash_{\text{Ax}} \psi'(a)$$

then

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\text{Ax}} \varphi'(a) \wp \psi'(a)$$

*Proof.* Assuming  $\varphi(a) \dashv\vdash_{\text{Ax}} \gamma(a) \wp \varphi'(a)$ , Lemma 6 gives

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\text{Ax}} (\gamma(a) \wp \varphi'(a)) \otimes \psi(a)$$

from which Lemma 6 together with Proposition 11 implies

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\text{Ax}} \varphi'(a) \wp (\psi(a) \otimes \gamma(a))$$

Since  $\psi(a) \otimes \gamma(a) \dashv\vdash_{\text{Ax}} \psi'(a)$ , Lemma 6 together with Proposition 11 give the result.  $\square$

– The cases of

$$\frac{\bar{\varphi} \vdash \bar{\psi}}{\bar{\varphi}, \mathbf{I} \vdash \bar{\psi}} \quad \frac{}{\vdash \mathbf{I}} \quad \frac{}{\perp \vdash} \quad \frac{\bar{\varphi} \vdash \bar{\psi}}{\bar{\varphi} \vdash \perp, \bar{\psi}}$$

directly follow from Lemma 6 together with Proposition 11.

– The cases of

$$\frac{\bar{\varphi}, \varphi_0, \varphi_1 \vdash \bar{\varphi}'}{\bar{\varphi}, \varphi_0 \otimes \varphi_1 \vdash \bar{\varphi}'} \quad \frac{\bar{\varphi} \vdash \varphi_0, \varphi_1, \bar{\varphi}'}{\bar{\varphi} \vdash \varphi_0 \wp \varphi_1, \bar{\varphi}'}$$

are tautological.

– Case of

$$\frac{\bar{\varphi} \vdash \varphi, \bar{\varphi}' \quad \bar{\psi} \vdash \psi, \bar{\psi}'}{\bar{\varphi}, \bar{\psi} \vdash \varphi \otimes \psi, \bar{\varphi}', \bar{\psi}'}$$

We show that if

$$\varphi(a) \dashv\vdash_{\mathbf{Ax}} \gamma_0(a) \wp \varphi'(a) \quad \text{and} \quad \psi(a) \dashv\vdash_{\mathbf{Ax}} \gamma_1(a) \wp \psi'(a)$$

then

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\mathbf{Ax}} (\gamma_0(a) \otimes \gamma_1(a)) \wp \varphi'(a) \wp \psi'(a)$$

*Proof.* By Lemma 6, we have

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\mathbf{Ax}} (\gamma_0(a) \wp \varphi'(a)) \otimes (\gamma_1(a) \wp \psi'(a))$$

from which we deduce by Lemma 6 and Proposition 11

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\mathbf{Ax}} (\gamma_0(a) \otimes (\gamma_1(a) \wp \psi'(a))) \wp \varphi'(a)$$

and then

$$\varphi(a) \otimes \psi(a) \dashv\vdash_{\mathbf{Ax}} ((\gamma_0(a) \otimes \gamma_1(a)) \wp \psi'(a)) \wp \varphi'(a)$$

□

– Case of

$$\frac{\bar{\varphi}, \varphi \vdash \bar{\varphi}' \quad \bar{\psi}, \psi \vdash \bar{\psi}'}{\bar{\varphi}, \bar{\psi}, \varphi \wp \psi \vdash \bar{\varphi}', \bar{\psi}'}$$

We show that if

$$\varphi(a) \otimes \gamma_0(a) \dashv\vdash_{\mathbf{Ax}} \varphi'(a) \quad \text{and} \quad \psi(a) \otimes \gamma_1(a) \dashv\vdash_{\mathbf{Ax}} \psi'(a)$$

then

$$\varphi(a) \otimes \psi(a) \otimes (\gamma_0(a) \wp \gamma_1(a)) \dashv\vdash_{\mathbf{Ax}} \varphi'(a) \wp \psi'(a)$$

*Proof.* By Lemma 6 we have

$$(\varphi(a) \otimes \gamma_0(a)) \wp (\psi(a) \otimes \gamma_1(a)) \dashv\vdash_{\mathbf{Ax}} \varphi'(a) \wp \psi'(a)$$

On the other hand, by Lemma 6 we have

$$\varphi(a) \otimes \psi(a) \otimes (\gamma_0(a) \wp \gamma_1(a)) \dashv\vdash_{\mathbf{Ax}} ((\varphi(a) \otimes \gamma_0) \wp \gamma_1(a)) \otimes \psi(a)$$

so that Lemma 6 together with Proposition 11 give

$$\varphi(a) \otimes \psi(a) \otimes (\gamma_0(a) \wp \gamma_1(a)) \dashv\vdash_{\mathbf{Ax}} (\varphi(a) \otimes \gamma_0) \wp (\gamma_1(a) \otimes \psi(a))$$

and we conclude by Proposition 11.

□

– Lemma 7 (together with Lemma 6 and Proposition 11) handles the rules:

$$\frac{\bar{\psi} \vdash \bar{\psi}'}{\bar{\psi}, !\varphi \vdash \bar{\psi}'} \quad \frac{\bar{\psi}, !\varphi, !\varphi \vdash \bar{\psi}'}{\bar{\psi}, !\varphi \vdash \bar{\psi}'} \quad \frac{\bar{\varphi}, \varphi \vdash \bar{\varphi}'}{\bar{\varphi}, !\varphi \vdash \bar{\varphi}'}$$

$$\frac{\bar{\psi} \vdash \bar{\psi}'}{\bar{\psi} \vdash ?\varphi, \bar{\psi}'} \quad \frac{\bar{\psi} \vdash ?\varphi, ?\varphi, \bar{\psi}'}{\bar{\psi} \vdash ?\varphi, \bar{\psi}'} \quad \frac{\bar{\varphi} \vdash \varphi, \bar{\psi}}{\bar{\varphi} \vdash ?\varphi, \bar{\psi}}$$

– Lemma 8 handles the rules

$$\frac{!\bar{\varphi} \vdash \varphi, ?\bar{\psi}}{!\bar{\varphi} \vdash !\varphi, ?\bar{\psi}} \quad \frac{!\bar{\varphi}, \varphi \vdash ?\bar{\psi}}{!\bar{\varphi}, ?\varphi \vdash ?\bar{\psi}}$$

– The rules

$$\frac{\bar{\varphi} \vdash \varphi[t/x], \bar{\varphi}'}{\bar{\varphi} \vdash (\exists x)\varphi, \bar{\varphi}'} \quad \frac{\bar{\varphi}, \varphi[t/x] \vdash \bar{\varphi}'}{\bar{\varphi}, (\forall x)\varphi \vdash \bar{\varphi}'}$$

follow from Lemma 10 (together with Lemma 6 and Proposition 11).

– The cases of

$$\frac{\bar{\varphi}, \varphi \vdash \bar{\varphi}'}{\bar{\varphi}, (\exists z)\varphi \vdash \bar{\varphi}'} \quad \frac{\bar{\varphi} \vdash \varphi, ?\bar{\psi}}{\bar{\varphi} \vdash (\forall z)\varphi, ?\bar{\psi}}$$

are tautological (Lemma 11).

– Lemma 9 handles the rules

$$\frac{\bar{\varphi}, \varphi \vdash \psi}{\bar{\varphi} \vdash \varphi \multimap \psi} \quad \text{and} \quad \frac{\bar{\varphi}, !\varphi \vdash \psi, ?\bar{\psi}}{\bar{\varphi} \vdash !\varphi \multimap \psi, ?\bar{\psi}}$$

– It remains to deal with

$$\frac{\bar{\varphi} \vdash \varphi, \bar{\varphi}' \quad \bar{\psi}, \psi \vdash \bar{\psi}'}{\bar{\varphi}, \bar{\psi}, \varphi \multimap \psi \vdash \bar{\varphi}', \bar{\psi}'}$$

Since

$$\frac{\frac{\frac{\varphi \multimap \psi \vdash \varphi \multimap \psi \quad \varphi \vdash \varphi}{\varphi \multimap \psi, \varphi \vdash \psi} \quad \bar{\psi}, \psi \vdash \bar{\psi}'}{\bar{\varphi} \vdash \varphi, \bar{\varphi}'} \quad \bar{\psi}, \varphi \multimap \psi, \varphi \vdash \bar{\psi}'}{\bar{\varphi}, \bar{\psi}, \varphi \multimap \psi \vdash \bar{\varphi}', \bar{\psi}'}$$

We are left with showing

$\gamma(a) \multimap_{Ax} \varphi(a) \multimap \psi(a)$  and  $\gamma'(a) \multimap_{Ax} \varphi(a)$  imply  $\gamma \otimes \gamma' \multimap_{Ax} \psi(a)$

*Proof.* Assume  $\gamma(a) \multimap_{Ax} \varphi(a) \multimap \psi(a)$  and  $\gamma'(a) \multimap_{Ax} \varphi(a)$ . Lemma 9 gives

$$\gamma(a) \otimes \varphi(a) \multimap_{Ax} \psi(a)$$

while Lemma 6 gives

$$\gamma(a) \otimes \gamma'(a) \multimap_{Ax} \gamma(a) \otimes \varphi(a)$$

and we conclude by Proposition 11.  $\square$

This concludes the proof of Theorem 8.  $\square$

## B.2 Realization of Additional Axioms (Proposition 4)

We decompose Proposition 4 as follows.

**Lemma 12.** *The axioms (LSIP) are realized in FS:*

- (1)  $(\forall a)(\varphi^-(a) \otimes \psi^-) \leftrightarrow (\forall a)\varphi^-(a) \otimes \psi^-$
- (2)  $(\forall a)(\varphi^-(a) \wp \psi^-) \leftrightarrow (\forall a)\varphi^-(a) \wp \psi^-$
- (3)  $(\exists a)\varphi^-(a) \wp \psi \leftrightarrow (\exists a)(\varphi^-(a) \wp \psi)$
- (4)  $\psi^- \multimap (\exists a)\varphi^-(a) \leftrightarrow (\exists a)(\psi^- \multimap \varphi^-(a))$
- (5)  $((\forall a)\varphi^\pm(a) \multimap \psi^\pm) \leftrightarrow (\exists a)(\varphi^\pm(a) \multimap \psi^\pm)$

*Proof.* Let

$$\varphi^D(a) = (\exists u)(\forall x)\varphi_D(u, x, a) \quad \text{and} \quad \psi^D(a) = (\exists v)(\forall y)\psi_D(v, y, a)$$

In each case we are done by taking terms similar to those of Lemma 5:

(1) We have

$$\begin{aligned} \left( (\forall a)(\varphi^-(a) \otimes \psi^-) \right)^D &= (\forall x, y, a)(\varphi_D(-, x, a) \otimes \psi_D(-, y)) \\ \left( (\forall a)\varphi^-(a) \otimes \psi^- \right)^D &= (\forall a, x, y)(\varphi_D(-, x, a) \otimes \psi_D(-, y)) \end{aligned}$$

(2) We have

$$\begin{aligned} \left( (\forall a)(\varphi^-(a) \wp \psi^-) \right)^D &= (\forall x, y, a)(\varphi_D(-, x, a) \wp \psi_D(-, y)) \\ \left( (\forall a)\varphi^-(a) \wp \psi^- \right)^D &= (\forall a, x, y)(\varphi_D(-, x, a) \wp \psi_D(-, y)) \end{aligned}$$

(3) We have

$$\begin{aligned} \left( (\exists a)\varphi^-(a) \wp \psi \right)^D &= (\exists a, v)(\forall x, y)(\varphi_D(-, x, a) \wp \psi_D(v, y)) \\ \left( (\exists a)(\varphi^-(a) \wp \psi) \right)^D &= (\exists a, v)(\forall x, y)(\varphi_D(-, x, a) \wp \psi_D(v, y)) \end{aligned}$$

(4) We have

$$\begin{aligned} \left( \psi^- \multimap (\exists a)\varphi^-(a) \right)^D &= (\exists a, y)(\forall x)(\psi_D(-, (y)x) \multimap \varphi_D(-, x, a)) \\ \left( (\exists a)(\psi^- \multimap \varphi^-(a)) \right)^D &= (\exists a, y)(\forall x)(\psi_D(-, (y)x) \multimap \varphi_D(-, x, a)) \end{aligned}$$

(5) We have

$$\begin{aligned} \left( (\forall a)\varphi^\pm(a) \multimap \psi^\pm \right)^D &= (\exists a)(\varphi_D(-, -, a) \multimap \psi_D(-, -)) \\ \left( (\exists a)(\varphi^\pm(a) \multimap \psi^\pm) \right)^D &= (\exists a)(\varphi_D(-, -, a) \multimap \psi_D(-, -)) \end{aligned}$$

□

**Lemma 13.** *The axiom (LAC) is realized in FS:*

$$(\forall a^\sigma)(\exists b^\tau)\varphi(a, b, c) \dashv\vdash (\exists f^{(\tau)\sigma})(\forall a^\sigma)\varphi(a, (f)a, c)$$

*Proof.* Let

$$\varphi^D(a, b, c) = (\exists u)(\forall x)\varphi_D(u, x, \langle a, b, c \rangle)$$

We thus have

$$\left( (\forall c)(\forall a)(\exists b)\varphi(a, b, c) \right)^D = (\exists b, u)(\forall x, a, c)\varphi_D((u)ac, x, \langle a, (b)ac, c \rangle)$$

and

$$\left( (\forall c)(\exists f)(\forall a)\varphi(a, (f)a, c) \right)^D = (\exists f, u)(\forall x, a, c)\varphi_D((u)ac, x, \langle a, (f)ac, c \rangle)$$

Hence we are done by taking terms similar to those of Lemma 5.  $\square$

**Lemma 14.** *The axioms (DEXP) are realized in Ax+(DEXP), in the sense that for all deterministic  $\delta$  we have*

$$\delta \dashv\vdash_{(\text{DEXP})} !\delta \quad \text{and} \quad ?\delta \dashv\vdash_{(\text{DEXP})} \delta$$

*Proof.* If  $\delta(a)$  is deterministic, then  $\delta^D(a) = \delta_D(-, -, a)$  with  $\delta_D$  deterministic. It follows that the axioms reduce to themselves:

$$\text{and} \quad \begin{array}{l} \delta_D(-, -, a) \vdash_{\text{FS}+(\text{DEXP})} !\delta_D(-, -, a) \\ ?\delta_D(-, -, a) \vdash_{\text{FS}+(\text{DEXP})} \delta_D(-, -, a) \end{array}$$

$\square$

### B.3 The Characterization Theorem (Theorem 3)

Theorem 3 can be split in two statements, one for polarized formulae and one for all FS-formulae.

