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# Geometric constructions in a dynamic environment (GeoGebra): The case of in-service teachers

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*A unit on dynamic geometric construction was included in a professional development course for in-service mathematics teachers. As a final task in that unit 28 teachers were required to construct a rhombus based on their own choice of given objects and tools, using the dynamic geometry software GeoGebra. Their responses were analysed according to: the choice of given objects; the choice of tools; the explanation and validity test; and the number of different rhombuses they claimed to have obtained. The teachers were found to have different concept images of a rhombus and different conceptions of what constitutes a valid geometric construction. While many claimed to have obtained an infinite number of different rhombuses, differences were observed in the "type of infinite". Recommendations are given for improving the task design to strengthen teachers' mathematical and pedagogical knowledge.*

*Keywords: Geometric construction, dynamic geometry, GeoGebra, concept image, in-service teachers.*

## Introduction

From the use of a dynamic environment for geometric construction arise new frames of reference for the idea of construction. In this paper we present data accumulated during an in-service teachers' professional development course on dynamic geometric construction. We report on the data from two viewpoints:

1. The practical viewpoint – we will describe teachers' construction methods in the final task in which they were required to construct a rhombus in different ways (each construction according to different given objects). We will discuss issues arising from the construction, both those concerning the nature and validity of the construction and those concerning the teachers' images of a rhombus.
2. The pedagogical viewpoint – we will suggest possible ideas for improving the task design in order to hone and strengthen teachers' mathematical and pedagogical knowledge.

## Theoretical background

Construction in geometry has a specific meaning: the drawing of geometric figures using only compass and straightedge without measuring angles or lengths (Hartshorne, 2000a). Geometric constructions have been a popular part of mathematics throughout history. Euclid documented them in his book entitled "Elements", which is still regarded as the authoritative geometry reference. In that work, he uses these constructions widely and extensively, and so they have become a part of the

geometry field of study. Geometric constructions also provide insight into geometric concepts and give us tools to draw figures when direct measurement is not appropriate (Hartshorne, 2000b).

For about a decade, geometric constructions were removed from the Israeli curriculum. In 2014 geometric constructions were again included in the geometry curriculum for middle school in Israel. Curriculum decision makers claim (INMMC, 2013) that geometric construction integrates geometric content knowledge with a deductive way of thinking that is essential for geometry proofs.

The professional literature indicates that teaching mathematics using technological tools helps in the process of constructing an abstract knowledge of mathematics, and geometry in particular (Lagrange et al., 2003). GeoGebra is a powerful dynamic environment that allows testing of countless number of examples; provides effective and convenient tools for confirming or contradicting conjectures; and provides a wide selection of different tools for geometric constructions – from digital analogs of compass and straightedge, to shortcut tools, such as automatically drawing a line parallel to a given line (Fahlberg-Stojanovska & Stojanovski, 2010). GeoGebra is also a tool of assessment (Bu et al., 2012); thinking processes can be observed by examining the construction protocol which can replay step-by-step onscreen.

Vinner and Hershkowitz (1983) focused on the cognitive development of mathematical concepts, and proposed a model of two components: the *concept definition* – the verbal description of the mathematical concept, which characterizes the concept mathematically; and the *concept image* – the cognitive structure that includes all the examples and the processes related to the concept in the learner's mind. Geometric concepts have a special status: Fischbein (1993) coined the term "figural concepts" and argued (Fischbein & Nachlieli, 1998) that geometrical figures are characterized by both conceptual and sensorial properties. A geometrical figure is a mental abstract which is governed by a definition. At the same time, it is an image. In geometrical reasoning the two categories of properties should merge absolutely.

In the context of this research, geometric constructions, which are carried out in a GeoGebra dynamic environment, can be seen as figural concepts. We used our personal, not formal definition (Vinner, 1991) for geometric construction. To construct a geometric figure means to draw the figure on a computer screen using GeoGebra digital tools in such a way that essential properties of the figure remain invariant under dragging.

### **The purpose of the study**

The aim of the present study is to:

1. Characterise the in-service teachers' geometric constructions on the final task of the unit.
2. Improve the design of the construction tasks in order to increase their contribution to the development of in-service teachers' mathematical and pedagogical knowledge.

