Learning sine and cosine in French secondary schools

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The purpose of this paper is to present a questionnaire devoted to trigonometry and its results. More specifically, the study focuses on curriculum and cognitive aspects of learning and teaching sine and cosine over the final five years in French secondary schools, from grade 8 to grade 12. We ask: what are the difficulties of students in learning sine and cosine concepts and making the transition from the geometric setting to the functional setting. We identify four major types of error committed by students. This raises the question of how to make students effectively assimilate the concepts of sine and cosine.

Keywords: Trigonometry, sine and cosine, secondary school, available knowledge.

Introduction

In secondary schools in France, trigonometry notions are introduced progressively from grade 8 to grade 11 (science option), and end with sine and cosine functions in grade 12 (science option). We ask the key questions: How to articulate the passage from trigonometry in right triangles to the trigonometric circle and to sine and cosine functions? And, how to make the students assimilate these notions? To answer these questions, we investigate the status of the educational system: curriculum and learned knowledge.

In connection with our analysis of institutional texts and manuals for the study of the mathematical organization (Anthropological Theory of the Didactic, Chevallard, 1999; Bosch & Gascón, 2014), we elaborated a questionnaire for grade 12 students. The purpose of this questionnaire is to test the knowledge acquired by students in the final five years of study on trigonometry and specifically the sine and cosine functions in grade 12 (science option), to identify implicit notions in learning and teaching of trigonometry, to clarify the set of issues in our research work on what the students have learned, and to generate further research questions.

The questionnaire consists of six exercises, mainly concerning the cosine and sine of an acute angle in a right triangle, the cosine and sine of an oriented angle in the trigonometric circle, and some properties of the cosine or sine function like periodicity. In this paper, we analyze the first three exercises of the questionnaire and describe its results.

Methodology

For the curriculum study, we use the Anthropological Theory of the Didactic (Chevallard, 1999; Bosch & Gascón, 2014) as a theoretical framework in order to identify the mathematical organization (MO) of the French institutions that is developed in French mathematics textbooks. It is rather an exploratory analysis of French textbooks which gives us an overview of the teaching of trigonometry and of trigonometric functions in secondary schools.

We do not present in details the curriculum study but the identified MO allowed us to construct a questionnaire. It is grounded in the most used types of tasks and techniques in the French teaching institutions, from grade 8 to grade 12 related to trigonometry in right triangles (Exercise I, grades 8-
9), trigonometric circle (Exercise II, grades 10-11) and trigonometric functions (Exercise III, grade 12).

To analyze the tasks in the questionnaire, we choose tools from the Theory of Double didactic and ergonomic Approach (Robert, 2008; Robert & Hache, 2013) which provides us with fine cognitive tools to analyze students’ knowledge. The questionnaire is a test given to grade 12 students outside the context of any particular chapter in order to avoid any didactical contract influence. We study the knowledge adaptation in these tasks (see the following section) and especially, in this paper, the available level of knowledge application. This will allow us to have an idea of the cognitive complexity of the questionnaire and what students have learned (particularly in terms of available knowledge).

When tasks require adaptations of knowledge that are at least partly indicated, we speak of the level of knowledge application that can be mobilized. Students’ work is not effectively analogous, depending on whether they must look for the knowledge to use (questions of why or what), or apply and adapt the indicated knowledge (question of how). If it is up to the student to recognize the knowledge to use, we speak of the available level of knowledge application. […] We also distinguish combinations, links, or changes among elements such as frameworks, and work further on different types of intellectual activities that are specific to mathematics. (Robert & Hache 2013, p.37)

The questionnaire was given in two versions in order to avoid any influence of neighbor students. Variations between them were cognitively irrelevant to our study: different lengths in Exercise I, different coordinates in Exercise II, and cosine vs. sine functions in Exercise III.

**Presentation of the questionnaire, a priori analysis**

In this section we specify the aims of each question (of each exercise), the available knowledge (AK), the correct methods, possible erroneous methods and possible errors made by students. Most of the tasks are not “simple and isolated tasks” in the following sense:

We first distinguish simple and isolated tasks, or immediate applications of piece of knowledge without adaptation or combination. A single piece of knowledge is used, potentially with simple replacement of general inputs by the given information in the context of the exercise. (Robert & Hache 2013, p.36)

**Exercise I: Trigonometry in a triangle**

![Figure 1: Exercise I – Trigonometry in a triangle](image)

Exercise I (see Figure 1) asks for the values of sine and cosine of the angles of the given triangle.