**Characterization for The Full System FS.** We begin by the full system. We show by induction on formulae that

$$\vdash_{\text{FS}+(\text{LAC})+(\text{LSIP})+(\text{DEXP})} \varphi(a) \circ\circ \varphi^D(a)$$

- If  $\varphi$  is atomic, then the result is trivial since  $\varphi^D = \varphi$ .
- Consider the case of  $\varphi(a) \square \psi(a)$  with  $\square$  is either  $\otimes$ ,  $\wp$  or  $\multimap$ . By induction hypothesis, we have

$$\begin{array}{l} \varphi(a) \circ\circ (\exists u)(\forall x)\varphi_D(u, x, a) \\ \psi(a) \circ\circ (\exists v)(\forall y)\psi_D(v, y, a) \end{array}$$

and in each case we are left with showing

$$(\varphi(a) \square \psi(a))^D \circ\circ \varphi^D(a) \square \psi^D(a)$$

- *Case of  $\varphi \otimes \psi$ .* We first show

$$(\varphi \otimes \psi)^D \quad \multimap \quad \varphi^D \otimes \psi^D$$

using the axiom

$$(\forall a)(\varphi^-(a) \otimes \psi^-) \quad \multimap \quad (\forall a)\varphi^-(a) \otimes \psi^-$$

$$\frac{\frac{\frac{(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y)) \vdash (\forall x)\varphi_D(u, x) \otimes (\forall y)\psi_D(v, y)}{(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x) \otimes (\exists v)(\forall y)\psi_D(v, y)}}{(\exists v)(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x) \otimes (\exists v)(\forall y)\psi_D(v, y)}}{(\exists uv)(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x) \otimes (\exists v)(\forall y)\psi_D(v, y)} \quad \frac{\vdots}{\mathcal{D}}$$

where  $\mathcal{D}$  is

$$\frac{\frac{\frac{(\forall x)\varphi_D(u, x) \vdash (\forall x)\varphi_D(u, x)}{(\forall x)\varphi_D(u, x) \vdash (\exists u)(\forall x)\varphi_D(u, x)} \quad \frac{(\forall y)\psi_D(v, y) \vdash (\forall y)\psi_D(v, y)}{(\forall y)\psi_D(v, y) \vdash (\exists v)(\forall y)\psi_D(v, y)}}{\frac{\forall x.\varphi_D(u, x), \forall y.\psi_D(v, y) \vdash \exists u.\forall x.\varphi_D(u, x) \otimes \exists v.\forall y.\psi_D(v, y)}{\forall x.\varphi_D(u, x) \otimes \forall y.\psi_D(v, y) \vdash \exists u.\forall x.\varphi_D(u, x) \otimes \exists v.\forall y.\psi_D(v, y)}}$$

The other direction is trivial:

$$\frac{\frac{\frac{\frac{\varphi_D(u, x), \psi_D(v, y) \vdash \varphi_D(u, x) \otimes \psi_D(v, y)}{\varphi_D(u, x), (\forall y)\psi_D(v, y) \vdash \varphi_D(u, x) \otimes \psi_D(v, y)}}{(\forall x)\varphi_D(u, x), (\forall y)\psi_D(v, y) \vdash \varphi_D(u, x) \otimes \psi_D(v, y)}}{(\forall x)\varphi_D(u, x), (\forall y)\psi_D(v, y) \vdash (\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y))}}{\frac{(\forall x)\varphi_D(u, x), (\forall y)\psi_D(v, y) \vdash (\exists uv)(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y))}{(\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y) \vdash (\exists uv)(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y))}}{\frac{(\exists u)(\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y) \vdash (\exists uv)(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y))}{(\exists u)(\forall x)\varphi_D(u, x) \otimes (\exists v)(\forall y)\psi_D(v, y) \vdash (\exists uv)(\forall xy)(\varphi_D(u, x) \otimes \psi_D(v, y))}}$$

- *Case of  $\varphi \wp \psi$ .* We first show

$$(\varphi \wp \psi)^D \quad \multimap \quad \varphi^D \wp \psi^D$$

using axiom

$$(\forall a)(\varphi^-(a) \wp \psi^-) \quad \multimap \quad (\forall a)\varphi^-(a) \wp \psi^-$$

$$\frac{\frac{\frac{(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y)) \vdash (\forall x)\varphi_D(u, x) \wp (\forall y)\psi_D(v, y)}{(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y)) \vdash (\forall x)\varphi_D(u, x), (\forall y)\psi_D(v, y)}}{(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y)) \vdash (\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y)}}{(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y)}}{\frac{(\exists uv)(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y)}{(\exists uv)(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x) \wp (\exists v)(\forall y)\psi_D(v, y)}} \quad \frac{\vdots}{\mathcal{D}}$$

where  $\mathcal{D}$  is

$$\frac{\frac{\frac{(\forall x)\varphi_D(u, x) \wp (\forall y)\psi_D(v, y) \vdash (\forall x)\varphi_D(u, x), (\forall y)\psi_D(v, y)}{(\forall x)\varphi_D(u, x) \wp (\forall y)\psi_D(v, y) \vdash (\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y)}}{(\forall x)\varphi_D(u, x) \wp (\forall y)\psi_D(v, y) \vdash (\exists u)(\forall x)\varphi_D(u, x), (\exists v)(\forall y)\psi_D(v, y)}}$$

For the converse implication, we use the axiom

$$(\exists a)\varphi^-(a) \wp \psi \quad \multimap \quad (\exists a)(\varphi^-(a) \wp \psi)$$

$$\frac{\frac{\frac{\frac{\exists u.\forall x.\varphi_D(u, x) \wp \exists v.\forall y.\psi_D(v, y) \vdash \exists u.\forall x.\varphi_D(u, x) \wp \exists v.\forall y.\psi_D(v, y)}{(\exists u)(\forall x)\varphi_D(u, x) \wp (\exists v)(\forall y)\psi_D(v, y) \vdash (\exists uv)((\forall x)\varphi_D(u, x) \wp (\forall y)\psi_D(v, y))}}{\vdots \quad \mathcal{D}'}}{\frac{\vdots \quad \mathcal{D}}{(\exists u)(\forall x)\varphi_D(u, x) \wp (\exists v)(\forall y)\psi_D(v, y) \vdash (\exists uv)(\forall xy)(\varphi_D(u, x) \wp \psi_D(v, y))}}$$

where  $\mathcal{D}$  is obtained by proceeding as in

$$\frac{\frac{\frac{(\forall a)\varphi(a) \wp \psi \vdash \varphi(a), \psi}{(\forall a)\varphi(a) \wp \psi \vdash \varphi(a) \wp \psi}}{(\forall a)\varphi(a) \wp \psi \vdash \forall a(\varphi(a) \wp \psi)}}$$

and where  $\mathcal{D}'$  is obtained using the axiom

$$(\exists a)\varphi^-(a) \wp \psi \quad \multimap \quad (\exists a)(\varphi^-(a) \wp \psi)$$

- *Case of  $\varphi \multimap \psi$ .* We first show

$$(\varphi \multimap \psi)^D \quad \multimap \quad \varphi^D \multimap \psi^D$$

$$\frac{\frac{\frac{\frac{\frac{\frac{\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y), \varphi_D(u, (F)uy) \vdash \varphi_D((f)u, y)}{\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y), (\forall x)\varphi_D(u, x) \vdash \varphi_D((f)u, y)}}{(\forall uy)(\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y)), (\forall x)\varphi_D(u, x) \vdash \varphi_D((f)u, y)}}{(\forall uy)(\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y)), (\forall x)\varphi_D(u, x) \vdash (\forall y)\varphi_D((f)u, y)}}{(\forall uy)(\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y)), (\forall x)\varphi_D(u, x) \vdash (\exists v)(\forall y)\varphi_D(v, y)}}{(\exists f, F)(\forall uy)(\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y)), (\exists u)(\forall x)\varphi_D(u, x) \vdash (\exists v)(\forall y)\varphi_D(v, y)}}{(\exists f, F)(\forall uy)(\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y)) \vdash (\exists u)(\forall x)\varphi_D(u, x) \multimap (\exists v)(\forall y)\varphi_D(v, y)}}$$

For the converse direction, we use the axioms (LAC) as well as

- \*  $\psi^- \multimap (\exists a)\varphi^-(a) \quad \multimap \quad \exists a.(\psi^- \multimap \varphi^-(a))$
- \*  $((\forall a)\varphi^\pm(a) \multimap \psi^\pm) \quad \multimap \quad (\exists a)(\varphi^\pm(a) \multimap \psi^\pm)$



We have to show

$$(\exists u)(\forall x)\varphi_D(u, x) \multimap (\exists v)(\forall y)\psi_D(v, y) \vdash (\exists f, F)(\forall uy)(\varphi_D(u, (F)uy) \multimap \psi_D((f)u, y))$$

Since

$$\frac{\frac{\frac{\varphi(a) \vdash (\exists a)\varphi(a)}{\quad} \quad \psi \vdash \psi}{(\exists a)\varphi(a) \multimap \psi, \varphi(a) \vdash \psi}}{(\exists a)\varphi(a) \multimap \psi \vdash \varphi(a) \multimap \psi}}{(\exists a)\varphi(a) \multimap \psi \vdash (\forall a)(\varphi(a) \multimap \psi)} \quad \text{and} \quad \frac{\frac{\frac{\frac{\varphi(a) \vdash \varphi(a)}{\quad} \quad (\forall a)\varphi(a) \vdash \varphi(a)}{\psi \vdash \psi} \quad \psi \multimap (\forall a)\varphi(a), \psi \vdash \varphi(a)}{\psi \multimap (\forall a)\varphi(a) \vdash \psi \multimap \varphi(a)}}{\psi \multimap (\forall a)\varphi(a) \vdash (\forall a)(\psi \multimap \varphi(a))}$$

and using the axioms

$$\begin{aligned} * \psi^- \multimap (\exists a)\varphi^-(a) &\multimap \exists a.(\psi^- \multimap \varphi^-(a)) \\ * ((\forall a)\varphi^\pm(a) \multimap \psi^\pm) &\multimap (\exists a)(\varphi^\pm(a) \multimap \psi^\pm) \end{aligned}$$

we derive

$$(\exists u)(\forall x)\varphi_D(u, x) \multimap (\exists v)(\forall y)\psi_D(v, y) \vdash (\forall u)(\exists v)(\forall y)(\exists x)(\varphi_D(u, x) \multimap \psi_D(v, y))$$

We can then conclude with (LAC).

– *Case of*  $(\exists a)\varphi(a)$ . Let

$$\varphi^D(a) = (\exists u)(\forall x)\varphi_D(u, x, a)$$

and by induction hypothesis assume

$$\varphi^D(a) \multimap \varphi(a)$$

Hence we are done if we show

$$(\exists a.\varphi(a))^D \multimap (\exists a)\varphi^D(a)$$

But this is trivial since  $(\exists a.\varphi(a))^D = (\exists a)\varphi^D(a)$  by definition.

– *Case of*  $(\forall a)\varphi(a)$ . Let

$$\varphi^D(a) = (\exists u)(\forall x)\varphi_D(u, x, a)$$

and by induction hypothesis assume

$$\varphi^D(a) \multimap \varphi(a)$$

We have to show

$$(\exists u)(\forall x)(\forall a)\varphi_D((u)a, x, a) \multimap (\forall a)\varphi^D(a)$$

The right-to-left implication is given by (LAC) and the left-to-right implication follows from

$$\frac{\frac{\frac{\frac{\frac{\varphi_D((u)a, x, a) \vdash \varphi_D((u)a, x, a)}{\quad} \quad (\forall a)\varphi_D((u)a, x, a) \vdash \varphi_D((u)a, x, a)}{\quad} \quad (\forall xa)\varphi_D((u)a, x, a) \vdash \varphi_D((u)a, x, a)}{\quad} \quad (\forall xa)\varphi_D((u)a, x, a) \vdash (\forall x)\varphi_D((u)a, x, a)}{\quad} \quad (\forall xa)\varphi_D((u)a, x, a) \vdash (\exists u)(\forall x)\varphi_D(u, x, a)}}{(\exists u)(\forall xa)\varphi_D((u)a, x, a) \vdash (\exists u)(\forall x)\varphi_D(u, x, a)}}{(\exists u)(\forall xa)\varphi_D((u)a, x, a) \vdash (\forall a)(\exists u)(\forall x)\varphi_D(u, x, a)}$$

– We finally deal with the exponentials. By induction hypothesis, assume

$$\varphi(a) \multimap (\exists u)(\forall x)\varphi_D(u, x, a)$$

In each relevant case we show

$$(!\varphi)^D \multimap !\varphi^D \quad \text{and} \quad (? \varphi)^D \multimap ?\varphi^D$$

- *Cases of  $!(\varphi^-)$  and  $?( \varphi^+)$ .* Both cases are trivial since

$$(!(\varphi^-))^D = !(\forall x)\varphi_D(-, x, a) = !(\varphi^-)^D \quad \text{and} \quad (?( \varphi^+))^D = ?(\exists u)\varphi_D(u, -, a) = ?(\varphi^+)^D$$

- *Cases of  $!(\varphi^+)$  and  $?( \varphi^-)$ .* We have

$$(!(\varphi^+))^D = (\exists u)!\varphi_D(u, -, a) \quad \text{and} \quad (?( \varphi^-))^D = (\forall x)?\varphi_D(-, x, a)$$

