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# Dialogic processes in collective geometric thinking: A case of defining and classifying quadrilaterals

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*In this paper, we report on how a group of UK 12-13-year-old students work with hierarchical defining and classifying quadrilaterals, an area which students find very difficult to understand. We implemented a geometry test to 9 groups of students. Quantitative data suggest that the students found very difficult to undertaking hierarchical defining and classifying quadrilaterals even in collaborative learning situations. Our qualitative video-recorded data from the four groups suggest that we find that even in collaborative learning settings prototypical images of geometrical figures strongly influence students' ways of hierarchically defining and classifying quadrilaterals. In addition, groups often had opportunities to examine their ideas, but they did not explore these opportunities because each member did not see what others were saying 'as if through the eyes of another'*

*Keywords: Dialogic, collective geometric thinking, defining and classifying quadrilaterals.*


## Introduction

Geometry has been recognised as one of the most important topics in school mathematics as it provides students with learning opportunities for developing their spatial thinking, reasoning and sense making of this world. Sinclair et al (2016) reviewed over 200 research papers in geometry published since 2008, and identified six themes, including the understanding of the teaching and learning of definitions. They state that one of their research questions is about students' understanding of hierarchically defining and classifying shapes (p. 706). Our paper is concerned with this issue, in the context of hierarchically defining and classifying shapes in collaborative learning settings which can be productive ways to develop mathematical thinking and understanding (e.g. Martin and Towers, 2014). We chose this topic because the research has reported that students find the understanding of hierarchically defining and classifying shapes difficult, but how students tackle this in collaborative settings has not been sufficiently investigated.

The purpose of this paper is to examine the following research question 'What obstacles will be identified when students are working together with geometrical problems?' In order to answer this question, we first propose our theoretical framework for emergence of collective geometrical thinking in the context of hierarchical defining and classifying quadrilaterals. We then investigate students' collaborative learning process in defining and classifying quadrilaterals. In this paper, we focus on collaborative group work where teachers or instructors' interventions are minimal. Therefore, while we acknowledge that teachers' roles are highly important to support learners' mathematical thinking and understanding (e.g. Martin and Towers, 2014), we do not consider this issue in this paper.

## Theoretical framework

### Concept images and definitions and prototypical examples of geometrical figures

In order to study students' thinking with geometrical shapes, the terms 'concept image' and 'concept definition', introduced by Vinner and Hershkowitz (1980) are useful. A concept definition is defined as 'a form of words used to specify that concept' and concept image as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process' (ibid, p. 152). When considering a parallelogram, at least two types of concept images are considered, i.e. one is 'conceptual images' such as 'parallelograms have two sets of parallel lines', 'opposite sides of a parallelograms are equal', and the other is 'visual images' (e.g. ). In relation to learners' concept image and definition, another useful idea is the prototype phenomenon (Hershkowitz, 1990). This theoretical idea claims that students' difficulty in seeing geometrical shapes flexibly is caused by the prototype example, which students often encounter in their initial stages of learning of geometrical figures. For example, as a concept definition, parallelograms are introduced as 'a quadrilateral with two pairs of parallel sides', but a typical 'slanted' visual image is often used. This 'visual images' will stay strongly in students' minds, and as a result their 'conceptual images' become "the subset of examples that had the "longest" list of attributes – all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics." (ibids., p. 82). Thus, for many students, when the hierarchical relationships between quadrilaterals are required, they cannot accept that rectangles can be a member of parallelogram group as, on the one hand, rectangles have 90 degree angles, and on the other hand, parallelograms should be 'slanted' ('visual geometrical images') and not have such angles, and therefore rectangle are not member of parallelograms, stating 'rectangles have 90 degree angles and so they are not a member of parallelograms' as their 'conceptual image' (e.g. Fujita, 2012).

### Collective geometrical thinking process

Collaborative learning has been recognised as a key topic in mathematics education research, and the difficulties in geometric thinking described above might be overcome if students' undertake problems collaboratively. Martin and Towers (2014) apply Pirie and Kieren's model (1994), which describes the growth of mathematical understanding with eight potential layers; Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Investing. The learners' developmental paths from Primitive knowing to Investing would not be straightforward. For example, when an individual/a group of learners had difficulty in noticing properties during problem solving, they might examine their already made images, and as a result they re-make new images for exploring new paths for problem solving. This is what Pirie and Kieren call Folding back. Martin and Tower also suggest that this process is crucial in collective thinking process. For example, suppose a group of students are discussing whether a 'rectangle' can also be seen as a 'parallelogram'. In order to solve this (under a curriculum hierarchical relationships of geometrical shapes are assumed), they have to collectively make and have their conceptual and visual images of rectangle and parallelograms including their definitions, examine their properties collectively, and then formulate their reasoning etc. In this process, they might fail to collectively have useful conceptual and visual images of parallelograms and in this case they have to fold back to their collective image making stage in order to continue to examine this problem.