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1 We focus on the knowledge effectively used by students, for example, we do not discuss methods using scalar product of vectors (seen in grade 11).
Note that in this figure, there is no coding denoting that $\beta$ is a right angle. We want to know the available knowledge of students, and particularly if the generalized Pythagorean theorem effectively is mobilized by students.

The first Method, M-I.1, relies on the definitions of the cosine and sine of an acute angle in a right triangle (seen in grades 8 and 9) which require first the reciprocal of Pythagorean theorem (AK, seen in grade 8) or the visual recognition of the right angle of the given triangle without proof.

Note that trigonometry in a right triangle only allows to calculate the cosine and sine of an acute angle whose measure is strictly between $0^\circ$ and $90^\circ$ but does not give $\cos \beta$ nor $\sin \beta$, which require another knowledge (AK).

There are three steps in M-I.1, where steps a and b form a non-simple task, and step c reinforces the complexity:

a. Recognition of a right triangle with the reciprocal of Pythagorean theorem (AK) or by visual inspection.

b. Definitions of the cosine and sine of an acute angle of the right triangle to find independently $\cos \alpha$, $\sin \alpha$, $\cos \gamma$ and $\sin \gamma$.

c. Property of the cosine and sine of the right angle to find $\cos \beta$ and $\sin \beta$.

Note that there are two other possible pieces of knowledge that could be used: finding, for example, $\cos \alpha$ with $\sin \alpha$ via the formula $\cos^2 + \sin^2 = 1$ (AK, seen in grade 9) or using a right triangle property (AK, seen in grade 9) that ensures the equalities $\cos \alpha = \sin \gamma$ and $\sin \alpha = \cos \gamma$.

Possible errors of the student are confusion between cosine and sine or an error in formulas.

The second method, M-I.2, relies on the generalized Pythagorean theorem (or Al-Kashi’s formula, it allows to calculate the cosine of an angle whose measure is strictly between $0^\circ$ and $180^\circ$ or between $0$ and $\pi$ radians, seen in grade 11) to find the cosine of an angle of any triangle. In this case, what is the reaction of the student when finding $\cos \beta = 0$? Does he/she conclude that $\beta$ is a right angle? Noticing that the given triangle is in fact a right triangle, will he/she change the strategy to M-I.1 method to determine the remaining values?

There are two related tasks, consisting in finding straightaway cosine, then sine. Apply the generalized Pythagorean theorem (AK) to independently find the three cosines and then, with the results obtained, use the fundamental relation $\cos^2 + \sin^2 = 1$ (AK, seen in grade 11) to find the three sines. It is a non-simple task to find the cosine of an angle of a triangle. And, it is a non-simple and non-isolated task to find the sine of angle: there is the introduction of steps and also a combination of settings (numerical and algebraic) - transformation into an equation of the type $x^2 = a$, determining the sign of the sine of the angle and deducing its value.

Possible errors of the student are in the application of the formulas, in algebraic transformation, and in numerical calculations.
Exercise II: Trigonometry in the trigonometric circle

From Exercise II, we only discuss here the question 1 (see Figure 2) which consists in asking for the cosine and sine of $\alpha$ (an acute angle) and of $\beta$ (an obtuse angle). In the context, we gave the coordinates of the points $M(4/5; 3/5)$ and $N(-24/25; 7/25)$. We want to know the available knowledge of students such as definitions of the cosine and sine of an oriented angle (seen in grade 11). It is asked similarly to the Exercise I for the cosine and sine values but in another setting, and moreover, with different notions of angles.

Note that in textbooks corresponding to the 2010 program in grade 11 (science option), one begins by defining measures of an oriented angle, then the cosine and sine of an oriented angle in this way: The cosine and the sine of an oriented angle are the cosine and the sine of any of its measures.

