WELL-POSEDNESS FOR SOME NON-LINEAR DIFFUSION PROCESSES AND RELATED PDE ON THE WASSERSTEIN SPACE

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Abstract. In this paper, we investigate the well-posedness of the martingale problem associated to non-linear stochastic differential equations (SDEs) in the sense of McKean-Vlasov under mild assumptions on the coefficients as well as classical solutions for a class of associated linear partial differential equations (PDEs) defined on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, for any $T>0$, $\mathcal{P}_2(\mathbb{R}^d)$ being the Wasserstein space (*i.e.* the space of probability measures on $\mathbb{R}^d$ with a finite second-order moment). In this case, the derivative of a map along a probability measure is understood in the Lions' sense. The martingale problem is addressed by a fixed point argument on a suitable complete metric space, under some mild regularity assumptions on the coefficients that covers a large class of interaction. Also, new well-posedness results in the strong sense are obtained from the previous analysis. Under additional assumptions, we then prove the existence of the associated density and investigate its smoothness property. In particular, we establish some Gaussian type bounds for its derivatives. We eventually address the existence and uniqueness for the related linear Cauchy problem with irregular terminal condition and source term.

Contents

1. Introduction

In this work, we are interested in some non-linear Stochastic Differential Equations (SDEs for short):

$$X_t^\xi = \xi + \int_0^t b(s, X_s^\xi, [X_s^\xi])ds + \int_0^t \sigma(s, X_s^\xi, [X_s^\xi])dW_s, \quad [\xi]\in\mathcal{P}(\mathbb{R}^d), \tag{1.1}$$

driven by a q-dimensional $W=(W^1,\cdots,W^q)$ Brownian motion with coefficients $b:\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$ and $\sigma:\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d\otimes\mathbb{R}^q$. Here and throughout, we denote by $[\theta]$ the law of the random variable θ. This kind of dynamics are also referred to as distribution dependent SDEs or mean-field or McKean-Vlasov SDEs as it describes the limiting behaviour of an individual particle evolving within a large system of particles interacting through its empirical measure, as the size of the population grows to infinity. More generally, the behaviour of the particle system is ruled by the so-called propagation of chaos phenomenon as originally studied by McKean [McK67] and then investigated by Sznitman [Szn91]. Roughly speaking, it says that if the initial conditions of a finite subset of the original system of particles become independent of each other, as the size of the whole system grows to infinity, then the dynamics of the particles of the finite subset synchronize and also become independent.

Since the original works of Kac [Kac56] in kinetic theory and of McKean [McK66] in non-linear parabolic partial differential equations (PDEs for short), many authors have investigated theoretical and numerical aspects of McKean-Vlasov SDEs under various settings such as: the well-posedness of related martingale problem, the propagation of chaos phenomenom and other limit theorems, probabilistic

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representations to non-linear parabolic PDEs and their numerical approximation schemes. We refer to Tanaka [Tan78], Gärtner [Gär88], [Szn91] among others.

On the well posedness of (1.1)**.** Well-posedness in the weak or strong sense of McKean-Vlasov SDEs have been intensively investigated under various settings by many authors during the last decades, see e.g. Funaki [Fun84], Oelschläger [Oel84], [Gär88], [Szn91], Jourdain [Jou97], and more recently, Li and Min [LM16], Chaudru de Raynal [CdR15], Mishura and Veretenikov [MV18], Lacker [Lac18] and Hammersley et al. [HvS18] for a short sample.

Classical well-posedness results usually rely on the Cauchy-Lipschitz theory when both coefficients b and σ are Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$ equipped with the product metric, the distance on $\mathcal{P}_p(\mathbb{R}^d)$ being the Wasserstein distance of order p, see *e.g.* [Szn91].

It actually turns out to be a challenging question to go beyond the aforementioned framework. Indeed, as it has been highlighted by Scheutzow in [Sch87], uniqueness may fail for a simple version of (1.1): when $p = q = 1$, $\sigma \equiv 0$, for all (t, x, m) in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$, $b(t, x, m) = \int \bar{b}(y)dm(y)$, for some bounded and locally Lipschitz function $\bar{b} : \mathbb{R} \to \mathbb{R}$, the SDE (1.1) with random initial condition have several solutions. Note that in this case the drift, seen as a function of the law, is only Lipschitz with respect to the total variation distance. Nevertheless, still in this setting, it has been shown by Shiga and Tanaka in [ST85] that pathwise uniqueness holds when $\sigma \equiv 1$. In that case, one may also relax the local Lispchitz assumption of the function $\bar{b}$ and only assume that it is bounded and measurable. Such a result has been extended by Jourdain [Jou97] where uniqueness is shown to hold for more general measurable and bounded drift b satisfying only a Lipschitz assumption with respect to the total variation distance and diffusion coefficient σ independent of the measure argument. These results have been recently revisited and extended to other non degenerate frameworks (allowing the diffusion coefficient to depend on the time and space variables) in Mishura and Veretenikov [MV18], Lacker [Lac18] and to possibly singular interaction of first order type by Röckner and Zhang in [RZ18]. We importantly emphasize that in all the aforementioned works, the diffusion coefficient only depends on the time and space variables and that the Lipschitz assumption of the drift coefficient with respect to the total variation distance as well as the non-degeneracy of the noise play a crucial role.

We start our work by revisiting the problem of the unique solvability of the SDE (1.1) by tackling the corresponding formulation of the martingale problem. Our main idea consists in a fixed point argument applied on a suitable complete metric space. To do so, we rely on a mild formulation of the transition density of the unique weak solution to the SDE (1.1) with coefficients frozen with respect to the measure argument. This formulation may be seen as the first step of a perturbation method for Markov semigroups, known as the parametrix technique, such as exposed in Friedman [Fri64], McKean and Singer [MS67]. We also refer to Konakov and Mammen [KM00], for the expansion in infinite series of a transition density and Delarue and Menozzi [DM10] or Frikha and Li [FL17] for some extensions of this technique in other directions.

Compared to the aforementioned results, our approach allows to deal with coefficients satisfying mild regularity assumption with respect to the space and measure variables. In particular, the diffusion coefficient may not be Lipschitz with respect to the Wasserstein distance which, to the best of our knowledge, appear to be new. Let us however mention the recent work [CdR19] of the first author where such a framework is handled for a particular class of interaction (of scalar type) and under stronger regularity assumptions on the coefficients. Then, by adding a Lipschitz continuity assumption in space on the diffusion coefficient, we derive through usual strong uniqueness results on linear SDE the well-posedness in the strong sense of the SDE (1.1).

Existence of a density for (1.1) **and associated Cauchy problem on the Wasserstein space.**

The well-posedness of the martingale problem then allows us to investigate in turn the regularity properties of the transition density associated to equation (1.1) and to establish some Gaussian type estimates for its derivatives. Some partial results related to the smoothing properties of McKean-Vlasov SDEs have been obtained by Chaudru de Raynal [CdR19], Baños [Bañ18], Crisan and McMurray [CM17]. In [CdR19], such type of bounds have been obtained in a regularized framework for McKean-Vlasov SDE (uniformly on the regularization procedure) with scalar interaction only. In [Bañ18], a Bismut-Elworthy-Li formula is proved for a similar equation (with scalar type interaction) under the assumption that both the drift and the diffusion matrix are continuously differentiable with bounded Lipschitz derivatives in both variables and the diffusion matrix is uniformly elliptic. In [CM17], in the uniform elliptic setting, using Malliavin calculus techniques, the authors proved several integration by parts formulae for the decoupled dynamics associated to the equation (1.1) from which stem several estimates on the associated

density and its derivatives when the coefficients b, σ are smooth and when the initial law in (1.1) is a Dirac mass.

Here, we will investigate the smoothness properties of the density of both random variables X_t^ξ and $X_t^{x,[\xi]}$ (given by the unique weak solution of the associated decoupled flow once the well-posedness for (1.1) has been established) under mild regularity assumptions on the coefficients, namely b and $a = \sigma\sigma^*$ are assumed to be continuous, bounded and Hölder continuous in space and a is uniformly elliptic. In this case, both the drift and diffusion coefficients are also assumed to have two bounded and Hölder continuous linear functional (or flat) derivatives with respect to its measure argument. We briefly present this notion of differentiation in Section 2.1 and refer to Carmona and Delarue [CD18] and Cardaliaguet & al. [CDLL15] for more details. Within this framework, we are able to take advantage of the smoothing property of the underlying heat kernel and to bring to light the regularity properties of the density with respect to its measure argument for a coarser topology. Namely, the coefficients admit two linear functional derivatives but the density admits two derivatives in the sense of Lions (see section 2.1 for definitions), which appears to be a stronger notion of differentiation. As a consequence, we recover an ad hoc version of the theory investigated in the linear case in the monograph of Friedman [Fri64], [Fri11]. In particular, we establish some Gaussian type estimates for both densities and their derivatives with respect to the time, space and measure arguments.

Finally, the previous smoothing properties of the densities enable us to investigate classical solutions for a class of linear parabolic PDEs on the Wasserstein space, namely

$$\begin{cases} (\partial_t + \mathcal{L}_t)U(t,x,\mu) = f(t,x,\mu) & \text{for } (t,x,\mu) \in [0,T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \\ U(T,x,\mu) = h(x,\mu) & \text{for } (x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{1.2}$$

where the source term $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and the terminal condition $h : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ are some given functions and the operator $\mathcal{L}_t$ is defined by

$$\begin{aligned} \mathcal{L}_t g(x,\mu) &= \sum_{i=1}^d b_i(t,x,\mu)\partial_{x_i} g(x,\mu) + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(t,x,\mu)\partial^2_{x_i,x_j} g(x,\mu) \\ &\quad + \int \left\{ \sum_{i=1}^d b_i(t,z,\mu)[\partial_\mu g(x,\mu)(z)]_i + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(t,z,\mu)\partial_{z_i}[\partial_\mu g(x,\mu)(z)]_j \right\} \mu(dz) \end{aligned} \tag{1.3}$$

and acts on sufficiently smooth test functions $g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $a = \sigma\sigma^*$ is uniformly elliptic. Though the first part of the operator appearing in the right-hand side of (1.3) is quite standard, the second part is new and involves the Lions' derivative of the test function with respect to the measure variable μ, as introduced by P.-L. Lions in his seminal lectures at *the Collège de France*, see [Lio14]. We briefly present this notion of differentiation on the Wasserstein space in Section 2.1 together with the chain rule formula established in Chassagneux et al. [CCD14], see also Carmona and Delarue [CD18], for the flow of measures generated by the law of an Itô process. Classical solutions for PDEs of the form (1.2) have already been investigated in the literature using different methods and under various settings, e.g. Buckdhan et al. [BLPR17] (for $f \equiv 0$), [CCD14] and very recently [CM17] (for $f \equiv 0$). We also refer the reader to the pedagogical paper Bensoussan et al. [BFY17] for a discussion of the different point of views in order to derive PDEs on the Wasserstein space and their applications.

In the classical diffusion setting, provided the coefficients b and σ and the terminal condition h are smooth enough (with bounded derivatives), it is now well-known that the solution to the related linear Kolmogorov PDE is smooth (see e. g. Krylov [Kry99]). In [BLPR17], the authors proved a similar result in the case of the linear PDE (1.2) (with $f \equiv 0$) and Chassagneux et al. [CCD14] reached the same conclusion for a non-linear version also known as the Master equation. In this sense, the solution of the considered PDE preserves the regularity of the terminal condition. Still in the standard diffusion setting, it is known that one can weaken the regularity assumption on h if one can benefit from the smoothness of the underlying transition density. Indeed in this case, $u(t,x) = \int h(y)\, p(t,T,x,y)\, dy$, $y \mapsto p(t,T,x,y)$ being the density of the (standard) SDE taken at time T and starting from x at time t. However, in order to benefit from this regularizing property, one has to assume that the associated operator $\mathcal{L}$ satisfies some non-degeneracy assumption. When the coefficients b, $a = \sigma\sigma^*$ are bounded measurable and Hölder continuous in space (uniformly in time) and if a is unformly elliptic, it is known (see e.g. [Fri64]) that the linear Kolmogorov PDE admits a fundamental solution so that the unique classical solution exists when the terminal condition h is not differentiable but only continuous. In the seminal paper [Hör67], Hörmander gave a sufficient condition for a second order linear Kolmogorov PDE

with smooth coefficients to be hypoelliptic. Thus, if Hörmander's condition is satisfied then the unique classical solution exists even if the terminal condition is not smooth. Note that this condition is known to be nearly necessary since in the non-hypoelliptic regime, even in the case of smooth coefficients, there exists counterexample to the regularity preservation of the terminal condition, see e.g. Hairer and al. [HHJ15].

The recent paper [CM17] provides the first result in this direction for the PDE (1.2) without source term and for non differentiable terminal condition h using Malliavin calculus techniques under the assumption that the time-homogeneous coefficients b, σ are smooth with respect to the space and measure variables. In particular, the function h has to belong to a certain class of (possibly non-smooth) functions for which Malliavin integration by parts can be applied in order to retrieve the differentiability of the solution in the measure direction. This kind of condition appears to be natural since one cannot expect the solution of the PDE (1.2) to preserve regularity in the measure variable in full generality as it is the case for the spatial argument, see Example 5.1 in [CM17] for more details on this loss of regularity.

Under the aforementioned regularity assumptions on the coefficients b and a and if the data f and h admit a linear functional derivative satisfying some mild regularity and growth assumptions, we derive a theory on the existence and uniqueness of classical solutions for the PDE (1.2) which is analogous to the one considered in Chapter 1 [Fri64] for linear parabolic PDEs.

Organization of the paper. The paper is organized as follows. The basic notions of differentiation on the Wasserstein space with an emphasis on the chain rule and on the regularization property of a map defined on $\mathcal{P}_2(\mathbb{R}^d)$ by a smooth flow of probability measures that will play a central role in our analysis are presented in Section 2. The general set-up together with the assumptions and the main results are described in Section 3. The well-posedness of the martingale problem associated to the SDE (1.1) is tackled in Section 4. The existence and the smoothness properties of its transition density are investigated in Section 5. Finally, classical solutions to the Cauchy problem related to the PDE (1.2) are studied in Section 6. The proof of some useful technical results are given in Appendix.

Notations: In the following we will denote by C and K some generic positive constants that may depend on the coefficients b and σ. We reserve the notation c for constants depending on $|\sigma|_\infty$ and λ (see assumption (**HE**) in Section 3) but not on the time horizon T. Moreover, the value of both C, K or c may eventually change from line to line.

We will denote by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures on $\mathbb{R}^d$ and by $\mathcal{P}_2(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$ the space of probability measures with finite second moment.

For a positive variance-covariance matrix Σ, the function $y \mapsto g(\Sigma, y)$ stands for the d-dimensional Gaussian kernel with Σ as covariance matrix $g(\Sigma, x) = (2\pi)^{-\frac{d}{2}}(\det \Sigma)^{-\frac{1}{2}} \exp(-\frac{1}{2}\langle \Sigma^{-1}x, x\rangle)$. We also define the first and second order Hermite polynomials: $H_1^i(\Sigma, x) := -(\Sigma^{-1}x)_i$ and $H_2^{i,j}(\Sigma, x) := (\Sigma^{-1}x)_i(\Sigma^{-1}x)_j - (\Sigma^{-1})_{i,j}$, $1 \le i, j \le d$ which are related to the previous Gaussian density as follows $\partial_{x_i} g(\Sigma, x) = H_1^i(\Sigma, x)g(\Sigma, x)$, $\partial^2_{x_i,x_j} g(\Sigma, x) = H_2^{i,j}(\Sigma, x)g(\Sigma, x)$. Also, when $\Sigma = cI_d$, for some positive constant c, the latter notation is simplified to $g(c, x) := (1/(2\pi c))^{d/2} \exp(-|x|^2/(2c))$.

One of the key inequality that will be used intensively in this work is the following: for any $p, q > 0$ and $x \in \mathbb{R}$, $|x|^p e^{-qx^2} \le (p/(2qe))^{p/2}$. As a direct consequence, we obtain the *space-time inequality*,

$$\forall p,\, c > 0, \quad |x|^p g(ct, x) \le C t^{p/2} g(c't, x) \tag{1.4}$$

which in turn gives the *standard Gaussian estimates* for the first and second order derivatives of Gaussian density, namely

$$\forall c > 0,\, |H_1^i(ct, x)|g(ct, x) \le \frac{C}{t^{\frac{1}{2}}} g(c't, x) \quad \text{and} \quad |H_2^{i,j}(ct, x)|g(ct, x) \le \frac{C}{t} g(c't, x) \tag{1.5}$$

for some positive constants C, c'. Since we will employ it quite frequently, we will often omit to mention it explicitly at some places. We finally define the Mittag-Leffler function $E_{\alpha,\beta}(z) := \sum_{n\ge 0} z^n/\Gamma(\alpha n + \beta)$, $z \in \mathbb{R}$, $\alpha,\ \beta > 0$.

2. Preliminaries: Differentiation on the Wasserstein space and Smoothing properties of McKean-Vlasov equations

2.1. **Differentiation on the Wasserstein space.** In this section, we present the reader with a brief overview of the regularity notions used when working with mappins defined on $\mathcal{P}_2(\mathbb{R}^d)$. We refer the

reader to Lions' seminal lectures [Lio14], to Cardaliaguet's lectures notes [Car13], to the recent work Cardaliaguet et al. [CDLL15] or to Chapter 5 of Carmona and Delarue's monograph [CD18] for a more complete and detailed exposition. The space $\mathcal{P}_2(\mathbb{R}^d)$ is equipped with the 2-Wasserstein metric

$$W_2(\mu,\nu) = \inf_{\pi\in\mathcal{P}(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, \pi(dx,dy) \right)^{\frac{1}{2}}$$

where, for given $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\mathcal{P}(\mu,\nu)$ denotes the set of measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν.

In what follows, we will work with two notions of differentiation of a continuous map U defined on $\mathcal{P}_2(\mathbb{R}^d)$. The first one, called the *linear functional derivative* and denoted by $\delta U/\delta m$, will be intensively employed in our linearization procedure to tackle the martingale problem and to study the smoothing properties of McKean-Vlasov SDEs. The second one is the *Lions' derivative*, L-derivative in short, and will be denoted by $\partial_\mu U$.

Linear functional derivative.

Definition 2.1. The continuous map $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to have a continuous linear functional derivative if there exists a continuous function $\delta U/\delta m : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that $y \mapsto [\delta U/\delta m](m)(y)$ has at most quadratic growth in y, uniformly in m for $m \in \mathcal{K}$, $\mathcal{K}$ being any compact subset of $\mathcal{P}_2(\mathbb{R}^d)$ and such that for any $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\lim_{\varepsilon\downarrow 0} \frac{U((1-\varepsilon)m+\varepsilon m') - U(m)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m)(y)\, d(m'-m)(y).$$

The map $y \mapsto [\delta U/\delta m](m)(y)$ being defined up to an additive constant, we will follow the usual normalization convention $\int_{\mathbb{R}^d} [\delta U/\delta m](m)(y)\, dm(y) = 0$.

Remark 2.2. Such a notion of derivative is linked to the regularity property with respect to the total variation distance. Observe indeed from the above definition that

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \quad U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-\lambda)m + \lambda m')(y)\, d(m'-m)(y)\, d\lambda,$$

so that if $(m,y) \mapsto [\delta U/\delta m](m)(y)$ is bounded, then one has

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \quad |U(m) - U(m')| \le \sup_{m''\in\mathcal{P}_2(\mathbb{R}^d)} \| \frac{\delta U}{\delta m}(m'')(.)\|_\infty \, d_{TV}(m,m') \tag{2.1}$$

where d_{TV} is the total variation metric. Therefore, if the map U admits a bounded linear functional derivative then it is Lipschitz continuous with respect to the distance d_{TV}.

With the above definition in mind, one may again investigate the smoothness of $m \mapsto [\delta U/\delta m](m)(y)$ for a fixed $y \in \mathbb{R}^d$. We will say that U has two continuous linear functional derivative and denote $[\delta^2 U/\delta m^2](m)(y)$ its second derivative taken at (m,y) if $m \mapsto [\delta U/\delta m](m)(y)$ has a continuous linear functional derivative in the sense of Definition 2.1. As a consequence,

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \quad \frac{\delta U}{\delta m}(m')(y) - \frac{\delta U}{\delta m}(m)(y) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta^2 U}{\delta m^2}((1-\lambda)m+\lambda m')(y,y')\, d(m'-m)(y')\, d\lambda$$

and if $\mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \ni (m,y,y') \mapsto [\delta^2 U/\delta m^2](m)(y,y')$ is continuous then $[\delta^2 U/\delta m^2](m)(y,y') = [\delta^2 U/\delta m^2](m)(y',y)$ for all $(m,y,y') \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$. Again, for more details on the above notion of derivative, we refer to [CDLL15] and [CD18].

Example 2.3. We here provide some examples of maps that admit a linear functional derivative. In the following, h denotes a map from $\mathcal{P}_2(\mathbb{R}^d)$ to $\mathbb{R}$. We can straightforwardly consider their multidimensional version.

(1) First order interaction. We say that h satisfies a first order interaction if it is of following form: for some continuous function with at most quadratic growth $\bar{h} : \mathbb{R}^d \to \mathbb{R}$, one has

$$h(\mu) = \int \bar{h}(y)\mu(dy).$$

(2) N order interaction. We say that h satisfies an N order interaction if it is of following form: for some continuous function with at most quadratic growth $\bar{h} : \mathbb{R}^N \to \mathbb{R}$, one has

$$h(\mu) = \int \cdots \int \bar{h}(y_1, \cdots, y_N)\, \mu(dy_1) \cdots \mu(dy_N).$$

(3) Polynomials on the Wasserstein space. We say that a function h is a polynomial on the Wasserstein space if there exist some real-valued continuous functions with at most quadratic growth $\bar{h}_1, \cdots, \bar{h}_N$ defined on $\mathbb{R}^d$ such that

$$h(\mu) = \prod_{i=1}^{N} \int \bar{h}_i(z)\mu(dz).$$

(4) Scalar interaction. We say that a function h satisfies a scalar interaction if there exist a continuously differentiable real-valued function $\bar{h}$ defined on $\mathbb{R}^N$ as well as some real-valued continuous functions with at most quadratic growth $\bar{h}_1, \cdots, \bar{h}_N$ defined on $\mathbb{R}^d$ such that

$$h(\mu) = \bar{h}\left(\int \bar{h}_1(y)\,\mu(dy), \cdots, \int \bar{h}_N(y)\,\mu(dy)\right).$$

(5) Sum, product and more generally any smooth composition of N order interactions, polynomials on Wasserstein space or scalar interaction.

The L-derivative. We now briefly present the second notion of derivatives we will employ as originally introduced by Lions [Lio14]. His strategy consists in considering the canonical lift of the real-valued function $U : \mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto U(\mu)$ into a function $\mathcal{U} : \mathbb{L}_2 \ni Z \mapsto \mathcal{U}(Z) = U([Z]) \in \mathbb{R}$, $(\Omega, \mathcal{F}, \mathbb{P})$ standing for an atomless probability space, with Ω a Polish space, $\mathcal{F}$ its Borel σ-algebra, $\mathbb{L}_2 := \mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d)$ standing for the space of $\mathbb{R}^d$-valued random variables defined on Ω with finite second moment and Z being a random variable with law μ. Taking advantage of the Hilbert structure of the $\mathbb{L}_2$ space, the function U is then said to be differentiable at $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ if its canonical lift $\mathcal{U}$ is Fréchet differentiable at some point Z such that $[Z] = \mu$. In that case, its gradient is denoted by $D\mathcal{U}$. Thanks to Riezs' representation theorem, we can identify $D\mathcal{U}$ as an element of $\mathbb{L}^2$. It then turns out that $D\mathcal{U}$ is a random variable which is $\sigma(Z)$-measurable and given by a function $DU(\mu)(.)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$, which depends on the law μ of Z and satisfying $DU(\mu)(.) \in \mathbb{L}^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$. Since we will work with mappings U depending on several variables, we will adopt the notation $\partial_\mu U(\mu)(.)$ in order to emphasize that we are taking the derivative of the map U with respect to its measure argument. Thus, inspired by [CD18], the L-derivative (or L-differential) of U at μ is the map $\partial_\mu U(\mu)(.) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$, satisfying $D\mathcal{U} = \partial_\mu U(\mu)(Z)$.