We first have

$$\frac{\frac{\frac{\varphi_D(u, -, a) \vdash \varphi_D(u, -, a)}{!\varphi_D(u, -, a) \vdash \varphi_D(u, -, a)}}{!\varphi_D(u, -, a) \vdash (\exists u)\varphi_D(u, -, a)}}{!\varphi_D(u, -, a) \vdash !(\exists u)\varphi_D(u, -, a)}}{(\exists u)!\varphi_D(u, -, a) \vdash !(\exists u)\varphi_D(u, -, a)} \quad \text{and} \quad \frac{\frac{\frac{\varphi_D(-, x, a) \vdash \varphi_D(-, x, a)}{\varphi_D(-, x, a) \vdash ?\varphi_D(-, x, a)}}{(\forall x)\varphi_D(-, x, a) \vdash ?\varphi_D(-, x, a)}}{?\varphi_D(-, x, a) \vdash ?\varphi_D(-, x, a)}}{?\varphi_D(-, x, a) \vdash ?(\forall x)\varphi_D(-, x, a)}$$

For the converse implications, we use the exponential axioms (DEXP):

$$\frac{\frac{\frac{\varphi_D(u, -, a) \vdash !\varphi_D(u, -, a)}{\varphi_D(u, -, a) \vdash (\exists u)!\varphi_D(u, -, a)}}{(\exists u)\varphi_D(u, -, a) \vdash (\exists u)!\varphi_D(u, -, a)}}{!(\exists u)\varphi_D(u, -, a) \vdash (\exists u)!\varphi_D(u, -, a)} \quad \text{and} \quad \frac{\frac{\frac{?\varphi_D(-, x, a) \vdash \varphi_D(-, x, a)}{(\forall x)?\varphi_D(-, x, a) \vdash \varphi_D(-, x, a)}}{(\forall x)?\varphi_D(-, x, a) \vdash (\forall x)\varphi_D(-, x, a)}}{(\forall x)?\varphi_D(-, x, a) \vdash ?(\forall x)\varphi_D(-, x, a)}$$

**Characterization for Polarized Formulae.** There are two ways to see characterization for polarized formulae. The first one, stated in Theorem 3,

$$\vdash_{\text{FS}+(\text{LSIP})+(\text{DEXP})} \varphi(a) \multimap \varphi^D(a) \quad (\varphi \text{ PF-formula})$$

amounts to the following.

**Lemma 15.** FS + (LSIP) *proves*

$$\begin{aligned} \varphi \multimap \varphi^D, \psi \multimap \psi^D \vdash (\varphi \multimap \psi) \multimap (\varphi \multimap \psi)^D & \quad (\varphi(a) \multimap \psi(a) \text{ polarized}) \\ \varphi(a, b) \multimap \varphi^D(a, b) \vdash (\forall b.\varphi(a, b)) \multimap (\forall b.\varphi(a, b))^D & \quad ((\forall b)\varphi(a, b) \text{ negative}) \end{aligned}$$

The second one is to notice that characterization for polarized formulae is provable within the polarized fragment PF augmented with the following polarized weakening of (LSIP) (with polarities as displayed):

$$\begin{aligned} (\forall a)(\varphi^\pm(a) \otimes \psi^-) & \multimap (\forall a)\varphi^\pm(a) \otimes \psi^- \\ (\forall a)(\varphi^\pm(a) \wp \psi^-) & \multimap (\forall a)\varphi^\pm(a) \wp \psi^- \\ (\exists a)\varphi^\pm(a) \wp \psi^+ & \multimap (\exists a)(\varphi^\pm(a) \wp \psi^+) & \quad (\text{PLSIP}) \\ \psi^- \multimap (\exists a)\varphi^\pm(a) & \multimap (\exists a)(\psi^- \multimap \varphi^\pm(a)) \\ ((\forall a)\varphi^\pm(a) \multimap \psi^\pm) & \multimap (\exists a)(\varphi^\pm(a) \multimap \psi^\pm) \end{aligned}$$

We only detail Lemma 15, as it corresponds to the statement of Theorem 3.

*Proof of Lemma 15.* Consider first the case of  $\varphi(a) \multimap \psi(a)$ . By assumption, for  $\theta$  either  $\varphi$  or  $\psi$ , we have

$$\begin{aligned}\theta^+(a) &\multimap \exists u.\theta_D(u, -, a) \\ \theta^-(a) &\multimap \forall x.\theta_D(-, x, a) \\ \theta^\pm(a) &\multimap \theta_D(-, -, a)\end{aligned}$$

The case of  $\varphi^\pm(a) \multimap \psi^\pm(a)$  is trivial. The other cases are given by the following derivations (where we did not display the free variable  $a$ ).

– *Case of  $(\varphi^- \multimap \psi^+)^D$ .*

We first show

$$\begin{array}{c}(\varphi^- \multimap \psi^+)^D \multimap ((\varphi^-)^D \multimap (\psi^+)^D) \\ \hline \frac{\frac{\frac{\frac{\overline{\varphi_D(x) \vdash \varphi_D(x)}}{(\forall x)\varphi_D(x) \vdash \varphi_D(x)}{\varphi_D(x) \multimap \psi_D(v), (\forall x)\varphi_D(x) \vdash (\exists v)\psi_D(v)}}{\varphi_D(x) \multimap \psi_D(v) \vdash (\forall x)\varphi_D(x) \multimap (\exists v)\psi_D(v)}}{(\exists v, x)(\varphi_D(x) \multimap \psi_D(v)) \vdash (\forall x)\varphi_D(x) \multimap (\exists v)\psi_D(v)}}\end{array}$$

For the converse implication, we use the axioms

- $\psi^- \multimap (\exists a)\varphi^\pm(a) \multimap (\exists a)(\psi^- \multimap \varphi^\pm(a))$
- $((\forall a)\varphi^\pm(a) \multimap \psi^\pm) \multimap (\exists a)(\varphi^\pm(a) \multimap \psi^\pm)$

$$\frac{\mathcal{D} \quad \frac{(\exists v, x)(\varphi_D(x) \multimap \psi_D(v)) \vdash (\exists v, x)(\varphi_D(x) \multimap \psi_D(v))}{(\exists v)((\forall x)\varphi_D(x) \multimap \psi_D(v)) \vdash (\exists v, x)(\varphi_D(x) \multimap \psi_D(v))}}{(\forall x)\varphi_D(x) \multimap (\exists v)\psi_D(v) \vdash (\exists v, x)(\varphi_D(x) \multimap \psi_D(v))}}{\mathcal{D}'}$$

where  $\mathcal{D}$  is obtained from the axiom

$$((\forall a)\varphi^\pm(a) \multimap \psi^\pm) \multimap (\exists a)(\varphi^\pm(a) \multimap \psi^\pm)$$

and  $\mathcal{D}'$  is obtained from the axiom

$$\psi^- \multimap (\exists a)\varphi^\pm(a) \multimap (\exists a)(\psi^- \multimap \varphi^\pm(a))$$

– *Case of  $(\varphi^+ \multimap \psi^-)^D$ .*

We first show

$$\begin{array}{c}(\varphi^+ \multimap \psi^-)^D \multimap ((\varphi^+)^D \multimap (\psi^-)^D) \\ \hline \frac{\frac{\frac{\frac{\overline{\varphi_D(u) \vdash \varphi_D(u)}}{\varphi_D(u) \multimap \psi_D(y), \varphi_D(u) \vdash \psi_D(y)}}{(\forall y, u)(\varphi_D(u) \multimap \psi_D(y)), \varphi_D(u) \vdash \psi_D(y)}}{(\forall y, u)(\varphi_D(u) \multimap \psi_D(y)), (\exists u)\varphi_D(u) \vdash (\forall y)\psi_D(y)}}{(\forall y, u)(\varphi_D(x) \multimap \psi_D(u)) \vdash (\exists u)\varphi_D(u) \multimap (\forall y)\psi_D(y)}}\end{array}$$

The converse direction is given by

$$\frac{\frac{\frac{\varphi_D(u) \vdash \varphi_D(u)}{\varphi_D(u) \vdash (\exists u)\varphi_D(u)} \quad \frac{\frac{\psi_D(y) \vdash \psi_D(y)}{(\forall y)\psi_D(y) \vdash \psi_D(y)}}{(\exists u)\varphi_D(u) \multimap (\forall y)\psi_D(y), \varphi_D(u) \vdash \psi_D(y)}}{(\exists u)\varphi_D(u) \multimap (\forall y)\psi_D(y) \vdash \varphi_D(u) \multimap \psi_D(y)}}{(\exists u)\varphi_D(u) \multimap (\forall y)\psi_D(y) \vdash (\forall u, y)(\varphi_D(u) \multimap \psi_D(y))}$$

Consider now the case of  $(\forall a)\varphi^-(a)$ . Let

$$\varphi^D(a) = (\forall x)\varphi_D(-, x, a)$$

and by assumption

$$\varphi^D(a) \multimap \varphi(a)$$

We have to show

$$(\forall x)(\forall a)\varphi_D(-, x, a) \multimap (\forall a)\varphi^D(a)$$

But this is trivial since for negative  $\varphi^-$ , we have  $(\forall a.\varphi(a))^D = (\forall a)\varphi^D(a)$  by definition.  $\square$

#### B.4 Extraction (Corollary 1)

Let recall the statement of Corollary 1

**Corollary 5 (Extraction (Cor. 1)).** *Consider a closed formula  $\varphi := (\forall x^\sigma)(\exists u^\tau)\delta(u, x)$  with  $\delta$  deterministic. From a proof of  $\varphi$  in  $\text{FS} + (\text{LAC}) + (\text{LSIP}) + (\text{DEXP})$  one can extract a term  $\mathbf{t}(x)$  such that  $\models (\forall x^\sigma)[\delta](\mathbf{t}(x), x)$ .*

*Proof.* Note that

$$\varphi^D = (\exists f^{(\tau)\sigma})(\forall x^\sigma)\delta_D(-, -, \langle (f)x, x \rangle)$$

By Theorem 2 (Thm. 8) and Proposition 4 (Lem. 12 & 13), from a proof of  $\varphi$  in  $\text{FS} + (\text{LAC}) + (\text{LSIP}) + (\text{DEXP})$  we get an eager term  $\mathbf{u}^{(\tau)\sigma}(x^\sigma)$  such that

$$\vdash_{\text{FS}+(\text{DEXP})} \delta_D(-, -, \langle \mathbf{e}(\mathbf{u})(x), x \rangle)$$

Since  $\delta$  deterministic, we have

$$\delta^D(a) = \delta_D(-, -, a)$$

and from Characterization for polarized formulae (Theorem 3) we get

$$\vdash_{\text{FS}+(\text{DEXP})+(\text{PEXP})} \delta(\mathbf{e}(\mathbf{u})(x), x)$$

We thus obtain

$$\models (\forall x^\sigma)[\delta](\mathbf{t}(x), x) \quad (\text{where } \mathbf{t}^\tau(x) := \mathbf{e}(\mathbf{u})(x))$$

$\square$

### B.5 Proofs of §3.4 (Translations of Classical Logic)

While the usual  $(-)^T$  and  $(-)^Q$  translations target resp. negative and positive formulae [9] (see also [16]), one can consider the following deterministic variants:

$$\begin{array}{ll}
\top^{T\pm} := ?\mathbf{I} & \top^{Q\pm} := \mathbf{!I} \\
\perp^{T\pm} := ?\perp & \perp^{Q\pm} := \mathbf{!}\perp \\
(\mathbf{t} \doteq \mathbf{u})^{T\pm} := ?(\mathbf{t} \doteq \mathbf{u}) & (\mathbf{t} \doteq \mathbf{u})^{Q\pm} := \mathbf{!}(\mathbf{t} \doteq \mathbf{u}) \\
(\varphi \rightarrow \psi)^{T\pm} := ?(!\varphi^{T\pm} \multimap \psi^{T\pm}) & (\varphi \rightarrow \psi)^{Q\pm} := \mathbf{!}(\varphi^{Q\pm} \multimap ?\psi^{Q\pm}) \\
(\varphi \wedge \psi)^{T\pm} := ?(!\varphi^{T\pm} \otimes \mathbf{!}\psi^{T\pm}) & (\varphi \wedge \psi)^{Q\pm} := \mathbf{!}(\varphi^{Q\pm} \otimes \psi^{Q\pm}) \\
(\varphi \vee \psi)^{T\pm} := ?(\varphi^{T\pm} \wp \psi^{T\pm}) & (\varphi \vee \psi)^{Q\pm} := \mathbf{!}(?\varphi^{Q\pm} \wp ?\psi^{Q\pm}) \\
(\exists x.\varphi)^{T\pm} := ?(\exists x)\mathbf{!}\varphi^{T\pm} & (\exists x.\varphi)^{Q\pm} := \mathbf{!}?( \exists x)\varphi^{Q\pm} \\
(\forall x.\varphi)^{T\pm} := ?!(\forall x)\varphi^{T\pm} & (\forall x.\varphi)^{Q\pm} := \mathbf{!}(\forall x)?\varphi^{Q\pm}
\end{array}
\quad \text{and}$$

**Soundness (Proposition 6).** The soundness of  $(-)^{T\pm}$  and  $(-)^{Q\pm}$  proceeds as that of  $(-)^T$  and  $(-)^Q$ . We then easily deduce the soundness of  $(-)^L$ .