The first method, M-II.1, relies on the expression of the Cartesian coordinates of a point of the trigonometric circle with the cosine and sine. As $M(a;b)$ is a point of the trigonometric circle in a direct orthonormal frame $(O;I,J)$ of the plane and as $\alpha$ designates the oriented angle $(OI,OM)$, so we have $\cos\alpha = a$ and $\sin\alpha = b$ (AK). It is a non-simple task. As $\alpha$ does not designate a measure of the oriented angle $(OI,OM)$, do students use the M-II.1 method, namely that the coordinates of the point $M$ are $(\cos\alpha;\sin\alpha)$? If not, with the graph, will they think of using other methods, for example, like M-II.2 in the following?

The second method, M-II.2, relies on the relations between algebraic writing of a complex number and its trigonometric writing (AK) and on the characterization of equality of two complex numbers via equality of real and imaginary parts (AK, seen in grade 12). It is a non-simple task - introduction of steps: mark, in the given graph, the angles $\alpha$, $\beta$ - consider the points $M$ and $N$ as respective image points of complex numbers of module 1, then write the two complex numbers in two forms: algebraic writing and trigonometric writing - deduce the exact values of the cosine and sine of $\alpha$.
and of $\beta$. Compared to M-II.1 method, M-II.2 method requires the change of settings, of registers and of point of view in the reasoning (Duval, 2006).

Possible errors of the student are the confusion of sine and cosine in the expression of Cartesian coordinates of a point of the trigonometric circle with cosine and sine, error in the application of formulas and in numerical calculations.

**Exercise III: Graph of trigonometric functions**

![Figure 3: Exercise III – Graph of cosine function](image)

From Exercise III, we only present here the question 1.c. We want to know the available knowledge of students like the existence and the nonexistence of a point on the curve of a trigonometric function (cosine or sine) and the possibility of placing a point on the curve in the given graph, and particularly the property of periodicity.

The third method, M-III.1: Placing the point C of abscissa $\frac{11 + 6 \times 2016}{6}$ on the curve $\mathcal{C}$ within the graphic is an impossible task: recognize the existence of a point on a curve justifying it using the domain of a function and the given graph. It is a non-simple and non-isolated task - combination of graphical and functional settings.

Comparing the coordinates of A and of C relies on the property of a point on a curve and that of the cosine of a real number (AK) (or rather that of periodicity of the cosine function). It is a non-simple task. Besides, one could use the geometric property of periodicity of a function (AK, using graph of the function with translation).

Note that for comparing the coordinates of A and of C, there is another possible method based on the property of a point on a curve and the calculator in radian mode (an incomplete method with an erroneous conclusion starting from the numerical results from the calculator).

Possible errors of the student are a numerical error in the calculation; the periodicity of cosine function might not be an available knowledge.
A posteriori analysis of the three exercises shown and their results

The questionnaire was given to 40 students in March 2016 in two grade 12 classes of two different high schools.

**Exercise I**: 37 students used M-I.1 method (22 by reciprocal Pythagorean theorem (see Figure 4) and 15 by visual inspection), only 1 student used M-I.2 method, and 2 students did not answer this exercise.

We identify three major types of error committed by students, denoted TE1 (5 students), TE2 (9 students), TE3 (18 students). TE1 and TE2 are two types of more general errors, and TE3 consists in attempting to give different values of the *cosine and sine of the right angle of the right triangle* in the geometric setting.

TE1: “Confusion of the values of the cosine and sine of an angle with the value of the angle”.

For instance, \(\cos \alpha = 5/13 = 67.4\) (with or without the sign “°” designating the measure of the angle \(\alpha\) obtained in degree; here, the second “=” sign would designate “it gives”). We did not meet the error committed like \(\alpha = 5/13\)signifying that the angle \(\alpha\) was the value of the cosine of \(\alpha\).

TE2: “Confusion of the value of the cosine (or sine) of an angle with the value of the cosine (or sine) of a real number”. Students mobilize correctly, for example, the definition of the cosine of an acute angle of the right triangle but they do not stop there, they continue putting the sign “=” and conclude with an approximate value of the cosine of the real number which designates the value of the cosine of the acute angle obtained using the calculator either in degree mode or in radian mode. In this case, their result would be an approximate value of the “cosine of the cosine of the angle”.