## Methodology

### The research population

This research was carried out in the framework of a 30-hour in-service professional development course whose aim was to acquaint middle-school and high-school teachers with a digital program for ninth-grade mathematics students. A part of this course was the geometric construction unit. Since our assumption (which subsequently proved unfounded) was that the teachers were already acquainted with traditional geometric constructions using straightedge and compass, the unit dealt mainly with the use of the dynamic geometry software GeoGebra. When necessary, traditional methods were referred to. The focus of the research was on the responses of 28 teachers to the final task of the unit.

### About the final task

Throughout the course the in-service teachers tried out different geometric construction activities using GeoGebra and thus learned to create a valid construction using dynamic tools. Instructions for the final task included the following reminder.

Constructing with dynamic tools is not the same as drawing on a page since the objects (points, segments, etc.) are moveable: you can drag them and observe how other objects change accordingly. Each figure must be constructed so that it retains its characteristics even after other objects are dragged. For example, on constructing a rhombus according to its diagonals, check that after changing the lengths of the diagonals the figure remains a rhombus. This type of construction is called a valid construction.

In the final task the teachers were required to:

1. Construct a rhombus by three different methods, each construction according to different given objects (according to your choice), for example, according to its diagonals.
2. Describe each construction process and prove that it results in a rhombus.
3. State and explain how many different (non-congruent) rhombuses can be obtained by each method.

Note that no instruction was given as to whether the given object is fixed or dynamic. A fixed side will yield an infinite number of rhombuses (same side length, different angles) while fixed diagonals will yield only one rhombus. Clearly if the objects are dynamic an infinite number of rhombuses are possible, but this is a feature of the software and not of the underlying mathematics.

### Data collecting and analyzing

Each teacher sent a solution which comprised a picture of the final construction of the rhombus, the GeoGebra file, and explanations and justifications for the construction process. We used interpretative methods for analyzing these data.

### Findings: Characteristics of geometric constructions

A qualitative analysis of the teachers' constructions yielded four main categories.

1. The given objects on which the construction of the rhombus was based
2. The construction process itself according to tools used

3. The teacher's explanation and validity test
4. The number of different rhombuses the teacher claimed to have obtained

We present some examples of teachers' constructions, according to the above categories. The teachers' choices of given objects on which to base their construction were varied; for example, one side, two diagonals, one diagonal, an angle, a combination of the already mentioned objects, and other objects (such as area). We will present just two of these choices: one side and two diagonals. Here are examples from three different teachers who constructed a rhombus according to its side.

### Tzila's construction

1. "The construction is according to 4 equal sides, each 5 cm long."
2. Used segments of fixed length, parallel line through a point, intersection point.
3. "I created a rhombus from two adjacent sides each 5 cm long, using parallel lines; that is a parallelogram with all 4 sides equal 5 cm."
4. Infinite number. "Using a circling movement, with point A fixed, in a sort of circling round each time getting another rhombus whose diagonals are changing."

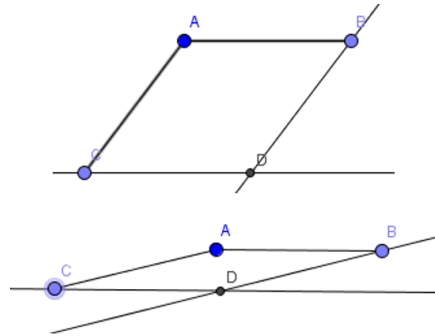


Figure 1: Tzila's construction before and after dragging

Instead of just writing that the side of the rhombus was given, Tzila added a definition of rhombus, which, like every definition, provided a sufficient condition. Possibly she misunderstood the task, and thought that she had to state the conditions for creating a rhombus, or perhaps her concept image of rhombus is a parallelogram with four equal sides, or perhaps she misunderstood the components of deductive geometry.

### Anat's construction

1. A quadrilateral of equal sides – each side of length 2 cm.
2. Used segment of fixed length, circle with fixed radius.
3. "According to the theorem: a rhombus is a quadrilateral with all sides equal."
4. No answer.

Anat's explanation shows that she, like many of the teachers, was confused about the components of deductive geometry (stating she was using a theorem when in fact she was using a definition).