**Proposition 12.** *If many-sorted first-order logic proves a sequent*

$$\varphi_1, \dots, \varphi_n \vdash \psi_1, \dots, \psi_m$$

then

$$\text{and} \quad \begin{array}{l}
!\varphi_1^{T\pm}, \dots, \mathbf{!}\varphi_n^{T\pm} \vdash_{\text{FS}} \psi_1^{T\pm}, \dots, \psi_m^{T\pm} \\
\varphi_1^{Q\pm}, \dots, \varphi_n^{Q\pm} \vdash_{\text{FS}} ?\psi_1^{Q\pm}, \dots, ?\psi_m^{Q\pm}
\end{array}$$

*Proof.* The proof goes on as in the case of  $(-)^T$  and  $(-)^Q$ . We obtain sequents provable in FS because both translations produce sequents of the form  $\mathbf{!}(-) \vdash ?(-)$ , so that the right  $\multimap$  and  $\forall$  rules of FS

$$\frac{\overline{\varphi}, \mathbf{!}\varphi \vdash \psi, ?\overline{\psi}}{\overline{\varphi} \vdash \mathbf{!}\varphi \multimap \psi, ?\overline{\psi}} \quad \frac{\overline{\varphi} \vdash \varphi, ?\overline{\psi}}{\overline{\varphi} \vdash (\forall z)\varphi, ?\overline{\psi}}$$

can be used for the right  $\rightarrow$  and  $\forall$  rules of classical logic

$$\frac{\overline{\varphi}, \varphi \vdash \psi, \overline{\psi}}{\overline{\varphi} \vdash \varphi \multimap \psi, \overline{\psi}} \quad \frac{\overline{\varphi} \vdash \varphi, \overline{\psi}}{\overline{\varphi} \vdash (\forall z)\varphi, \overline{\psi}}$$

We only detail the cases which differs from [16]. As usual we write

$$\varphi^{T\pm} = ?\varphi_{T\pm} \quad \text{and} \quad \varphi^{Q\pm} = ?\varphi_{Q\pm}$$

– Case of the  $(-)^{T\pm}$  translation of the rule

$$\frac{\overline{\varphi}, \varphi_0, \varphi_1 \vdash \overline{\psi}}{\overline{\varphi}, \varphi_0 \wedge \varphi_1 \vdash \overline{\psi}}$$



– Case of the  $(-)^{T^\pm}$  translation of the rule

$$\frac{\bar{\varphi} \vdash \varphi, \bar{\psi}}{\bar{\varphi} \vdash (\forall z)\varphi, \bar{\psi}}$$

The result follows from

$$\frac{\frac{\frac{\vdots}{!?\bar{\varphi}_{T^\pm} \vdash ?\varphi_{T^\pm}, ?\bar{\psi}_{T^\pm}}{!?\bar{\varphi}_{T^\pm} \vdash (\forall z)?\varphi_{T^\pm}, ?\bar{\psi}_{T^\pm}}{!?\bar{\varphi}_{T^\pm} \vdash !(\forall z)?\varphi_{T^\pm}, ?\bar{\psi}_{T^\pm}}}{!?\bar{\varphi}_{T^\pm} \vdash ?!(\forall z)?\varphi_{T^\pm}, ?\bar{\psi}_{T^\pm}}$$

– Case of the  $(-)^{Q^\pm}$  translation of the rule

$$\frac{\bar{\varphi}, \varphi \vdash \bar{\varphi}' \quad \bar{\psi}, \psi \vdash \bar{\psi}'}{\bar{\varphi}, \bar{\psi}, \varphi \vee \psi \vdash \bar{\varphi}', \bar{\psi}'}$$

The result follows from

$$\frac{\frac{\frac{\vdots}{!\bar{\varphi}_{Q^\pm}, !\varphi_{Q^\pm} \vdash ?!\bar{\varphi}'_{Q^\pm}}{!\bar{\varphi}_{Q^\pm}, ?!\varphi_{Q^\pm} \vdash ?!\bar{\varphi}'_{Q^\pm}}{!\bar{\varphi}_{Q^\pm}, !\bar{\psi}_{Q^\pm}, ?!\varphi_{Q^\pm} \wp ?!\psi_{Q^\pm} \vdash ?!\bar{\varphi}'_{Q^\pm}, ?!\bar{\psi}'_{Q^\pm}}}{!\bar{\varphi}_{Q^\pm}, !\bar{\psi}_{Q^\pm}, !(?!\varphi_{Q^\pm} \wp ?!\psi_{Q^\pm}) \vdash ?!\bar{\varphi}'_{Q^\pm}, ?!\bar{\psi}'_{Q^\pm}}$$

– Case of the  $(-)^{Q^\pm}$  translation of the rule

$$\frac{\bar{\varphi} \vdash \varphi_0, \varphi_1, \bar{\varphi}'}{\bar{\varphi} \vdash \varphi_0 \vee \varphi_1, \bar{\varphi}'}$$

The result follows from

$$\frac{\frac{\frac{\vdots}{!\bar{\varphi}_{Q^\pm} \vdash ?!\varphi_{0Q^\pm}, ?!\varphi_{1Q^\pm}, ?!\bar{\varphi}'_{Q^\pm}}{!\bar{\varphi}_{Q^\pm} \vdash ?!\varphi_{0Q^\pm} \wp ?!\varphi_{1Q^\pm}, ?!\bar{\varphi}'_{Q^\pm}}{!\bar{\varphi}_{Q^\pm} \vdash ?(?!\varphi_{0Q^\pm} \wp ?!\varphi_{1Q^\pm}), ?!\bar{\varphi}'_{Q^\pm}}$$

– Case of the  $(-)^{Q^\pm}$  translation of the rule

$$\frac{\bar{\varphi}, \varphi \vdash \bar{\psi}}{\bar{\varphi}, (\exists z)\varphi \vdash \bar{\psi}}$$

The result follows from

$$\begin{array}{c}
\vdots \\
\hline
! \bar{\varphi}_{Q\pm}, !\varphi_{Q\pm} \vdash ?!\bar{\psi}_{Q\pm} \\
\hline
! \bar{\varphi}_{Q\pm}, (\exists z)!\varphi_{Q\pm} \vdash ?!\bar{\psi}_{Q\pm} \\
\hline
! \bar{\varphi}_{Q\pm}, ?(\exists z)!\varphi_{Q\pm} \vdash ?!\bar{\psi}_{Q\pm} \\
\hline
! \bar{\varphi}_{Q\pm}, !?(\exists z)!\varphi_{Q\pm} \vdash ?!\bar{\psi}_{Q\pm}
\end{array}$$

– Case of the  $(-)^{Q\pm}$  translation of the rule

$$\begin{array}{c}
\bar{\varphi} \vdash \varphi[\mathbf{t}/x], \bar{\psi} \\
\hline
\bar{\varphi} \vdash (\exists x)\varphi, \bar{\psi}
\end{array}$$

The result follows by cutting

$$\begin{array}{c}
\hline
!\varphi_{Q\pm} \vdash !\varphi_{Q\pm} \\
\hline
!\varphi_{Q\pm} \vdash (\exists x)!\varphi_{Q\pm} \\
\hline
!\varphi_{Q\pm} \vdash !(\exists x)!\varphi_{Q\pm} \\
\hline
!\varphi_{Q\pm} \vdash !?(\exists x)!\varphi_{Q\pm} \\
\hline
!\varphi_{Q\pm} \vdash !!?(\exists x)!\varphi_{Q\pm} \\
\hline
?! \varphi_{Q\pm} \vdash !!?(\exists x)!\varphi_{Q\pm} \\
\hline
(\exists x)?! \varphi_{Q\pm} \vdash !!?(\exists x)!\varphi_{Q\pm} \\
\hline
!(\exists x)?! \varphi_{Q\pm} \vdash !!?(\exists x)!\varphi_{Q\pm} \\
\hline
?!(\exists x)?! \varphi_{Q\pm} \vdash !!?(\exists x)!\varphi_{Q\pm}
\end{array}$$

with

$$\begin{array}{c}
\vdots \\
\hline
! \bar{\varphi}_{Q\pm} \vdash ?!\varphi_{Q\pm}[\mathbf{t}/x], ?!\bar{\psi}_{Q\pm} \\
\hline
! \bar{\varphi}_{Q\pm} \vdash (\exists x)?! \varphi_{Q\pm}, ?!\bar{\psi}_{Q\pm} \\
\hline
! \bar{\varphi}_{Q\pm} \vdash !(\exists x)?! \varphi_{Q\pm}, ?!\bar{\psi}_{Q\pm} \\
\hline
! \bar{\varphi}_{Q\pm} \vdash ?!(\exists x)?! \varphi_{Q\pm}, ?!\bar{\psi}_{Q\pm}
\end{array}$$

□

The soundness of  $(-)^L$  (Proposition 6) then follows by noticing that  $\text{FS} + (\text{DEXP})$  proves that  $(-)^{T\pm}$ ,  $(-)^{Q\pm}$  and  $(-)^L$  are all equivalent.

**Lemma 16.** *For all formula  $\varphi$  of classical logic with equality, we have*

$$\vdash_{\text{FS}+(\text{DEXP})} \varphi^{T\pm} \quad \circ\text{-}\circ \quad \varphi^L \quad \circ\text{-}\circ \quad \varphi^{Q\pm}$$



**Corollary 6 (Prop. 6).** *If many-sorted first-order logic proves a sequent*

$$\varphi_1, \dots, \varphi_n \vdash \psi_1, \dots, \psi_m$$

then

$$\varphi_1^L, \dots, \varphi_n^L \vdash_{\text{FS}+(\text{DEXP})} \psi_1^L, \dots, \psi_m^L$$

*Proof.* It remains to deal with the equality axioms (7). But they follow from the fact that  $\alpha^L = \alpha$  for each atomic formula  $\alpha \in \text{At}$ .  $\square$

**Proof of Proposition 5.** We now turn to Prop. 5, namely the equivalence of (DEXP) + (PEXP) with

$$?\varphi^+ \circ\text{-} [\varphi^+]^L \quad \delta^\pm \circ\text{-} [\delta^\pm]^L \quad !\psi^- \circ\text{-} [\psi^-]^L$$

First, notice that

$$\begin{array}{ll} [\mathbf{I}]^{T^\pm} = ?\mathbf{I} & [\mathbf{I}]^{Q^\pm} = \mathbf{I!} \\ [\perp]^{T^\pm} = ?\perp & [\perp]^{Q^\pm} = !\perp \\ [(\mathbf{t} \doteq \mathbf{u})]^{T^\pm} = ?(\mathbf{t} \doteq \mathbf{u}) & [(\mathbf{t} \doteq \mathbf{u})]^{Q^\pm} = !(\mathbf{t} \doteq \mathbf{u}) \\ [\varphi \multimap \psi]^{T^\pm} = ?(![\varphi]^{T^\pm} \multimap [\psi]^{T^\pm}) & [\varphi \multimap \psi]^{Q^\pm} = !([\varphi]^{Q^\pm} \multimap ?[\psi]^{Q^\pm}) \\ [\varphi \otimes \psi]^{T^\pm} = ?(![\varphi]^{T^\pm} \otimes ![\psi]^{T^\pm}) & [\varphi \otimes \psi]^{Q^\pm} = !([\varphi]^{Q^\pm} \otimes [\psi]^{Q^\pm}) \\ [\varphi \wp \psi]^{T^\pm} = ?([\varphi]^{T^\pm} \wp [\psi]^{T^\pm}) & [\varphi \wp \psi]^{Q^\pm} = !(?[\varphi]^{Q^\pm} \wp ?[\psi]^{Q^\pm}) \\ [!\varphi]^{T^\pm} = [\varphi]^{T^\pm} & [!\varphi]^{Q^\pm} = [\varphi]^{Q^\pm} \\ [?\varphi]^{T^\pm} = [\varphi]^{T^\pm} & [?\varphi]^{Q^\pm} = [\varphi]^{Q^\pm} \\ [\exists x.\varphi]^{T^\pm} = ?(\exists x)![\varphi]^{T^\pm} & [\exists x.\varphi]^{Q^\pm} = !?( \exists x)[\varphi]^{Q^\pm} \\ [\forall x.\varphi]^{T^\pm} = ?!(\forall x)[\varphi]^{T^\pm} & [\forall x.\varphi]^{Q^\pm} = !(\forall x)?[\varphi]^{Q^\pm} \end{array}$$

Hence (DEXP) proves

$$[\varphi]^{T^\pm} \circ\text{-} [\varphi]^L \circ\text{-} [\varphi]^{Q^\pm}$$

Moreover, (DEXP) follows from the equivalence

$$[\delta]^L \circ\text{-} \delta \quad (\delta \text{ deterministic})$$

So we can as well prove Proposition 5 with  $(-)^{T^\pm}$  instead of  $(-)^L$ . We split this into two statements.

**Lemma 17.** *FS augmented with the axioms*

$$?\varphi^+ \circ\text{-} [\varphi^+]^{T^\pm} \quad \delta^\pm \circ\text{-} [\delta^\pm]^{T^\pm} \quad !\psi^- \circ\text{-} [\psi^-]^{T^\pm}$$

(for formulae  $\varphi^+, \psi^-, \delta^\pm$  with the displayed equalities), proves all instances of (DEXP) and (PEXP).

*Proof.* We show that FS augmented with the axioms

$$?\varphi^+ \multimap [\varphi^+]^{T^\pm} \quad \delta^\pm \multimap [\delta^\pm]^{T^\pm} \quad !\psi^- \multimap [\psi^-]^{T^\pm}$$

proves

- (1)  $\delta^\pm \multimap !\delta^\pm$ .
- (2)  $?\delta^\pm \multimap \delta^\pm$ .
- (3)  $?( \varphi^+ ) \multimap ?!( \varphi^+ )$ .
- (4)  $!( \psi^- ) \multimap !(\psi^-)$ .
- (5)  $!( \varphi^- \multimap ?(\psi^+) ) \multimap ?(\varphi^- \multimap \psi^+)$ .
- (6)  $(?( \varphi^+ ) \multimap !(\psi^-)) \multimap !( \varphi^+ \multimap \psi^- )$ .
- (7)  $?( \varphi^+ ) \otimes ?(\psi^+) \multimap ?(\varphi^+ \otimes \psi^+)$ .
- (8)  $!( \varphi^- \otimes \psi^- ) \multimap !( \varphi^- ) \otimes !(\psi^-)$ .
- (9)  $?( \varphi^+ ) \wp ?(\psi^+) \multimap ?(\varphi^+ \wp \psi^+)$ .
- (10)  $!( \varphi^- \wp \psi^- ) \multimap !( \varphi^- ) \wp !(\psi^-)$ .

- (1) We have  $[\delta]^{T^\pm} = [\delta]^{T^\pm}$  and we obtain the result by following the chain of implications

$$\delta^\pm \multimap [\delta^\pm]^{T^\pm} = [!\delta^\pm]^{T^\pm} \multimap !\delta^\pm$$

- (2) Similar.