It is important to note that this representation holds irrespectively of the choice of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows, we will only consider functions which are $\mathcal{C}^1$, that is, functions for which the associated canonical lift is $\mathcal{C}^1$ on $\mathbb{L}^2$. We will also restrict our consideration to the class of functions which are $\mathcal{C}^1$ and for which there exists a continuous version of the mapping $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, v) \mapsto \partial_\mu U(\mu)(v) \in \mathbb{R}^d$. It then appears that this version is unique. We straightforwardly extend the above discussion to $\mathbb{R}^d$-valued or $\mathbb{R}^d \otimes \mathbb{R}^d$-valued maps U defined on $\mathcal{P}_2(\mathbb{R}^d)$, component by component.

Remark 2.4. Let us point out the link between this notion of derivative and the regularity property with respect to the Wasserstein distance. Observe indeed that if a map U is continuously L-differentiable and if the Fréchet derivative of its lift $D\mathcal{U}$ is bounded in $\mathbb{L}_2$ then

$$\begin{aligned} \forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d), \quad |U(\mu) - U(\mu')| &= \int_0^1 d\lambda |\mathbb{E}[\partial_\mu U([\lambda X + (1-\lambda)X'])(\lambda X + (1-\lambda)X')(X - X')] \\ &\leq \|D\mathcal{U}\|_{\mathbb{L}_2} W_2(\mu, \mu'), \end{aligned}$$

thanks to Cauchy-Schwarz's inequality and where above X and X' denote two independent random variables in $\mathbb{L}_2$ with respective law μ and μ'. In comparison with Remark 2.2, if one now assumes that the L-differential $\partial_\mu U$, viewed as the map $\mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, y) \mapsto \partial_\mu U(\mu)(y)$, is bounded in supremum norm then, from the above computations one readily sees that

$$\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d), \quad |U(\mu) - U(\mu')| \leq \sup_{\mu'' \in \mathcal{P}_2(\mathbb{R}^d)} \|\partial_\mu U(\mu)(\cdot)\|_\infty W_1(\mu, \mu').$$

To conclude, as underlined in Proposition 5.48 of [CD18], the following relation holds between the linear functional and the L-derivative. If a map h admits a linear functional derivative $\delta h/\delta m$ such that for any μ in $\mathcal{P}_2(\mathbb{R}^d)$, the map $y \mapsto [\delta h/\delta m](\mu)(y)$ is differentiable and its derivative is jointly continuous in y and m and at most of linear growth in y uniformly in μ for any μ in compact subset of $\mathcal{P}_2(\mathbb{R}^d)$ then one has

$$\partial_\mu h(\mu)(\cdot) = \partial_y [\frac{\delta h}{\delta m}](\mu)(\cdot). \tag{2.2}$$

Class of smooth functions of time, space and probability measure and associated Itô's formula on the Wasserstein space. In order to tackle the PDE (1.2) on the Wasserstein space, we need a chain rule formula for $(U(t,Y_t,[X_t]))_{t\geq 0}$, where $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are two Itô processes defined for sake of simplicity on the same probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ assumed to be equipped with a right-continuous and complete filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$. Their dynamics are given by

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s,\ X_0\in\mathbb{L}_2, \tag{2.3}$$

$$Y_t = Y_0 + \int_0^t \eta_s\,ds + \int_0^t \gamma_s\,dW_s \tag{2.4}$$

where $W=(W_t)_{t\geq 0}$ is an $\mathbb{F}$-adapted d-dimensional Brownian, $(b_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$, $(\sigma_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ are $\mathbb{F}$-progressively measurable processes, with values in $\mathbb{R}^d$, $\mathbb{R}^d$, $\mathbb{R}^d\otimes\mathbb{R}^d$ and $\mathbb{R}^{d\times q}$ respectively, satisfying the following conditions

$$\forall T>0,\quad \mathbb{E}\Big[\int_0^T (|b_t|^2+|\sigma_t|^4)\,dt\Big]<\infty \text{ and } \mathbb{P}\left(\int_0^T (|\eta_t|+|\gamma_t|^2)\,dt<+\infty\right)=1. \tag{2.5}$$

We now introduce two classes of functions we will work with throughout the paper.

Definition 2.5. (The space $\mathcal{C}^{p,2,2}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, for $p=0$, 1) Let $T>0$ and $p\in\{0,1\}$. The continuous function $U:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ is in $\mathcal{C}^{p,2,2}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ if the following conditions hold:

(i) For any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, the mapping $[0,T]\times\mathbb{R}^d\ni(t,x)\mapsto U(t,x,\mu)$ is in $\mathcal{C}^{p,2}([0,T]\times\mathbb{R}^d)$ and the functions $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,x,\mu)\mapsto\partial_t^p U(t,x,\mu)$, $\partial_x U(t,x,\mu)$, $\partial_x^2 U(t,x,\mu)$ are continuous.

(ii) For any $(t,x)\in[0,T]\times\mathbb{R}^d$, the mapping $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto U(t,x,\mu)$ is continuously L-differentiable and for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, we can find a version of the mapping $\mathbb{R}^d\ni v\mapsto\partial_\mu U(t,x,\mu)(v)$ such that the mapping $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\ni(t,x,\mu,v)\mapsto\partial_\mu U(t,x,\mu)(v)$ is locally bounded and is continuous at any (t,x,μ,v) such that $v\in\mathrm{Supp}(\mu)$.

(iii) For the version of $\partial_\mu U$ mentioned above and for any (t,x,μ) in $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, the mapping $\mathbb{R}^d\ni v\mapsto\partial_\mu U(t,x,\mu)(v)$ is continuously differentiable and its derivative $\partial_v[\partial_\mu U(t,x,\mu)](v)\in\mathbb{R}^{d\times d}$ is jointly continuous in (t,x,μ,v) at any point (t,x,μ,v) such that $v\in\mathrm{Supp}(\mu)$.

Remark 2.6. We will also consider the space $\mathcal{C}^{1,p}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ for $p=1$, 2, where we adequately remove the space variable in the Definition 2.5. We will say that $U\in\mathcal{C}^{1,1}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ if U is continuous, $t\mapsto U(t,\mu)\in\mathcal{C}^1([0,T])$ for any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $(t,\mu)\mapsto\partial_t U(t,\mu)$ being continuous and if for any $t\in[0,T]$, $\mu\mapsto U(t,\mu)$ is continuously L-differentiable such that we can find a version of $v\mapsto\partial_\mu U(t,\mu)(v)$ satisfying: $(t,\mu,v)\mapsto\partial_\mu U(t,\mu)(v)$ is locally bounded and continuous at any (t,μ,v) satisfying $v\in\mathrm{Supp}(\mu)$.

We will say that $U\in\mathcal{C}^{1,2}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ if $U\in\mathcal{C}^{1,1}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$ and for the version of $\partial_\mu U$ previously considered, for any $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, the mapping $\mathbb{R}^d\ni v\mapsto\partial_\mu U(t,\mu)(v)$ is continuously differentiable and its derivative $\partial_v[\partial_\mu U(t,\mu)](v)\in\mathbb{R}^{d\times d}$ is jointly continuous in (t,μ,v) at any point (t,μ,v) such that $v\in\mathrm{Supp}(\mu)$.

With the above definitions, we can now provide the chain rule formula on the Wasserstein space that will be play a central role in our analysis.

Proposition 2.1 ([CD18], Proposition 5.102)**.** Let X and Y be two Itô processes, with respective dynamics (2.3) and (2.4), satisfying (2.5). Assume that $U\in\mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ in the sense of Definition 2.5 such that for any compact set $\mathcal{K}\subset\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$,

$$\sup_{(t,x,\mu)\in[0,T]\times\mathcal{K}}\left\{\int_{\mathbb{R}^d}|\partial_\mu U(t,x,\mu)(v)|^2\,\mu(dv)+\int_{\mathbb{R}^d}|\partial_v[\partial_\mu U(t,x,\mu)](v)|^2\,\mu(dv)\right\}<\infty. \tag{2.6}$$

Then, $\mathbb{P}$-a.s., $\forall t\in[0,T]$, one has

$$\begin{aligned} U(t,Y_t,[X_t]) &= U(0,Y_0,[X_0]) + \int_0^t \partial_x U(s,Y_s,[X_s])\,.\gamma_s\,dW_s \\ &+ \int_0^t\left\{\partial_s U(s,Y_s,[X_s]) + \partial_x U(s,Y_s,[X_s]).\eta_s + \frac{1}{2}Tr(\partial_x^2 U(s,Y_s,[X_s])\gamma_s\gamma_s^T)\right\}ds \\ &+ \int_0^t\left\{\widetilde{\mathbb{E}}\big[\partial_\mu U(s,Y_s,[X_s])(\widetilde{X}_s).\widetilde{b}_s\big] + \frac{1}{2}\widetilde{\mathbb{E}}\big[Tr(\partial_v[\partial_\mu U(s,Y_s,[X_s])](\widetilde{X}_s)\widetilde{a}_s)\big]\right\}ds \end{aligned} \tag{2.7}$$

where the Itô process $(\widetilde{X}_t, \widetilde{b}_t, \widetilde{\sigma}_t)_{0\leq t\leq T}$ is a copy of the original process $(X_t, b_t, \sigma_t)_{0\leq t\leq T}$ defined on a copy $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2. **Smoothing properties of McKean-Vlasov semigroup.** One of the central feature of our analysis is that it involves Lions' derivatives of maps defined on $\mathcal{P}_2(\mathbb{R}^d)$ along smooth flow of probability measures. Nevertheless, we here only assume that such maps admit linear functional derivatives, which is a weaker notion of differentiation. To overcome such an issue, we take advantage of the smoothing properties of the composition with the flow. Namely, we will benefit from the smoothing effect of the McKean-Vlasov semigroup which, in our current setting, translates into a weakening of the topology w.r.t. which maps are, a priori, smooth. This particular feature will allow us to directly differentiate the density associated to a McKean-Vlasov SDE with respect to its measure argument without assuming any *intrinsic smoothness* of the coefficients, i.e. smoothness in the sense of Lions. We will then take advantage of this important property to provide a well-posedness theory for the SDE (1.1) and for classical solutions to the related PDE (1.2).

In order to foster the understanding of the key idea, let us consider a map $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ which is assumed to admit a bounded and continuous linear functional derivative $\delta h/\delta m$. Recall also from (2.1) that the map h is Lipschitz continuous w.r.t the total variation distance. Consider the simplest version of (1.1) (i.e. with $d = q = 1$, $b \equiv 0$ and $\sigma \equiv 1$ therein) that is the process $X_t^\xi = \xi + W_t$ and recall that W is a Brownian motion independent of ξ.

Let us first show how the noise regularizes the map $\mu \mapsto h([X_t^\xi])$ in the sense that it is now Lipschitz w.r.t. a weaker topology and differentiable in a stronger sense (i.e. in the sense of Lions). Note first that in that setting, $\mu \mapsto h([X_t^\xi])$ rewrites $\mu \mapsto h(\mu \star g_t)$, g_t being the Gaussian density with variance t and where $\star$ stands for the usual convolution product that is for all A in $\mathcal{B}(\mathbb{R}^d)$, $(\mu \star g_t)(A) = \int \int_A g_t(y-x)\, dy d\mu(x)$. We have, for all ξ, ξ' in $\mathbb{L}_2$:

$$\begin{aligned} & h([X_t^\xi]) - h([X_t^{\xi'}]) = h([\xi] \star g_t) - h([\xi'] \star g_t) \\ = & \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta h}{\delta m}([\lambda\xi + (1-\lambda)\xi'] \star g_t)(y) g_t(y-x)\, d([\xi] - [\xi'])(x) dy d\lambda \\ = & \int_0^1 \int_{\mathbb{R}^d} \frac{\delta h}{\delta m}([\lambda\xi + (1-\lambda)\xi'] \star g_t)(y) \mathbb{E}[g_t(y-\xi) - g_t(y-\xi')] dy d\lambda. \end{aligned} \tag{2.8}$$

At the one hand, we readily obtain from the previous identity and the mean-value theorem as well as (1.5) that

$$|h([X_t^\xi]) - h([X_t^{\xi'}])| \leq \frac{C}{t^{\frac{1}{2}}} \inf_{\pi \in \Pi_2(m,m')} \int \{|x-y| \wedge 1\} d\pi(x,y) =: \frac{C}{t^{\frac{1}{2}}} d([\xi], [\xi']),$$

where $\Pi_2(m, m')$ denotes the set of probability measures on $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ with m and m' as respective marginals. Hence, starting at time 0 with a map h being Lipschitz in total variation distance, we end up at any time $t > 0$ with a map which is Lipschitz w.r.t. the distance d, which is well seen to be less than the total variation distance and the Wasserstein metric W_1.

On the other hand, coming back to (2.8) and choosing $\xi' = \xi + \varepsilon Y$ for some Y in $\mathbb{L}_2$ and $\varepsilon > 0$ we have, from Lebesgue's dominated convergence theorem, continuity of the integrands in the right hand side and then Fubini's theorem:

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(h(X_t^{[\xi+\varepsilon Y]}) - h(X_t^{[\xi]})) \\ = & \lim_{\varepsilon \downarrow 0} \int_0^1 \int_{\mathbb{R}^d} \frac{\delta h}{\delta m}([\lambda[\xi + \varepsilon Y] + (1-\lambda)[\xi]]) \star g_t)(y) \mathbb{E}[\int_0^1 (-H_1 \cdot g_t)(y - \lambda'(\xi + \varepsilon Y) - (1-\lambda')(\xi)) \cdot Y d\lambda'] dy d\lambda \\ = & \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \frac{\delta h}{\delta m}(\mu \star g_t)(y)(-H_1 \cdot g_t)(y-\xi) dy\right) \cdot Y\right]. \end{aligned}$$

Therefore, the map $\mu \mapsto h([X_t^\xi])$ is now L-differentiable for any non negative time t.

Let us eventually conclude this illustration in view of the relation (2.2) between the flat and Lions derivatives. We readily have from (2.8) and the previous computation that

$$\partial_\mu h(\mu)(\cdot) = \partial_y[\frac{\delta h}{\delta m}](\mu)(\cdot),$$

so that the map $y \mapsto [\delta h/\delta m]([X_t^\xi])(y)$ is clearly a smooth function with bounded derivatives and $y \mapsto \partial_\mu h([X_t^\xi])(y)$ is also a bounded function. This immediately gives that $\mu \mapsto h([X_t^\xi])$ is Lipschitz continuous w.r.t. the W_2 metric.

From the above simple illustration, it is then naturally expected that such regularizing effect along smooth probability flows of probability measures holds in a more general way. Let us recast the above discussion in our framework with the following Proposition. It will be a major tool in our analysis.

Proposition 2.2. Assume that the continuous map $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ admits a bounded continuous linear functional derivative. Consider a map $(t,x,\mu) \mapsto p(\mu,t,T,x,z) \in \mathcal{C}^{1,2,2}([0,T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, for some prescribed $T > 0$, $z \mapsto p(\mu,t,T,x,z)$ being a density function, such that $z \mapsto (p(\mu,t,T,.,z)\sharp\mu) \in \mathcal{P}_2(\mathbb{R}^d)$, locally uniformly with respect to $(t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$, i.e. uniformly in $(t,\mu) \in \mathcal{K}$, $\mathcal{K}$ being any compact subset of $[0,T) \times \mathcal{P}_2(\mathbb{R}^d)$, such that the mappings $\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} |\partial_v^n[\partial_\mu p(\mu,t,T,x,z)](v)|\, dz$, $\int_{\mathbb{R}^d} |\partial_t p(\mu,t,T,x,z)|\, dz$, $n = 0,1$, are at most of quadratic growth, uniformly in (t,μ,v) in compact subsets of $[0,T) \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ and such that for any compact set $\mathcal{K}' \subset [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$, for any $n = 0,1$,

$$\int \sup_{(t,\mu,x,y)\in\mathcal{K}'} \big\{ |\partial_t^n p(\mu,t,T,x,z)| + |\, \partial_x^{1+n} p(\mu,t,T,x,z)| + |\partial_y^n[\partial_\mu p(\mu,t,T,x,z)](y)| \big\}\, dz < \infty, \tag{2.9}$$

Let $\Theta(t,\mu) : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,\mu) \mapsto \Theta(t,\mu)(dz) = (p(\mu,t,T,.,z)\sharp\mu)(dz) = \int_{\mathbb{R}^d} p(\mu,t,T,x,z)\mu(dx)\, dz$. Then, one has:

- the map $[0,T) \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,\mu) \mapsto h(\Theta(t,\mu)) \in \mathcal{C}^{1,2}([0,T) \times \mathcal{P}_2(\mathbb{R}^d))$,
- the Lions and time derivatives satisfy for $n = 0,1$:

$$\begin{aligned}
\partial_y^n[\partial_\mu h(\Theta(t,\mu))](y) &= \partial_v^n\Big[\partial_\nu\Big[\int\int \frac{\delta h}{\delta m}(\Theta(t,\mu))(z)\, p(\nu,t,T,x,z)\, dz\, \nu(dx)\Big]_{|\nu=\mu}\Big](y) \\
&= \int_{\mathbb{R}^d} \Big[\frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(y)\Big]\partial_x^{1+n} p(\mu,t,T,y,z)\, dz \\
&\quad + \int_{(\mathbb{R}^d)^2} \Big[\frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x)\Big]\partial_y^n[\partial_\mu p(\mu,t,T,x,z)](y)\, dz\, \mu(dx)
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\partial_t h(\Theta(t,\mu)) &= \partial_s\Big[\int_{(\mathbb{R}^d)^2} \frac{\delta h}{\delta m}(\Theta(t,\mu))(z)\, p(\mu,s,T,x,z)\, dz\, \mu(dx)\Big]_{|s=t} \\
&= \int_{(\mathbb{R}^d)^2} \Big[\frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x)\Big]\, \partial_t p(\mu,t,T,x,z)\, dz\, \mu(dx).
\end{aligned} \tag{2.11}$$

Proof. Let $(t_n,\mu_n)_{n\geq 1}$ be a sequence of $[0,T) \times \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\lim_n |t_n - t| = \lim_n W_2(\mu_n,\mu) = 0$. In order to prove that $[0,T) \times \mathcal{P}_2(\mathbb{R}^d)(\mathbb{R}^d) \ni (t,\mu) \mapsto h(\Theta(t,\mu))$ is continuous, it is sufficient to prove that $\lim_n W_2(\Theta(t_n,\mu_n),\Theta(t,\mu)) = 0$. Let $\bar{h}$ be a real-valued continuous function defined on $\mathbb{R}^d$. We decompose the difference $\langle \bar{h}, \Theta(t_n,\mu_n)\rangle - \langle \bar{h}, \Theta(t,\mu)\rangle = \int_{(\mathbb{R}^d)^2} \bar{h}(z) p(\mu_n,t_n,T,x,z)\, dz\, \mu_n(dx) - \int_{(\mathbb{R}^d)^2} \bar{h}(z) p(\mu,t,T,x,z)\, dz\, \mu(dx)$ as the sum of the two terms

$$\mathrm{I}_n := \int\int \bar{h}(z) p(\mu_n,t_n,T,x,z)\, dz(\mu_n - \mu)(dz), \quad \mathrm{II}_n := \int\int \bar{h}(z)(p(\mu_n,t_n,T,x,z) - p(\mu,t,T,x,z))\, dz\mu(dx)$$

and prove that each term goes to zero as $n \uparrow \infty$. Let us note that from condition (2.9) and Lebesgue's dominated convergence theorem, one directly gets $\lim_n \mathrm{II}_n = 0$. In order to prove that $\lim_n \mathrm{I}_n = 0$, we again decompose I_n as the sum of two terms namely

$$\mathrm{I}_n^1 := \int\int \bar{h}(z) p(\mu_n,t_n,T,x,z)\, dz\, \eta_R(x)(\mu_n - \mu)(dx),\ \mathrm{I}_n^2 := \int\int \bar{h}(z) p(\mu_n,t_n,T,x,z)\, dz\, (1-\eta_R)(x)(\mu_n - \mu)(dx)$$

where η_R is a non-negative smooth cutoff function such that $\eta_R(x) = 1$ for $|x| \leq R$ and $\eta_R(x) = 0$ for $|x| \geq 2R$, with $R > 0$. The uniform continuity of the map $\mathcal{K} \times B_{2R} \ni (t,\mu,x) \mapsto \int \bar{h}(z) p(\mu,t,T,x,z)\, dz\, \eta_R(x)$, $\mathcal{K}$ being any compact subset of $[0,T) \times \mathcal{P}_2(\mathbb{R}^d)$ and B_{2R} being the closed ball of radius $2R$ around the origin, implies that the family of maps $\big\{\int \bar{h}(z) p(\mu,t,T,x,z)\, dz\, \eta_R(x),\ (t,\mu) \in \mathcal{K}\big\}$ is equicontinuous and (2.9) implies its boundedness. By weak convergence of $(\mu_n)_{n\geq 1}$, we thus deduce

$$\lim_n \sup_{(t,\mu')\in\mathcal{K}} \Big| \int\int \bar{h}(z) p(\mu',t,T,x,z)\, dz\, \eta_R(x)\, (\mu_n - \mu)(dx)\Big| = 0$$

so that $\lim_n \mathrm{I}_n^1 = 0$. From the boundedness of $\bar{h}$ and the weak convergence of $(\mu_n)_{n\geq 1}$, we also obtain $\limsup_n |\mathrm{I}_n^2| \leq |\bar{h}|_\infty(\limsup_n \int_{|x|\geq 2R} \mu_n(dx) + \int_{|x|\geq 2R} \mu(dx)) \leq 2|\bar{h}|_\infty \int_{|x|\geq 2R} \mu(dx)$ so that by letting R goes to infinity in the previous inequality we deduce $\lim_n \mathrm{I}_n^2 = 0$. We thus conclude that $\lim_n \langle \bar{h}, \Theta(t_n,\mu_n)\rangle = \langle \bar{h}, \Theta(t,\mu)\rangle$. Similar arguments yield $\lim_n \int_{(\mathbb{R}^d)^2} |z|^2 p(\mu_n, t_n, T, x, z)\, dz\, \mu_n(dx) = \int_{(\mathbb{R}^d)^2} |z|^2 p(\mu, t, T, x, z)\, dz\, \mu(dx)$. We thus conclude that $\lim_n W_2(\Theta(t_n,\mu_n), \Theta(t,\mu)) = 0$.

Let us set $\Theta_{\varepsilon,\lambda}(t,\mu) := (1-\lambda)\Theta(t,\mu) + \lambda\Theta(t+\varepsilon,\mu)$, for a fixed $(t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$ and $\varepsilon > 0$ small enough. Then,

$$
\begin{aligned}
& h(\Theta(t+\varepsilon,\mu)) - h(\Theta(t,\mu)) \\
= & \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta h}{\delta m}(\Theta_{\varepsilon,\lambda}(t,\mu))(y) \left\{ p(\mu, t+\varepsilon, T, x, y) - p(\mu, t, T, x, y) \right\} dy \mu(dx) d\lambda \\
= & \int_0^1 \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta h}{\delta m}(\Theta_{\varepsilon,\lambda}(t,\mu))(y) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x) \right\} \left[p(\mu, t+\varepsilon, T, x, y) - p(\mu, t, T, x, y) \right] dy \mu(dx) d\lambda
\end{aligned}
$$

so that dividing by ε and letting ε goes to zero yields that $t \mapsto h(\Theta(t,\mu))$ is right-differentiable (and also continuously differentiable since the limit is continuous on $[0,T)$) and also the identity (2.11) thanks to the dominated convergence theorem as well as the continuity and the boundedness of $[\delta h/\delta m]$. The continuity of the map $[0,T) \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,\mu) \mapsto \partial_t h(\Theta(t,\mu))$ then follows from (2.11) and arguments similar to those employed above.