### **Dialogic process in collective geometrical thinking**

The framework for collective thinking process by Martin and Towers (2014) discussed above can offer useful ways of analysing collective thinking process which collectively made or had conceptual and visual images, held back, noticed properties and so on, but this approach can be strengthened by considering some of the dialogic processes involved in thinking. For example, Mercer and Sams (2006) studied how certain types of talk, which mediate conceptual knowledge, affect students' ways of collective thinking and problem solving. They particularly consider that the roles of exploratory talk, described as being critical friends each other and using explicit reasoning during problem solving, showing how it is crucial for developing understanding, comparing to the other types of talk such as disputational (being competitive or disagreeing with each other in egoistic ways) or cumulative talk (agreeing each other without constructive criticisms). Extending this talk type approach further, Kazak, Wegerif and Fujita (2015) report that an 'Aha!' moment occurred after learners had engaged in productive 'dialogues'. 'Dialogues' to which we refer include more than exchanging recognisable utterances, but it is in a Bakhtinian sense, which Barwell (2016) recently described as "a theoretical idea that defines the nature of many aspects of the relationality of language." (p. 6). Our view is that such 'dialogues' elucidate differences and gaps, and encourage learners to see their learning from a different perspective, which is based on Bakhtinian dialogic theory (1963; 1984).

From this point of view, in addition to effective collaborative practice such as building effective collective conceptual and visual images of geometrical figures for problem solving, seeing a problem 'as if through the eyes of another' is important for emergence and development of collective group thinking and understanding. This includes, for example, recognising multiple 'voices' in mathematical concepts, seeing ideas from an 'outside' perspective, establishing dialogic space, learners' attitudes to each other, laughter, and so on. This is what Wegerif (2011) refers to as dialogic process of conceptual growth. Barwell (2016) also states, in the context of the development of the concept 'polygon': "the process of making sense of the word and the concept 'polygon' arises through the differences between the two groups of shapes on the blackboard, between the different ways of classifying shapes that preceded this moment, and so on." (p. 9). Let us take again the example whether a 'rectangle' can also be seen as a 'parallelogram'. Here, in their utterances students will use 'rectangle' or 'parallelogram', but they will contain 'multiple perspectives and agencies, i.e. rectangles for their own definitions and conceptual and visual images, for peers' definitions and images, for the formal definitions which appear in the textbook or for definitions used by teachers, and so on. In their talk they might agree or disagree with their thinking and if the group of students do not see a rectangle from an 'outside' perspective, they might not be able to reach mutual agreements or reasonable answers, or extend their discussions and apply other contexts such as 'is a square a type of parallelogram?', etc.

### **Methodology**

The participants of our study were 27 Y7 students (12-13 year old) in a lower secondary school in South West England. Their abilities are recognised by their class teacher as the second highest group in the year group, meaning that their achievements are higher than the average students in the UK school context. They have also studied formal definitions of basic 2D shapes including parallelograms. The participants undertook the following tests in 2015-16, summarised in fig. 1.

Group thinking measure test	Geometry group test
<p>General group thinking test: We first identify groups' thinking by using the group thinking test which consists of two tests A and B of 15 graphical puzzles each, carefully matched to be of equal difficulty. We used this test as it was reported that the test provides useful insights into how groups work well in general. Students do one test working in groups of three and the other test working individually, assuming a measure of individual thinking correlated to a measure of group thinking with a measure of the difference between the individual scores and the group score (Wegerif et al, in press). In our study, the half of students in the chosen class individually undertook Test A, and then we formed groups of three in accordance with their test scores. Then each group undertook Test B. The other half did Test B as their individual and Test A as group test. All groups' work was video-recorded.</p> <div data-bbox="181 1032 687 1368"> </div> <p>For each question, students have to choose which graphical image should fit into '?' based on patterns and properties of the other 8. For the left the answer is 5, and the right one it is 4 (by seeing 'outside' as addition and 'inside' as subtraction).</p>	<p>Geometrical thinking tests: The same groups of students undertook geometry test which are derived by Fujita (2012) about hierarchical relationships between parallelograms. 4 of 9 groups were video-recorded for further analysis, informed by their group thinking test performances. Students undertake the following questions related to inclusion relations between quadrilaterals. For each question, 3 points will be given if students' choice is based on the hierarchical classification, but if it is prototypical then 1 point will be given. For example, for Q1, 3 points for '1, 2, 4, 5, 6, 7, 9, 11, 13, 14, 15' but 1 point for '1, 6, 9, 14' or for Q3, 1 point for choosing (b) and (c) are correct.</p> <p>Q1. Which of the quadrilaterals 1-15 above are members of the Parallelogram family?</p> <div data-bbox="842 770 1315 1016"> </div> <p>Q2. What is a parallelogram? Please write its definition.</p> <p>Q3. Read the following sentences carefully, and circle the statements which you think are correct.</p> <ul style="list-style-type: none"> <li>(a) There is a type of parallelogram which has right angles.</li> <li>(b) The lengths of the opposite sides of parallelograms are equal.</li> <li>(c) The diagonally opposite angles of parallelograms are equal.</li> <li>(d) There is a type of parallelogram which has 4 sides of equal length.</li> <li>(e) Some parallelograms have more than two lines of symmetry.</li> </ul> <p>Q4. Is it possible to draw a parallelogram whose four vertices are on the circumference of a circle?</p>