- (3) We have  $[!\varphi]^{T^\pm} = [\varphi]^{T^\pm}$  and we obtain the result by following the chain of implications

$$?( \varphi^+ ) \multimap [\varphi^+]^{T^\pm} = [!(\varphi^+)]^{T^\pm} \multimap ?!(\varphi^+)$$

- (4) Similar.

- (5) We have

$$!(\varphi^-) \multimap [\varphi^-]^{T^\pm} \quad ?(\psi^+) \multimap [\psi^+]^{T^\pm} \quad \text{and} \quad ?(\varphi^- \multimap \psi^+) \multimap [\varphi^- \multimap \psi^+]^{T^\pm}$$

Using the ! and ?-properties of deterministic formulae, it follows that we have the following chain of implications:

$$\begin{aligned} ( !(\varphi^-) \multimap ?(\psi^+) ) &\multimap ( [\varphi^-]^{T^\pm} \multimap [\psi^+]^{T^\pm} ) \\ &\multimap ?( [!\varphi^-]^{T^\pm} \multimap [\psi^+]^{T^\pm} ) \\ &= [\varphi^- \multimap \psi^+]^{T^\pm} \\ &\multimap ?(\varphi^- \multimap \psi^+) \end{aligned}$$

- (6) Similar.

- (7) We have

$$?( \varphi^+ ) \multimap [\varphi^+]^{T^\pm} \quad ?(\psi^+) \multimap [\psi^+]^{T^\pm} \quad \text{and} \quad ?(\varphi^+ \otimes \psi^+) \multimap [\varphi^+ \otimes \psi^+]^{T^\pm}$$

Using the ! and ?-properties of deterministic formulae, it follows that we have the following chain of implications:

$$\begin{aligned} ( ?(\varphi^+) \otimes ?(\psi^+) ) &\multimap ( [\varphi^+]^{T^\pm} \otimes [\psi^+]^{T^\pm} ) \\ &\multimap ?( [!\varphi^+]^{T^\pm} \otimes [!\psi^+]^{T^\pm} ) \\ &= [\varphi^+ \otimes \psi^+]^{T^\pm} \\ &\multimap ?(\varphi^+ \otimes \psi^+) \end{aligned}$$

(8) Similar.

(9) We have

$$?(\varphi^+) \circ\text{-} [\varphi^+]^{T^\pm} \quad ?(\psi^+) \circ\text{-} [\psi^+]^{T^\pm} \quad \text{and} \quad ?(\varphi^+ \wp \psi^+) \circ\text{-} [\varphi^+ \wp \psi^+]^{T^\pm}$$

Using the ?-property of deterministic formulae, it follows that we have the following chain of implications:

$$\begin{aligned} (?(\varphi^+) \wp ?(\psi^+)) &\text{-} ([\varphi^+]^{T^\pm} \wp [\psi^+]^{T^\pm}) \\ &\text{-} ?([\varphi^+]^{T^\pm} \wp [\psi^+]^{T^\pm}) \\ &= [\varphi^+ \wp \psi^+]^{T^\pm} \\ &\text{-} ?(\varphi^+ \wp \psi^+) \end{aligned}$$

(10) Similar. □

**Lemma 18.** *The equivalences*

$$?\varphi^+ \circ\text{-} [\varphi^+]^{T^\pm} \quad \delta^\pm \circ\text{-} [\delta^\pm]^{T^\pm} \quad !\psi^- \circ\text{-} [\psi^-]^{T^\pm}$$

(where  $\varphi^+$ ,  $\delta^\pm$  and  $\psi^-$  have the displayed polarities) are provable in FS + (DEXP) + (PEXP).

*Proof.* First note that using the ! and ?-axioms on deterministic formulae, for a deterministic  $\delta^\pm$ , the equivalence

$$\delta \circ\text{-} [\delta]^{T^\pm}$$

follows from the series of equivalences

$$[\delta^+]^{T^\pm} \circ\text{-} ?(\delta^+) \circ\text{-} \delta^\pm \circ\text{-} !(\delta^-) \circ\text{-} [\delta^-]^{T^\pm}$$

Conversely, the equivalence

$$\delta \circ\text{-} [\delta]^{T^\pm}$$

entails

$$?(\delta^+) \circ\text{-} [\delta^+]^{T^\pm} \quad \text{and} \quad !(\delta^-) \circ\text{-} [\delta^-]^{T^\pm}$$

We proceed by simultaneous induction on formulae.

– *Case of  $\varphi$  and atomic formula  $\alpha$ .* Since  $[\alpha]^{T^\pm} = ?\alpha$ , we have to show

$$?\alpha^+ \circ\text{-} ?\alpha \quad \text{and} \quad !\alpha^- \circ\text{-} ?\alpha$$

and we are done by the ! and ?-axioms on deterministic formulae.

– *Case of  $(\varphi^- \text{-} \psi^+)^+$ .* We have

$$[\varphi^- \text{-} \psi^+]^{T^\pm} = ?(![\varphi^-]^{T^\pm} \text{-} [\psi^+]^{T^\pm})$$

so that we have to show

$$?(\varphi^- \text{-} \psi^+) \circ\text{-} ?(![\varphi^-]^{T^\pm} \text{-} [\psi^+]^{T^\pm})$$

By induction hypothesis we have

$$!(\varphi^-) \circ\text{-} [\varphi^-]^{T^\pm} \quad \text{and} \quad ?(\psi^+) \circ\text{-} [\psi^+]^{T^\pm}$$

We are thus left with showing

$$?(\varphi^- \text{-} \psi^+) \circ\text{-} ?(!!(\varphi^-) \text{-} ?(\psi^+))$$

The left-to-right implication follows from

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\varphi^- \vdash \varphi^-}{\varphi^- \text{-} \psi^+, \varphi^- \vdash \psi^+}}{\varphi^- \text{-} \psi^+, !(\varphi^-) \vdash \psi^+}}{\varphi^- \text{-} \psi^+, !!(\varphi^-) \vdash \psi^+}}{\varphi^- \text{-} \psi^+, !!(\varphi^-) \vdash ?(\psi^+)}}{\varphi^- \text{-} \psi^+ \vdash !!(\varphi^-) \text{-} ?(\psi^+)}}{\varphi^- \text{-} \psi^+ \vdash ?(!!(\varphi^-) \text{-} ?(\psi^+))}}{?(\varphi^- \text{-} \psi^+) \vdash ?(!!(\varphi^-) \text{-} ?(\psi^+))}}$$

The right-to-left implication follows from the  $?/ \text{-} \circ$ -axiom for positive implications:

$$\frac{\frac{\frac{\frac{\frac{!(\varphi^-) \vdash !(\varphi^-)}{!(\varphi^-) \vdash !!(\varphi^-)}{!!(\varphi^-) \text{-} ?(\psi^+), !(\varphi^-) \vdash ?(\psi^+)}}{!!(\varphi^-) \text{-} ?(\psi^+) \vdash !(\varphi^-) \text{-} ?(\psi^+)}}{!(\varphi^-) \text{-} ?(\psi^+) \vdash ?(\varphi^- \text{-} \psi^+)}}{!!(\varphi^-) \text{-} ?(\psi^+) \vdash ?(\varphi^- \text{-} \psi^+)}}{?(!!(\varphi^-) \text{-} ?(\psi^+)) \vdash ?(\varphi^- \text{-} \psi^+)}}$$

– *Case of  $(\varphi^+ \text{-} \psi^-)^-$ .* We have

$$[\varphi \text{-} \psi]^{T^\pm} = ?(![\varphi]^{T^\pm} \text{-} [\psi]^{T^\pm})$$

so that we have to show

$$!(\varphi^+ \text{-} \psi^-) \circ\text{-} ?(![\varphi]^{T^\pm} \text{-} [\psi]^{T^\pm})$$

By induction hypothesis we have

$$?(\varphi^+) \circ\text{-} [\varphi^+]^{T^\pm} \quad \text{and} \quad !(\psi^-) \circ\text{-} [\psi^-]^{T^\pm}$$

We are thus left with showing

$$!(\varphi^+ \text{-} \psi^-) \circ\text{-} ?(!(\varphi^+) \text{-} !(\psi^-))$$

The left-to-right implication follows from the !-axiom for deterministic formulae. We cut

$$\frac{\frac{\frac{\overline{\varphi^+ \vdash \varphi^+}}{\varphi^+ \vdash ?(\varphi^+)}}{?(\varphi^+) \multimap !(\psi^-), \varphi^+ \vdash \psi^-}}{?(\varphi^+) \multimap !(\psi^-) \vdash \varphi^+ \multimap \psi^-}}{\frac{!(\varphi^+) \multimap !(\psi^-) \vdash \varphi^+ \multimap \psi^-}{!(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+ \multimap \psi^-)}} \quad \frac{!(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+ \multimap \psi^-)}{?(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+ \multimap \psi^-)}}$$

with

$$\frac{\frac{\frac{?(\varphi^+) \vdash !(\varphi^+) \quad !(\psi^-) \vdash !(\psi^-)}{!(\varphi^+) \multimap !(\psi^-), ?(\varphi^+) \vdash !(\psi^-)}}{!(\varphi^+) \multimap !(\psi^-) \vdash ?(\varphi^+) \multimap !(\psi^-)} \quad ?(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+ \multimap \psi^-)}{!(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+ \multimap \psi^-)}}{!(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+ \multimap \psi^-)}}$$

The right-to-left implication follows from the !/  $\multimap$ -axiom for negative implications:

$$\frac{\frac{\frac{\frac{?(\varphi^+) \vdash ?(\varphi^+) \quad !(\psi^-) \vdash !(\psi^-)}{?(\varphi^+) \multimap !(\psi^-), ?(\varphi^+) \vdash !(\psi^-)}}{?(\varphi^+) \multimap !(\psi^-), !(\varphi^+) \vdash !(\psi^-)}}{?(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+) \multimap !(\psi^-)}}{!(\varphi^+ \multimap \psi^-) \vdash ?(\varphi^+) \multimap !(\psi^-)} \quad ?(\varphi^+) \multimap !(\psi^-) \vdash !(\varphi^+) \multimap !(\psi^-)}}{!(\varphi^+ \multimap \psi^-) \vdash !(\varphi^+) \multimap !(\psi^-)}}$$

– *Case of  $\varphi^+ \otimes \psi^+$ . We have*

$$[\varphi \otimes \psi]^{T^\pm} = ?([\varphi]^{T^\pm} \otimes ![\psi]^{T^\pm})$$

so that we have to show

$$?(\varphi^+ \otimes \psi^+) \multimap ?([\varphi^+]^{T^\pm} \otimes ![\psi^+]^{T^\pm})$$

The induction hypothesis gives

$$?(\varphi^+) \multimap [\varphi^+]^{T^\pm} \quad \text{and} \quad ?(\psi^+) \multimap ![\psi^+]^{T^\pm}$$

We are thus left with showing

$$?(\varphi^+ \otimes \psi^+) \multimap ?(!(\varphi^+) \otimes !(\psi^+))$$

The left-to-right implication follows from the !-axiom on deterministic formulae:

$$\begin{array}{c}
\frac{\overline{\varphi^+ \vdash \varphi^+}}{\varphi^+ \vdash ?(\varphi^+)} \quad ?(\varphi^+) \vdash !?(\varphi^+) \quad \frac{\overline{\psi^+ \vdash \psi^+}}{\psi^+ \vdash ?(\psi^+)} \quad ?(\psi^+) \vdash !?(\psi^+) \\
\hline
\frac{\varphi^+ \vdash !?(\varphi^+) \quad \psi^+ \vdash !?(\psi^+)}{\varphi^+, \psi^+ \vdash !?(\varphi^+) \otimes !?(\psi^+)} \\
\frac{\varphi^+, \psi^+ \vdash !?(\varphi^+) \otimes !?(\psi^+)}{\varphi^+ \otimes \psi^+ \vdash ?(!?(\varphi^+) \otimes !?(\psi^+))} \\
\frac{\varphi^+ \otimes \psi^+ \vdash ?(!?(\varphi^+) \otimes !?(\psi^+))}{?(\varphi^+ \otimes \psi^+) \vdash ?(!?(\varphi^+) \otimes !?(\psi^+))}
\end{array}$$

The right-to-left implication follows from the ?/⊗-axiom for positive formulae:

$$\begin{array}{c}
?(\varphi^+), ?(\psi^+) \vdash ?(\varphi^+ \otimes \psi^+) \\
\frac{!?( \varphi^+ ), !?( \psi^+ ) \vdash ?(\varphi^+ \otimes \psi^+)}{!?( \varphi^+ ) \otimes !?( \psi^+ ) \vdash ?(\varphi^+ \otimes \psi^+)} \\
\frac{!?( \varphi^+ ) \otimes !?( \psi^+ ) \vdash ?(\varphi^+ \otimes \psi^+)}{?( !?( \varphi^+ ) \otimes !?( \psi^+ ) ) \vdash ?(\varphi^+ \otimes \psi^+)}
\end{array}$$

– *Case of  $\varphi^- \otimes \psi^-$ .* We have

$$[\varphi \otimes \psi]^{T^\pm} = ?([\varphi]^{T^\pm} \otimes ![\psi]^{T^\pm})$$

so that we have to show

$$!(\varphi^- \otimes \psi^-) \circ\!\circ ?([\varphi^-]^{T^\pm} \otimes ![\psi^-]^{T^\pm})$$

The induction hypothesis gives

$$!(\varphi^-) \circ\!\circ [\varphi^-]^{T^\pm} \quad \text{and} \quad !(\psi^-) \circ\!\circ [\psi^-]^{T^\pm}$$

We are thus left with showing

$$!(\varphi^- \otimes \psi^-) \circ\!\circ !( ([\varphi^-]^{T^\pm} \otimes !([\psi^-]^{T^\pm})) )$$