We now prove that $\mu \mapsto h(\Theta(t,\mu))$ is continuously L-differentiable for any $t \in [0,T]$. Let us set $\Theta_{\varepsilon,\lambda}(t,\mu) := (1-\lambda)\Theta(t,\mu) + \lambda\Theta(t,(1-\varepsilon)\mu + \varepsilon\mu')$ and $\mu_{\lambda',\varepsilon} := (1-\lambda')\mu + \lambda'[(1-\varepsilon)\mu + \varepsilon\mu']$, for a fixed $(t,\mu,\mu') \in [0,T) \times (\mathcal{P}_2(\mathbb{R}^d))^2$. Then,

$$
\begin{aligned}
& h(\Theta(t,(1-\varepsilon)\mu + \varepsilon\mu')) - h(\Theta(t,\mu)) \\
& = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta h}{\delta m}(\Theta_{\varepsilon,\lambda}(t,\mu))(y) \left\{ p((1-\varepsilon)\mu + \varepsilon\mu', t, T, y) - p(\mu, t, T, y) \right\} dy\, d\lambda \\
& = \int_0^1 \int_{\mathbb{R}^d)^2} \frac{\delta h}{\delta m}(\Theta_{\varepsilon,\lambda}(t,\mu))(y) \left[p((1-\varepsilon)\mu + \varepsilon\mu', t, T, x, y)[(1-\varepsilon)\mu + \varepsilon\mu'](dx) - p(\mu, t, T, x, y)\mu(dx) \right] dy\, d\lambda \\
& = \varepsilon \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta h}{\delta m}(\Theta_{\varepsilon,\lambda}(t,\mu))(y) p((1-\varepsilon)\mu + \varepsilon\mu', t, T, x, y)\, dy\, (\mu' - \mu)(dx)\, d\lambda \\
& \quad + \varepsilon \int_0^1 \int_0^1 \int_{(\mathbb{R}^d)^3} \left\{ \frac{\delta h}{\delta m}(\Theta_{\varepsilon,\lambda}(t,\mu))(y) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x) \right\} \frac{\delta}{\delta m} p(\mu_{\varepsilon,\lambda'}, t, T, x, y)(x')\, dz \mu(dx)\, (\mu' - \mu)(dx')\, d\lambda d\lambda'
\end{aligned}
$$

so that dividing by ε and letting ε goes to zero yields that $\mu \mapsto h(\Theta(t,\mu))$ admits a continuous linear functional derivative thanks to the dominated convergence theorem as well as the continuity and the boundedness of $[\delta h/\delta m]$. Moreover, one has

$$
\begin{aligned}
\frac{\delta}{\delta m} h(\Theta(t,\mu))(y) = & \int_{\mathbb{R}^d} \frac{\delta h}{\delta m}(\Theta(t,\mu))(z)\, p(\mu, t, T, y, z)\, dz \\
& + \int_{(\mathbb{R}^d)^2} \left\{ \frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x) \right\} \frac{\delta}{\delta m} p(\mu, t, T, x, z)(y)\, dz d\mu(x).
\end{aligned}
$$

Each term appearing in the right-hand side, seen as a function of y, is continuously differentiable on $\mathbb{R}^d$ so that for any $y_0 \in \mathbb{R}^d$

$$\begin{aligned}
\partial_y \frac{\delta}{\delta m} h(\Theta(t,\mu))(y) &= \partial_y \Big(\int_{\mathbb{R}^d} \frac{\delta h}{\delta m}(\Theta(t,\mu))(z)\, p(\mu,t,T,y,z)\, dz \Big) \\
&\quad + \int_{(\mathbb{R}^d)^2} \Big\{ \frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x) \Big\} \partial_y \frac{\delta}{\delta m} p(\mu,t,T,x,z)(y)\, dz d\mu(x) \\
&= \int_{\mathbb{R}^d} \Big[\frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(y_0) \Big] \partial_y p(\mu,t,T,y,z)\, dz \\
&\quad + \int_{(\mathbb{R}^d)^2} \Big\{ \frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x) \Big\} \partial_y \frac{\delta}{\delta m} p(\mu,t,T,x,z)(y)\, dz d\mu(x) \\
&\quad + \partial_y \Big(\int_{\mathbb{R}^d} \frac{\delta h}{\delta m}(\Theta(t,\mu))(y_0)\, p(\mu,t,T,y,z)\, dz \Big) \\
&= \int_{\mathbb{R}^d} \Big[\frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(y_0) \Big] \partial_y p(\mu,t,T,y,z)\, dz \qquad (2.12) \\
&\quad + \int_{(\mathbb{R}^d)^2} \Big\{ \frac{\delta h}{\delta m}(\Theta(t,\mu))(z) - \frac{\delta h}{\delta m}(\Theta(t,\mu))(x) \Big\} \partial_y \frac{\delta}{\delta m} p(\mu,t,T,x,z)(y)\, dz d\mu(x) \qquad (2.13)
\end{aligned}$$

where we used the fact that the last term appearing in the last but one equality is 0 since $z \mapsto p(\mu,t,T,y,z)$ is a density function. The identity (2.10) for $n = 0$ then follows by taking $y_0 = y$. The continuity of the map $[0,T) \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,\mu,y) \mapsto \partial_\mu h(\Theta(t,\mu))(y)$ then follows from (2.10) and arguments similar to those employed above. One may again differentiate (2.12) and (2.13) with respect to y for a fixed $(t,\mu) \in [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$ and then select $y_0 = y$ in order to obtain (2.10) for $n = 1$. The continuity of the map $(t,\mu,y) \mapsto \partial_y[\partial_\mu h(\Theta(t,\mu))](y)$ finally follows again from arguments similar to those employed before.

□

Remark 2.7. ∘ Importantly, we note that in the above proposition we do not impose the *intrinsic smoothness* (i.e. smoothness in the sense of Lions) of the map h but only require the existence of a bounded *linear or flat* derivative. In this regard, the composition with the smooth flow $(t,\mu) \mapsto \Theta(t,\mu)$ of probability measures of $\mathcal{P}_2(\mathbb{R}^d)$ allows to regularize the map h, the regularity being understood for a coarser topology. As already mentioned before, in what follows, the map Θ will be the one generated by the unique weak solution of the SDE (1.1), i.e. we will be interested in the smoothness of $[0,T) \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,\mu) \mapsto h([X_T^{t,\xi}])$.

∘ For functions $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$, we will straightforwardly extend the previous proposition to each component and still denote $[\delta h/\delta m] : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $[\delta h/\delta m] : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ the corresponding maps.

To conclude we again emphasize that the main advantage of the previous result is that it allows to take advantage of the usual smoothing effect of Gaussian-like kernels. We will intensively exploit this property to establish our regularity results for the density associated to solutions of McKean-Vlasov SDEs. We end this section with a simple result that illustrates this central idea.

The two representation formulas (2.10) and (2.11) are crucial for the analysis of the regularity properties of densities associated to McKean-Vlasov SDEs. Indeed, under the additional assumption that the map $y \mapsto [\delta h/\delta m](m)(y)$ is Hölder continuous and if $(t,\mu,x) \mapsto p(\mu,t,x,z)$ as well as its derivatives satisfy suitable Gaussian-type bounds, they allow thanks to the space-time inequality (1.4) to match the diagonal regime of the underlying heat kernel and to benefit from the so-called smoothing property of Gaussian kernels. At this stage, we stop from elaborating on this central idea in our analysis and postpone the discussion to the appropriate place.

3. Overview, assumptions and main results

3.1. **On the well-posedness of the martingale problem related to the SDE** (1.1)**.** We first present the martingale problem associated to equation (1.1).

Definition 3.1. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$. We say that the probability measure $\mathbb{P}$ on the canonical space $\mathcal{C}([0,\infty),\mathbb{R}^d)$ (endowed with the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$) with time marginals $(\mathbb{P}(t))_{t\geq 0}$, solves the non-linear martingale problem associated to the SDE (1.1) with initial distribution μ at time 0 if the canonical process $(y_t)_{t\geq 0}$ satisfies the following two conditions:

(i) $\mathbb{P}(y_0 \in \Gamma) = \mu(\Gamma)$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$.
(ii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, the process

$$f(y_t) - f(y_0) - \int_0^t \left\{ \sum_{i=1}^d b_i(s, y_s, \mathbb{P}(s))\partial_{x_i} f(y_s) + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(s, y_s, \mathbb{P}(s))\partial^2_{x_i,x_j} f(y_s) \right\} ds \tag{3.1}$$

is a square integrable martingale under $\mathbb{P}$.

Remark 3.2. A similar definition holds by letting the canonical process starts from time t_0 with initial distribution μ, in which case we say that the initial condition is (t_0, μ) and (i) is replaced by the condition: $\mathbb{P}(y(s) \in \Gamma; 0 \leq s \leq t_0) = \mu(\Gamma)$.

Having this definition at hand we now introduce some assumptions on the coefficients:

(**HR**) (i) The drift coefficient $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ is a bounded and continuous function. Moreover, for any $(t, x) \in \mathbb{R}_\times \mathbb{R}^d$, the map $m \mapsto b(t, x, m)$ is Lipschitz for the total variation metric, uniformly with respect to t, x, that is, there exists a positive constant C such that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, for all $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$

$$|b(t, x, m) - b(t, x, m')| \leq C d_{TV}(m, m')$$

where d_{TV} is the total variation metric.

(ii) The function $\mathbb{R}^d \ni x \mapsto a(t, x, m) \in \mathbb{R}^d \otimes \mathbb{R}^d$ is uniformly η-Hölder continuous for some $\eta \in (0, 1]$,

$$[a]_H := \sup_{t \geq 0,\, x \neq y,\, m \in \mathcal{P}_2(\mathbb{R}^d)} \frac{|a(t, x, m) - a(t, y, m)|}{|x - y|^\eta} < \infty.$$

(iii) For every $(i, j) \{1, \cdots, d\}^2$, for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the map $\mathcal{P}_2(\mathbb{R}^d) \ni m \mapsto a_{i,j}(t, x, m)$ has a continuous linear functional derivative.

(iv) For every $(i, j) \{1, \cdots, d\}^2$, for every $(t, m) \in \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d)$, the map $(\mathbb{R}^d)^2 \ni (x, y) \mapsto \frac{\delta}{\delta m} a_{i,j}(t, x, m)(y)$ is a bounded and η-Hölder continuous function, for some $\eta \in (0, 1]$, uniformly with respect to the other variables.

(**HE**) The diffusion coefficient is uniformly elliptic, that is, there exists $\lambda \geq 1$ such that for every $(t, m) \in [0, \infty) \times \mathcal{P}_2(\mathbb{R}^d)$ and $(x, \xi) \in (\mathbb{R}^d)^2$, $\lambda^{-1}|\xi|^2 \leq \langle a(t, x, m)\xi, \xi \rangle \leq \lambda |\xi|^2$ where $a(t, x, m) = (\sigma\sigma^*)(t, x, m)$.

Remark 3.3. Assumption (**HR**)(i) may be reformulated as the following stronger assumption: the map $m \mapsto b(t, x, m)$ has a bounded continuous linear functional derivative. We also point out that under assumption (**HR**)(iii) and (iv), the map $\mathcal{P}(\mathbb{R}^d) \ni m \mapsto a_{i,j}(t, x, m)$ is Lipschitz continuous with respect to the distance

$$d_\eta(m, m') = \inf_{\pi \in \Pi_2(m, m')} \int_{(\mathbb{R}^d)^2} |x - y|^\eta \wedge 1 \, \pi(dx, dy)$$

where $\Pi_2(m, m')$ is the set of all transference plan from m to m'. Indeed, for all $m, m' \in \mathcal{P}_2(\mathbb{R}^d)$, for any transference plan $\pi \in \Pi_2(m, m')$

$$\begin{aligned}
& |a_{i,j}(t, x, m) - a_{i,j}(t, x, m')| \\
= \; & \Big| \int_0^1 \int_{\mathbb{R}^d} \frac{\delta a_{i,j}}{\delta m}(t, x, (1 - \lambda)m + \lambda m')(y') d(m - m')(y') \, d\lambda \Big| \\
= \; & \Big| \int_0^1 \int_{\mathbb{R}^d} \frac{\delta a_{i,j}}{\delta m}(t, x, (1 - \lambda)m + \lambda m')(x) - \frac{\delta}{\delta m} a_{i,j}(t, x, (1 - \lambda)m + \lambda m')(y) \, d\lambda \, \pi(dx, dy) \Big| \\
\leq \; & \sup_{t,x,m} \Big[\frac{\delta a_{i,j}}{\delta m}(t, x, m)(.) \Big]_H \int_{(\mathbb{R}^d)^2} |x - y|^\eta \wedge 1 \, \pi(dx, dy)
\end{aligned}$$

where $\sup_{t,x,m} \left[[\delta a_{i,j}/\delta m](t, x, m)(.) \right]_H$ denotes the uniform Hölder modulus of the map $y \mapsto [\delta a_{i,j}/\delta m](t, x, m)(y)$. Finally, the claim follows by taking the infimum in the previous inequality with respect to $\pi \in \Pi_2(m, m')$.

Our first main result concerns the well-posedness of the martingale problem associated to the SDE (1.1).

Theorem 3.4. Under (**HR**) and (**HE**), the martingale problem associated with (1.1) is well-posed for any initial distribution $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. In particular, weak uniqueness in law holds for the SDE (1.1).

When investigating strong well-posedness of non-linear SDE an interesting fact is that, combining uniqueness in law for the non-linear SDE together with strong uniqueness result for the associated linear SDE, *i.e.* the same SDE with time inhomogeneous coefficients, the law argument being now treated as a time-inhomogeneity, immediately yields to strong uniqueness. To be more specific, from the previous well-posedness result we have that any strong solution Y of the SDE (1.1) (if it exists) writes

$$Y_t = \xi + \int_0^t b(s, Y_s, [X_s^\xi])ds + \int_0^t \sigma(s, Y_s, [X_s^\xi])dW_s, \quad [\xi] \in \mathcal{P}_2(\mathbb{R}^d) \tag{3.2}$$

implying that, setting $\widehat{b} : \mathbb{R}^+ \times \mathbb{R}^d \ni (t, y) \mapsto b(t, y, [X_t^\xi]) \in \mathbb{R}^d$ and $\widehat{\sigma} : \mathbb{R}^+ \times \mathbb{R}^d \ni (t, y) \mapsto \widehat{\sigma}(t, y, [X_t^\xi]) \in \mathbb{R}^d \times \mathbb{R}^d$, it solves

$$Y_t = \xi + \int_0^t \widehat{b}(s, Y_s)ds + \int_0^t \widehat{\sigma}(s, Y_s)dW_s, \quad [\xi] \in \mathcal{P}_2(\mathbb{R}^d). \tag{3.3}$$

But this linear SDE is well posed in the strong sense under the additional assumption that the diffusion coefficient $\widehat{\sigma}$ is Lipschitz in space (see [Ver80]). Hence, any strong solutions of (3.2) are equals $\mathbb{P}$-a.s. so that strong well-posedness follows from the Yamada-Watanabe theorem. This gives the following corollary.

Corollary 3.5. *Assume that the assumptions of Theorem 3.4 hold and, that for all (t, m) in $\mathbb{R}^+ \times \mathcal{P}_2(\mathbb{R}^d)$, the mapping $x \mapsto \sigma(t, x, m)$ is Lispchitz continuous uniformly with respect to t and m. Then, strong uniqueness holds for the SDE* (1.1).

Here are some examples for which our weak and strong uniqueness results apply.

Example 3.6. (First order interaction) We consider the following non-linear SDE with coefficients $b : \mathbb{R}_+ \times (\mathbb{R}^d)^2 \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times (\mathbb{R}^d)^2 \to \mathbb{R}^d \otimes \mathbb{R}^q$:

$$X_t^\xi = \xi + \int_0^t \widetilde{\mathbb{E}}[b(s, X_s^\xi, \widetilde{X}_s^\xi)]ds + \int_0^t \widetilde{\mathbb{E}}[\sigma(s, X_s^\xi, \widetilde{X}_s^\xi)]dW_s, \quad [\xi] \in \mathcal{P}_2(\mathbb{R}^d). \tag{3.4}$$

where the process $(\widetilde{X}_t^\xi)_{t\geq 0}$ is a copy of $(X_t^\xi)_{t\geq 0}$ defined on a copy $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let b and $a = \sigma\sigma^*$ are bounded measurable functions. Assume that $(x, z) \mapsto \sigma(t, x, z)$ is η-Hölder continuous uniformly with respect to t and that $(\int \sigma(t, x, z)\mu(dz))(\int \sigma(t, x, z)\mu(dz))^*$ is uniformly elliptic, uniformly with respect to the variables t, x, μ.

Then, assumptions of Theorem 3.4 are fulfilled and the SDE (3.4) is well posed in the weak sense. If in addition $x \mapsto \sigma(t, x, z)$ is Lipschitz continuous uniformly with respect to t and z then assumptions of Corollary 3.5 are satisfied and strong well posedness holds for (3.4).

Example 3.7. (N order interaction) For some positive integer N, we consider the following non-linear SDE with coefficients $b : \mathbb{R}_+ \times (\mathbb{R}^d)^{N+1} \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times (\mathbb{R}^d)^{N+1} \to \mathbb{R}^{d\times q}$:

$$X_t^\xi = \xi + \int_0^t \widetilde{\mathbb{E}}[b(s, X_s^\xi, X_s^{\xi,(1)}, \cdots, X_s^{\xi,(N)})]ds + \int_0^t \widetilde{\mathbb{E}}[\sigma(s, X_s^\xi, X_s^{\xi,(1)}, \cdots, X_s^{\xi,(N)})]dW_s, \tag{3.5}$$

with $[\xi] \in \mathcal{P}_2(\mathbb{R}^d)$ and where the processes $\Big\{(X_t^{\xi,(i)})_{t\geq 0},\ 1 \leq i \leq N\Big\}$ are mutually independent copies of the process $(X_t^\xi)_{t\geq 0}$ defined on a copy $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let b and σ be bounded and measurable functions. Assume that $(x, z) \mapsto \sigma(t, x, z)$ is η-Hölder continuous uniformly with respect to t, denoting by μ_N the N-fold product measure of μ, and that $a(t, x, \mu) := (\int \sigma(t, x, z)\, \mu_N(dz))(\int \sigma(t, x, z)\, \mu_N(dz))^*$ is uniformly elliptic, uniformly w.r.t to the variables t, x, μ.

Then, assumptions of Theorem 3.4 are fulfilled and the SDE (1.1) is well posed in the weak sense. If in addition $x \mapsto \sigma(t, x, z)$ is Lipschitz continuous uniformly with respect to t and z, assumptions of Corollary 3.5 are satisfied and strong well-posedness holds.

Example 3.8. (Scalar interaction(s)) For some $N \geq 0$, for maps $\psi_1, \varphi_1 \cdots, \psi_N, \varphi_N : \mathbb{R}^d \to \mathbb{R}$, we consider the following non-linear SDE with coefficients $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^q$:

$$\begin{aligned} X_t^\xi = \xi &+ \int_0^t b\Big(s, X_s^\xi, \widetilde{\mathbb{E}}\big[\psi_1(\widetilde{X}_s^\xi)\big], \cdots, \widetilde{\mathbb{E}}\big[\psi_N(\widetilde{X}_s^\xi)\big]\Big)ds \\ &+ \int_0^t \sigma\Big(s, X_s^\xi, \widetilde{\mathbb{E}}\big[\varphi_1(\widetilde{X}_s^\xi)\big], \cdots, \widetilde{\mathbb{E}}\big[\varphi_N(\widetilde{X}_s^\xi)\big]\Big)dW_s \end{aligned} \tag{3.6}$$

with $[\xi] \in \mathcal{P}_2(\mathbb{R}^d)$ and where the process $(\widetilde{X}_t^\xi)_{t\geq 0}$ is a copy of $(X_t^\xi)_{t\geq 0}$ defined on a copy $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Assume that $\psi_1, \varphi_1, \cdots, \psi_N, \varphi_N$ are bounded and measurable functions such that each φ_i, $i = 1, \cdots, N$, is η-Hölder continuous for some $\eta \in (0, 1]$. Assume that b is a bounded and measurable function such that $z \mapsto b(t, x, z)$ is Lipschitz continuous uniformly with respect to t, x. Suppose that $a = \sigma\sigma^*$ is uniformly elliptic and is a bounded continuous function that satisfies: $x \mapsto a(t, x, z)$ is η-Hölder continuous uniformly with respect to t and z; $z \mapsto a_{i,j}(t, x, z)$ is continuously differentiable with a bounded derivative for any $1 \leq i, j \leq d$; $x \mapsto \partial_{z_k} a_{i,j}(t, x, z)$ is η-Hölder, uniformly with respect to the other variables for all $(i, j, k) \in \{1, \cdots, d\}^2 \times \{1, \cdots, N\}$.

Then, assumptions of Theorem 3.4 are fulfilled and the SDE (3.6) is well posed in the weak sense. If in addition $x \mapsto \sigma(t, x, z)$ is Lipschitz continuous, assumptions of Corollary 3.5 are satisfied and strong well posedness holds.

Example 3.9. (Polynomials on the Wasserstein space) We consider the following scalar non-linear SDE

$$X_t^\xi = \xi + \int_0^t \prod_{i=1}^N \widetilde{\mathbb{E}}\Big[\psi_i\big(t, X_s^\xi, \widetilde{X}_s^\xi\big)\Big] ds + \int_0^t \prod_{i=1}^N \widetilde{\mathbb{E}}\Big[\varphi_i\big(t, X_s^\xi, \widetilde{X}_s^\xi\big)\Big] dW_s, \quad [\xi] \in \mathcal{P}_2(\mathbb{R}^d), \tag{3.7}$$

where $\psi_1, \varphi_1 \cdots, \psi_N, \varphi_N : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ and where the process $(\widetilde{X}_t^\xi)_{t\geq 0}$ is a copy of $(X_t^\xi)_{t\geq 0}$ defined on a copy $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Assume that $\psi_1, \varphi_1, \cdots, \psi_N, \varphi_N$ are bounded and measurable functions and that the functions φ_i are η-Hölder continuous in space (uniformly in time) for some $\eta \in (0, 1]$ for any $i = 1, \cdots, N$. Assume for sake of simplicity that there exists $\lambda > 0$ such that, for any $i \in \{1, \cdots, N\}$, for all $(t, x, z) \in \mathbb{R}_+ \times \mathbb{R}^2$, $\lambda < \varphi_i(t, x, z)$.

Then, assumptions of Theorem 3.4 are fulfilled and the SDE (3.7) is well posed in the weak sense. If in addition, $x \mapsto \prod_{i=1}^N \int \varphi_i(t, x, z)\mu(dz)$ is Lipschitz continuous uniformly with respect to t and μ, assumptions of Corollary 3.5 are satisfied and strong well posedness holds.

A multi-dimensional version of (3.7) can be described as follows. We consider functions $\varphi_i : \mathbb{R}_+ \times (\mathbb{R}^d)^2 \to \mathbb{R}^{q_{i-1}} \otimes \mathbb{R}^{q_i}$, $i = 1, \cdots, N$, for some positive integers $q_0, \cdots, q_N$ satisfying $q_0 = d$ and $q_N = q$, where each $(x, z) \mapsto \varphi_i(t, x, z)$ is a bounded and η-Hölder continuous (uniformly with respect to t) function. For sake of simplicity, assume that each $a_i(t, x, \mu) := (\int \varphi_i(t, x, z)\mu(dz))(\int \varphi_i(t, x, z)\mu(dz))^*$, $i = 1, \cdots, N$, is uniformly elliptic so that $a(t, x, \mu) := (\prod_{i=1}^N \int \varphi_i(t, x, z)\mu(dz))(\prod_{i=1}^N \int \varphi_i(t, x, z)\mu(dz))^*$ is also uniformly elliptic.

Then, $(t, x, \mu) \mapsto b(t, x, \mu)$, with $b_j(t, x, \mu) := \prod_{i=1}^N \int \psi_{i,j}(t, x, z)\, \mu(dz)$, $j = 1, \cdots, d$, each $\psi_{i,j} : \mathbb{R}_+ \times (\mathbb{R}^d)^2 \to \mathbb{R}$ being a bounded and measurable function, and $(t, x, \mu) \mapsto \sigma(t, x, \mu) := \prod_{i=1}^N \int \varphi_i(t, x, z)\mu(dz)$ satisfy (**HR**) and (**HE**). Hence, the SDE (3.7) is well-posed in the weak sense. If in addition, each $x \mapsto \prod_{i=1}^N \int \varphi_i(t, x, z)\mu(dz)$ is Lipschitz continuous, then assumptions of Corollary 3.5 are satisfied and strong well posedness holds.