For instance, \(\cos \alpha = 5/13 = 0.99\) with the calculator in degree mode, and in radian mode, \(\cos \alpha = 5/13 = 0.93\) (see Figure 4).

TE3: “Inventing values of the cosine and sine of the right angle of the right triangle”. It seems that students do not remember the values of the cosine and sine of the right angle which are the respective particular values 0 and 1. Thus, they invent these values in the right triangle committing TE3 (see Figure 4).

![Figure 4: Exercise I – TE2 and TE3](image)

We can conclude that the cosine and sine of an acute angle (\(\alpha\) or \(\gamma\)) of the right triangle are available knowledge only for half of the students while those of the right angle (\(\beta\)) are not available knowledge because only about a quarter of students correctly gave the values 0 and 1.
Exercise II: 11 students out of 40 did not do this exercise. 17 students out of 29 correctly gave the values of the cosine and sine of the angle $\alpha$: 11 students recognized the cosine and sine of the angle $\alpha$ as the $x$ and $y$ coordinates of the point M (M-II.1); 2 students exploited the properties of complex numbers (M-II.2); 4 students used previous knowledge seen in the 8th and 9th grades (M-I.1). 13 of these 17 students also correctly gave those of cosine and sine of the angle $\beta$ (11 used M-II.1 & 2 used M-II.2) while 4 other students who used M-I.1 method to give the cosine and sine of the angle $\alpha$ (acute angle) had any difficulties to calculate the cosine and sine of the angle $\beta$ (obtuse angle). 3 students out of 12 who incorrectly gave the values of the cosine and sine of the angles $\alpha$ and $\beta$ committed TE2, and 5 students did not calculate those of $\beta$. Thus, cosine and sine of an oriented angle are not available knowledge because only about a quarter of students effectively mobilized this knowledge.

Exercise III: Among 36 students who did this exercise, 11 committed TE4 (see below) and 14 did not answer question 1.c. And, 6 students out of 22 who answered question 1.c committed TE4 (see Figure 5) and they placed inside of the graphic the point C on the curve either at the point A or elsewhere.

Figure 5: Exercise III-1.c – TE4

TE4: “Confusion between the position of the two points on the trigonometric circle, images of two real numbers of the difference $2k\pi$ ($k$ is an integer) and that of the points on the cosine/sine curve with abscissas these two real numbers”. The TE4 is an error amounting to say, for instance, that the points A and C might have the same abscissa while the two points have in fact the same ordinate by the periodicity of the cosine function (see Figure 5a).

Conclusion

Through our analysis of this questionnaire, we clearly see the difficulties of students in using their knowledge on cosine and sine of an angle (seen in grades 8, 9, 11) and on those of a real number (seen in grades 10, 12) in the geometric and functional settings. Considering French institutional texts and manuals, and the work produced by students in answering the questionnaire, we identify the implicit notions related to the learning and teaching of trigonometry as follows.

In the case of trigonometry in right triangles (grades 8, 9), the notion of cosine and sine of an acute angle is an available knowledge for students while that of cosine and sine of the right angle in right triangles is not an available knowledge. To give the values of the cosine and sine of the right
angle, about half of the students tried to use the ratio of two lengths of the right triangle but it is not adapted to the case of the right angle, and other knowledge is required (seen in grades 10, 11).

In the case of **trigonometry in the trigonometric circle** (grade 11, science option), a remarkable number of students have difficulties seeing the link between the coordinates of a point on the trigonometric circle and the cosine and sine of the oriented angle that is defined in this circle. This knowledge is an available knowledge for only about a quarter of the students.

In the case of **sine and cosine functions** (grade 12, science option), some students have difficulties distinguishing between two real numbers of difference $2k\pi$ ($k$ is an integer) denoting two measurements in radian of the same oriented angle.

Overall, there is available knowledge for students to solve mathematical tasks on the trigonometry and on trigonometric functions, yet there is a blur or confusion in using their learned knowledge: between the value of cosine (or sine) and the angle or a measure of the angle, between an angle and its measurements, between a measure of an angle and a real number.

We can undoubtedly find epistemological, didactic, and curricular reasons; and this constitutes our research questions.

**References**