The left-to-right implication follows from the !/⊗-axiom for negative formulae:

$$\begin{array}{c}
\frac{\overline{!(\varphi^-) \vdash !(\varphi^-)}}{!(\varphi^-) \vdash !!(\varphi^-)} \quad \frac{\overline{!(\psi^-) \vdash !(\psi^-)}}{!(\psi^-) \vdash !!(\psi^-)} \\
\frac{!(\varphi^-), !(\psi^-) \vdash !!(\varphi^-) \otimes !!(\psi^-)}{!(\varphi^-) \otimes !(\psi^-) \vdash !!(\varphi^-) \otimes !!(\psi^-)} \\
\frac{!(\varphi^- \otimes \psi^-) \vdash !(\varphi^-) \otimes !(\psi^-) \quad !(\varphi^-) \otimes !(\psi^-) \vdash !!(\varphi^-) \otimes !!(\psi^-)}{!(\varphi^- \otimes \psi^-) \vdash !!(\varphi^-) \otimes !!(\psi^-)} \\
\frac{!(\varphi^- \otimes \psi^-) \vdash !!(\varphi^-) \otimes !!(\psi^-)}{!(\varphi^- \otimes \psi^-) \vdash !( ([\varphi^-]^{T^\pm} \otimes !([\psi^-]^{T^\pm})) )}
\end{array}$$

The right-to-left implication is given as follows:

$$\frac{\frac{\frac{\overline{\varphi^- \vdash \varphi^-}}{!(\varphi^-) \vdash \varphi^-}}{!!(\varphi^-) \vdash \varphi^-} \quad \frac{\frac{\overline{\psi^- \vdash \psi^-}}{!(\psi^-) \vdash \psi^-}}{!!(\psi^-) \vdash \psi^-}}{\frac{!!(\varphi^-), !!(\psi^-) \vdash \varphi^- \otimes \psi^-}{!!(\varphi^-) \otimes !!(\psi^-) \vdash \varphi^- \otimes \psi^-}}}{\frac{!(!!(\varphi^-) \otimes !!(\psi^-)) \vdash \varphi^- \otimes \psi^-}{!(!!(\varphi^-) \otimes !!(\psi^-)) \vdash !(\varphi^- \otimes \psi^-)}}$$

– *Case of  $\varphi^+ \wp \psi^+$ . We have*

$$[\varphi \wp \psi]^{T^\pm} = ?([\varphi]^{T^\pm} \wp [\psi]^{T^\pm})$$

so that we have to show

$$?(\varphi^+ \wp \psi^+) \circ\text{-} ?([\varphi^+]^{T^\pm} \wp [\psi^+]^{T^\pm})$$

The induction hypothesis gives

$$?(\varphi^+) \circ\text{-} [\varphi^+]^{T^\pm} \quad \text{and} \quad ?(\psi^+) \circ\text{-} [\psi^+]^{T^\pm}$$

We are thus left with showing

$$?(\varphi^+ \wp \psi^+) \circ\text{-} ?(?(\varphi^+) \wp ?(\psi^+))$$

The left-to-right implication is given by

$$\frac{\frac{\frac{\overline{\varphi^+ \vdash \varphi^+}}{\varphi^+ \vdash ?(\varphi^+)} \quad \frac{\overline{\psi^+ \vdash \psi^+}}{\psi^+ \vdash ?(\psi^+)}}{\frac{\varphi^+ \wp \psi^+ \vdash ?(\varphi^+), ?(\psi^+)}{\varphi^+ \wp \psi^+ \vdash ?(\varphi^+) \wp ?(\psi^+)}}}{\frac{\varphi^+ \wp \psi^+ \vdash ?(?(\varphi^+) \wp ?(\psi^+))}{?(\varphi^+ \wp \psi^+) \vdash ?(?(\varphi^+) \wp ?(\psi^+)}}}$$

The right-to-left implication follows from the  $?\wp$ -axiom on positive formulae:

$$\frac{?(\varphi^+) \wp ?(\psi^+) \vdash ?(\varphi^+ \wp \psi^+)}{?(?(\varphi^+) \wp ?(\psi^+)) \vdash ?(\varphi^+ \wp \psi^+)}$$

– *Case of  $\varphi^- \wp \psi^-$ . We have*

$$[\varphi \wp \psi]^{T^\pm} = ?([\varphi]^{T^\pm} \wp [\psi]^{T^\pm})$$

so that we have to show

$$!(\varphi^- \wp \psi^-) \circ\text{-} ?([\varphi^-]^{T^\pm} \wp [\psi^-]^{T^\pm})$$

The induction hypothesis gives

$$!(\varphi^-) \circ\!\circ [\varphi^-]^{T^\pm} \quad \text{and} \quad !(\psi^-) \circ\!\circ [\psi^-]^{T^\pm}$$

We are thus left with showing

$$!(\varphi^- \wp \psi^-) \circ\!\circ ?!(\varphi^-) \wp !(\psi^-)$$

The left-to-right implication follows from the  $!/\wp$ -axiom for negative formulae:

$$\frac{!(\varphi^- \wp \psi^-) \vdash !(\varphi^-) \wp !(\psi^-)}{!(\varphi^- \wp \psi^-) \vdash ?!(\varphi^-) \wp !(\psi^-)}$$

The right-to-left implication follows from the  $!$  and  $?$ -axioms on deterministic formulae:

$$\frac{\frac{\frac{\frac{\overline{\varphi^- \vdash \varphi^-}}{!(\varphi^-) \vdash \varphi^-} \quad \frac{\overline{\psi^- \vdash \psi^-}}{!(\psi^-) \vdash \psi^-}}{!(\varphi^-) \wp !(\psi^-) \vdash \varphi^-, \psi^-}}{!(\varphi^-) \wp !(\psi^-) \vdash \varphi^- \wp \psi^-} \quad \frac{}{\varphi^- \wp \psi^- \vdash !(\varphi^- \wp \psi^-)}}{?!(\varphi^-) \wp !(\psi^-) \vdash !(\varphi^-) \wp !(\psi^-)} \quad \frac{}{!(\varphi^-) \wp !(\psi^-) \vdash !(\varphi^- \wp \psi^-)}}{?!(\varphi^-) \wp !(\psi^-) \vdash !(\varphi^-) \wp !(\psi^-) \quad !(\varphi^-) \wp !(\psi^-) \vdash !(\varphi^- \wp \psi^-)}$$

– *Case of  $(!(\varphi^+))^+$ .* Since  $[\varphi]^{T^\pm} = [\varphi]^{T^\pm}$  we have to show

$$?!\varphi \circ\!\circ [\varphi]^{T^\pm}$$

We have  $[\varphi^+]^{T^\pm} \circ\!\circ ?(\varphi^+)$  by induction hypothesis. We are thus left with

$$?!(\varphi^+) \circ\!\circ ?(\varphi^+)$$

and the result follows from the  $!$ -axiom on positive formulae.

– *Case of  $(!(\varphi^-))^\pm$ .* Since  $[\varphi]^{T^\pm} = [\varphi]^{T^\pm}$  we have to show

$$!\varphi \circ\!\circ [\varphi]^{T^\pm}$$

We have  $[\varphi^-]^{T^\pm} \circ\!\circ !(\varphi^-)$  by induction hypothesis. We are thus left with

$$!\varphi \circ\!\circ !\varphi$$

and we are done.

– *Case of  $(?(\varphi^-))^-$ .* Since  $[?\varphi]^{T^\pm} = [\varphi]^{T^\pm}$  we have to show

$$!?\varphi \circ\!\circ [\varphi]^{T^\pm}$$

We have  $[\varphi^-]^{T^\pm} \circ\!\circ !(\varphi^-)$  by induction hypothesis. We are thus left with showing

$$!?\varphi \circ\!\circ !\varphi$$

and the result follows from the  $?$ -axiom on negative formulae.



– *Case of  $(\varphi^+)^{\pm}$ .* Since  $[\varphi^+]^{T\pm} = [\varphi]^{T\pm}$  we have to show

$$?\varphi \circ\!\circ [\varphi]^{T\pm}$$

We have  $[\varphi^+]^{T\pm} \circ\!\circ ?(\varphi^+)$  by induction hypothesis. We are thus left with

$$?\varphi \circ\!\circ ?\varphi$$

and we are done.

– *Case of  $(\exists x)\varphi^+$ .* Since

$$[(\exists x)\varphi]^{T\pm} = ?(\exists x)![\varphi]^{T\pm}$$

we have to show

$$?(\exists x)\varphi \circ\!\circ ?(\exists x)![\varphi]^{T\pm}$$

The induction hypothesis gives

$$?(\varphi^+) \circ\!\circ [\varphi^+]^{T\pm}$$

and it follows from the !-axioms on deterministic formulae that we have the sequence of equivalences

$$?(\exists x)![\varphi]^{T\pm} \circ\!\circ ?(\exists x)[\varphi]^{T\pm} \circ\!\circ ?(\exists x)?\varphi$$

We are thus left with showing

$$?(\exists x)\varphi \circ\!\circ ?(\exists x)?\varphi$$

and the result follows from

$$\frac{\frac{\frac{\overline{\varphi \vdash \varphi}}{\varphi \vdash (\exists x)\varphi}}{\varphi \vdash ?(\exists x)\varphi}}{?\varphi \vdash ?(\exists x)\varphi}}{(\exists x)?\varphi \vdash ?(\exists x)\varphi}}{?(\exists x)?\varphi \vdash ?(\exists x)\varphi}} \quad \text{and} \quad \frac{\frac{\frac{\overline{\varphi \vdash \varphi}}{\varphi \vdash ?\varphi}}{\varphi \vdash (\exists x)?\varphi}}{\varphi \vdash ?(\exists x)?\varphi}}{(\exists x)\varphi \vdash ?(\exists x)?\varphi}}{?(\exists x)\varphi \vdash ?(\exists x)?\varphi}}$$

– *Case of  $(\forall x)\varphi^-$ .* Since

$$[(\forall x)\varphi]^{T\pm} = ?!(\forall x)[\varphi]^{T\pm}$$

we have to show

$$!(\forall x)\varphi \circ\!\circ ?!(\forall x)[\varphi]^{T\pm}$$

The induction hypothesis gives

$$!(\varphi^-) \circ\!\circ [\varphi^-]^{T\pm}$$

and it follows from the  $?$ -axioms on deterministic formulae that we have the sequence of equivalences

$$?!(\forall x)[\varphi]^{T\pm} \circ\!\!\!\circ \ !(\forall x)[\varphi]^{T\pm} \circ\!\!\!\circ \ !(\forall x)\!\varphi$$

We are thus left with showing

$$!(\forall x)\varphi \circ\!\!\!\circ \ !(\forall x)\!\varphi$$

and the result follows from

$$\frac{\frac{\frac{\overline{\varphi \vdash \varphi}}{(\forall x)\varphi \vdash \varphi}}{!(\forall x)\varphi \vdash \varphi}}{!(\forall x)\varphi \vdash !\varphi} \quad \text{and} \quad \frac{\frac{\frac{\overline{\varphi \vdash \varphi}}{!\varphi \vdash \varphi}}{(\forall x)\!\varphi \vdash \varphi}}{!(\forall x)\!\varphi \vdash \varphi}}{!(\forall x)\!\varphi \vdash (\forall x)\varphi}}{!(\forall x)\!\varphi \vdash !(\forall x)\varphi}$$

□

**Proof of Proposition 7 (Extraction from FS + (DEXP) + (PEXP)).** We first show that FS + (LSIP) + (DEXP) + (PEXP) realises all instances of (PEXP).

**Lemma 19.** *The axioms (PEXP) are realized in FS+(LSIP)+(DEXP)+(PEXP):*

- (1)  $!(\varphi^+) \circ\!\!\!\circ \ ?!(\varphi^+)$
- (2)  $!?( \psi^-) \circ\!\!\!\circ \ !(\psi^-)$
- (3)  $!(\varphi^-) \circ\!\!\!\circ \ ?(\psi^+) \circ\!\!\!\circ \ ?(\varphi^- \circ\!\!\!\circ \ \psi^+)$
- (4)  $!(\varphi^+) \circ\!\!\!\circ \ !(\psi^-) \circ\!\!\!\circ \ !(\varphi^+ \circ\!\!\!\circ \ \psi^-)$
- (5)  $!(\varphi^+) \otimes ?(\psi^+) \circ\!\!\!\circ \ ?(\varphi^+ \otimes \psi^+)$
- (6)  $!(\varphi^- \otimes \psi^-) \circ\!\!\!\circ \ !(\varphi^-) \otimes !(\psi^-)$
- (7)  $!(\varphi^+) \wp ?(\psi^+) \circ\!\!\!\circ \ ?(\varphi^+ \wp \psi^+)$
- (8)  $!(\varphi^- \wp \psi^-) \circ\!\!\!\circ \ !(\varphi^-) \wp !(\psi^-)$

*Proof.* Let

$$\varphi^D = (\exists u)(\forall x)\varphi_D \quad \text{and} \quad \psi^D = (\exists v)(\forall y)\psi_D(v, y)$$

(1) We have to show

$$!(\exists u)\varphi_D(u, -) \circ\!\!\!\circ \ ?(\exists u)\!\varphi_D(u, -)$$

This follows from (DEXP):

$$\frac{\frac{\frac{\varphi_D(u) \vdash !\varphi_D(u)}{\varphi_D(u) \vdash (\exists u)\!\varphi_D(u)}}{\varphi_D(u) \vdash ?(\exists u)\!\varphi_D(u)}}{(\exists u)\varphi_D(u) \vdash ?(\exists u)\!\varphi_D(u)}}{!(\exists u)\varphi_D(u) \vdash ?(\exists u)\!\varphi_D(u)}$$

(2) We have to show

$$!(\forall y)?\psi_D(-, y) \multimap !(\forall y)\psi_D$$

This follows from (DEXP):

$$\frac{\frac{\frac{?\psi_D(y) \vdash \psi_D(y)}{(\forall y)?\psi_D(y) \vdash \psi_D(y)}}{!(\forall y)?\psi_D(y) \vdash \psi_D(y)}}{!(\forall y)?\psi_D(y) \vdash (\forall y)\psi_D(y)}}{!(\forall y)?\psi_D(y) \vdash !(\forall y)\psi_D(y)}$$

(3) We have to show

$$!(\forall x)\varphi_D(-, x) \multimap ?(\exists v)\psi_D(v, -) \multimap ?(\exists v)(\exists x)(\varphi_D(-, x) \multimap \psi_D(v, -))$$

With (LSIP) we have

$$(\forall x)\varphi_D(x) \multimap (\exists v)\psi_D(v) \vdash (\exists v)(\exists x)(\varphi_D(x) \multimap \psi_D(v))$$

and we conclude with (PEXP).