3.2. **On the density of the solution of the SDE** (1.1) **and its regularity properties.** Under the assumption of Theorem 3.4, by weak uniqueness, the law of the process $(X_t^{s,\xi})_{t\geq s}$ given by the unique solution to the SDE (1.1) starting from the initial distribution $\mu = [\xi]$ at time s only depends upon ξ through its law μ. Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, it thus makes sense to consider $([X_t^{s,\xi}])_{t\geq s}$ as a function of μ (and also of the time variable s) without specifying the choice of the lifted random variable ξ that has μ as distribution. We then introduce, for any $x \in \mathbb{R}^d$, the following *decoupled stochastic flow* associated to the SDE (1.1)

$$X_t^{s,x,\mu} = x + \int_s^t b(r, X_r^{s,x,\mu}, [X_r^{s,\xi}])\, dr + \int_s^t \sigma(r, X_r^{s,x,\mu}, [X_r^{s,\xi}])\, dW_r. \tag{3.8}$$

We note that the previous equation is not a McKean-Vlasov SDE since the law appearing in the coefficients is not $[X_r^{s,x,\mu}]$ but rather $[X_r^{s,\xi}]$, that is, the law of the solution to the SDE (1.1) (starting at time s from the initial distribution μ) at time r. Under the assumptions of Theorem 3.4, the time-inhomogeneous martingale problem associated to the SDE (3.8) is well-posed, see e.g. Stroock and Varadhan [SV79]. In particular, weak existence and uniqueness in law holds for the SDE (3.8).

Moreover, from Friedman [Fri64], see also McKean and Singer [MS67], it follows that the transition density of the SDE (3.8) exists[1]. In particular, the random variable $X_t^{s,x,\mu}$ has a density that we denote by $z \mapsto p(\mu, s, t, x, z)$ which admits a representation in infinite series by means of the parametrix method that we now briefly describe. We refer the reader to [Fri64] or Konakov and Mammen [KM00] for a more complete exposition. We first introduce the approximation process $(\widehat{X}_{t_2}^{t_1,x,\mu})_{t_2 \geq t_1}$ obtained from the dynamics (3.8) by removing the drift and freezing the diffusion coefficient in space at a fixed point y, namely

$$\widehat{X}_{t_2}^{t_1,x,\mu} = x + \int_{t_1}^{t_2} \sigma(r, y, [X_r^{s,\xi}])\, dW_r. \tag{3.9}$$

The process $(\widehat{X}_{t_2}^{t_1,x,\mu})_{t_2 \geq t_1}$ is a simple Gaussian process with transition density given explicitly by

$$\widehat{p}^y(\mu, s, t_1, t_2, x, z) := g\left(\int_{t_1}^{t_2} a(r, y, [X_r^{s,\xi}])\, dr, z - x \right).$$

To make the notation simpler, we will write $\widehat{p}(\mu, s, t_1, t_2, x, y) := \widehat{p}^y(\mu, s, t_1, t_2, x, y)$ and $\widehat{p}^y(\mu, s, t_2, x, z) = \widehat{p}^y(\mu, s, s, t_2, x, z)$. Note importantly that the variable y acts twice since it appears as a terminal point where the density is evaluated and also as the point where the diffusion coefficient is frozen. Note also that in what follows we need to separate between the starting time t_1 of the approximation process and the starting time s of the original McKean-Vlasov dynamics. We now introduce the two infinitesimal generators associated to the dynamics (3.8) and (3.9), namely

$$\mathcal{L}_{s,t} f(\mu, t, x) = \sum_{i=1}^d b_i(t, x, [X_t^{s,\xi}]) \partial_{x_i} f(\mu, t, x) + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, x, [X_t^{s,\xi}]) \partial^2_{x_i, x_j} f(\mu, t, x),$$

$$\widehat{\mathcal{L}}_{s,t} f(\mu, t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(t, y, [X_t^{s,\xi}]) \partial^2_{x_i, x_j} f(\mu, t, x)$$

and define the parametrix kernel $\mathcal{H}$ for $(\mu, r, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times [s, t) \times (\mathbb{R}^d)^2$

$$\begin{aligned}
\mathcal{H}(\mu, s, r, t, x, y) &:= (\mathcal{L}_{s,r} - \widehat{\mathcal{L}}_{s,r}) \widehat{p}(\mu, s, r, t, x, y) \\
&= \sum_{i=1}^d b_i(r, x, [X_r^{s,\xi}]) \partial_{x_i} \widehat{p}(\mu, s, r, t, x, y) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d (a_{i,j}(r, x, [X_r^{s,\xi}]) - a_{i,j}(r, y, [X_r^{s,\xi}])) \partial^2_{x_i, x_j} \widehat{p}(\mu, s, r, t, x, y).
\end{aligned}$$

Now we define the following space-time convolution operator

$$(f \otimes g)(\mu, s, r, t, x, y) := \int_r^t \int_{\mathbb{R}^d} f(\mu, s, r, v, x, z) g(\mu, s, v, t, z, y)\, dz\, dv$$

and to simplify the notation we will write $(f \otimes g)(\mu, s, t, x, y) := (f \otimes g)(\mu, s, s, t, x, y)$, $\mathcal{H}(\mu, s, t, x, z) = \mathcal{H}(\mu, s, s, t, x, z)$ and proceed similarly for other maps. We also define $f \otimes \mathcal{H}^{(k)} = (f \otimes \mathcal{H}^{(k-1)}) \otimes \mathcal{H}$ for $k \geq 1$ with the convention that $f \otimes \mathcal{H}^{(0)} \equiv f$. With these notations, the following parametrix expansion in infinite series of the transition $p(\mu, s, t, x, z)$ holds. Let $T > 0$. For any $0 \leq s < t \leq T$ and any $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$p(\mu, s, t, x, y) = \widehat{p}(\mu, s, t, x, y) + p \otimes \mathcal{H}(\mu, s, t, x, y). \tag{3.10}$$

so that, by induction

$$p(\mu, s, t, x, y) = \sum_{k \geq 0} (\widehat{p} \otimes \mathcal{H}^{(k)})(\mu, s, t, x, y). \tag{3.11}$$

Moreover, the above infinite series converge absolutely and uniformly for $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$ and satisfies the following Gaussian upper-bound: for any $0 \leq s < t \leq T$ and any $(\mu, x, y) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$p(\mu, s, t, x, y) \leq E_{\eta/2,1}(C(|b|_\infty + 1))\, g(c(t - s), y - x) \tag{3.12}$$

[1]In [Fri64], it is proved that if $x \mapsto \bar{b}(r, x) = b(r, x, [X_r^{s,\xi}])$ is bounded and Hölder-continuous then the fundamental solution associated to the infinitesimal generator of (3.8) exists and is unique by means of the parametrix method. However, existence of the transition density as well as weak existence and weak uniqueness can be derived under the sole assumption that the drift is bounded and measurable and the diffusion matrix is uniformly elliptic and Hölder continuous.

where $C := C(T, \lambda, \eta)$ and $c := c(\lambda)$ are two positive constants. We refer to [MS67] for a proof based on Kolmogorov's backward and forward equations satisfied by p, see also Frikha [Fri17] for a proof based on probabilistic arguments.

Under the additional assumption that $x \mapsto b(t, x, \mu)$ is η-Hölder continuous, $x \mapsto p(\mu, s, t, x, z)$ is two times differentiable. Moreover, the following Gaussian type estimates for its derivatives hold:

$$|\partial_x^n p(\mu, s, t, x, y)| \leq \frac{C}{(t-s)^{\frac{n}{2}}} g(c(t-s), y-x), \quad n = 0, 1, 2 \tag{3.13}$$

and

$$\forall \beta \in [0, \eta),\ |\partial_x^2 p(\mu, s, t, x_1, y) - \partial_x^2 p(\mu, s, t, x_2, y)| \leq C \frac{|x_1 - x_2|^\beta}{(t-s)^{1+\frac{\beta}{2}}} \Big[g(c(t-s), y-x_1) + g(c(t-s), y-x_2) \Big] \tag{3.14}$$

for some positive constants $C := C(T, |b|_\infty, [b]_H, [a]_H, \lambda, \eta)$ and $c := c(\lambda)$. Let us point out that the differentiability of the map $[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, x, z)$ is the main question that we want to address here. We refer again to [Fri64] for a proof of the above estimates.

A similar representation in infinite series is also valid for the density of the random variable $X_t^{s,\mu}$, denoted by $z \mapsto p(\mu, s, t, z)$, but we will not use it explicitly. Actually, we will make use of the following key relation

$$p(\mu, s, t, z) = \int_{\mathbb{R}^d} p(\mu, s, t, x, z)\, \mu(dx). \tag{3.15}$$

The representation in infinite series of $p(\mu, s, t, z)$ is thus obtained by integrating $x \mapsto p(\mu, s, t, x, z)$ against the initial distribution μ, in other words, $z \mapsto p(\mu, s, t, z)$ is the density of the image measure of the map $x \mapsto p(\mu, s, t, x, z)$ by the measure μ.

We now introduce the following additional assumption on the coefficients.

($\mathbf{HR}_+$) The coefficients b and a satisfy ($\mathbf{HR}$) and the following additional assumptions :

(i) The function $\mathbb{R}^d \ni x \mapsto b(t, x, \mu) \in \mathbb{R}^d$ is uniformly η-Hölder continuous for some $\eta \in (0, 1]$,

$$[b]_H := \sup_{t \geq 0,\, x \neq y,\, m \in \mathcal{P}_2(\mathbb{R}^d)} \frac{|b(t, x, m) - b(t, y, m)|}{|x - y|^\eta} < \infty.$$

(ii) For any $(i, j) \in \{1, \cdots, d\}^2$, for any $(t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}^d)^2$, the map $m \mapsto [\delta/\delta m] a_{i,j}(t, x, m)(y)$ has a bounded continuous linear functional derivative, such that $(x, y') \mapsto [\delta^2/\delta m^2] a_{i,j}(t, x, m)(y, y')$ is η-Hölder continuous uniformly with respect to the other variables.

(iii) For any $i \in \{1, \cdots, d\}$, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the map $m \mapsto b_i(t, x, m)$ has a bounded continuous linear functional derivative, such that $y \mapsto [\delta/\delta m] b_i(t, x, m)(y)$ is η-Hölder continuous uniformly with respect to the other variables. Moreover, for any $i \in \{1, \cdots, d\}$, for any $(t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}^d)^2$, the map $m \mapsto [\delta/\delta m] b_i(t, x, m)(y)$ has a bounded continuous linear functional derivative, such that $y' \mapsto [\delta^2/\delta m^2] b_i(t, x, m)(y, y')$ is η-Hölder continuous uniformly with respect to the other variables.

Our next result concerns the regularity properties of the two maps $(s, \mu) \mapsto p(\mu, s, t, z)$ and $(s, \mu, x) \mapsto p(\mu, s, t, x, z)$ and also important estimates on its derivatives. As mentioned above under the assumptions of Theorem 3.4 and ($\mathbf{HR}_+$), $x \mapsto p(\mu, s, t, x, z)$ is two times continuously differentiable. In view of the relation (3.15), it thus suffices to investigate the smoothness of the map $(s, \mu, x) \mapsto p(\mu, s, t, x, z)$.

Theorem 3.10. Assume that ($\mathbf{HE}$) and ($\mathbf{HR}_+$) hold. Then, the mapping $[0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, x, \mu) \mapsto p(\mu, s, t, x, z)$ is in $\mathcal{C}^{1,2,2}([0, t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. Moreover, for any fixed $T > 0$, there exist two positive constants $C := C(b, a, [\delta b/\delta m], [\delta a/\delta m], T, \eta)$, $c := c(\lambda)$, such that for any $(\mu, s, x, x', z, v, v') \in \mathcal{P}_2(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^5$ and any $0 \leq s < t \leq T$

$$|\partial_v^n [\partial_\mu p(\mu, s, t, x, z)](v)| \leq \frac{C}{(t-s)^{\frac{1+n-\eta}{2}}} g(c(t-s), z-x),\ n = 0, 1, \tag{3.16}$$

$$|\partial_s p(\mu, s, t, x, z)| \leq \frac{C}{t-s} g(c(t-s), z-x), \tag{3.17}$$

$$\begin{aligned} &|\partial_v^n [\partial_\mu p(\mu, s, t, x, z)](v) - \partial_v^n [\partial_\mu p(\mu, s, t, x', z)](v)| \\ &\qquad \leq C \frac{|x - x'|^\beta}{(t-s)^{\frac{1+n+\beta-\eta}{2}}} \left\{ g(c(t-s), z-x) + g(c(t-s), z-x') \right\}, \end{aligned} \tag{3.18}$$

where $\beta \in [0,1]$ for $n=0$ and $\beta \in [0,\eta)$ for $n=1$,

(3.19)

$$\forall \beta \in [0,\eta), \quad |\partial_v[\partial_\mu p(\mu,s,t,x,z)](v) - \partial_v[\partial_\mu p(\mu,s,t,x,z)](v')| \leq C\frac{|v-v'|^\beta}{(t-s)^{1+\frac{\beta-\eta}{2}}} g(c(t-s),z-x),$$

There exist positive constants $C := C(b,a,[\delta b/\delta m],[\delta^2 b/\delta m^2],[\delta a/\delta m],[\delta^2 a/\delta m^2],T,\eta)$, $c := c(\lambda)$, such that for any $(\mu,\mu',s,x,z,v) \in (\mathcal{P}_2(\mathbb{R}^d))^2 \times [0,t) \times (\mathbb{R}^d)^3$,

$$|\partial_x^n p(\mu,s,t,x,z) - \partial_x^n p(\mu',s,t,x,z)](v)| \leq C\frac{W_2^\beta(\mu,\mu')}{(t-s)^{\frac{n+\beta-\eta}{2}}}\, g(c(t-s),z-x), \tag{3.20}$$

where $\beta \in [0,1]$ for $n=0$, 1 and $\beta \in [0,\eta)$ for $n=2$,

$$|\partial_v^n[\partial_\mu p(\mu,s,t,x,z)](v) - \partial_v^n[\partial_\mu p(\mu',s,t,x,z)](v)| \leq C\frac{W_2^\beta(\mu,\mu')}{(t-s)^{\frac{1+n+\beta-\eta}{2}}} g(c(t-s),z-x) \tag{3.21}$$

where $\beta \in [0,1]$ for $n=0$ and $\beta \in [0,\eta)$ for $n=1$ and for all $(s_1,s_2) \in [0,t)$

$$\begin{aligned} &|\partial_x^n p(\mu,s_1,t,x,z) - \partial_x^n p(\mu,s_2,t,x,z)| \\ &\leq C\left\{ \frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{n}{2}+\beta}}\, g(c(t-s_1),z-x) + \frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{n}{2}+\beta}}\, g(c(t-s_2),z-x)\right\}, \end{aligned} \tag{3.22}$$

where $\beta \in [0,1]$ for $n=0$, $\beta \in [0,\frac{1+\eta}{2})$ for $n=1$ and $\beta \in [0,\frac{\eta}{2})$ for $n=2$ and

$$\begin{aligned} &|\partial_v^n[\partial_\mu p(\mu,s_1,t,x,z)](v) - \partial_v[\partial_\mu p(\mu,s_2,t,x,z)](v)| \\ &\leq C\left\{ \frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{1+n-\eta}{2}+\beta}}\, g(c(t-s_1),z-x) + \frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{1+n-\eta}{2}+\beta}}\, g(c(t-s_2),z-x)\right\}, \end{aligned} \tag{3.23}$$

where $\beta \in [0,\frac{1+\eta}{2})$ for $n=0$ and $\beta \in [0,\frac{\eta}{2})$ for $n=1$.

3.3. **On the Cauchy problem related to the PDE** (1.2)**.** The previous regularity properties on the density of the random variables $X_t^{s,\xi}$ and $X_t^{s,x,\mu}$ allow us in turn to tackle the Cauchy problem in the strip $0 \leq t \leq T$ related to the PDE (1.2) on the Wasserstein space. The two real-valued maps f and h appearing in (1.2) will be assumed to satisfy the following conditions:

(**HST**) (i) The two maps $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,m) \mapsto f(t,x,m)$ and $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,m) \mapsto h(x,m)$ are continuous, the two maps $m \mapsto f(t,x,m)$ and $m \mapsto h(x,m)$ have a continuous linear functional derivative. Moreover, the maps $[0,T] \times (\mathbb{R}^d)^2 \times \mathcal{P}_2(\mathbb{R}^d) \ni (t,x,y,m) \mapsto [\delta/\delta m]f(t,x,m)(y)$, $(\mathbb{R}^d)^2 \times \mathcal{P}_2(\mathbb{R}^d) \ni (x,y,m) \mapsto [\delta/\delta m]h(x,m)(y)$ are continuous.

(ii) The two functions $x \mapsto f(t,x,m)$ and $y \mapsto [\delta/\delta m]f(t,x,m)(y)$ are locally Hölder continuous with exponent η, uniformly with respect to the other variables.

(iii) The maps f, h, $[\delta/\delta m]f$ and $[\delta/\delta m]h$ satisfy the following growth assumptions:

$$|f(t,x,m)| + |h(x,m)| \leq C\exp\left(\alpha\frac{|x|^2}{T}\right)(1+M_2^q(m)), \tag{3.24}$$

$$|\frac{\delta}{\delta m}f(t,x,m)(y)| + |\frac{\delta}{\delta m}h(x,m)(y)| \leq C\exp\left(\alpha\frac{|x|^2}{T}\right)(1+|y|^2+M_2^q(m)) \tag{3.25}$$

where $M_2(m) := \int_{\mathbb{R}^d} |x|^2 m(dx)$, for some positive constants $C := C(T)$, α and $q \geq 1$.

Theorem 3.11. Assume that the assumptions of Theorem 3.10 and that (**HST**) hold. Then, there exists a positive constant $c := c(\lambda)$ such that for any $\alpha < c$, the function U defined by

$$\begin{aligned} U(t,x,\mu) &:= \int_{\mathbb{R}^d} h(z,[X_T^{t,\xi}])\, p(\mu,t,T,x,z)\, dz - \int_t^T \int_{\mathbb{R}^d} f(s,z,[X_s^{t,\xi}])\, p(\mu,t,s,x,z)\, dz\, ds \\ &= \mathbb{E}\left[h(X_T^{t,x,\mu},[X_T^{t,\xi}]) - \int_t^T f(s,X_s^{t,x,\mu},[X_s^{t,\xi}])\, ds \right], \end{aligned} \tag{3.26}$$

where $\xi \in \mathbb{L}^2$ with $[\xi] = \mu$, is the unique solution of the Cauchy problem (1.2) (in the strip $0 \leq t \leq T$) and

$$|U(t,x,\mu)| \leq C \exp\Big(\frac{k|x|^2}{T}\Big)(1 + M_2^q(\mu)), \quad \text{for } (t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \tag{3.27}$$

where $C := C(T, b, a, [\delta b/\delta m], [\delta a/\delta m], \eta)$, $k := k(\lambda, \alpha)$ are positive constants.

Moreover, U is unique among all of the classical solutions to the PDE (1.2) satisfying (2.6), T being replaced by any $T' \in [0,T)$, as well as the exponential growth assumption (3.27) and with h and f satisfying (3.24) and (3.25) for some positive constants k and α.

4. Well-posedness of the martingale problem

In this section, we tackle the martingale problem associated to the SDE (1.1). We thus assume that (**HE**) and (**HR**) are in force throughout this section. As mentioned in the introduction, the proof follows from a fixed point argument, more precisely from the Banach fixed point theorem applied to a suitable map and complete metric space.

4.1. **Proof of Theorem 3.4.** Let $(s,\mu) \in \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d)$ be the initial condition of the martingale problem of Definition 3.1. For a fixed $T > s$ (to be chosen later in the proof), we consider the following set

$$\begin{aligned}\mathscr{A}_{s,T,\mu} = \big\{ & P \in \mathcal{C}([s,T], \mathcal{P}(\mathbb{R}^d)) : P(s) = \mu, \\ & \forall t \in (s,T],\ P(t) \text{ is absolutely continuous w.r.t. the Lebesgue measure}\big\}\end{aligned}$$

which is a complete metric space for the distance $d_{s,T}(P,P') := \sup_{t\in[s,T]} \int_{\mathbb{R}^d} |p - p'|(\mu,s,t,z)dz$ where p and p' stands for measurable versions for the densities of P and P' respectively. We define the map $\mathscr{T} : \mathscr{A}_{s,T,\mu} \to \mathscr{A}_{s,T,\mu}$ which to a probability measure $Q \in \mathscr{A}_{s,T,\mu}$ associates the measure $\mathscr{T}(Q) \in \mathscr{A}_{s,T,\mu}$ induced by the unique weak solution to the following SDE with dynamics

$$\bar{X}_t^{s,\xi} = \xi + \int_s^t b(r, \bar{X}_r^{s,\xi}, Q(r))\, dr + \int_s^t \sigma(r, \bar{X}_r^{s,\xi}, Q(r))\, dW_r, \quad t \in [s,T],$$

that is, $\mathscr{T}(Q)(t) = [\bar{X}_t^{s,\xi}]$, $t \in [s,T]$. Note that any fixed point of $\mathscr{T}$ is a solution to the martingale problem (on the considered time interval $[s,T]$). For P_1, $P_2 \in \mathscr{A}_{s,T,\mu}$, we consider the two following sequences of SDEs $(\bar{X}^{1,(\ell)})_{\ell\geq 0}$ and $(\bar{X}^{2,(\ell)})_{\ell\geq 0}$ constructed by induction on ℓ as the unique weak solution to the following SDEs:

$$\forall \ell \geq 0, \forall t \in [s,T], \quad \bar{X}_t^{i,(\ell+1)} = \xi + \int_s^t b(r, \bar{X}_r^{i,(\ell+1)}, [\bar{X}_r^{i,(\ell)}])dr + \int_s^t \sigma(r, \bar{X}_r^{i,(\ell+1)}, [\bar{X}_r^{i,(\ell)}])dW_r \tag{4.1}$$

where $[\bar{X}_t^{i,(0)}] = P_i(t)$ for i=1, 2. Note that under our assumptions, for any $\ell \geq 0$, the above SDE admits a unique weak solution so that $([\bar{X}_t^{i,(\ell)}])_{t\in[s,T]}$ is uniquely determined. We denote by $P_i^{(\ell)} = \mathscr{T}^{(\ell)}(P_i) = \mathscr{T}(\mathscr{T}^{(\ell-1)}(P_i)) = (P_i^{(\ell)}(t))_{t\in[s,T]} \in \mathscr{A}_{s,T,\mu}$, $\mathscr{T}^{(0)}(P_i) = P_i$, the time marginals induced by $(\bar{X}_t^{i,(\ell)})_{t\in[s,T]}$. The density function of the random vector $\bar{X}_t^{i,(\ell+1)}$ is given by $z \mapsto p_i^{(\ell+1)}(\mu,s,t,z) = \int p_i^{(\ell+1)}(\mu,s,t,x,z)\,\mu(dx)$, where $z \mapsto p_i^{(\ell+1)}(\mu,r,t,x,z)$ is density function of the random variable $\bar{X}_t^{i,(\ell+1),x,\mu}$, given by the unique weak solution to the decoupling field associated to (4.1):

Observe that the notation $p_i^{(\ell+1)}(\mu,s,t,x,z)$ (and also $p_i^{(\ell+1)}(\mu,s,t,z)$) makes sense since, by weak uniqueness, $[\bar{X}_t^{i,(\ell)}]$ only depends on ξ through its law μ. We then proceed using the first step of the parametrix expansion (3.10) (with $\widehat{p}$ and $\mathcal{H}$ defined accordingly), namely, for any $i = 1, 2$, for any fixed $(t,z) \in \mathbb{R}_+^* \times \mathbb{R}^d$, we write

$$\begin{aligned} p_i^{(\ell+1)}(\mu,s,t,x,z) &= \widetilde{p}_i^{(\ell+1)}(\mu,s,t,x,z) \\ &\quad + \int_s^t \int_{\mathbb{R}^d} p_i^{(\ell+1)}(\mu,s,r,x,y)(\mathcal{L}_r^{(\ell),i} - \widetilde{\mathcal{L}}_r^{(\ell),i})\widetilde{p}_i^{(\ell+1)}(\mu,r,t,y,z)\, dy\, dr \end{aligned} \tag{4.2}$$

where

$$\widetilde{p}_i^{(\ell+1)}(\mu,s,t,x,z) = g(\int_s^t a(r,z,[\bar{X}_r^{i,(\ell)}])\, dr, z - x), \quad \widetilde{\mathcal{L}}_r^{(\ell),i} h(x) = \frac{1}{2}\sum_{k,l=1}^d a_{k,l}(r,z,[\bar{X}_r^{i,(\ell)}])\partial^2_{x_k,x_l} h(x).$$

Introducing the notations

$$\delta(\mathcal{L}_r^{(\ell)} - \widetilde{\mathcal{L}}_r^{(\ell)})\widetilde{p}^{(\ell+1)}(\mu,r,t,y,z) := (\mathcal{L}_r^{(\ell),1} - \widetilde{\mathcal{L}}_r^{(\ell),1})\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z) - (\mathcal{L}_r^{(\ell),2} - \widetilde{\mathcal{L}}_r^{(\ell),2})\widetilde{p}_2^{(\ell+1)}(\mu,r,t,y,z)$$

and

$$\delta\widetilde{p}^{(\ell+1)}(\mu,r,t,x,z) := (\widetilde{p}_1^{(\ell+1)} - \widetilde{p}_2^{(\ell+1)})(\mu,r,t,x,z),$$

from (4.2) we thus see that $\delta p^{(\ell+1)}(\mu,r,t,x,z) := (p_1^{(\ell+1)} - p_2^{(\ell+1)})(\mu,r,t,x,z)$ satisfies the following relation

$$\delta p^{(\ell+1)}(\mu,s,t,x,z) = \delta\widetilde{p}^{(\ell+1)}(\mu,s,t,x,z) + \int_s^t \int_{\mathbb{R}^d} \delta p^{(\ell+1)}(\mu,s,r,x,y)(\mathcal{L}_r^{(\ell),1} - \widetilde{\mathcal{L}}_r^{(\ell),1})\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z)\,dy\,dr$$

$$+ \int_s^t \int_{\mathbb{R}^d} p_2^{(\ell+1)}(\mu,s,r,x,y)\delta(\mathcal{L}_r^{(\ell)} - \widetilde{\mathcal{L}}_r^{(\ell)})\widetilde{p}^{(\ell+1)}(\mu,r,t,y,z)\,dy\,dr. \tag{4.3}$$

We now quantify the contribution of each term appearing in the previous decomposition.