**Figure 1. Tests for group thinking and geometrical thinking**

In the data analysis, we first examined general relationships between general group thinking test and geometry test, focusing on whether geometrical thinking can be predicted from group thinking test performances. We then analysed the video data by considering what types of talk (disputational, cumulative, or explorative) can be recognised in their group work, what kind of 'voice' can be recognised in their collective Image Having/Making, Property Noticing and Folding Back processes (see the next section for the examples).

## Findings and analysis

### Overall performance in their group thinking test and collective geometric thinking

The test results for our sample of 27 students are as follows:

Test	Mean	S.D.
Group thinking test (Individual, N=27)	9.3 (out of 15)	2.07
Group thinking test (Group, N=9)	10.4 (out of 15)	1.24
Geometry test (N=9)	3.67 (out of 12)	1.66

There is no statistical significant difference between individual and group scores in the Group thinking tests (Mann-Whitney U test, p-value is 0.1063,  $p > .05$ ), indicating in this class in general collaborative learning did not positively affect test scores. Also, low scores from geometry tests indicate that the students' collective geometric thinking are also governed by prototypical examples of parallelograms, despite being given opportunities to share their ideas and to work collaboratively to solve the geometry test. Furthermore, statistical analysis of the data, using linear regression modelling, showed that the ability to predict geometry test scores from individual thinking scores, and group maths test scores was very weak ( $R^2$  of 0.046). Likewise, the relationship between individual group thinking scores and geometry test scores was very weak (Spearman Correlation 0.224). This might suggest that collective geometric thinking as 'measured' by the geometry test is different from the thinking 'measured' by the group thinking test at least in our sample (we will explain relationships between general group thinking and mathematical thinking in more detail in our presentation.)

### Examples of students' collective thinking process

Although quantitative analysis did not suggest strong relationships between general group thinking and collective geometric thinking, the video data suggest some interesting features relating to why students could not do well in the geometry test in their collaborative work. In total 340 interactions from students were examined in terms of stages of collective thinking process and dialogic theory. In this section we select examples from Group 1 and 5, whose obstacles were particularly related to not only their conceptual and visual images of quadrilaterals but also their dialogic relationships in their collaborative learning.

In the individual test, the three students BS, AC, and JC in Group 1 scored 14, 10, 11 but their group score was 12. This means their group work did not benefit very positively (in the context of group thinking measure test). In their geometry test, their interactions were rather disputational and they could not see their peers' ideas from the others' point of view in addition to influences from prototypical examples. For example, in their collective Image Making/Having stage, they discussed what a parallelogram was conceptually and visually, one of them questioned if rectangle or square can be a parallelograms based on the statement voiced by BS (line G1 49), but immediately after AC said "And a square and a rectangle. It's trash". This indicate in the line 54, the word 'square' or 'rectangle' by AC were very personal, and not accepting the 'voice' by BS or JC:

G1 47. JC      What is a parallelogram? Write the definition.

G1 48. AC      A squashed up rectangle.

- G1 49. JC      No both sides are parallel.  
 G1 50. AC      A squashed up rectangle.  
 G1 51. BS      So that would mean thirteen as well and two.  
 G1 52. AC:      And a square.  
 G1 53. BS:      And one and...  
 G1 54. AC:      And a square and a rectangle. It's trash.