(4) We have to show

$$?(\exists u)\varphi_D(u, -) \multimap !(\forall y)\psi_D(-, y) \multimap !(\forall u)(\forall y)(\varphi_D(u, -) \multimap \psi_D(-, y))$$

First, with (PEXP) we have

$$?(\exists u)\varphi_D(u, -) \multimap !(\forall y)\psi_D(-, y) \multimap !((\exists u)\varphi_D(u, -) \multimap (\forall y)\psi_D(-, y))$$

We then conclude from

$$(\exists u)\varphi_D(u, -) \multimap (\forall y)\psi_D(-, y) \multimap (\forall u)(\forall y)(\varphi_D(u, -) \multimap \psi_D(-, y))$$

(see the case of  $(-)^D \multimap (-)^D \multimap (- \multimap -)^D$  in the proof of Theorem 3, §B.3.)

(5) We have to show

$$?(\exists u)\varphi_D(u, -) \otimes ?(\exists v)\psi_D(v, -) \multimap ?(\exists u, v)(\varphi_D(u, -) \otimes \psi_D(v, -))$$

First, with (PEXP) we have

$$?(\exists u)\varphi_D(u, -) \otimes ?(\exists v)\psi_D(v, -) \multimap ?((\exists u)\varphi_D(u, -) \otimes (\exists v)\psi_D(v, -))$$

We then conclude as follows:

$$\frac{\frac{\frac{\varphi_D(u), \psi_D(v) \vdash \varphi_D(u) \otimes \psi_D(v)}{\varphi_D(u), \psi_D(v) \vdash \exists u, v. \varphi_D(u) \otimes \psi_D(v)}}{(\exists u)\varphi_D(u), (\exists v)\psi_D(v) \vdash \exists u, v. \varphi_D(u) \otimes \psi_D(v)}}{(\exists u)\varphi_D(u) \otimes (\exists v)\psi_D(v) \vdash \exists u, v. \varphi_D(u) \otimes \psi_D(v)}$$

(6) We have to show

$$!(\forall x, y)(\varphi_D(-, x) \otimes \psi_D(-, y)) \quad \multimap \quad !(\forall x)\varphi_D(-, x) \otimes !(\forall y)\psi_D(-, y)$$

First, by (LSIP) we have

$$(\forall x, y)(\varphi_D(-, x) \otimes \psi_D(-, y)) \quad \multimap \quad (\forall x)\varphi_D(-, x) \otimes (\forall y)\psi_D(-, y)$$

and we conclude with (PEXP).

(7) We have to show

$$?(\exists u)\varphi_D(u, -) \wp ?(\exists v)\psi_D(v, -) \quad \multimap \quad ?(\exists u, v)(\varphi_D(u, -) \wp \psi_D(v, -))$$

First, by (PEXP) we have

$$?(\exists u)\varphi_D(u, -) \wp ?(\exists v)\psi_D(v, -) \quad \multimap \quad ?((\exists u)\varphi_D(u, -) \wp (\exists v)\psi_D(v, -))$$

and we conclude with (LSIP).

(8) We have to show

$$!(\forall x, y)(\varphi_D(-, x) \wp \psi_D(-, y)) \quad \multimap \quad !(\forall x)\varphi_D(-, x) \wp !(\forall y)\psi_D(-, y)$$

First, by (LSIP) we have

$$(\forall x, y)(\varphi_D(-, x) \wp \psi_D(-, y)) \quad \multimap \quad (\forall x)\varphi_D(-, x) \wp (\forall y)\psi_D(-, y)$$

and we conclude with (PEXP). □

The corresponding extension of Corollary 1 (Cor. 5) is the following. The proof is exactly that of Cor. 5, but invoking Lemma 19 in addition to Theorem 2 (Thm. 8) and Proposition 4 (Lem. 12 & 13).

**Corollary 7.** *Consider a closed formula  $\varphi := (\forall x^\sigma)(\exists u^\tau)\delta(u, x)$  with  $\delta$  deterministic. From a proof of  $\varphi$  in  $\text{FS} + (\text{LAC}) + (\text{LSIP}) + (\text{DEXP}) + (\text{PEXP})$  one can extract a term  $\mathfrak{t}(x)$  such that  $\models (\forall x^\sigma)[\delta(\mathfrak{t}(x), x)]$ .*

## C Proofs of §4 (Completeness)

### C.1 Completeness of MSO(M) (Theorem 4)

Thanks to the equational theory of MSO(M), we reduce Theorem. 4 to the completeness of  $\text{MSO}^+$  [24, Thm. 2.11]:

**Theorem 9 (Completeness of  $\text{MSO}^+$ ).** *For each closed formula  $\varphi$  of  $\text{MSO}^+$ ,*

$$\models \varphi \quad \iff \quad \vdash_{\text{MSO}^+} \varphi$$

First, recall that Proposition 1 implies that  $\text{MSO}(\mathbf{M})$  proves (3). In particular,  $\text{MSO}(\mathbf{M})$  proves

$$\mathbf{t} \doteq \langle \pi_1 \mathbf{t}, \dots, \pi_{2^p} \mathbf{t} \rangle \quad (\llbracket \sigma \rrbracket = \mathbf{2}^p)$$

Hence each term  $\mathbf{t}$  of sort  $(\sigma_1, \dots, \sigma_n; \tau)$  with  $\llbracket \tau \rrbracket = \mathbf{2}^p$  is provably equal to a tuple of terms  $(\mathbf{t}_i)_{1 \leq i \leq 2^p}$  with each  $\mathbf{t}_i$  of sort  $(\sigma_1, \dots, \sigma_n; o)$ . Now, we turn terms  $\mathbf{t}$  of sort  $(\sigma_1, \dots, \sigma_n; o)$  with

$$\llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket = \mathbf{2}^q$$

to terms  $\tilde{\mathbf{t}}$  of sort

$$\underbrace{(\mathbf{2}, \dots, \mathbf{2})}_q; \mathbf{2}$$

Moreover, we replace each such term  $\tilde{\mathbf{t}}$  with the  $\text{MSO}^+$ -term  $\langle \mathbf{t} \rangle$  representing the Mealy machine  $\mathbf{2}^q \rightarrow \mathbf{2}$  for  $\tilde{\mathbf{t}}$ . To summarize, we have

$$\vdash_{\text{MSO}(\mathcal{M})(\omega)} \mathbf{t} \doteq_{\sigma} \mathbf{u} \quad \longleftrightarrow \quad \bigwedge_{1 \leq i \leq 2^p} \langle \mathbf{t}_i \rangle \doteq_o \langle \mathbf{u}_i \rangle \quad (\mathbf{t}, \mathbf{u} \text{ of sort } (\sigma_1, \dots, \sigma_n; \tau))$$

Consider now an  $\text{MSO}(\mathbf{M})$ -formula  $\varphi$  with free variables among  $x^{\sigma_1}, \dots, x^{\sigma_n}$ . To  $\varphi$  we can associate an  $\text{MSO}^+$ -formula  $\langle \varphi \rangle$  with free variables among  $x_{1,1}^o, \dots, x_{1,m_1}^o, \dots, x_{n,1}^o, \dots, x_{n,m_n}^o$  (where  $\llbracket \sigma_i \rrbracket = \mathbf{2}^{m_i}$ ) and such that

$$\bigwedge_{1 \leq i \leq n} x^{\sigma_i} \doteq \langle x_{i,1}^o, \dots, x_{i,m_i}^o \rangle \quad \vdash_{\text{MSO}(\mathcal{M})(\omega)} \varphi \quad \longleftrightarrow \quad \langle \varphi \rangle$$

Theorem 4 then follows from Theorem 9.

## C.2 Proof of §4.2 (The Logic LMSO)

We split Proposition 8 into two statements.

**Proposition 13.** *If  $\vdash_{\text{MSO}(\mathbf{M})} \varphi$  then  $\vdash_{\text{LMSO}} \varphi^L$ .*

*Proof.* The result directly follows from Proposition 6 (Cor. 6) and the fact that LMSO proves the  $(-)^L$  translation of all the axioms of  $\text{MSO}(\mathbf{M})$ . They are dealt with exactly as in [24].

- The arithmetic rules of Fig. 5 follow from the fact that  $\alpha^L = \alpha$  for each atomic formula  $\alpha \in \text{At}$ .
- The induction scheme of LMSO requires one hypothesis to be under an exponential modality  $!(-)$  to accommodate arbitrary negative formulae; the situation is resolved by cutting with the LMSO axiom enabling to remove exponentials over deterministic formulae.

By the induction hypothesis (and since  $(-)^L$  commutes over substitution), we have proofs of

$$\overline{\varphi}^L, 0(z) \vdash \varphi^L[z/x], \overline{\psi}^L \quad \text{and} \quad \overline{\varphi}^L, \mathbf{S}(y, z), \varphi^L[y/x] \vdash \varphi^L[z/x], \overline{\psi}^L$$

Using (DEXP) we thus derive

$$!\overline{\varphi}^L, 0(z) \vdash \varphi^L[z/x], ?\overline{\psi}^L \quad \text{and} \quad !\overline{\varphi}^L, \mathbf{S}(y, z), !\varphi^L[y/x] \vdash \varphi^L[z/x], ?\overline{\psi}^L$$

from which the induction scheme of **LMSO** gives

$$!\overline{\varphi}^L \vdash (\forall x^t)\varphi^L, ?\overline{\psi}^L$$

and we directly get

$$!\overline{\varphi}^L \vdash !(\forall x^t)\varphi^L, ?\overline{\psi}^L$$

- The translation of each instance of the comprehension scheme of **MSO(M)** is an instance of the comprehension scheme of **LMSO**.
- The axiom scheme defining terms in **MSO(M)** is as follows

$$\overline{\vdash (\forall \overline{X}^o)(\forall x^t) (x \in \mathbf{f}_{\mathcal{M}}(\overline{X}) \iff \delta_{\mathcal{M}}(x, \overline{X}))}$$

Clearly, it is equivalent to the following scheme where we make the universal quantification implicit by using formulae with free variables

$$\overline{\vdash \mathbf{N}(x) \rightarrow (x \in \mathbf{f}_{\mathcal{M}}(\overline{X}) \iff \delta_{\mathcal{M}}(x, \overline{X}))}$$

which translate to the following, which is then clearly derivable from the corresponding scheme in **LMSO** by instantiating the universal quantifiers by the free variables

$$\overline{\vdash \mathbf{N}(x) \multimap (x \in \mathbf{f}_{\mathcal{M}}(\overline{X}) \multimap \delta_{\mathcal{M}}(x, \overline{X})^L)}$$

□

**Proposition 14.** *Given a realizable instance of Church's synthesis  $(\forall x^\sigma)(\exists u^\tau)\varphi(u, x)$ , we have  $\vdash_{\text{LMSO}} (\forall x^\sigma)(\exists u^\tau)\varphi^L(u, x)$ .*

*Proof.* Assume that there is a Mealy term  $\mathbf{u}^\tau(x^\sigma)$  such that

$$\models (\forall x^\sigma)\varphi(\mathbf{u}(x), x)$$

It then follows from the completeness of **MSO(M)** (Theorem 4) that

$$\vdash_{\text{MSO(M)}} (\forall x^\sigma)\varphi(\mathbf{u}(x), x)$$

so that Proposition 8 (Proposition 13) gives

$$\vdash_{\text{LMSO}} !(\forall x^\sigma)\varphi^L(\mathbf{u}(x), x)$$

We then easily derive

$$\vdash_{\text{LMSO}} \varphi^L(\mathbf{u}(x), x)$$

whence the result. □

**Theorem 10 (Soundness of  $(-)^D$  (Theorem 5)).** *Let  $\varphi$  be closed with  $\varphi^D = (\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x)$ . From a proof of  $\varphi$  in  $\text{LMSO}(\mathfrak{C})$  one can extract an eager term  $u(x)$  such that  $\text{LMSO}$  proves  $(\forall x^\sigma)\varphi_D(u(x), x)$ .*

*Proof.* The result follows from the soundness of  $(-)^D$  in  $\text{FS} + (\text{LAC}) + (\text{LSIP}) + (\text{DEXP}) + (\text{PEXP})$  given by Theorem 2 (Thm. 8), Proposition 4 (Lem. 12 & 13) and Lemma 19, and the fact that the axioms of  $\text{LMSO}$  are realized.

- For the arithmetic rules of Fig. 5, this follows from the fact that atomic formulae are interpreted by themselves, so these axioms are interpreted by instances of themselves.
- For comprehension, this follows from the fact that the axiom is a deterministic formula, so its realizers are trivial, and the fact that each instance is interpreted by an instance of the axiom.
- For the axiom giving definition definitions of Mealy machines, since atomic formulae are interpreted by themselves and since  $(-)^L$ -formulae are deterministic, each instance is interpreted by a formula of the form

$$x \in \mathbf{f}_{\mathcal{M}}(\bar{X}) \multimap (\delta_{\mathcal{M}}^L)^D(x, \bar{X})$$

The result then follows from Characterization (Theorem 3).