For $\ell \geq 0$, by the mean-value theorem, one has

$$\delta\widetilde{p}^{(\ell+1)}(\mu,s,t,x,z) = \sum_{i,j=1}^d \int_0^1 \frac{1}{2}(H_2^{i,j}.g)(\int_s^t (\lambda a(r,z,[\bar{X}_r^{1,(\ell)}]) + (1-\lambda)a(r,z,[\bar{X}_r^{2,(\ell)}])dr, z-x) \tag{4.4}$$

$$\times \int_s^t (a_{i,j}(r,z,[\bar{X}_r^{1,(\ell)}]) - a_{i,j}(r,z,[\bar{X}_r^{2,(\ell)}]))dr\,d\lambda.$$

Since $m \mapsto a_{i,j}(r,z,m)$ has a continuous linear functional derivative, again by the mean-value theorem,

$$\begin{aligned}
&a_{i,j}(r,z,[\bar{X}_r^{1,(\ell)}]) - a_{i,j}(r,z,[\bar{X}_r^{2,(\ell)}]) \\
&= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta a_{i,j}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}] + (1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')\,\delta p^{(\ell)}(\mu,s,r,y')\,dy'\,d\lambda \\
&= \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta a_{i,j}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}] + (1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')\,\delta p^{(\ell)}(\mu,s,r,x',y')\,dy'\mu(dx')d\lambda \\
&= \int_0^1 \int_{(\mathbb{R}^d)^2} \Big[\frac{\delta a_{i,j}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}] + (1-\lambda)[\bar{X}_r^{2,(\ell)}])(y') - \frac{\delta a_{i,j}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}] + (1-\lambda)[\bar{X}_r^{2,(\ell)}])(x')\Big] \\
&\quad \times \delta p^{(\ell)}(\mu,s,r,x',y')\,dy'\mu(dx')d\lambda
\end{aligned}$$

where we used the important fact that $\int_{\mathbb{R}^d} \delta p^{(\ell)}(\mu,s,r,x',y')\,dy' = 0$ for the last equality. Using the fact that $y' \mapsto [\delta/\delta m]a_{i,j}(t,x,m)(y')$ is a bounded η-Hölder continuous function

$$|a_{i,j}(r,z,[\bar{X}_r^{1,(\ell)}]) - a_{i,j}(r,z,[\bar{X}_r^{2,(\ell)}])| \leq C \int_{(\mathbb{R}^d)^2} (|y'-x'|^\eta \wedge 1)\,|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')$$

for $\ell \geq 1$ where C is a positive constant independent of ℓ that may change from line to line. For $\ell = 0$ using the fact that $y \mapsto [\delta/\delta m]a_{i,j}(t,x,m)(y)$ is uniformly bounded

$$|a_{i,j}(r,z,[\bar{X}_r^{1,(0)}]) - a_{i,j}(r,z,[\bar{X}_r^{2,(0)}])| = |a_{i,j}(r,z,P_1(r)) - a_{i,j}(r,z,P_2(r))| \leq C d_{s,r}(P_1,P_2)$$

Plugging the previous estimate into (4.4) and using (1.5), one gets

$$|\delta\widetilde{p}^{(\ell+1)}|(\mu,s,t,x,z) \leq \frac{C}{t-s}\int_s^t \int_{(\mathbb{R}^d)^2} (|y'-x'|^\eta \wedge 1)\,|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)$$

for $\ell \geq 1$ and

$$|\delta\widetilde{p}^{(1)}|(\mu,s,t,x,z) \leq C d_{s,t}(P_1,P_2) g(c(t-s),z-x).$$

Note that similar lines of reasonings allow to derive

$$\begin{aligned}
&|\partial_x^2 \delta\widetilde{p}^{(\ell+1)}|(\mu,s,t,x,z) \\
&\leq \frac{C}{(t-s)^2}\int_s^t \int_{(\mathbb{R}^d)^2} (|y'-x'|^\eta \wedge 1)\,|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)
\end{aligned} \tag{4.5}$$

and technical details are omitted. The previous estimate will be useful in the sequel.

From the uniform η-Hölder regularity of $x \mapsto a(t,x,\mu)$ and the space-time inequality (1.4) one gets for all $\ell \geq 0$

$$|(\mathcal{L}_r^{(\ell),1} - \widetilde{\mathcal{L}}_r^{(\ell),1})\widetilde{p}_1^{(\ell+1)}|(\mu,r,t,y,z) \leq \frac{C}{(t-r)^{1-\frac{\eta}{2}}} g(c(t-r),z-y),$$

This in turn implies

$$|\int_{\mathbb{R}^d}\delta p^{(\ell+1)}(\mu,s,r,x,y)(\mathcal{L}_r^{(\ell),1}-\widetilde{\mathcal{L}}_r^{(\ell),1})\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z)\,dy|$$
$$\leq C\int_{\mathbb{R}^d}|\delta p^{(\ell+1)}|(\mu,s,r,x,y)\frac{1}{(t-r)^{1-\frac{\eta}{2}}}g(c(t-r),z-y)\,dy.$$

We then make use of the following decomposition

$$\delta(\mathcal{L}_r^{(\ell)}-\widetilde{\mathcal{L}}_r^{(\ell)})\widetilde{p}^{(\ell+1)}(\mu,r,t,y,z)=:\mathrm{I}+\mathrm{II},$$

with

$$\mathrm{I}:=[(\mathcal{L}_r^{(\ell),1}-\widetilde{\mathcal{L}}_r^{(\ell),1})-(\mathcal{L}_r^{(\ell),2}-\widetilde{\mathcal{L}}_r^{(\ell),2})]\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z)=\mathrm{I}_1+\mathrm{I}_2,$$
$$\mathrm{II}:=(\mathcal{L}_r^{(\ell),2}-\widetilde{\mathcal{L}}_r^{(\ell),2})\delta\widetilde{p}^{(\ell+1)}(\mu,r,t,y,z)$$

and

$$\mathrm{I}_1=\sum_{l=1}^d[b_l(r,y,[\bar{X}_r^{1,(\ell)}])-b_l(r,y,[\bar{X}_r^{2,(\ell)}])]\partial_{y_l}\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z),$$

$$\mathrm{I}_2=\frac{1}{2}\sum_{k,l=1}^d\Big\{[a_{k,l}(r,y,[\bar{X}_r^{1,(\ell)}])-a_{k,l}(r,y,[\bar{X}_r^{2,(\ell)}])]-[a_{k,l}(r,z,[\bar{X}_r^{1,(\ell)}])-a_{k,l}(r,z,[\bar{X}_r^{2,(\ell)}])]\Big\}$$
$$\times\partial^2_{y_k,y_l}\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z).$$

Then, from (**HR**)(i), one has

$$|\mathrm{I}_1|\ \leq\frac{C}{(t-r)^{\frac{1}{2}}}\int_{(\mathbb{R}^d)^2}|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\mu(dx')\,g(c(t-r),z-y)$$

and, by the mean-value theorem,

$$\mathrm{I}_2=\frac{1}{2}\sum_{k,l=1}^d\int_0^1\int_{\mathbb{R}^d}\Big\{\frac{\delta a_{k,l}}{\delta m}(r,y,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')-\frac{\delta a_{k,l}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')\Big\}$$
$$\times\delta p^{(\ell)}(\mu,s,r,y')\,dy'\,d\lambda\ \times\partial^2_{y_k,y_l}\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z)$$
$$=\frac{1}{2}\sum_{k,l=1}^d\int_0^1\int_{(\mathbb{R}^d)^2}\Big\{\frac{\delta a_{k,l}}{\delta m}(r,y,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')-\frac{\delta a_{k,l}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')\Big\}$$
$$\times\delta p^{(\ell)}(\mu,s,r,x',y')\,dy'\,\mu(dx')\,d\lambda\times\partial^2_{y_k,y_l}\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z)$$
$$=\frac{1}{2}\sum_{k,l=1}^d\int_0^1\int_{(\mathbb{R}^d)^2}\Big\{\frac{\delta a_{k,l}}{\delta m}(r,y,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')-\frac{\delta a_{k,l}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(y')$$
$$-\Big[\frac{\delta a_{k,l}}{\delta m}(r,y,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(x')-\frac{\delta a_{k,l}}{\delta m}(r,z,\lambda[\bar{X}_r^{1,(\ell)}]+(1-\lambda)[\bar{X}_r^{2,(\ell)}])(x')\Big]\Big\}$$
$$\times\delta p^{(\ell)}(\mu,s,r,x',y')\,dy'\,\mu(dx')d\lambda\times\partial^2_{y_k,y_l}\widetilde{p}_1^{(\ell+1)}(\mu,r,t,y,z),$$

where we again used the fact that $\int_{\mathbb{R}^d}\delta p^{(\ell)}(\mu,s,r,x',y')\,dy'=0$. We now bound the integrand appearing in the right-hand side of the last equality, namely using the boundedness as well as the uniform joint η-Hölder regularity of $(y,y')\mapsto[\delta/\delta m]a_{k,l}(r,y,\mu)(y')$, we get

$$\Big|\frac{\delta a_{k,l}}{\delta m}(r,y,(1-\lambda)[\bar{X}_r^{1,(\ell)}]+\lambda[\bar{X}_r^{2,(\ell)}])(y')-\frac{\delta a_{k,l}}{\delta m}(r,z,(1-\lambda)[\bar{X}_r^{1,(\ell)}]+\lambda[\bar{X}_r^{2,(\ell)}])(y')$$
$$-\Big[\frac{\delta a_{k,l}}{\delta m}(r,y,(1-\lambda)[\bar{X}_r^{1,(\ell)}]+\lambda[\bar{X}_r^{2,(\ell)}])(x')-\frac{\delta a_{k,l}}{\delta m}(r,z,(1-\lambda)[\bar{X}_r^{1,(\ell)}]+\lambda[\bar{X}_r^{2,(\ell)}])(x')\Big]\Big|$$
$$\leq\sup_{(k,l),(t,m)}[\frac{\delta a_{k,l}}{\delta m}(t,.,m)(.)]_H(|y'-x'|^\eta\wedge|z-y|^\eta\wedge1).$$

From the last estimate, we thus obtain

$$|\mathrm{I}_2|\leq\frac{C}{t-r}\int_{(\mathbb{R}^d)^2}(|y'-x'|^\eta\wedge|z-y|^\eta\wedge1)|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,g(c(t-r),z-y).$$

We may break the integral appearing in the right-hand side of the above inequality into two parts J1 and J2 by dividing the domain of dy' integration into the two corresponding domains, namely

$\{y' \in \mathbb{R}^d : |y'-x'| \geq |z-y|\}$ and $\{y' \in \mathbb{R}^d : |y'-x'| < |z-y|\}$. For J1, we bound $(|y'-x'|^\eta \wedge |z-y|^\eta \wedge 1)$ by $|z-y|^\eta \leq |y'-x'|^{\eta/2}|z-y|^{\eta/2}$ while for J2, we bound $(|y'-x'|^\eta \wedge |z-y|^\eta \wedge 1)$ by $|y'-x'|^\eta \leq |y'-x'|^{\eta/2}|z-y|^{\eta/2}$. We thus obtain

$$\begin{aligned}|\mathrm{I}_2| &\leq C\frac{|z-y|^{\frac{\eta}{2}}}{t-r}\int_{(\mathbb{R}^d)^2}(|y'-x'|^{\frac{\eta}{2}}\wedge 1)|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,g(c(t-r),z-y)\\ &\leq \frac{C}{(t-r)^{1-\frac{\eta}{4}}}\int_{(\mathbb{R}^d)^2}(|y'-x'|^{\frac{\eta}{2}}\wedge 1)|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,g(c(t-r),z-y),\end{aligned}$$

where we used the space-time inequality (1.4) for the last inequality.

In order to deal with II, we make use of the uniform η-Hölder regularity of $x \mapsto a_{k,l}(r,x,\mu)$, the estimate (4.5) and the space-time inequality (1.4). We obtain

$$|\mathrm{II}| \leq \frac{C}{(t-r)^{2-\frac{\eta}{2}}}\int_r^t\int_{(\mathbb{R}^d)^2}(|y'-x'|^\eta\wedge 1)\,|\delta p^{(\ell)}|(\mu,s,u,x',y')\,dy'\,\mu(dx')\,du\,g(c(t-r),z-y),$$

so that, coming back to (4.3) and gathering the previous estimates, we obtain

$$\begin{aligned}&|\delta p^{(\ell+1)}|(\mu,s,t,x,z)\\ &\leq \frac{C}{t-s}\int_s^t\int_{(\mathbb{R}^d)^2}(|y'-x'|^\eta\wedge 1)\,|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)\\ &\quad + C\int_s^t\int_{\mathbb{R}^d}|\delta p^{(\ell+1)}|(\mu,s,r,x,y)\frac{1}{(t-r)^{1-\frac{\eta}{2}}}g(c(t-r),z-y)\,dy\,dr\\ &\quad + C\int_s^t\frac{1}{(t-r)^{1-\frac{\eta}{4}}}\int_{(\mathbb{R}^d)^2}(|y'-x'|^{\frac{\eta}{2}}\wedge 1)|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)\\ &\quad + C\int_s^t\frac{1}{(t-r)^{\frac{1}{2}}}\int_{(\mathbb{R}^d)^2}|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)\\ &\quad + C\int_s^t\frac{1}{(t-r)^{2-\frac{\eta}{2}}}\int_r^t\int_{(\mathbb{R}^d)^2}(|y'-x'|^{\frac{\eta}{2}}\wedge 1)\,|\delta p^{(\ell)}|(\mu,s,u,x',y')\,dy'\,\mu(dx')\,du\,dr\,g(c(t-s),z-x)\end{aligned}$$

so that, by Fubini's theorem

$$\begin{aligned}&|\delta p^{(\ell+1)}|(\mu,s,t,x,z)\\ &\quad\leq \frac{C}{t-s}\int_s^t\int_{(\mathbb{R}^d)^2}(|y'-x'|^\eta\wedge 1)\,|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)\\ &\quad + C\int_s^t\int_{\mathbb{R}^d}|\delta p^{(\ell+1)}|(\mu,s,r,x,y)\frac{1}{(t-r)^{1-\frac{\eta}{2}}}\,g(c(t-r),z-y)\,dy\,dr\\ &\quad + C\int_s^t\frac{1}{(t-r)^{\frac{1}{2}}}\int_{(\mathbb{R}^d)^2}|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x)\\ &\quad + C\int_s^t\frac{1}{(t-r)^{1-\frac{\eta}{4}}}\int_{(\mathbb{R}^d)^2}(|y'-x'|^{\frac{\eta}{2}}\wedge 1)|\delta p^{(\ell)}|(\mu,s,r,x',y')\,dy'\,\mu(dx')\,dr\,g(c(t-s),z-x).\end{aligned}\tag{4.6}$$

We now claim: there exists $T := T(a,b,\eta) > 0$ and some non-decreasing positive function $t \mapsto K(t)$ such that for all integer ℓ

$$\sup_{s\leq t\leq T+s}\int_{(\mathbb{R}^d)^2}(|y'-x'|^{\frac{\eta}{2}}\wedge 1)|\delta p^{(\ell+1)}|(\mu,s,t,x',y')\,dy'\mu(dx') \leq (K(T)T^{\frac{\eta}{4}})^{\ell+1}d_{s,s+T}(P_1,P_2)\tag{4.7}$$

and

$$\sup_{s\leq t\leq T+s}\int_{(\mathbb{R}^d)^2}|\delta p^{(\ell+1)}|(\mu,s,t,x',y')\,dy'\mu(dx') \leq (K(T)T^{\frac{\eta}{4}})^{\ell}d_{s,s+T}(P_1,P_2).\tag{4.8}$$

Before proving (4.7) and (4.8), we observe that the latter estimate yields

$$\begin{aligned}d_{s,s+T}(\mathscr{T}^{(\ell+1)}(P_1),\mathscr{T}^{(\ell+1)}(P_2)) &:= \sup_{t\in[s,s+T]}\int|p_1^{(\ell+1)}-p_2^{(\ell+1)}|(\mu,s,t,z)\,dz\\ &\leq K(KT^{\frac{\eta}{4}})^\ell\,d_{s,s+T}(P_1,P_2)\end{aligned}\tag{4.9}$$

and then, taking T sufficiently small so that $\sum_{\ell\geq 0}(K(T)T^{\frac{\eta}{4}})^\ell < \infty$, the Banach fixed point theorem guarantees that the map $\mathscr{T}$ admits a unique fixed point $\mathbb{P}^* \in \mathscr{A}_{s,s+T,\mu}$. Hence, the martingale problem

associated to the SDE (1.1) is well-posed on the time interval $[s, s+T]$. The well-posedness then extends to $[0, \infty)$.

We now prove (4.7) and (4.8). We may assume without loss of generality that $T \leq 1$. We proceed by induction. For $\ell = 0$, we integrate w.r.t. $dz\mu(dx)$ both sides of the inequality (4.6), after some simplifications we obtain

$$\begin{aligned}\int_{(\mathbb{R}^d)^2} |\delta p^{(1)}|(\mu, s, t, x, z)\, dz\mu(dx) &\leq \frac{C}{t-s}\int_s^t \int_{(\mathbb{R}^d)^2} (|z-x|^{\frac{\eta}{2}} \wedge 1)\, |\delta p^{(0)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\quad + C\int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(1)}|(\mu, s, r, x, z)\, dz\mu(dx)\, dr \\ &\quad + C\int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{4}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(0)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\leq C \sup_{s\leq t\leq s+T} \int_{(\mathbb{R}^d)^2} |\delta p^{(0)}|(\mu, s, t, x, z)\, dz\, \mu(dx) \\ &\quad + CT^{\frac{\eta}{2}} \sup_{s\leq t\leq s+T} \int_{(\mathbb{R}^d)^2} |\delta p^{(1)}|(\mu, s, t, x, z)\, dz\mu(dx)\end{aligned}$$

so that, taking $T := T(a, b, \eta)$ sufficiently small,

$$\sup_{s\leq t\leq s+T} \int_{(\mathbb{R}^d)^2} |\delta p^{(1)}|(\mu, s, t, x, z)\, dz\mu(dx) \leq K(T) \sup_{s\leq t\leq T+s} \int_{(\mathbb{R}^d)^2} |\delta p^{(0)}|(\mu, s, t, x, z)\, dz\, \mu(dx)$$

where $[0, T] \ni t \mapsto K(t)$ is some positive non-decreasing function. This concludes the proof of (4.8) for $\ell = 0$.

Similarly, we multiply by $|z-x|^{\frac{\eta}{2}} \wedge 1$ and then integrate w.r.t $dz\, \mu(dx)$ both sides of the inequality (4.6), using the space-time inequality (1.4) and some standard computations, we obtain

$$\begin{aligned}&\int_{(\mathbb{R}^d)^2} |\delta p^{(1)}|(\mu, s, t, x, z)(|z-x|^{\frac{\eta}{2}} \wedge 1)\, dz\mu(dx) \\ &\quad \leq \frac{C}{(t-s)^{1-\frac{\eta}{4}}}\int_s^t \int_{(\mathbb{R}^d)^2} (|z-x|^{\eta} \wedge 1)\, |\delta p^{(0)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\qquad + C\int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(1)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\qquad + CT^{\frac{\eta}{4}}\int_s^t \frac{1}{(t-r)^{\frac{1}{2}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(0)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\qquad + C\int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{4}}} \int_{(\mathbb{R}^d)^2} (|z-x|^{\frac{\eta}{2}} \wedge 1)|\delta p^{(0)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\quad \leq CK(T)T^{\frac{\eta}{4}} \sup_{s\leq t\leq T+s} \int_{(\mathbb{R}^d)^2} |\delta p^{(0)}|(\mu, s, t, x, z)\, dz\, \mu(dx)\end{aligned}$$

so that, taking the supremum over $t \in [s, s+T]$ on the left-hand side of the previous inequality concludes the proof of (4.7) for $\ell = 0$. Assuming that both (4.8) and (4.7) hold at step ℓ, we proceed similarly, namely we integrate w.r.t. $dz\mu(dx)$ both sides of the inequality (4.6)

$$\begin{aligned}\int_{(\mathbb{R}^d)^2} |\delta p^{(\ell+1)}|(\mu, s, t, x, z)\, dz\mu(dx) &\leq \frac{C}{t-s}\int_s^t \int_{(\mathbb{R}^d)^2} (|z-x|^{\eta} \wedge 1)\, |\delta p^{(\ell)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\quad + C\int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell+1)}|(\mu, s, r, x, z)\, dz\mu(dx)\, dr \\ &\quad + C\int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{4}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell)}|(\mu, s, r, x, z)\, dz\, \mu(dx)\, dr \\ &\leq C(K(T)T^{\frac{\eta}{4}})^{\ell} d_{s,s+T}(P_1, P_2) \\ &\quad + CT^{\frac{\eta}{4}} \sup_{s\leq t\leq s+T} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell+1)}|(\mu, s, t, x, z)\, dz\, \mu(dx)\end{aligned}$$

which in turn implies (4.8). Similarly, we multiply both sides of the inequality (4.6) by $|z-x|^\eta \wedge 1$ and then integrate w.r.t. $dz\mu(dx)$

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^2} |\delta p^{(\ell+1)}|(\mu,s,t,x,z)(|z-x|^{\frac{\eta}{2}} \wedge 1)\, dz\, \mu(dx) \\
&\quad \le \frac{C}{(t-s)^{1-\frac{\eta}{4}}} \int_s^t \int_{(\mathbb{R}^d)^2} (|z-x|^{\frac{\eta}{2}} \wedge 1)\, |\delta p^{(\ell)}|(\mu,s,r,x,z)\, dz\, \mu(dx)\, dr \\
&\qquad + C \int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell+1)}|(\mu,s,r,x,z)\, dz\, \mu(dx)\, dr \\
&\qquad + CT^{\frac{\eta}{4}} \int_s^t \frac{1}{(t-r)^{\frac{1}{2}}} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell)}|(\mu,s,r,x,z)\, dz\, \mu(dx)\, dr \\
&\qquad + C \int_s^t \frac{1}{(t-r)^{1-\frac{\eta}{4}}} \int_{(\mathbb{R}^d)^2} (|z-x|^{\frac{\eta}{2}} \wedge 1) |\delta p^{(\ell)}|(\mu,s,r,x,z)\, dz\, \mu(dx)\, dr \\
&\quad \le CT^{\frac{\eta}{4}} \sup_{s\le t\le s+T} \int_{(\mathbb{R}^d)^2} (|z-x|^\eta \wedge 1) |\delta p^{(\ell)}|(\mu,s,t,x,z)\, dz\, \mu(dx) \\
&\qquad + CT^{\frac{\eta}{2}} \sup_{s\le t\le s+T} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell+1)}|(\mu,s,t,x,z)\, dz\, \mu(dx) \\
&\qquad + CT^{\frac{\eta}{4}+\frac{1}{2}} \sup_{s\le t\le s+T} \int_{(\mathbb{R}^d)^2} |\delta p^{(\ell)}|(\mu,s,t,x,z)\, dz\, \mu(dx) \\
&\quad \le C(K(T)T^{\frac{\eta}{4}})^{\ell+1} d_{s,s+T}(P_1,P_2)
\end{aligned}$$

which allows to establish (4.7). The proof is now complete. □

5. Existence and regularity properties of the transition density

This section is dedicated to the proof of Theorem 3.10. Hence, throughout this section, we assume that (**HC**), (**HE**) and ($\mathbf{HR}_+$) are in force.