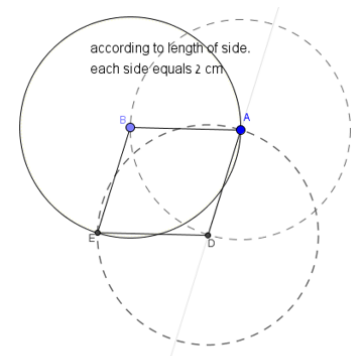


Figure 2: Anat's construction

### Yaron's first construction

1. Rhombus with side of length  $a$ .
2. Used circle with given radius, intersection point.

3. "The length of the side of the rhombus is  $a$  since all the sides and the radius are length  $a$ . The construction is valid according to the dragging test. The construction relies on the principal of the length of the radius of a circle".
4. "There can be an infinite number of rhombuses since although the sides remain equal lengths the diagonals and angles can be changed."

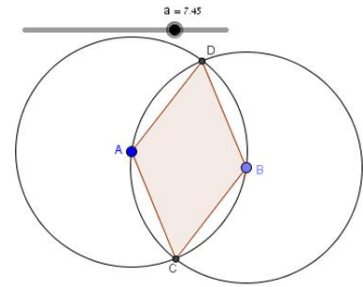


Figure 3: Yaron's first construction

Despite Yaron's claim that the diagonals and angles can be changed, he in fact built a rhombus in which the shorter diagonal was equal to the side and so the acute angle of the rhombus was  $60^\circ$ . He claimed that there are an infinite number of possible rhombuses given the length of a side, which is correct, but does not correspond to what he constructed. This error was repeated by many of the teachers. Interestingly, in another construction – described below – Yaron erroneously claimed an infinite number of possible rhombuses because of the dynamic nature of the given attributes.

Here are examples of two teachers who constructed a rhombus according to its diagonals.

**Yaron's second construction**

1. Rhombus according to two diagonals.
2. Used segment, perpendicular bisector, circle, intersection point.
3. "I based this on the fact that in a rhombus the diagonals are perpendicular and bisect each other, and on the principal of the length of the radius of a circle. The construction is valid according to the dragging test."
4. "There can be an infinite number of rhombuses since the lengths of the diagonals are not fixed. can lengthen or shorten them (or even only one of them). The rhombus can change – both its sides and its angles."

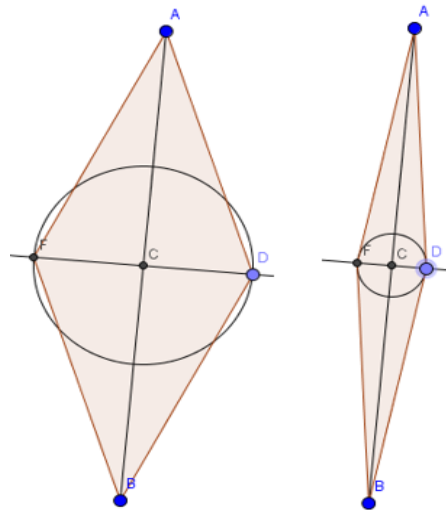


Figure 4: Yaron's second construction before and after dragging

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Yaron constructed diagonals whose length changed dynamically. He did not relate to the fact that each pair of diagonals determines only one possible rhombus.

**Nora's construction**

1. According to two diagonals.
2. Used segments of fixed length. By trial and error changing the angles between the segments.
3. Did not check validity.
4. No answer.

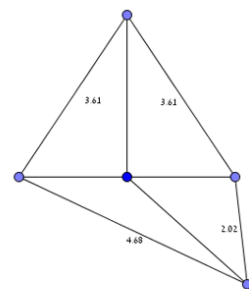


Figure 5: Nora's construction after dragging a vertex

If Nora had checked her construction she would have realised that it did not pass the validity test, as seen in figure 5.

**Discussion and conclusion**

In the discussion we will relate to the two viewpoints mentioned earlier: the practical and the pedagogical.

### **The practical viewpoint**

In the context of the practical viewpoint we will characterise the findings in each of the construction categories. The first category is: **Given objects for constructing a rhombus**. We have presented two of the teachers' choices of given objects: one side or two diagonals. All the teachers in the course chose the size of their given objects in one of two ways – either fixed size (a number) or dynamic (a parameter). This choice seems to be related to the teacher's conception of geometric construction – what is permitted in such a construction. It should be noted that these teachers had not previously learned constructions in a rigorous manner, and in particular, not dynamic constructions. Therefore we relate to their answers as based on intuitions about what is a construction and on geometric knowledge relevant to the specific topic. We suggest that teachers who chose fixed numerical givens have a concrete conception of construction that is related to particular sizes during the construction process, and did not refer to the characters of the figure which is constructed with dynamic tools. Teachers who chose dynamic givens have a broader conception of construction, not connected to size. This choice of dynamic givens corresponds more closely to the instructions – see above description of the final task. In addition, there were some teachers who were confused about defining the given objects, for example, Tzila, as described above.