- Consider now induction:

$$\frac{!\bar{\varphi}, 0(z) \vdash \varphi^-[z/x], ?\bar{\varphi}' \quad !\bar{\varphi}, \mathbf{S}(y, z), !\varphi^-[y/x] \vdash \varphi^-[z/x], ?\bar{\varphi}'}{!\bar{\varphi} \vdash (\forall x^\iota)\varphi^-, ?\bar{\varphi}'}$$

First, note that the conclusion is of the form  $(-)^+ \vdash (-)^-$ , so no realizers have to be provided and we just have to show that the axiom is interpreted by an instance of itself. Now,  $(-)_D$  takes  $!$ -formulae to  $!$ -formulae and  $?$ -formulae to  $?$ -formulae. So we are left with showing that from a proof of

$$(!\bar{\varphi})_D, \mathbf{S}(y, z), (!\varphi^-)_D[y/x] \vdash \varphi_D(-, \tilde{x}, z)(?\varphi_{D'})_D$$

one can get a proof of

$$(!\bar{\varphi})_D, \mathbf{S}(y, z), !\varphi_D(-, \tilde{x}, [y/z]) \vdash \varphi_D(-, \tilde{x}, z)(?\varphi_{D'})_D$$

But this is trivial since

$$(!\varphi^-)_D(a) = !(\forall \tilde{x})\varphi_D(-, \tilde{x}, a) \multimap !\varphi_D(-, \tilde{x}, a)$$

□

### C.3 Proofs of §4.3 (Completeness of $\text{LMSO}(\mathfrak{C})$ )

**Lemma 20 (Lem. 1).** *For each  $\text{LMSO}$ -formula  $\varphi$ , we have*

$$(\varphi(a) \multimap \perp) \multimap \perp \vdash_{\text{LMSO}(\mathfrak{C})} \varphi(a)$$

Lemma 20 relies on a combinatorial property already used in [24].

**Lemma 21.** *Let  $\mathsf{P}$  and  $\mathsf{O}$  be sets such that  $\mathsf{O}$  may be well-ordered. Then the following is true*

$$\forall F \in \mathsf{P}^{\mathsf{O}^{\mathsf{P}}}. \exists p \in \mathsf{P}. \forall o \in \mathsf{O}. \exists f \in \mathsf{O}^{\mathsf{P}}. [F(f) = p \text{ and } f(p) = o]$$

*Proof.* Fix  $F \in \mathsf{P}^{\mathsf{O}^{\mathsf{P}}}$ . The negation of our statement is equivalent to

$$\forall p \in \mathsf{P}. \exists o \in \mathsf{O}. \neg (\exists f \in \mathsf{O}^{\mathsf{P}}. [F(f) = p \wedge f(p) = o])$$

Using an instance of choice available thanks to the fact that  $\mathsf{O}$  may be well-ordered, this is in turn equivalent to

$$\exists \tilde{o} \in \mathsf{O}^{\mathsf{P}}. \forall p \in \mathsf{P}. \neg (\exists f \in \mathsf{O}^{\mathsf{P}}. [F(f) = p \wedge f(p) = \tilde{o}(p)])$$

It follows that we only need to prove

$$\forall F \in \mathsf{P}^{\mathsf{O}^{\mathsf{P}}}. \forall \tilde{o} \in \mathsf{O}^{\mathsf{P}}. \exists p \in \mathsf{P}. \exists f \in \mathsf{O}^{\mathsf{P}}. [F(f) = p \text{ and } f(p) = \tilde{o}(p)]$$

But this is now easy: given  $F \in \mathsf{P}^{\mathsf{O}^{\mathsf{P}}}$  and  $\tilde{o} \in \mathsf{O}^{\mathsf{P}}$ , we can take  $f := \tilde{o}$  and  $p := F(\tilde{o})$  to conclude.  $\square$

In particular, given alphabets  $\Sigma$  and  $\Gamma$ , there are functions

$$u : (\Gamma)((\Sigma)\Gamma) \longrightarrow \Gamma \quad \text{and} \quad g : (\Gamma)((\Sigma)\Gamma) \times \Sigma \longrightarrow (\Sigma)\Gamma$$

such that for all  $F : (\Sigma)\Gamma \rightarrow \Gamma$  and all  $x \in \Sigma$ ,

$$\textcircled{\@}(F, g(F, x)) = u(F) \quad \text{and} \quad \textcircled{\@}(g(F, x), u(F)) = x$$

We can now prove Lemma 20. Recall that

$$\vdash_{\text{LMSO}} \delta, \delta \multimap \perp \quad (\delta \text{ deterministic})$$

so that

$$(\delta \multimap \perp) \multimap \perp \vdash_{\text{LMSO}} \delta \quad (\delta \text{ deterministic})$$

*Proof (of Lemma 20).* Let

$$\psi(a) := (\varphi(a) \multimap \perp) \multimap \perp$$

Thanks to the Characterization Theorem 3, we are done if we show

$$\psi^D(a) \vdash_{\text{LMSO}(\mathfrak{c})} \varphi^D(a)$$

Assume

$$\varphi^D(a) = (\exists u^\tau)(\forall x^\sigma)\varphi_D(u, x, a)$$

We thus have

$$(\varphi(a) \multimap \perp)^D = (\exists g^{(\sigma)\tau})(\forall u^\tau)(\varphi_D(u, (g)u, a) \multimap \perp)$$



so that

$$((\varphi(a) \multimap \perp) \multimap \perp)^D \multimap (\exists F^{(\tau)((\sigma)\tau})(\forall g^{(\sigma)\tau})\varphi_D((F)g, (g)((F)g), a))$$

Then we are done if we show that LMSO proves

$$(\forall F^{(\tau)((\sigma)\tau})(\exists u^\tau)(\forall x^\sigma)(\exists g^{(\sigma)\tau})\left(\varphi_D((F)g, (g)((F)g), a) \multimap \varphi_D(u, x, a)\right))$$

Consider the functions

$$u : \llbracket (\tau)((\sigma)\tau) \rrbracket \longrightarrow \llbracket \tau \rrbracket \quad \text{and} \quad g : \llbracket (\tau)((\sigma)\tau) \rrbracket \times \llbracket \sigma \rrbracket \longrightarrow \llbracket (\sigma)\tau \rrbracket$$

given by Lemma 21. Let  $\mathbf{u}(F)$  and  $\mathbf{g}(F, x)$  be terms representing their pointwise lift to  $\mathbb{M}$ . We then have, in LMSO

$$\mathbf{u}(F) \doteq @(\mathbf{F}, \mathbf{g}(F, x)) \quad \text{and} \quad @(\mathbf{g}(F, x), \mathbf{u}(F)) \doteq x$$

and we conclude as follows:

$$\frac{\frac{\frac{\vdash_{\text{LMSO}} \varphi_D\left(@(\mathbf{F}, \mathbf{g}(F, x)), @(\mathbf{g}(F, x), @(\mathbf{F}, \mathbf{g}(F, x))), a\right) \multimap \varphi_D(\mathbf{u}(F), x)}{\vdash_{\text{LMSO}} (\forall x^\sigma)(\exists g^{(\sigma)\tau})\left(\varphi_D((F)g, (g)((F)g), a) \multimap \varphi_D(\mathbf{u}(F), x)\right)}}{\vdash_{\text{LMSO}} (\forall F^{(\tau)((\sigma)\tau)})(\exists u^\tau)(\forall x^\sigma)(\exists g^{(\sigma)\tau})\left(\varphi_D((F)g, (g)((F)g), a) \multimap \varphi_D(u, x)\right)}}{\square}$$

**Lemma 22 (Lemma 2).**

$$(\forall x^\sigma)\varphi(\mathbf{t}^\tau(x), x) \vdash_{\text{LMSO}(\mathfrak{e})} (\exists u^\tau)(\forall x^\sigma)\varphi(u, x) \quad (\mathbf{t} \text{ eager})$$

*Proof.* Thanks to Corollary 3 we are done if we show

$$(\forall x^\sigma)\varphi(\mathbf{t}^\tau(x), x) \vdash_{\text{LMSO}(\mathfrak{e})} (\forall f^{(\sigma)\tau})(\exists u^\tau)\varphi(u, (f)u) \quad (\mathbf{t} \text{ eager})$$

We use fixpoints for finite-state eager functions (Proposition 3 and §A.1). Let

$$\Sigma := \Delta := \llbracket \sigma \rrbracket \quad \Gamma := \llbracket (\sigma)\tau \rrbracket \quad \Theta := \llbracket \tau \rrbracket$$

We apply Corollary 4 to

$$F := \llbracket \mathbf{t} \rrbracket : \Sigma \times \Gamma \longrightarrow_{\text{EM}} \Theta \quad \text{and} \quad G := @ : \Gamma \times \Theta \longrightarrow_{\text{M}} \Delta$$

We thus obtain a f.s. eager

$$H : \Sigma \times \Gamma \longrightarrow_{\text{EM}} \Delta^\Sigma$$

such that

$$@(\mathbf{H}(B, C), C) = @(\mathbf{C}, \llbracket \mathbf{t} \rrbracket(B)) \quad (\text{for all } B \in \Sigma^\omega, C \in \Gamma^\omega)$$

We now apply Proposition 3 and obtain a finite-state fixpoint of  $H$ , that is a finite-state eager

$$\text{fix}(H) : \Gamma \longrightarrow_{\text{EM}} \Sigma^\Gamma$$

such that

$$\begin{aligned} @(\text{fix}(H)(C), C) &= \mathbf{e}(\text{fix}(H))(C) \\ &= @ (H(\mathbf{e}(\text{fix}(H))(C), C), C) \quad (\text{for all } C \in \Gamma^\omega) \\ &= @ (C, \llbracket \mathbf{t} \rrbracket (\mathbf{e}(\text{fix}(H))(C))) \end{aligned}$$

Let  $\mathbf{v}(f^{(\sigma)\tau})$  be the (Mealy) term for  $\mathbf{e}(\text{fix}(H)) : \llbracket (\sigma)\tau \rrbracket \rightarrow_{\mathbb{M}} \llbracket \sigma \rrbracket$ . We thus have

$$\vdash_{\text{LMSO}} \mathbf{v}(f) \doteq @ (f, \mathbf{t}(\mathbf{v}(f)))$$

and it follows that

$$\begin{aligned} (\forall x^\sigma) \varphi(\mathbf{t}(x), x) &\vdash_{\text{LMSO}} \varphi(\mathbf{t}(\mathbf{v}(f)), \mathbf{v}(f)) \\ &\vdash_{\text{LMSO}} \varphi(\mathbf{t}(\mathbf{v}(f)), @ (f, \mathbf{t}(\mathbf{v}(f)))) \\ &\vdash_{\text{LMSO}} (\exists u^\tau) \varphi(u, (f)u) \\ &\vdash_{\text{LMSO}} (\forall f^{(\sigma)\tau}) (\exists u^\tau) \varphi(u, (f)u) \end{aligned}$$

□

**Theorem 11 (Completeness of LMSO( $\mathfrak{C}$ ) (Theorem 6)).** *For each closed formula  $\varphi$ , either  $\vdash_{\text{LMSO}(\mathfrak{C})} \varphi$  or  $\vdash_{\text{LMSO}(\mathfrak{C})} \varphi \rightarrow \perp$ .*

*Proof.* Let  $\varphi$  be a closed LMSO-formula and  $\varphi_D(u^\tau, x^\sigma)$  be the body of its Dialectica interpretation. We apply Büchi-Landweber Theorem 1 to the MSO( $\mathbf{M}$ )-formula  $[\varphi_D(u^\tau, x^\sigma)]$ . There are two cases.

- Either there exists an eager term  $\mathbf{u}(x)$  of sort  $(\sigma, \tau)$  such that  $(\forall x^\sigma) [\varphi_D(\mathbf{t}(x), x)]$  holds. We then proceed as follows.

MSO( $\mathbf{M}$ ) $\vdash$ $[\varphi_D(\mathbf{u}(x), x)]$	By Completeness of MSO( $\mathbf{M}$ ) (Thm. 4)
LMSO( $\mathfrak{C}$ ) $\vdash$ $[\varphi_D(\mathbf{u}(x), x)]^L$	By Proposition 8
LMSO( $\mathfrak{C}$ ) $\vdash$ $\varphi_D(\mathbf{u}(x), x)$	By Proposition 5 ( $\varphi_D$ is always $\pm$ )
LMSO( $\mathfrak{C}$ ) $\vdash$ $(\forall x^\sigma) \varphi_D(\mathbf{u}(x), x)$	$\forall$ -right
LMSO( $\mathfrak{C}$ ) $\vdash$ $(\exists u^\tau) (\forall x^\sigma) \varphi_D(u, x)$	By Lemma 22, since $\mathbf{u}(x)$ is eager
LMSO( $\mathfrak{C}$ ) $\vdash$ $\varphi^D$	By definition
LMSO( $\mathfrak{C}$ ) $\vdash$ $\varphi$	By Characterization (Thm. 3)

- Otherwise, there exists a term  $\mathbf{x}(u)$  of sort  $(\tau; \sigma)$  such that  $(\forall u^\tau) \neg [\varphi_D(\mathbf{x}(u), u)]$  holds. Note that

$$\neg [\varphi_D(\mathbf{x}(u), u)] = [\varphi_D(\mathbf{x}(u), u) \rightarrow \perp]$$

We then conclude as follows.

$\text{MSO}(\mathbf{M})$	$\vdash \lfloor \varphi_D(u, \mathbf{x}(u)) \dashv\vdash \perp \rfloor$	By Completeness of $\text{MSO}(\mathbf{M})$ (Thm. 4)
$\text{LMSO}$	$\vdash \lfloor \varphi_D(u, \mathbf{x}(u)) \dashv\vdash \perp \rfloor^L$	By Proposition 8
$\text{LMSO}$	$\vdash \varphi_D(u, \mathbf{x}(u)) \dashv\vdash \perp$	By Proposition 5 ( $\varphi_D$ is always $\pm$ )
$\text{LMSO}$	$\vdash (\exists x^\sigma)(\varphi_D(u, x) \dashv\vdash \perp)$	$\exists$ -right
$\text{LMSO}$	$\vdash (\forall u^\tau)(\exists x^\sigma)(\varphi_D(u, x) \dashv\vdash \perp)$	$\forall$ -right
$\text{LMSO}(\mathfrak{C})$	$\vdash (\exists g^{(\sigma)\tau})(\forall u^\tau)(\varphi_D(u, (g)u) \dashv\vdash \perp)$	By (LAC)
$\text{LMSO}(\mathfrak{C})$	$\vdash (\varphi \dashv\vdash \perp)^D$	By definition (see Fig. 3)
$\text{LMSO}(\mathfrak{C})$	$\vdash \varphi \dashv\vdash \perp$	By Characterization (Thm. 3)

□

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