5.1. **Strategy of proof.** Our strategy is based on a Picard approximation scheme for the transition density $p(\mu,s,t,x,z)$ and sharp uniform estimates on its derivatives from which we can extract a uniformly convergent subsequence.

Step 1: Construction of an approximation sequence and related estimates

For a given initial condition $(s,\mu) \in \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d)$ and a probability measure $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\nu \neq \mu$, we let $\mathbb{P}^{(0)} = (\mathbb{P}^{(0)}(t))_{t\ge s}$ be the probability measure on $\mathcal{C}([s,\infty),\mathbb{R}^d)$, endowed with its canonical filtration, satisfying $\mathbb{P}^{(0)}(t) = \nu$, $t \ge s$, and we consider the following recursive sequence of probability measures $\{\mathbb{P}^{(m)}; m \ge 0\}$, with time marginals $(\mathbb{P}^{(m)}(t))_{t\ge s}$, where, $\mathbb{P}^{(m)}$ being given, $\mathbb{P}^{(m+1)}$ is the unique solution to the following martingale problem

(i) $\mathbb{P}^{(m+1)}(y(r) \in \Gamma; 0 \le r \le s) = \mu(\Gamma)$, for all $\Gamma \in \mathcal{B}(\mathbb{R}^d)$.

(ii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$f(y_t) - f(y_s) - \int_s^t \left\{ \sum_{i=1}^d b_i(r,y_r,\mathbb{P}^{(m)}(r))\partial_i f(y_r) + \sum_{i,j=1}^d \frac{1}{2} a_{i,j}(r,y_r,\mathbb{P}^{(m)}(r))\partial^2_{i,j} f(y_r) \right\} dr$$

is a continuous square-integrable martingale under $\mathbb{P}^{(m+1)}$.

Note that, under the considered assumptions, the well-posedness of the above martingale problem follows from standard results, see e.g. [SV79], so that there exists a unique weak solution to the SDE

$$X_t^{s,\xi,(m+1)} = \xi + \int_s^t b(r, X_r^{s,\xi,(m+1)}, [X_r^{s,\xi,(m)}])dr + \int_s^t \sigma(r, X_r^{s,\xi,(m+1)}, [X_r^{s,\xi,(m)}])dW_r. \tag{5.1}$$

We also associate to the above dynamics the decoupled stochastic flow given by the unique weak solution to SDE

$$X_t^{s,x,\mu,(m+1)} = x + \int_s^t b(r, X_r^{s,x,\mu,(m+1)}, [X_r^{s,\xi,(m)}])dr + \int_s^t \sigma(r, X_r^{s,x,\mu,(m+1)}, [X_r^{s,\xi,(m)}])dW_r. \tag{5.2}$$

We point out that the notation $X_t^{s,x,\mu,(m+1)}$ makes sense since by weak uniqueness of solution to the SDE (5.1), the law $[X_t^{s,\xi,(m)}]$ only depends on the initial condition ξ through its law μ.

From [Fri64], for any $m \geq 0$, the two random variables $X_t^{s,\xi,(m)}$ and $X_t^{s,x,\mu,(m)}$ admit a density respectively denoted by $p_m(\mu,s,t,z)$ and $p_m(\mu,s,t,x,z)$. Moreover, the following relation is satisfied

$$\forall z \in \mathbb{R}^d, \quad p_m(\mu,s,t,z) = \int p_m(\mu,s,t,x,z)\mu(dx), \tag{5.3}$$

where for all $m \geq 1$

$$p_m(\mu,s,t,x,z) = \sum_{k\geq 0} (\widehat{p}_m \otimes \mathcal{H}_m^{(k)})(\mu,s,t,x,z), \tag{5.4}$$

with

$$\begin{aligned}
\widehat{p}_m^y(\mu,s,r,t,x,z) &= g\left(\int_r^t a(v,y,[X_v^{s,\xi,(m-1)}])dv, z-x\right),\\
\widehat{p}_m(\mu,s,r,t,x,z) &= \widehat{p}_m^z(\mu,s,r,t,x,z) g\left(\int_r^t a(v,z,[X_v^{s,\xi,(m-1)}])dv, z-x\right),\\
\mathcal{H}_m(\mu,s,r,t,x,z) &= \left\{ -\sum_{i=1}^d b_i(r,x,[X_r^{s,\xi,(m-1)}]) H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m-1)}])dv, z-x\right)\right.\\
&\quad + \frac{1}{2}(a_{i,j}(r,x,[X_r^{s,\xi,(m-1)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m-1)}]))\\
&\quad \left. \times H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m-1)}])dv, z-x\right)\right\} \widehat{p}_m(\mu,s,r,t,x,z)
\end{aligned}$$

and $\mathcal{H}_m^{(k+1)}(\mu,s,t,x,z) = (\mathcal{H}_m^{(k)} \otimes \mathcal{H}_m)(\mu,s,t,x,z)$, $\mathcal{H}_m^{(0)} = I_d$, with the convention that $[X_t^{s,\xi,(0)}] = \mathbb{P}^{(0)}(t) = \nu$, $t \geq 0$. In what follows, we will often make use of the following estimates: there exist constants $c := c(\lambda) > 1$, $C := C(T,a,b,\eta) > 0$, such that for all $0 \leq s < t \leq T$, for all integer k, one has

$$\forall (x,z) \in (\mathbb{R}^d)^2, \quad |\widehat{p}_m \otimes \mathcal{H}_m^{(k)}(\mu,s,t,x,z)| \leq C^k (t-s)^{k\frac{\eta}{2}} \prod_{i=1}^k B\left(1 + \frac{(i-1)\eta}{2}, \frac{\eta}{2}\right) g(c(t-s), z-x) \tag{5.5}$$

where $B(k,\ell) = \int_0^1 (1-v)^{-1+k} v^{-1+\ell} dv$ stands for the Beta function. As a consequence of the asymptotics of the Beta function, the series (5.4) converges absolutely and uniformly for $(\mu,x,z) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$ and satisfies: for all $m \geq 1$, for any $0 \leq s < t \leq T$ and any $(\mu,x,z) \in \mathcal{P}_2(\mathbb{R}^d) \times (\mathbb{R}^d)^2$

$$|\partial_x^n p_m(\mu,s,t,x,z)| \leq C(t-s)^{-\frac{n}{2}} g(c(t-s), z-x), \; n = 0,1,2, \tag{5.6}$$

where $C := C(T,b,a,\eta)$ and $c := c(\lambda)$ are two positive constants and for all $(x,x') \in (\mathbb{R}^d)^2$, for any $m \geq 1$,

$$\begin{aligned}
&|\partial_x^n p_m(\mu,s,r,x,z) - \partial_x^n p_m(\mu,s,r,x',z)|\\
&\qquad \leq C \frac{|x-x'|^\beta}{(r-s)^{\frac{n+\beta}{2}}} \left\{ g(c(r-s), z-x) + g(c(r-s), z-x') \right\}, \; n = 0,1,2,
\end{aligned} \tag{5.7}$$

where $\beta \in [0,1]$ if $n = 0$, 1 and $\beta \in [0,\eta)$ if $n = 2$. We refer to Friedman [Fri64] for a proof of the above estimates.

Denote by $\Phi_m(\mu,s,r,t,x_1,x_2)$ the solution to the Volterra integral equation

$$\Phi_m(\mu,s,r,t,x_1,x_2) = \mathcal{H}_m(\mu,s,r,t,x_1,x_2) + (\mathcal{H}_m \otimes \Phi_m)(\mu,s,r,t,x_1,x_2). \tag{5.8}$$

From the space-time inequality (1.4), it is easily seen that the singular kernel $\mathcal{H}_m(\mu,s,r,t,x_1,x_2)$ induces an integrable singularity in time in the above space-time convolution so that the solution exists and is given by the (uniform) convergent series

$$\Phi_m(\mu,s,r,t,x_1,x_2) = \sum_{k\geq 1} \mathcal{H}_m^{(k)}(\mu,s,r,t,x_1,x_2) \tag{5.9}$$

so that (5.4) now writes

$$p_m(\mu,s,t,x,z) = \widehat{p}_m(\mu,s,t,x,z) + \int_s^t \int_{\mathbb{R}^d} \widehat{p}_m(\mu,s,r,x,y) \Phi_m(\mu,s,r,t,y,z)\, dy\, dr. \tag{5.10}$$

Moreover, from Theorem 7, Chapter 1 in [Fri64], for any $m \geq 1$, the map $x \mapsto \Phi_m(\mu,s,r,t,x,z)$ is Hölder-continuous. More precisely, for any $\beta \in [0,\eta)$, there exist two positive constants $C :=$

$C(T,a,b,\eta,\lambda)$, $c(\lambda)$, thus do not depending on m, such that

$$|\Phi_m(\mu,s,r,t,x,z)-\Phi_m(\mu,s,r,t,y,z)| \leq C\frac{|x-y|^\beta}{(t-r)^{1+\frac{\beta-\eta}{2}}}\left\{g(c(t-r),z-x)+g(c(t-r),z-y)\right\}. \tag{5.11}$$

With the above notations and properties, we prove the following key proposition whose proof is postponed to the next subsection.

Proposition 5.1. Let $T>0$. For any fixed $(t,z)\in(0,T]\times\mathbb{R}^d$, for all $m\geq 1$, the following properties hold:

- The mapping $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(s,x,\mu)\mapsto p_m(\mu,s,t,x,z)$ is in $\mathcal{C}^{1,2,2}([0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$.
- There exist two positive constants $C:=C(T,b,a,[\delta b/\delta m],[\delta a/\delta m],\eta)$, $c:=c(\lambda)$, thus do not depending on m, such that for any $(\mu,s,x,x',z,v,v')\in\mathcal{P}_2(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^5$,

$$|\partial_v^n[\partial_\mu p_m(\mu,s,t,x,z)](v)| \leq \frac{C}{(t-s)^{\frac{1+n-\eta}{2}}}\,g(c(t-s),z-x),\quad n=0,1, \tag{5.12}$$

$$|\partial_s p_m(\mu,s,t,x,z)| \leq \frac{C}{t-s}\,g(c(t-s),z-x), \tag{5.13}$$

$$|\partial_v^n[\partial_\mu p_m(\mu,s,t,x,z)](v)-\partial_v^n[\partial_\mu p_m(\mu,s,t,x',z)](v)| \leq C\frac{|x-x'|^\beta}{(t-s)^{\frac{1+n+\beta-\eta}{2}}}\left\{g(c(t-s),z-x)+g(c(t-s),z-x')\right\}, \tag{5.14}$$

where $\beta\in[0,1]$ for $n=0$ and $\beta\in[0,\eta)$ for $n=1$,

$$\forall\beta\in[0,\eta),\quad |\partial_v[\partial_\mu p_m(\mu,s,t,x,z)](v)-\partial_v[\partial_\mu p_m(\mu,s,t,x,z)](v')| \leq C\frac{|v-v'|^\beta}{(t-s)^{1+\frac{\beta-\eta}{2}}}\,g(c(t-s),z-x). \tag{5.15}$$

- There exist $C:=C(T,b,a,[\delta b/\delta m],[\delta^2 b/\delta m^2],[\delta a/\delta m],[\delta^2 a/\delta m^2],\eta)>0$, $c:=c(\lambda)>0$, thus do not depending on m, such that for any $(\mu,\mu',s,x,z,v)\in(\mathcal{P}_2(\mathbb{R}^d))^2\times[0,t)\times(\mathbb{R}^d)^3$,

$$|\partial_x^n p_m(\mu,s,t,x,z)-\partial_x^n p_m(\mu',s,t,x,z)]| \leq C\frac{W_2^\beta(\mu,\mu')}{(t-s)^{\frac{n+\beta-\eta}{2}}}\,g(c(t-s),z-x), \tag{5.16}$$

where $\beta\in[0,1]$ for $n=0$, 1 and $\beta\in[0,\eta)$ for $n=2$,

$$|\partial_v^n[\partial_\mu p_m(\mu,s,t,x,z)](v)-\partial_v^n[\partial_\mu p_m(\mu',s,t,x,z)](v)| \leq C\frac{W_2^\beta(\mu,\mu')}{(t-s)^{\frac{1+n+\beta-\eta}{2}}}\,g(c(t-s),z-x), \tag{5.17}$$

where $\beta\in[0,1]$ for $n=0$ and $\beta\in[0,\eta)$ for $n=1$, and for all $(s_1,s_2)\in[0,t)^2$,

$$|\partial_x^n p_m(\mu,s_1,t,x,z)-\partial_x^n p_m(\mu,s_2,t,x,z)| \leq C\left\{\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{n}{2}+\beta}}\,g(c(t-s_1),z-x)+\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{n}{2}+\beta}}\,g(c(t-s_2),z-x)\right\}, \tag{5.18}$$

where $\beta\in[0,1]$ for $n=0$, $\beta\in[0,\frac{1+\eta}{2})$ for $n=1$ and $\beta\in[0,\frac{\eta}{2})$ for $n=2$ and

$$|\partial_v^n[\partial_\mu p_m(\mu,s_1,t,x,z)](v)-\partial_v^n[\partial_\mu p_m(\mu,s_2,t,x,z)](v)| \leq C\left\{\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{1+n-\eta}{2}+\beta}}\,g(c(t-s_1),z-x)+\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{1+n-\eta}{2}+\beta}}\,g(c(t-s_2),z-x)\right\}, \tag{5.19}$$

where $\beta\in[0,\frac{1+\eta}{2})$ if $n=0$, $\beta\in[0,\frac{\eta}{2})$ if $n=1$.

The proof of the above result being rather long and technical it is postponed to the subsection 5.2.

Step 2: Extraction of a convergent subsequence

Our next step now is to extract from the following sequences $\big\{\mathbb{L}^2 \ni \xi \mapsto \tilde{p}_m(\xi,s,t,x,z), m \geq 0\big\}$ (the lifting of $\mu \mapsto p_m(\mu,s,t,x,z)$), $\big\{\mathbb{R}^d \ni v \mapsto \partial_\mu p_m(\mu,s,t,x,z)(v), m \geq 0\big\}$, $\big\{\mathbb{R}^d \ni v \mapsto \partial_v[\partial_\mu p_m(\mu,s,t,x,z)](v), m \geq 0\big\}$ the corresponding subsequences which converge locally uniformly using the Arzelà-Ascoli's theorem.

Since the coefficients b_i, $a_{i,j}$ are bounded and the initial condition $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the sequence $(\mathbb{P}^{(m)})_{m\geq 0}$ is tight. Relabelling the indices if necessary, we may assert that $(\mathbb{P}^{(m)})_{m\geq 0}$ converges weakly to a probability measure $\mathbb{P}^\infty$. From standard arguments that we omit (passing to the limit in the characterisation of the martingale problem solved by $\mathbb{P}^{(m)}$) we deduce that $\mathbb{P}^\infty$ is the probability measure $\mathbb{P}$ induced by the unique weak solution to the McKean-Vlasov SDE (1.1). As a consequence, every convergent subsequence converges to the same limit $\mathbb{P}$ and so does the original sequence $(\mathbb{P}^{(m)})_{m\geq 1}$.

By Lebesgue's dominated convergence, for any fixed $t > 0$ and $z \in \mathbb{R}^d$, using (5.5), one may pass to the limit as $m \uparrow \infty$ in the parametrix infinite series (5.4) and thus deduce that the sequence of functions $\{\mathcal{K} \ni (s,x,\mu) \mapsto p_m(\mu,s,t,x,z),\ m \geq 1\}$, $\mathcal{K}$ being a compact set of $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, converges to $(s,x,\mu) \mapsto p(\mu,s,t,x,z)$, for any fixed (s,x,μ) given by the infinite series (3.11). Moreover, it is clearly uniformly bounded and from (5.16), (5.18) and (5.7), C, c being two positive constants independent of m, it is equicontinuous. Relabelling the indices if necessary, from the Arzelà-Ascoli's theorem, we may assert that it converges uniformly. Hence, $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s,x,\mu) \mapsto p(\mu,s,t,x,z)$ is continuous.

For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and any integer m, the mapping $(s,x) \mapsto p_m(\mu,s,t,x,z)$ is in $\mathcal{C}^{0,2}([0,t) \times \mathbb{R}^d)$. Moreover, from the estimates (5.16), (5.18) and (5.7) (for $n = 1,2$), the sequence of functions $\{\mathcal{K} \ni (s,x,\mu) \mapsto \partial_x p_m(\mu,s,t,x,z),\ \partial_x^2 p_m(\mu,s,t,x,z), m \geq 0\}$, are uniformly bounded and equicontinuous. Hence, from Arzelà-Ascoli's theorem, we may assert that $(s,x) \mapsto p(\mu,s,t,x,z) \in \mathcal{C}^{0,2}([0,t) \times \mathbb{R}^d)$ and that the mappings $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s,x,\mu) \mapsto \partial_x p(\mu,s,t,x,z)$, $\partial_x^2 p(\mu,s,t,x,z)$ are continuous.

From (5.12), the sequence $\big\{L^2(\Omega,\mathcal{A},\mathbb{P}) \supset B(0,R) \ni \xi \mapsto D\tilde{p}_m(\xi,s,t,x,z) = \partial_\mu p_m(\mu,s,t,x,z)(\xi),\ m \geq 1\big\}$ is uniformly bounded and equicontinuous for any $R > 0$. Relabelling the indices if necessary, from the Arzelà-Ascoli's theorem, we may assert it converges uniformly. Hence, $L^2(\Omega,\mathcal{A},\mathbb{P}) \supset B(0,R) \ni \xi \mapsto \tilde{p}(\xi,s,t,x,z)$ is differentiable. As a consequence, $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto p(\mu,s,t,x,z)$ is continuously L-differentiable. From the estimates (5.14), (5.12), (5.17) and (5.19) (the three for $n = 0$) and (5.12) (for $n = 1$), the same conclusion holds for the sequence $\{\mathcal{K} \ni (s,x,\mu,v) \mapsto \partial_\mu p_m(\mu,s,t,x,z)(v),\ m \geq 1\}$, $\mathcal{K}$ being a compact set of $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$, that is, it is uniformly bounded and equicontinuous so that the map $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s,x,\mu,v) \mapsto \partial_\mu p(\mu,s,t,x,z)(v)$ is continuous.

From the estimates (5.12) and (5.15) (both for $n = 1$), the sequence $\{\mathbb{R}^d \supset B(0,R) \ni v \mapsto \partial_v[\partial_\mu p_m(\mu,s,t,x,z)](v),\ m \geq 1\}$, is bounded and equicontinuous. We thus conclude that $\mathbb{R}^d \ni v \mapsto \partial_\mu p(\mu,s,t,x,z)(v)$ is continuously differentiable.

The continuity of the map $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s,x,\mu,v) \mapsto \partial_v[\partial_\mu p(\mu,s,t,x,z)](v)$ can be deduced from the uniform convergence of the sequence of continuous mappings $\{\mathcal{K} \ni (s,x,\mu,v) \mapsto \partial_v[\partial_\mu p_m(\mu,s,t,x,z)](v),\ m \geq 1\}$, $\mathcal{K}$ being a compact set of $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$, along a subsequence, obtained from the estimates (5.14), (5.15), (5.17) and (5.19) for $n = 1$ combined with the Arzelà-Ascoli's theorem. The estimates (3.16) and (3.18) to (3.23) then follow by passing to the limit in the corresponding upper-bounds proved in the first step.

Step 3: $\mathcal{C}^{1,2,2}([0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ regularity and related time estimates.

Let us now prove that $(s,x,\mu) \mapsto p(\mu,s,t,x,z)$ is in $\mathcal{C}^{1,2,2}([0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$. From the Markov property satisfied by the SDE (1.1), stemming from the well-posedness of the related martingale problem, the following relation is satisfied for all $h > 0$

$$p(\mu,s-h,t,x,z) = \mathbb{E}[p([X_s^{s-h,\xi}],s,t,X_s^{s-h,x,\mu},z)].$$

Combining estimates (3.16) and (3.13) (for $n = 1$) with the chain rule formula of Proposition 2.1 (with respect to the space and measure variables only) we obtain

$$\mathbb{E}[p([X_s^{s-h,\xi}],s,t,X_s^{s-h,x,\mu},z)] = p(\mu,s,t,x,z) + \mathbb{E}\left[\int_{s-h}^s \mathcal{L}_r p([X_r^{s-h,\xi}],s,t,X_r^{s-h,x,\mu},z)\,dr\right]$$

with

$$\begin{aligned}\mathcal{L}_r h(x,\mu) &:= \sum_{i=1}^d b_i(r,x,\mu)\partial_{x_i} h(x,\mu) + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(r,x,\mu)\partial^2_{x_i,x_j} h(x,\mu)\\ &\quad + \int_{\mathbb{R}^d}\left\{\sum_{i=1}^d b_i(r,v,\mu)[\partial_\mu h(x,\mu)(v)]_i + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(r,v,\mu)\partial_{v_i}[\partial_\mu h(x,\mu)(v)]_j\right\}\mu(dv).\end{aligned}$$

Hence, one has

$$\frac{1}{h}(p(\mu,s-h,t,x,z)-p(\mu,s,t,x,z))=\frac{1}{h}\mathbb{E}\left[\int_{s-h}^{s}\mathcal{L}_r p([X_r^{s-h,\xi}],s,t,X_r^{s-h,x,\mu},z)\,dr\right]$$

so that, letting $h\downarrow 0$, from the boundedness and continuity of the coefficients as well as the continuity of the maps $(\mu,x)\mapsto p(\mu,s,t,x,z)$, $\partial_x^{1+n}p(\mu,s,t,x,z)$, $\partial_v^n[\partial_\mu p(\mu,s,t,x,z)]$, for $n=0,1$, we deduce that $[0,t)\ni s\mapsto p(\mu,s,t,x,z)$ is left differentiable. Still from the continuity of the coefficients and of the map $(s,x,\mu)\mapsto\mathcal{L}_s p(\mu,s,t,x,z)$, we then conclude that it is differentiable in time on the interval $[0,t)$ with a time derivative satisfying

$$\partial_s p(\mu,s,t,x,z)=-\mathcal{L}_s p(\mu,s,t,x,z)\quad\text{ on }[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d).$$

The time derivative estimate (3.17) now follows from the previous relation as well as the estimates (3.13) and (3.16).

We now move to the proof of Proposition 5.1.