They then continued their discussion, and it is evident that their understanding is influenced by the prototypical image of parallelogram (line G1 59, G1 60 or G1 70). In addition, it seemed that they could not see each other's positions. In the line 61, AC aggressively said 'That's what we got told in...', referring to authoritative voices. In the line 64, BS again held back to a definition "all the sides are parallel" and suggested rectangle can be a parallelogram (line G1 67). There was a dialogic gap between BS and AC/JC. However, JC and AC again referred to a (wrong) definition based on the prototypical image (line G1 68 and G1 69), and BS's voice was dismissed, and BS disappointingly said 'Oh no', and their collaborative explorations stopped.

- G1 58. AC      I think it's...  
 G1 59. BS      A squashed up rectangle.  
 G1 60. JC      No if it's a squashed up I need to know squeeze it.  
 G1 61. AC      That's what we got told in...  
 G1 62. BS      All sides are the same.  
 G1 63. AC      No they're not.  
 G1 64. BS      No, no, all the sides are parallel.  
 G1 65. JC      Yeah.  
 G1 66. AC      Yes so is a square.  
 G1 67. BS      So that will do one, two (pointing a rectangle image).  
 G1 68. JC      No because that's a quadrilateral not a parallelogram. A parallelograms are like...  
 G1 69. AC      A squashed up rectangle.  
 G1 70. JC      No parallelograms are like that they're like that they're messed up.  
 G1 71. BS      Oh no.

Let us see another group, Group 5. In the individual test, the three students JM, BH, and TF in Group 5 scored 9, 10, 7, but their group score was 11. This means their group work did not benefit either positively or negatively. In their collective Image Making stage of the geometry test, JM first voiced his own image and definition (line G5 12) which was influenced by the prototypical image and then TF agreed. Then BH added 'Two pairs of parallel sides' (line G5 14). This made JM question "a rectangle has two pairs of parallel sides as well?" (line G8 17), but after a moment he said "But it (parallelogram) doesn't have right angles" (line G5 18), indicating he could not see BH's point of view. TF then agreed with JM. BH did not argue back from here (a kind of cumulative talk), and they now had parallelogram as 'a rectangle without 90 degree angles' as their collective image of parallelogram.

- G5 7. JM      ...Ok what is a parallelogram?  
 G5 8. BH      Oh.  
 G5 9. JM      It's rectangle but...

- G5 10. BH     It's like...
- G5 11. TF     Erm like...
- G5 12. JM     It's like, it's a rectangle but it doesn't, we're not, it doesn't have all ninety degrees. It doesn't have all right angles.
- G5 13. TF     Yeah, yeah yes so it's a rectangle but it doesn't have...
- G5 14. BH     Two pairs of parallel sides.
- G5 15. JM     It's a rectangle.
- G5 16. BH     Two pairs of parallel sides.
- G5 17. JM     Yeah but that erm that a rectangle has two pairs of parallel sides as well.
- G5 18. JM     ... (a moment) But it doesn't have right angles so it's rectangle without ...
- G5 19. TF     Ninety degree angles.

After this, this shared definition used throughout the problem solving process in their Collective Property Noticing stage, resulting they only chose (b) and (c) of Q3 as true or in Q4 they formulated it would be impossible to draw a parallelogram whose four vertices are on the circumference of a circle because “the obtuse angles would not touch the circumference of the circle” (G5 line 56). The other groups (Group 2 and 8) also showed similar processes, i.e. definitions based on prototypical images were collectively made and had uncritically at first and then these were used to examine properties and formulate their answers.

## Discussion

In this paper we examined what obstacles will be identified when students are working together with geometrical problems. By answering our research question, our findings suggest that even collaborative learning settings prototypical images (Hershkowitz, 1990; Fujita, 2012) strongly influence when students were making/having conceptual and visual images of geometrical figures collaboratively, i.e. collective Image Making and Having stages (Pirie and Kieran, 1994; Martin and Towers, 2014). Also, when learners collectively had definitions based on prototypical images and missed opportunities to dialogically examine these (Bahktin, 1964; Wegerif, 2011; Barwell, 2016). Even if they shared ideas during their problem solving processes well, these students could not reach the correct answers by examining different ideas voiced in their collaborative work (Barwell, 2016). It is interesting to see that groups often had opportunities to examine their collective definitions (e.g. line G1 67 or G5 18-19), but they did not explore these opportunities because each member did not see what others were saying ‘as if through the eyes of another’ (e.g. line G1 67-71 or G5 16-19). Thus, in conclusion, in addition to prototype phenomenon, in collaborative learning settings it is necessary for students to dialogically examine their starting points of problem solving (in this case the definition of parallelogram).

In our research context, we did not find strong relationships between general group thinking and geometric thinking. As the sample size is relatively small, we would like to pursue this topic in our future research, together with developing effective pedagogical models for better collective geometric thinking.

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