Addendum: Non-cooperative Fisher–KPP systems: traveling waves and long-time behavior

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ADDENDUM: NON-COOPERATIVE FISHER–KPP SYSTEMS: TRAVELING WAVES AND LONG-TIME BEHAVIOR

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Abstract. The solution to a problem left open in the original paper [1] and solved afterwards is presented. Additionally, a few conflicting notations and a typo are corrected.

1. Extinction in the critical case $\lambda_{PF}(\mathbf{L}) = 0$

The critical case $\lambda_{PF}(\mathbf{L}) = 0$, for which extinction still occurs, was unsolved at the time of writing of [1]. It was solved later on thanks to an hint of Adrian Lam, who pointed out that the argument used to establish the upper estimates of [1, Theorem 1.2] can actually be used again to solve the extinction case.

Consequently, [1, Theorem 1.3] should be corrected as follows. The conjecture following [1, Theorem 1.3] and the discussion in [1, Section 4.1.1] can now safely be ignored.

Theorem. *[Extinction or persistence dichotomy]*

(1) *Assume $\lambda_{PF}(\mathbf{L}) < 0$. Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ converge asymptotically in time, exponentially fast, and uniformly in space to $\mathbf{0}$.*

(2) *Assume $\lambda_{PF}(\mathbf{L}) > 0$. Then there exists $\nu > 0$ such that all bounded positive classical solutions $\mathbf{u}$ of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ satisfy, for all bounded intervals $I\subset\mathbb{R}$,*

$$\left(\liminf_{t\to+\infty}\inf_{x\in I} u_i(t,x)\right)_{i\in[N]} \geq \nu\mathbf{1}_{N,1}.$$

Consequently, all bounded nonnegative nonzero classical solutions of (S_{KPP}) are valued in

$$\prod_{i=1}^{N}\left[\nu, g_i(0)\right].$$

(3) *Assume $\lambda_{PF}(\mathbf{L}) = 0$ and*

$$span\left(\mathbf{n}_{PF}(\mathbf{L})\right)\cap\mathsf{K}\cap\mathbf{c}^{-1}\left(\{\mathbf{0}\}\right) = \{\mathbf{0}\}.$$

Then all bounded nonnegative classical solutions of (E_{KPP}) set in $(0,+\infty)\times\mathbb{R}$ converge asymptotically in time and uniformly in space to $\mathbf{0}$.

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Proof. The third case is proved as follows.

Let $\mathbf{u}$ be a bounded nonnegative classical solution. Using its boundedness, we can define for all $T>0$

$$C_T = \inf\left\{C>0 \mid C\mathbf{n}_{PF}\left(\mathbf{L}\right) \geq \mathbf{u}\left(T,x\right) \text{ for all } x\in\mathbb{R}\right\}.$$

Subsequently, fix temporarily $T>0$ and let $\mathbf{v} = C_T\mathbf{n}_{PF}\left(\mathbf{L}\right)-\mathbf{u}$. Since $\lambda_{PF}\left(\mathbf{L}\right)=0$, $\mathbf{v}$ satisfies

$$\mathbf{v}\left(T,x\right)\geq\mathbf{0} \text{ for all } x\in\mathbb{R},$$
$$\partial_t\mathbf{v}-\mathbf{D}\partial_{xx}\mathbf{v}-\mathbf{L}\mathbf{v} = \mathbf{c}\left[\mathbf{u}\right]\circ\mathbf{u} \text{ in } \left(0,+\infty\right)\times\mathbb{R}.$$

Since the nonnegativity of $\mathbf{c}$ (H_2) implies $\mathbf{c}\left[\mathbf{u}\right]\circ\mathbf{u}\geq\mathbf{0}$, it follows from the maximum principle that

$$\mathbf{v}\geq\mathbf{0} \text{ in } \left[T,+\infty\right)\times\mathbb{R},$$

that is

$$C_T\mathbf{n}_{PF}\left(\mathbf{L}\right)\geq\mathbf{u} \text{ in } \left[T,+\infty\right)\times\mathbb{R},$$

whence $C_{T'}\leq C_T$ for all $T'\geq T$. In other words, $\left(C_T\right)_{T>0}$ is a nonincreasing family.

Next, let us verify that it is in fact a decreasing family if and only if $\mathbf{u}$ is nonzero.

Of course, if $\mathbf{u}=\mathbf{0}$, then $C_T=0$ for all $T>0$.