5.2. **Proof of Proposition 5.1.** We proceed by induction on m. For $m=1$, observe that $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(s,x,\mu)\mapsto p_1(\mu,s,t,x,z)=\sum_{k\geq 0}(\widehat{p}_1\otimes\mathcal{H}_1^{(k)})(\mu,s,t,x,z)$ is continuous, where we emphasise from the very definition of our iterative scheme that $\widehat{p}_1$ and $\mathcal{H}_1$ do not depend on the law μ but only on the initial probability measure $\mathbb{P}^{(0)}$ of the iterative scheme. Hence, $\mu\mapsto p_1(\mu,s,t,x,z)$ is continuously L-differentiable with $\partial_\mu p_1(\mu,s,t,x,z)(v)=\partial_v[\partial_\mu p_1(\mu,s,t,x,z)](v)\equiv 0$. For any $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, $(s,x)\mapsto p_1(\mu,s,t,x,z)\in\mathcal{C}^{1,2}([0,t)\times\mathbb{R}^d)$.
Moreover, differentiating n-times ($n=1,2$) with respect to the variable x the relation

$$\begin{aligned}\int_{\mathbb{R}^d}\widehat{p}_m(\mu,s,r,x,y)\Phi_m(\mu,s,r,t,y,z)\,dy&=\int_{\mathbb{R}^d}\widehat{p}_m(\mu,s,r,x,y)[\Phi_m(\mu,s,r,t,y,z)-\Phi_m(\mu,s,r,t,x',z)]\,dy\\&\quad+\Phi_m(\mu,s,r,t,x',z)\int_{\mathbb{R}^d}[\widehat{p}_m(\mu,s,r,x,y)-\widehat{p}_m^{x'}(\mu,s,r,x,y)]\,dy\\&\quad+\Phi_m(\mu,s,r,t,x',z)\end{aligned}$$

and then choosing $x'=x$, by Lebesgue's differentiation theorem, we get

$$\begin{aligned}\int_{\mathbb{R}^d}\partial_x^n\widehat{p}_m(\mu,s,r,x,y)\Phi_m(\mu,s,r,t,y,z)\,dy&=\int_{\mathbb{R}^d}\partial_x^n\widehat{p}_m(\mu,s,r,x,y)[\Phi_m(\mu,s,r,t,y,z)-\Phi_m(\mu,s,r,t,x,z)]\,dy\\&\quad+\Phi_m(\mu,s,r,t,x,z)\int_{\mathbb{R}^d}[\partial_x^n\widehat{p}_m^{y}(\mu,s,r,x,y)-\partial_x^n\widehat{p}_m^{x}(\mu,s,r,x,y)]\,dy.\end{aligned}$$

The η-Hölder regularity of $x\mapsto a(t,x,\mu)$ and the space-time inequality (1.4) imply

$$\begin{aligned}|\partial_x^n\widehat{p}_m^{y}(\mu,s,r,x,y)-\partial_x^n\widehat{p}_m^{x}(\mu,s,r,x,y)|&\leq K|y-x|^{\eta}(r-s)^{-\frac{n}{2}}g(c(r-s),y-x)\\&\leq K(r-s)^{\frac{-n+\eta}{2}}g(c(r-s),y-x)\end{aligned}\tag{5.20}$$

and, from (5.11) and the space-time inequality (1.4),

$$|\int_{\mathbb{R}^d}\partial_x^n\widehat{p}_m(\mu,s,r,x,y)[\Phi_m(\mu,s,r,t,y,z)-\Phi_m(\mu,s,r,t,x,z)]\,dy|\leq K(t-r)^{-1+\frac{\eta-\beta}{2}}(r-s)^{\frac{-n+\beta}{2}}g(c(t-r),z-x)$$

for any $\beta\in[0,\eta)$. Hence, differentiating (5.10) w.r.t the variable x, from Lebesgue's differentiation theorem, we obtain

$$\begin{aligned}\partial_x^n p_m(\mu,s,t,x,z)&=\partial_x^n\widehat{p}_m(\mu,s,t,x,z)\\&\quad+\int_s^t\int_{\mathbb{R}^d}\partial_x^n\widehat{p}_m(\mu,s,r,x,y)\Big[\Phi_m(\mu,s,r,t,y,z)-\Phi_m(\mu,s,r,t,x,z)\Big]\,dy\,dr\\&\quad+\int_s^t\Phi_m(\mu,s,r,t,x,z)\int_{\mathbb{R}^d}\Big[\partial_x^n\widehat{p}_m^{y}(\mu,s,r,x,y)-\partial_x^n\widehat{p}_m^{x}(\mu,s,r,x,y)\Big]\,dy\,dr,\ n=1,2.\end{aligned}\tag{5.21}$$

Observe that since $[X_v^{s,\xi,(0)}]=\nu$ we have that $\Phi_1(\mu,s,r,t,x,z)\equiv\Phi_1(r,t,x,z)$ and $\widehat{p}_1(\mu,s,s,t,x,z)\equiv\widehat{p}_1(s,t,x,z)$. In a completely analogous manner, one may differentiate w.r.t. the variable s the relation

(5.10) for $m=1$. We obtain

$$
\begin{aligned}
\partial_s p_1(\mu,s,t,x,z) &= \partial_s \widehat{p}_1(s,t,x,z) - \Phi_1(s,t,x,z) \\
&\quad + \int_s^t \int_{\mathbb{R}^d} \partial_s \widehat{p}_1(s,r,x,y)\Big[\Phi_1(r,t,y,z) - \Phi_1(r,t,x,z)\Big]\, dy\, dr \\
&\quad + \int_s^t \Phi_1(r,t,x,z) \int_{\mathbb{R}^d} \Big[\partial_s \widehat{p}_1^{\,y}(s,r,x,y) - \partial_s \widehat{p}_1^{\,x}(s,r,x,y)\Big]\, dy\, dr.
\end{aligned} \tag{5.22}
$$

Then, Lebesgue's dominated convergence theorem, the inequality $|\partial_s \widehat{p}_1^{\,y}(s,r,x,y) - \partial_s \widehat{p}_1^{\,x}(s,r,x,y)| \leq C|x-y|^{\eta}(r-s)^{-1}g(c(r-s),y-x) \leq C(r-s)^{-1+\frac{\eta}{2}}g(c(r-s),y-x)$, derived from the space-time inequality (1.4), (5.11) as well as the continuity of the mappings $(s,x,\mu) \mapsto \Phi_1(r,t,x,z)$, $a(s,x,\mu)$, $b(s,x,\mu)$, allow to conclude that the three maps $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \ni (s,x,\mu) \mapsto \partial_x p_1(\mu,s,t,x,z)$, $\partial_x^2 p_1(\mu,s,t,x,z)$, $\partial_s p_1(\mu,s,t,x,z)$ are continuous. From the previous computations, the estimates (5.12) to (5.17) and (5.19) are straightforward for $m=1$.

In order to obtain the estimate (5.18) for $m=1$, we proceed as follows. First, observe that we may assume without loss of generality that $|s_1-s_2| \leq t - s_1 \vee s_2$. Indeed, if $|s_1-s_2| > t - s_1 \vee s_2$, then from (5.6) one directly gets

$$
\begin{aligned}
&|\partial_x^n p_1(\mu,s_1,t,x,z) - \partial_x^n p_1(\mu,s_2,t,x,z)| \\
&\qquad = |\partial_x^n p_1(\mu,s_1\vee s_2,t,x,z) - \partial_x^n p_1(\mu,s_1\wedge s_2,t,x,z)| \\
&\qquad \leq C\left\{\frac{1}{(t-s_1\vee s_2)^{\frac{n}{2}}}g(c(t-s_1\vee s_2),z-x) + \frac{1}{(t-s_1\wedge s_2)^{\frac{n}{2}}}g(c(t-s_1\wedge s_2),z-x)\right\} \\
&\qquad \leq C\left\{\frac{|s_1-s_2|^{\beta}}{(t-s_1\vee s_2)^{\frac{n}{2}+\beta}}g(c(t-s_1\vee s_2),z-x) + \frac{(t-s_1\wedge s_2)^{\beta}}{(t-s_1\wedge s_2)^{\frac{n}{2}+\beta}}g(c(t-s_1\wedge s_2),z-x)\right\} \\
&\qquad \leq C\left\{\frac{|s_1-s_2|^{\beta}}{(t-s_1)^{\frac{n}{2}+\beta}}g(c(t-s_1),z-x) + \frac{|s_1-s_2|^{\beta}}{(t-s_2)^{\frac{n}{2}+\beta}}g(c(t-s_2),z-x)\right\},\ n=1,2,
\end{aligned}
$$

for any $\beta\in[0,1]$ which is the desired bound. Hence, for the rest of the proof of (5.18), we will assume that $|s_1-s_2| \leq t - s_1\vee s_2$.

From (5.21), for all $m\geq 1$, we easily obtain the following decomposition

$$
\begin{aligned}
\partial_x^n p_m(\mu,s,t,x,z) &= \partial_x^n \widehat{p}_m(\mu,s,t,x,z) \\
&\quad + \int_s^{\frac{t+s}{2}} \int_{\mathbb{R}^d} \partial_x^n \widehat{p}_m(\mu,s,r,x,y)\Big[\Phi_m(\mu,s,r,t,y,z) - \Phi_m(\mu,s,r,t,x,z)\Big]\, dy\, dr \\
&\quad + \int_s^{\frac{t+s}{2}} \Phi_m(\mu,s,r,t,x,z) \int_{\mathbb{R}^d} \Big[\partial_x^n \widehat{p}_m^{\,y}(\mu,s,r,x,y) - \partial_x^n \widehat{p}_m^{\,x}(\mu,s,r,x,y)\Big]\, dy\, dr \\
&\quad + \int_{\frac{t+s}{2}}^t \int_{\mathbb{R}^d} \partial_x^n \widehat{p}_m(\mu,s,r,x,y)\Phi_m(\mu,r,t,y,z)\, dy\, dr,
\end{aligned} \tag{5.23}
$$

which in turn implies

$$
\begin{aligned}
&\partial_x^n p_1(\mu, s_1\vee s_2, t, x, z) - \partial_x^n p_1(\mu, s_1\wedge s_2, t, x, z)\\
&\quad = \partial_x^n \widehat{p}_1(s_1\vee s_2, t, x, z) - \partial_x^n \widehat{p}_1(s_1\wedge s_2, t, x, z)\\
&\qquad + \int_{s_1\vee s_2}^{\frac{t+s_1\vee s_2}{2}} \int_{\mathbb{R}^d} [\partial_x^n \widehat{p}_1(s_1\vee s_2, r, x, y) - \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y)]\\
&\qquad\quad . \Big[\Phi_1(r, t, y, z) - \Phi_1(r, t, x, z)\Big]\, dy\, dr\\
&\qquad - \int_{s_1\wedge s_2}^{s_1\vee s_2} \int_{\mathbb{R}^d} \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y) \Big[\Phi_1(r, t, y, z) - \Phi_1(r, t, x, z)\Big]\, dy\, dr\\
&\qquad + \int_{\frac{t+s_1\wedge s_2}{2}}^{\frac{t+s_1\vee s_2}{2}} \int_{\mathbb{R}^d} \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y) \Big[\Phi_1(r, t, y, z) - \Phi_1(r, t, x, z)\Big]\, dy\, dr\\
&\qquad + \int_{s_1\vee s_2}^{\frac{t+s_1\vee s_2}{2}} \Phi_1(r, t, x, z) \int_{\mathbb{R}^d} \Big[\partial_x^n \widehat{p}_1^{\,y}(s_1\vee s_2, r, x, y) - \partial_x^n \widehat{p}_1^{\,y}(s_1\wedge s_2, r, x, y)\Big]\, dy\, dr\\
&\qquad - \int_{s_1\wedge s_2}^{s_1\vee s_2} \Phi_1(r, t, x, z) \int_{\mathbb{R}^d} \Big[\partial_x^n \widehat{p}_1^{\,y}(s_1\wedge s_2, r, x, y) - \partial_x^n \widehat{p}_1^{\,x}(s_1\wedge s_2, r, x, y)\Big]\, dy\, dr\\
&\qquad + \int_{\frac{t+s_1\wedge s_2}{2}}^{\frac{t+s_1\vee s_2}{2}} \Phi_1(r, t, x, z) \int_{\mathbb{R}^d} \Big[\partial_x^n \widehat{p}_1^{\,y}(s_1\wedge s_2, r, x, y) - \partial_x^n \widehat{p}_1^{\,x}(s_1\wedge s_2, r, x, y)\Big]\, dy\, dr\\
&\qquad + \int_{\frac{t+s_1\vee s_2}{2}}^{t} \int_{\mathbb{R}^d} \Big[\partial_x^n \widehat{p}_1(s_1\vee s_2, r, x, y) - \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y)\Big]\Phi_1(r, t, y, z)\, dy\, dr\\
&\qquad + \int_{\frac{t+s_1\wedge s_2}{2}}^{\frac{t+s_1\vee s_2}{2}} \int_{\mathbb{R}^d} \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y)\Phi_1(r, t, y, z)\, dy\, dr\\
&\quad =: \sum_{i=1}^{9} A_i^n.
\end{aligned}
$$

From the mean-value theorem and the inequality $|s_1 - s_2| \le t - s_1\vee s_2$, we obtain from standards computations

$$
\begin{aligned}
\forall \beta\in[0,1], \quad |A_1^n| &= |\partial_x^n \widehat{p}_1(s_1, t, x, z) - \partial_x^n \widehat{p}_1(s_2, t, x, z)|\\
&\le C\Big[\frac{|s_1 - s_2|^\beta}{(t-s_1)^{\frac{n}{2}+\beta}} g(c(t-s_1), z-x) + \frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{n}{2}+\beta}} g(c(t-s_2), z-x)\Big].
\end{aligned}
\tag{5.24}
$$

Combining (5.11), (5.24) and the space-time inequality (1.4), for all $\alpha\in[0,\eta)$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\partial_x^n \widehat{p}_1(s_1\vee s_2, r, x, y) - \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y)||\Phi_1(r, t, y, z) - \Phi_1(r, t, x, z)|\, dy\\
&\quad \le \frac{C}{(t-r)^{1+\frac{\alpha-\eta}{2}}}\Big[\frac{|s_1-s_2|^\beta}{(r-s_1)^{\frac{n}{2}+\beta-\frac{\alpha}{2}}} g(c(t-s_1), z-x) + \frac{|s_1-s_2|^\beta}{(r-s_2)^{\frac{n}{2}+\beta-\frac{\alpha}{2}}} g(c(t-s_2), z-x)\Big],
\end{aligned}
$$

so that,

$$
|A_2^n| \le C\Big[\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{n}{2}+\beta-\frac{\eta}{2}}} g(c(t-s_1), z-x) + \frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{n}{2}+\beta-\frac{\eta}{2}}} g(c(t-s_2), z-x)\Big],
$$

where $\beta\in[0,1]$ if $n=0$, $\beta\in[0,(1+\eta)/2)$ if $n=1$ and $\beta\in[0,\eta/2)$ if $n=2$. For A_3^n, similar arguments yield

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \partial_x^n \widehat{p}_1(s_1\wedge s_2, r, x, y)\Big[\Phi_1(r, t, y, z) - \Phi_1(r, t, x, z)\Big]\, dy\Big|\\
&\le C\frac{1}{(r - s_1\wedge s_2)^{\frac{n-\alpha}{2}}} \frac{1}{(t-r)^{1+\frac{\alpha-\eta}{2}}} g(c(t - s_1\wedge s_2), z-x),
\end{aligned}
$$

where we used the inequality $(t-r)^{-1} \leq (t-s_1\wedge s_2)^{-1}(1+(r-s_1\wedge s_2)/(t-r)) \leq 2(t-s_1\wedge s_2)^{-1}$, for $r < s_1 \vee s_2$ as well as $|s_1-s_2| \leq t - s_1 \vee s_2$. Hence, for all $\alpha \in [0,\eta)$

$$\begin{aligned}|A_3^n| &\leq C\int_{s_1\wedge s_2}^{s_1\vee s_2} \frac{1}{(r-s_1\wedge s_2)^{\frac{n-\alpha}{2}}}\frac{1}{(t-r)^{1+\frac{\alpha-\eta}{2}}}\,dr\, g(c(t-s_1\wedge s_2),z-x)\\ &\leq C\frac{|s_1-s_2|^{1+\frac{\alpha-n}{2}}}{(t-s_1\wedge s_2)^{1+\frac{\alpha-\eta}{2}}}\, g(c(t-s_1\wedge s_2),z-x)\\ &\leq C\frac{|s_1-s_2|^{\beta}}{(t-s_1\wedge s_2)^{\frac{n-\eta}{2}+\beta}}\, g(c(t-s_1\wedge s_2),z-x).\end{aligned}$$

For A_4^n, we remark that since $|s_1-s_2| \leq t - s_1 \vee s_2$, from (5.11) with $\beta = 0$, one gets

$$\begin{aligned}|A_4^n| &\leq C\int_{\frac{t+s_1\wedge s_2}{2}}^{\frac{t+s_1\vee s_2}{2}} \frac{1}{(r-s_1\wedge s_2)^{\frac{n}{2}}}\frac{1}{(t-r)^{1-\frac{\eta}{2}}}\,dr\, g(c(t-s_1\wedge s_2),z-x)\\ &\leq \frac{C}{(t-s_1\wedge s_2)^{\frac{n}{2}}}\int_{\frac{t+s_1\wedge s_2}{2}}^{\frac{t+s_1\vee s_2}{2}} \frac{1}{(t-r)^{1-\frac{\eta}{2}}}\,dr\, g(c(t-s_1\wedge s_2),z-x)\\ &\leq C\frac{|s_1-s_2|^{\alpha}}{(t-s_1\wedge s_2)^{\frac{n-\eta}{2}+\alpha}} g(c(t-s_1\wedge s_2),z-x),\end{aligned}$$

for any $\alpha \in [0,1]$.

We proceed similarly for A_5^n. If $|s_1-s_2| \leq r - s_1 \vee s_2$, we use the mean-value theorem, the uniform η-Hölder regularity of $x \mapsto a(s,x,\mu)$ and the space-time inequality (1.4) to get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\Big[\partial_x^n\widehat{p}_1^y(s_1\vee s_2,r,x,y)-\partial_x^n\widehat{p}_1^y(s_1\wedge s_2,r,x,y)\Big]\,dy\Big|\\ &\qquad= \Big|\int_{\mathbb{R}^d}\Big[\partial_x^n\widehat{p}_1^y(s_1\vee s_2,r,x,y)-\partial_x^n\widehat{p}_1^x(s_1\vee s_2,r,x,y)\\ &\qquad\qquad-(\partial_x^n\widehat{p}_1^y(s_1\wedge s_2,r,x,y)-\partial_x^n\widehat{p}_1^x(s_1\wedge s_2,r,x,y))\Big]\,dy\Big|\\ &\qquad\leq C\int_0^1 \frac{|s_1-s_2|}{(r-(\lambda s_1+(1-\lambda)s_2)^{1+\frac{n-\eta}{2}}}\,d\lambda\\ &\qquad\leq C\frac{|s_1-s_2|^{\alpha}}{(r-s_1\vee s_2)^{\frac{n-\eta}{2}+\alpha}},\end{aligned}$$

for any $\alpha \in [0,1]$. Otherwise, if $|s_1-s_2| \geq r - s_1 \vee s_2$ then one directly gets

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\Big[\partial_x^n\widehat{p}_1^y(s_1\vee s_2,r,x,y)-\partial_x^n\widehat{p}_1^y(s_1\wedge s_2,r,x,y)\Big]\,dy\Big|\\ &\qquad= \Big|\int_{\mathbb{R}^d}\Big[\partial_x^n\widehat{p}_1^y(s_1\vee s_2,r,x,y)-\partial_x^n\widehat{p}_1^x(s_1\vee s_2,r,x,y)\\ &\qquad\qquad-(\partial_x^n\widehat{p}_1^y(s_1\wedge s_2,r,x,y)-\partial_x^n\widehat{p}_1^x(s_1\wedge s_2,r,x,y))\Big]\,dy\Big|\\ &\qquad\leq C\frac{|s_1-s_2|^{\alpha}}{(r-s_1\vee s_2)^{\frac{n-\eta}{2}+\alpha}},\end{aligned}$$

for any $\alpha \in [0,1]$. Hence, we conclude

$$|A_5^n| \leq C\frac{|s_1-s_2|^{\alpha}}{(t-s_1\vee s_2)^{\frac{n}{2}+\alpha-\eta}}\, g(c(t-s_1\vee s_2),z-x),$$

with $\alpha \in [0,1]$ for $n=0$, $\alpha \in [0,(1+\eta)/2)$ for $n=1$ and $\alpha \in [0,\eta/2)$ for $n=2$. Similar arguments yield

$$|A_6^n| \leq C\frac{|s_1-s_2|^{1+\frac{\eta-n}{2}}}{(t-s_1\wedge s_2)^{1-\frac{\eta}{2}}}\, g(c(t-s_1\wedge s_2),z-x) \leq C\frac{|s_1-s_2|^{\alpha}}{(t-s_1\wedge s_2)^{\frac{n}{2}+\alpha-\eta}}\, g(c(t-s_1\wedge s_2),z-x)$$

and

$$|A_7^n| \leq C\frac{|s_1-s_2|^{\alpha}}{(t-s_1\wedge s_2)^{\frac{n}{2}+\alpha-\eta}}\, g(c(t-s_1\wedge s_2),z-x),$$

with $\alpha \in [0,1]$ for $n=0$, $\alpha \in [0,(1+\eta)/2)$ for $n=1$ and $\alpha \in [0,\eta/2)$ for $n=2$. From (5.24), for any $\alpha \in [0,1]$ we get

$$\begin{aligned}|A_8^n| &\le C\int_{\frac{t+s_1\vee s_2}{2}}^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}}\Big[\frac{|s_1-s_2|^\alpha}{(r-s_1\vee s_2)^{\frac{n}{2}+\alpha}}g(c(t-s_1\vee s_2),z-x)\\ &\quad+\frac{|s_1-s_2|^\alpha}{(r-s_1\wedge s_2)^{\frac{n}{2}+\alpha}}g(c(t-s_1\wedge s_2),z-x)\Big]\,dr\\ &\le C\Big[\frac{|s_1-s_2|^\alpha}{(t-s_1\vee s_2)^{\frac{n}{2}+\alpha-\frac{\eta}{2}}}g(c(t-s_1\vee s_2),z-x)\\ &\quad+\frac{|s_1-s_2|^\alpha}{(t-s_1\wedge s_2)^{\frac{n}{2}+\alpha-\frac{\eta}{2}}}g(c(t-s_1\wedge s_2),z-x)\Big].\end{aligned}$$

Finally, using computations similar as those employed before, for all $\alpha \in [0,1]$, we get

$$|A_9^n| \le C\frac{|s_1-s_2|}{(t-s_1\wedge s_2)^{\frac{n}{2}}(t-s_1\vee s_2)^{1-\frac{\eta}{2}}}g(c(t-s_1\wedge s_2),z-x) \le C\frac{|s_1-s_2|^\alpha}{(t-s_1\wedge s_2)^{\frac{n}{2}+\alpha-\frac{\eta}{2}}}g(c(t-s_1\wedge s_2),z-x).$$

This last bound concludes the proof of (5.18) at step $m=1$.

Assuming that the induction hypothesis is valid at step m, we then remark that if $(s_n,x_n,\mu_n)_{n\ge1}$ is a sequence of $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ satisfying $\lim_n|s_n-s|=\lim_n|x_n-x|=\lim_n W_2(\mu_n,\mu)=0$ for some $(s,x,\mu)\in[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, then, from the continuity of the map $[0,t)\times\mathcal{P}_2(\mathbb{R}^d)\ni(s,\mu)\mapsto p_m(\mu,s,t,z)$, (5.3), (5.4) and (5.5), we obtain $\lim_n W_2([X_t^{s_n,\xi_n,(m)}],[X_t^{s,\xi,(m)}])=0$, where $[\xi_n]=\mu_n$ and $[\xi]=\mu$, so that $\lim_n a(t,x_n,[X_t^{s_n,\xi_n,(m)}])=a(t,x,[X_t^{s,\xi,(m)}])$ and $\lim_n b(t,x_n,[X_t^{s_n,\xi_n,(m)}])=b(t,x,[X_t^{s,\xi,(m)}])$. From the representation in infinite series (5.4) (at step $m+1$) and the Lebesgue's dominated convergence theorem, we thus deduce that the map $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(s,x,\mu)\mapsto p_{m+1}(\mu,s,t,x,z)$ is continuous.