The second category is: **Choice of construction tools**. We identified three types of construction process: measuring; using the extensive range of tools supplied by GeoGebra; and using a limited range of tools (cf. Fahlberg-Stojanovska & Stojanovski, 2010).

Construction by measuring involves fixing the size of the required object (side, diagonal, or angle). For example, in Nora's construction she moved two rays to form a right angle. Another teacher created four equal segments using co-ordinate axes.

An example of construction with the extensive range of tools is using the parallel line tool to ensure a parallelogram, as in Tzila's construction. Another example is using the perpendicular bisector tool, as in Yaron's construction.

Construction with a limited range of tools imitates construction with straightedge and compass. For example, Yaron built a circle of variable radius. Anat used both measurement and the limited range of tools (using a circle of radius equal to the given segment of length 2 cm).

The third category is: **Explanation and validity check**. None of the teachers gave formal proofs of their constructions. However their explanations provided us with an idea of their concept images (Vinner, 1991) of a rhombus. They explained their constructions according to their concept image and not according to the definition as given in the national school curriculum: a quadrilateral with all sides equal. A problematic concept image seems to create confusion between the different components of a deductive geometric argument, for example, between a theorem and a definition and between a theorem and a property. This confusion can be seen in the explanations of Tzila and Anat.

We discerned two problematic areas in the teachers' explanations: an incoherent concept image related to mathematical knowledge; and an intuitive conception of construction in a dynamic environment, which seem to be related to the construction process in everyday life, for example

Nora's construction which produced a rhombus. This construction did not fulfill the demand that the construction will remain a rhombus also after dragging.

This conception resulted in most of the teachers producing an invalid construction. There is not enough information in the data to pinpoint the main causes of these problems – are they connected to missing technological knowledge, or to missing mathematical knowledge, or to a faulty conception of construction, or to a combination of all three? More research is required to clarify this.

The fourth category is: **Number of solutions**. We identified three types of responses, each claiming an infinite number of solutions: relating to the dynamic nature of the given object, as seen in Yaron's second construction; relating to the mathematical properties of the given object, as seen in Tzila's construction; relating to the position of the given object in the plane. This latter type was seen in the response of a teacher who wrote: "In the plane it is possible to position the rhombus in any place you want".

### **The pedagogical viewpoint**

In the light of these research findings we recommend investigating some aspects of task design in a dynamic environment. On one hand the task design can assist teachers to execute the task, and on the other hand it can persuade them to use this task in their classroom (Bu et al., 2012). We suggest that it would be worthwhile to design the task on three levels: mathematical-pedagogical, technological-pedagogical, and reflective.

The mathematical-pedagogical level requires the teacher to define the concept and to construct the figure accordingly. In line with Fischbein (1993) we relate to a geometrical concept as a figural concept and thus its construction has practical meaning. Subsequently the teacher is required to construct the figure based on different sufficient conditions. Such a construction task would enable identification of the teacher's mathematical knowledge and her concept image, and could help to bring that concept image closer to the concept definition.

The technological-pedagogical level requires the teacher to construct the figure at first using the limited range of GeoGebra tools (imitating straightedge and compass) and subsequently using the extensive range (enabling short cuts). We would like to investigate the connection between these two types of construction and the connection between the teacher's image of the geometric concept and her conception of geometric construction.

The reflective level requires the teacher to check the validity of her construction, while considering the meaning of validity. Giving a detailed account of the results of the validity test should be an inseparable part of the task. Such an execution of the task should strengthen the connection between the mathematical and technological aspects. For example, if the construction "collapses" – after dragging the dynamic objects the properties of the required figure are not preserved – it is important to understand whether the "collapse" is due to a mathematical failure or a technological failure, or a combination of both. Such an analysis should contribute to the development of mathematical knowledge, pedagogical knowledge, and technological knowledge.

We believe that the above characterisation may provide a starting point for further research and may contribute to the technology, pedagogy, and content knowledge framework (Koehler & Mishra, 2009) – to help teachers integrate technology into their teaching.



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