Now, assume that there exist $T>0$ and $T'>T$ such that, for all $t\in[T,T']$, $C_t=C_T$. Let $\mathbf{v}=C_T\mathbf{n}_{PF}\left(\mathbf{L}\right)-\mathbf{u}$. By optimality of $C_{T'}$,

$$\min_{i\in[N]}\inf_{x\in\mathbb{R}} v_i\left(T',x\right)=0.$$

If there exists $x\in\mathbb{R}$ such that

$$\mathbf{v}\left(T',x\right)=C_{T'}\mathbf{n}_{PF}\left(\mathbf{L}\right)-\mathbf{u}\left(T',x\right)\in\partial\mathsf{K},$$

then by the strong maximum principle

$$\mathbf{v}=\mathbf{0} \text{ in } \left[T,T'\right]\times\mathbb{R}.$$

This leads to $\mathbf{c}\left[\mathbf{u}\right]\circ\mathbf{u}=\mathbf{0}$ on one hand and to $\mathbf{u}=C_T\mathbf{n}_{PF}\left(\mathbf{L}\right)$ on the other hand, whence by assumption on $\mathbf{c}$ we deduce $C_T=0$. Therefore $\mathbf{u}=\mathbf{0}$ in $[T,T']\times\mathbb{R}$ and then in $(0,+\infty)\times\mathbb{R}$.

On the contrary, if such an $x\in\mathbb{R}$ does not exist, then by optimality of $C_{T'}$, there exists $\mathbf{w}\in\partial\mathsf{K}$ and $\left(x_n\right)_{n\in\mathbb{N}}$ such that, as $n\to+\infty$,

$$x_n\to\pm\infty \text{ and } \mathbf{v}\left(T',x_n\right)\to\mathbf{w}.$$

Defining the sequence

$$\mathbf{u}_n:(t,x)\mapsto\mathbf{u}\left(t,x+x_n\right)$$

and using classical parabolic estimates to extract a locally uniform limit $\mathbf{u}_\infty$, we find that $\mathbf{v}_\infty=C_T\mathbf{n}_{PF}\left(\mathbf{L}\right)-\mathbf{u}_\infty$ satisfies

$$\partial_t\mathbf{v}_\infty-\mathbf{D}\partial_{xx}\mathbf{v}_\infty-\mathbf{L}\mathbf{v}_\infty\geq\mathbf{0} \text{ in } \left[T,T'\right]\times\mathbb{R},$$
$$\mathbf{v}_\infty\geq\mathbf{0} \text{ in } \left[T,T'\right]\times\mathbb{R},$$
$$\mathbf{v}_\infty\left(T',0\right)=\mathbf{w}\in\partial\mathsf{K},$$

and then again by the strong maximum principle we find $\mathbf{v}_\infty=\mathbf{0}$ and subsequently $\mathbf{u}=\mathbf{0}$.

Hence either $\mathbf{u}=\mathbf{0}$ or the family $\left(C_T\right)_{T>0}$ is decreasing. Let

$$C_\infty=\lim_{T\to+\infty}C_T\geq 0.$$

Assuming by contradiction that $C_\infty > 0$, defining the sequence

$$\mathbf{u}_n : (t,x) \mapsto \mathbf{u}(t+n,x)$$

and its locally uniform limit $\mathbf{u}_\infty$, we can repeat the argument and obtain that the family $(D_T)_{T>0}$, where

$$D_T = \inf \{D > 0 \mid D\mathbf{n}_{PF}(\mathbf{L}) \geq \mathbf{u}_\infty(T,x) \text{ for all } x \in \mathbb{R}\},$$

is decreasing, which directly contradicts the fact that $D_T = C_\infty$ for all $T > 0$. This ends the proof. □

2. Conflicting notations, typos

2.1. Conflicting notations. In [1, Section 1.5.1], "The cane toads equation with non-local competition", the notation θ_N denotes the mesh size of the partition $\frac{\overline{\theta}-\underline{\theta}}{N-1}$. However in [1, Section 1.5.2], "The cane toads equation with non-local mutations and competition", it denotes both the mesh size and the last point of the partition $(\theta_i)_{i\in[N]}$. In order to solve this conflict, the mesh size should be denoted $\delta\theta$ instead.

Similarly, in [1, Section 1.5.3], "The Gurtin–MacCamy equation with diffusion and overcrowding effect", the mesh size should be denoted δa instead of a_{N+1}.

2.2. Typo. On page 153, first line, A_ε should be replaced by A.

References

[1] Girardin Léo, 2018, Non-cooperative Fisher–KPP systems: traveling waves and long-time behavior, *Nonlinearity* **31**, 108–164