We next apply Proposition 2.2 to the maps $(s,\mu,x)\mapsto p_m(\mu,s,t,x,z)\in\mathcal{C}^{1,2,2}([0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ and $\mu\mapsto a(t,x,\mu)$, $b(t,x,\mu)$. Note that from the estimates (5.12), (5.13) and (5.6), the map $[0,r)\times\mathcal{P}_2(\mathbb{R}^d)\times(s,x,\mu)\mapsto p_m(\mu,s,r,x,z)$ satisfies the conditions of Proposition 2.2, in particular the condition (2.9). We thus deduce that $(s,\mu)\mapsto a(r,x,[X_r^{s,\xi,(m)}])$, $b(r,x,[X_r^{s,\xi,(m)}])\in\mathcal{C}^{1,2}([0,r)\times\mathcal{P}_2(\mathbb{R}^d))$ with derivatives satisfying

(5.25)
$$\partial_s a(r,x,[X_r^{s,\xi,(m)}]) = \int\int(\frac{\delta a}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(x'))\,\partial_s p_m(\mu,s,r,x',y')\,dy'\,\mu(dx'),$$

$$\partial_v^n[\partial_\mu a(r,x,[X_r^{s,\xi,(m)}])](v) = \int(\frac{\delta a}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(v))\,\partial_v^{1+n}p_m(\mu,s,r,v,y')\,dy'$$

(5.26)
$$+\int\int(\frac{\delta a}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(x'))\,\partial_v^n[\partial_\mu p_m(\mu,s,r,x',y')](v)\,dy'\,\mu(dx')$$

and similarly

$$\partial_s b(r,x,[X_r^{s,\xi,(m)}]) = \int\int(\frac{\delta b}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta b}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(x'))\,\partial_s p_m(\mu,s,r,x',y')\,dy'\,\mu(dx'),$$

(5.27)
$$\begin{aligned}\partial_v^n[\partial_\mu b(r,x,[X_t^{s,\xi,(m)}])](v) &= \int(\frac{\delta b}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta b}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(v))\,\partial_v^{1+n}p_m(\mu,s,r,v,y')\,dy'\\ &\quad+\int\int(\frac{\delta b}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta b}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(x'))\,\partial_v^n[\partial_\mu p_m(\mu,s,r,x',y')](v)\,dy'\,\mu(dx').\end{aligned}$$

As a consequence, the maps $[0,r)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\mathbb{R}^d\ni(s,x,\mu,v)\mapsto\partial_s a(r,x,[X_r^{s,\xi,(m)}])$, $\partial_v^n[\partial_\mu a(r,x,[X_r^{s,\xi,(m)}])](v)$ and $[0,r]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\ni(s,\mu,v)\mapsto\partial_s b(r,x,[X_r^{s,\xi,(m)}])$, $\partial_v^n[\partial_\mu b(r,x,[X_r^{s,\xi,(m)}])](v)$ are continuous for $n=0,1$. Moreover, from (5.13) at step m and the space-time inequality (1.4), one also derives the following bound:

$$\begin{aligned}|\partial_s b(r,x,[X_r^{s,\xi,(m)}])|+|\partial_s a(r,x,[X_r^{s,\xi,(m)}])| &\le K\int\int(|y'-x'|^\eta\wedge1)|\partial_s p_m(\mu,s,r,x',y')|\,dy'\,\mu(dx')\\ &\le K(r-s)^{-1+\frac{\eta}{2}}.\end{aligned}\tag{5.28}$$

Similarly, from the estimates (5.12), (5.6) and the space-time inequality (1.4), we also obtain

$$
\begin{aligned}
&|\partial_v^n[\partial_\mu b(t,x,[X_t^{s,\xi,(m)}])](v)| + |\partial_v^n[\partial_\mu a(t,x,[X_t^{s,\xi,(m)}])](v)| \\
&\qquad \le K\left\{(t-s)^{-\frac{1+n-\eta}{2}} + \int\int (|y-x'|^\eta \wedge 1)|\partial_v^n[\partial_\mu p_m(\mu,s,t,x',y)](v)|\,\mu(dx')\,dy\right\} \qquad (5.29)\\
&\qquad \le K(t-s)^{-\frac{1+n-\eta}{2}}, \qquad (5.30)
\end{aligned}
$$

for $n = 0, 1$. As a consequence, the mapping $[0,r) \ni s \mapsto \mathcal{H}_{m+1}(\mu,s,r,t,x,z)$ is continuously differentiable for any $(\mu,x,z) \in \mathcal{P}_2(\mathbb{R}^d)\times(\mathbb{R}^d)^2$ with:

$$
\begin{aligned}
\partial_s\mathcal{H}_{m+1}(\mu,s,r,t,x,z) &:= \Big[-\sum_{i=1}^d \partial_s b_i\left(r,x,[X_r^{s,\xi,(m)}]\right) H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\\
&\quad - \sum_{i=1}^d b_i\left(r,x,[X_r^{s,\xi,(m)}]\right)\partial_s H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\\
&\quad + \frac{1}{2}\sum_{i,j=1}^d (\partial_s a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - \partial_s a_{i,j}(r,z,[X_r^{s,\xi,(m)}]))\\
&\qquad \times H_2^{i,j}(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x)\\
&\quad + \frac{1}{2}\sum_{i,j=1}^d \Big(a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])\Big)\\
&\qquad \times \partial_s H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\Big]\widehat{p}_{m+1}(\mu,s,r,t,x,z)\\
&\quad + \Big[-\sum_{i=1}^d b_i(r,x,[X_r^{s,\xi,(m)}])H_1^i(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x)\\
&\qquad + \frac{1}{2}\sum_{i,j=1}^d \Big(a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,x,[X_r^{s,\xi,(m)}])\Big)\\
&\qquad\quad \times H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)})\,dv, z-x\right)\Big]\partial_s\widehat{p}_{m+1}(\mu,s,r,t,x,z).
\end{aligned} \qquad (5.31)
$$

The previous expression with the previous continuity results also yield the continuity of the mapping $[0,r)\times\mathcal{P}_2(\mathbb{R}^d) \ni (s,\mu) \mapsto \partial_s\mathcal{H}_{m+1}(\mu,s,r,t,x,z)$. Moreover, from (5.25), using either the η-Hölder regularity of $x \mapsto [\delta a_{i,j}/\delta m](r,x,\mu)(y)$ with (5.13) or (5.28), we get

$$
|\partial_s a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - \partial_s a_{i,j}(r,z,[X_r^{s,\xi,(m)}])| \le K\left\{\frac{|z-x|^\eta}{(r-s)} \wedge \frac{1}{(r-s)^{1-\frac{\eta}{2}}}\right\}
$$

so that, from (5.28) and the space-time inequality (1.4), we get the following bound

$$
|\partial_s\mathcal{H}_{m+1}(\mu,s,r,t,x,z)| \le K\Big[\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)} \wedge \frac{1}{(t-r)(r-s)^{1-\frac{\eta}{2}}}\Big]g(c(t-r),z-x), \qquad (5.32)
$$

for two positive constants $K := K(T,a,b,\lambda,\eta)$, $c := c(\lambda)$. Then, standard computations based on the previous estimate imply the convergence of the series $\sum_{k\ge0}(\mathcal{H}_{m+1}^{(k)} \otimes \partial_s\mathcal{H}_{m+1})(\mu,s,r,t,x,z)$. Moreover, if we formally differentiate w.r.t. the variable s the relation $\Phi_{m+1}(\mu,s,r,t,x,z) = \mathcal{H}_{m+1}(\mu,s,r,t,x,z) + (\mathcal{H}_{m+1}\otimes\Phi_{m+1})(\mu,s,r,t,x,z)$, we get

$$
\begin{aligned}
\partial_s\Phi_{m+1}(\mu,s,r,t,x,z) &= \partial_s\mathcal{H}_{m+1}(\mu,s,r,t,x,z) + (\partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1})(\mu,s,r,t,x,z)\\
&\quad + (\mathcal{H}_{m+1}\otimes\partial_s\Phi_{m+1})(\mu,s,r,t,x,z)
\end{aligned}
$$

so that, iterating the previous relation, we deduce that the map $[0,r) \ni s \mapsto \Phi_{m+1}(\mu,s,r,t,x,z)$ is continuously differentiable with

$$
\begin{aligned}
\partial_s\Phi_{m+1}(\mu,s,r,t,x,z) &= \sum_{k\ge0}\Big(\mathcal{H}_{m+1}^{(k)}\otimes[\partial_s\mathcal{H}_{m+1} + \partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1}]\Big)(\mu,s,r,t,x,z)\\
&= (\partial_s\mathcal{H}_{m+1} + \partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1})(\mu,s,r,t,x,z)\\
&\quad + (\Phi_{m+1}\otimes[\partial_s\mathcal{H}_{m+1} + \partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1}])(\mu,s,r,t,x,z).
\end{aligned} \qquad (5.33)
$$

We importantly note that from the estimate (5.32) and the continuity of $(s,\mu)\mapsto \partial_s\mathcal{H}_{m+1}(\mu,s,r,t,x,z)$ each term appearing in the above expression makes sense and is continuous w.r.t. (s,μ). We thus deduce that $[0,r)\ni s\mapsto \Phi_{m+1}(\mu,s,r,t,x,z)$ is continuously differentiable and $\partial_s\Phi_{m+1}$ is globally continuous w.r.t. the variables s and μ. The same conclusion holds true for $[0,t)\ni s\mapsto p_{m+1}(\mu,s,t,x,z)$. More precisely, differentiating the relation (5.10) w.r.t. the variable s, then plugging (5.33) and using relation (5.10) again, we successively get

$$
\begin{aligned}
\partial_s p_{m+1}(\mu,s,t,x,z) &= \partial_s\widehat{p}_{m+1}(\mu,s,t,x,z)-\Phi_{m+1}(\mu,s,t,x,z)\\
&\quad+\int_s^t\int_{\mathbb{R}^d}\partial_s\widehat{p}_{m+1}(\mu,s,r,x,y)\Phi_{m+1}(\mu,s,r,t,y,z)\,dy\,dr\\
&\quad+\int_s^t\int_{\mathbb{R}^d}\widehat{p}_{m+1}(\mu,s,r,x,y)\partial_s\Phi_{m+1}(\mu,s,r,t,y,z)\,dy\,dr\\
&=\partial_s\widehat{p}_{m+1}(\mu,s,t,x,z)-\Phi_{m+1}(\mu,s,t,x,z)+\partial_s\widehat{p}_{m+1}\otimes\Phi_{m+1}(\mu,s,t,x,z)\\
&\quad+p_{m+1}\otimes\partial_s\mathcal{H}_{m+1}(\mu,s,t,x,z)+[(p_{m+1}\otimes\partial_s\mathcal{H}_{m+1})\otimes\Phi_{m+1}](\mu,s,t,x,z).
\end{aligned}
\tag{5.34}
$$

Again, we note that each term appearing in the above equality makes sense and is continuous with respect to (s,x,μ). Indeed, we make us of a similar decomposition as the one employed in (5.22), namely

$$
\begin{aligned}
\partial_s\widehat{p}_{m+1}\otimes\Phi_{m+1}(\mu,s,t,x,z)&=\int_s^t\int_{\mathbb{R}^d}\partial_s\widehat{p}_{m+1}(\mu,s,r,x,y)\Big[\Phi_{m+1}(\mu,s,r,t,y,z)-\Phi_{m+1}(\mu,s,r,t,x,z)\Big]\,dy\,dr\\
&\quad+\int_s^t\Phi_{m+1}(\mu,s,r,t,x,z)\int_{\mathbb{R}^d}\Big[\partial_s\widehat{p}^{\,y}_{m+1}(\mu,s,r,x,y)-\partial_s\widehat{p}^{\,x}_{m+1}(\mu,s,r,x,y)\Big]\,dy\,dr
\end{aligned}
\tag{5.35}
$$

with

$$
\begin{aligned}
\partial_s\widehat{p}^{\,z}_{m+1}(\mu,s,t,x,y)&=-\frac{1}{2}\sum_{i,j=1}^d a_{i,j}(s,z,\mu)H_2^{i,j}\left(\int_s^t a(r,z,[X_r^{s,\xi,(m)}])\,dr,y-x\right)\widehat{p}^{\,z}_{m+1}(\mu,s,t,x,y)\\
&\quad+Df_{y-x}\left(\int_s^t a(r,z,[X_r^{s,\xi,(m)}])dr\right)\\
&\quad\cdot\int_s^t\int\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(x'))\,\partial_s p_m(\mu,s,r,x',y')\,dy'\,\mu(dx')dr,
\end{aligned}
\tag{5.36}
$$

where, identifying any $d\times d$ matrix Γ as an $\mathbb{R}^{d^2}$ valued vector, we write for any z in $\mathbb{R}^d$

$$
f_z:\mathbb{R}^{d^2}\ni\Gamma\mapsto(2\pi)^{\frac{d}{2}}\det(\Gamma)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\langle\Gamma^{-1}z,z\rangle\right)\in\mathbb{R}.
$$

Now, for the first term appearing in the right-hand side of the previous identity, we combine (5.11) with the space-time inequality (1.4) to get rid off the singularity in time induced by $\partial_s\widehat{p}_{m+1}(\mu,s,r,x,y)$. For the second term appearing in the right-hand side of (5.35), from (5.36) and the η-Hölder regularity of the two maps $y\mapsto a_{i,j}(s,y,\mu),[\delta a/\delta m](r,z,\mu)(y)$, (5.13) (at step m) and the continuity of $(s,\mu)\mapsto a(r,x,[X_r^{s,\xi,(m)}])$, $[\delta a/\delta m](r,x,[X_r^{s,\xi,(m)}])(y)$, we deduce that $(s,x,\mu)\mapsto\partial_s\widehat{p}^{\,y}_{m+1}(\mu,s,r,x,y)-\partial_s\widehat{p}^{\,x}_{m+1}(\mu,s,r,x,y)$ is continuous and, by the space-time inequality (1.4), satisfies the inequality $|\partial_s\widehat{p}^{\,y}_{m+1}(\mu,s,r,x,y)-\partial_s\widehat{p}^{\,x}_{m+1}(\mu,s,r,x,y)|\le C(r-s)^{-1+\frac{\eta}{2}}g(c(r-s),y-x)$. From the representation formula (5.9) and the continuity of $(s,x,\mu)\mapsto\mathcal{H}_{m+1}(\mu,s,t,x,z)$, the same conclusion holds for the map $(s,x,\mu)\mapsto\Phi_{m+1}(\mu,s,t,x,z)$. We thus deduce that the second term appearing in the right-hand side of (5.35) is continuous with respect to (s,x,μ). We then conclude that the two maps: $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(s,x,\mu)\mapsto\partial_s\widehat{p}_{m+1}(\mu,s,t,x,y)$, $(\partial_s\widehat{p}_{m+1}\otimes\Phi_{m+1})(\mu,s,t,x,z)$ are also continuous. Combining (5.31), (5.32) and arguments similar to those previously employed, we derive in an analogous manner that the two maps $(s,x,\mu)\mapsto p_{m+1}\otimes\partial_s\mathcal{H}_{m+1}(\mu,s,t,x,z)$, $[(p_{m+1}\otimes\partial_s\mathcal{H}_{m+1})\otimes\Phi_{m+1}](\mu,s,t,x,z)$ are continuous. We thus finally conclude that $[0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(s,x,\mu)\mapsto\partial_s p_{m+1}(\mu,s,t,x,z)$ is continuous. From the relation (5.21), the same conclusion remains valid for the maps $(s,x,\mu)\mapsto\partial_x^n p_{m+1}(\mu,s,t,x,z)$, $n=1,2$.

We also deduce that the mappings $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto\mathcal{H}_{m+1}(\mu,s,r,t,x,z)$, $\widehat{p}^{\,y'}_{m+1}(\mu,s,r,t,x,z)$, $\widehat{p}^{\,y'}_{m+1}(\mu,s,t,x,z)$ are L-differentiable and that $\mathbb{R}^d\ni y\mapsto\partial_\mu\mathcal{H}_{m+1}(\mu,s,r,t,x,z)(y)$, $\partial_\mu\widehat{p}^{\,y'}_{m+1}(\mu,s,t,x,z)(y)$, $\partial_\mu\widehat{p}^{\,y'}_{m+1}(\mu,s,r,t,x,z)(y)$

are continuously differentiable satisfying for any fixed $0 \leq s \leq r < t$, $(x, y, y', z) \in (\mathbb{R}^d)^4$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$

(5.37)
$$\partial_y^n[\partial_\mu \widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)](y) = Df_{z-x}\Big(\int_r^t a(v, y', [X_v^{s,\xi,(m)}])\, dv\Big).\int_r^t \partial_y^n[\partial_\mu a(v, y', [X_v^{s,\xi,(m)}])](y)\, dv$$

and

(5.38)
$$\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu, s, r, t, x, z)](y) =: \mathrm{I}^n(y) + \mathrm{II}^n(y) + \mathrm{III}^n(y), \quad n = 0, 1,$$

with

$$\begin{aligned}
\mathrm{I}^n(y) &:= \left\{ -\sum_{i=1}^d H_1^i\left(\int_r^t a(v, z, [X_v^{s,\xi,(m)}])dv, z - x\right) \partial_y^n[\partial_\mu b_i(r, x, [X_r^{s,\xi,(m)}])](y)\right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\
&+ \left\{ -\sum_{i=1}^d b_i(r, x, [X_r^{s,\xi,(m)}])\partial_y^n\left[\partial_\mu H_1^i\left(\int_r^t a(v, z, [X_v^{s,\xi,(m)}])dv, z - x\right)\right](y)\right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\
&=: \mathrm{I}_1^n(y) + \mathrm{I}_2^n(y), \\
\mathrm{II}^n(y) &:= \Big\{\frac{1}{2}\sum_{i,j=1}^d \partial_y^n[\partial_\mu\Big(a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}])\Big)](y) \\
&\times H_2^{i,j}\left(\int_r^t a(v, z, [X_v^{s,\xi,(m)}])dv, z - x\right)\Big\}\widehat{p}_{m+1}(\mu, s, r, t, x, z) \\
&+ \left\{\frac{1}{2}\sum_{i,j=1}^d \Big(a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}])\Big)\right. \\
&\left.\times \partial_y^n\left[\partial_\mu H_2^{i,j}\left(\int_r^t a(v, z, [X_v^{s,\xi,(m)}])dv, z - x\right)\right](y)\right\} \widehat{p}_{m+1}(\mu, s, r, t, x, z) \\
&=: \mathrm{II}_1^n(y) + \mathrm{II}_2^n(y),
\end{aligned}$$

$$\begin{aligned}
\mathrm{III}^n(y) &:= \left\{ -\sum_{i=1}^d b_i(r, x, [X_r^{s,\xi,(m)}])H_1^i\left(\int_r^t a(v, z, [X_v^{s,\xi,(m)}])dv, z - x\right)\right. \\
&\left. + \frac{1}{2}\sum_{i,j=1}^d (a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}]))H_2^{i,j}\left(\int_r^t a(v, z, [X_v^{s,\xi,(m)}])dv, z - x\right)\right\} \\
&\times \partial_y^n[\partial_\mu \widehat{p}_{m+1}(\mu, s, r, t, x, z)](y).
\end{aligned}$$

From the above relations and the previous continuity results, we readily derive the continuity of the mappings $(s, \mu, y) \mapsto \partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu, s, r, t, x, z)](y)$ and $(s, x, \mu, y) \mapsto \partial_y^n[\partial_\mu \widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)](y)$. From (5.30) and (5.37), we directly get the following bounds

$$|\partial_y^n[\partial_\mu \widehat{p}_{m+1}^{y'}(\mu, s, t, x, z)](y)| \leq C(t - s)^{-\frac{1+n-\eta}{2}}\, g(c(t - s), z - x)$$

and

(5.39)
$$|\partial_y^n[\partial_\mu \widehat{p}_{m+1}^{y'}(\mu, s, r, t, x, z)](y)| \leq C(r - s)^{-\frac{1+n-\eta}{2}}\, g(c(t - r), z - x).$$

Combining the previous estimate with (5.30) and using the η-Hölder regularity of $x \mapsto a_{i,j}(r, x, \mu)$ with the space time inequality (1.4) yield

$$\begin{aligned}
|\mathrm{I}^n| &\leq \frac{K}{(t - r)^{\frac{1}{2}}(r - s)^{\frac{1+n-\eta}{2}}} g(c(t - r), z - x), \\
|\mathrm{III}^n| &\leq \frac{K}{(t - r)^{1-\frac{\eta}{2}}(r - s)^{\frac{1+n-\eta}{2}}} g(c(t - r), z - x).
\end{aligned}$$

From the key identity (5.26), the estimates (5.6), (5.12) at step m and using either the η-Hölder regularity of $x \mapsto [\delta a_{i,j}/\delta m](t, x, \mu)(y)$ or the η-Hölder regularity of $y \mapsto [\delta a_{i,j}/\delta m](t, x, \mu)(y)$, we get

$$|\partial_y^n[\partial_\mu\Big(a_{i,j}(r, x, [X_r^{s,\xi,(m)}]) - a_{i,j}(r, z, [X_r^{s,\xi,(m)}])\Big)](y)| \leq K\left\{\frac{|z - x|^\eta}{(r - s)^{\frac{1+n}{2}}} \wedge \frac{1}{(r - s)^{\frac{1+n-\eta}{2}}}\right\},$$

so that, by the space-time inequality (1.4), one finally obtains

$$|\mathrm{II}^n| \le K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right) g(c(t-r), z-x).$$

Gathering the three previous estimates on I_n, II_n and III_n, we get

$$|\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,x,z)](y)| \le K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right) g(c(t-r), z-x).$$

The previous estimates yield the (absolute) convergence of the two series $\sum_{k\ge 0} \partial_y^n[\partial_\mu \widehat{p}_{m+1}] \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(y)$ and $\sum_{k\ge 0}(p_{m+1} \otimes \partial_y^n[\partial_\mu \mathcal{H}_{m+1}]) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(y)$ as well as their global continuity with respect to the variables s, x, μ, y on $[0,r) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$.

Formally differentiating with respect to the variables μ and then y, the relation $p_{m+1}(\mu,s,t,x,z) = \widehat{p}_{m+1}(\mu,s,t,x,z) + (p_{m+1} \otimes \mathcal{H}_{m+1})(\mu,s,t,x,z)$, we obtain

$$\begin{aligned}\partial_y^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](y) &= \partial_y^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](y) + (p_{m+1} \otimes \partial_y^n[\partial_\mu \mathcal{H}_{m+1}])(\mu,s,t,x,z)(y)\\ &\quad + (\partial_y^n[\partial_\mu p_{m+1}] \otimes \mathcal{H}_{m+1})(\mu,s,t,x,z)(y)\end{aligned}$$

so that a direct iteration yields the following key relation

(5.40)
$$\partial_y^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](y) = \sum_{k\ge 0}(\partial_y^n[\partial_\mu \widehat{p}_{m+1}] + p_{m+1} \otimes \partial_y^n[\partial_\mu \mathcal{H}_{m+1}]) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(y),\ n = 0, 1.$$

Hence, $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto p_{m+1}(\mu,s,t,x,z)$ is L-differentiable and $\mathbb{R}^d \ni y \mapsto \partial_\mu p_{m+1}(\mu,s,t,x,z)(y)$ is continuously differentiable and the maps $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (s,x,\mu,y) \mapsto \partial_\mu p_{m+1}(\mu,s,t,x,z)(y)$, $\partial_y[\partial_\mu p_{m+1}(\mu,s,t,x,z)](y)$ are continuous. From (5.21) and similar arguments as those previously employed, we also derive the continuity of the maps $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s,x,\mu) \mapsto \partial_x^n p_{m+1}(\mu,s,t,x,z)$, $n = 1, 2$. We finally conclude that the mapping $[0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (s,x,\mu) \mapsto p_{m+1}(\mu,s,t,x,z)$ is in $\mathcal{C}^{1,2,2}([0,t) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$.

We now prove the estimates (5.12) to (5.19) at step $m+1$. Since their proofs are rather long, technical and relies on similar ideas and arguments, we will not prove all the announced estimates. We start with (5.12) and will deliberately omit the proofs of the estimates (5.13), (5.14) and (5.18). We introduce the following quantities for $n = 0$, 1:

$$\begin{aligned}u_m^n(s,t) &:= \sup_{v\in\mathbb{R}^d} \int \mu(dx) \int (|y-x|^\eta \wedge 1)|\partial_v^n[\partial_\mu p_m(\mu,s,t,x,y)](v)|\, dy,\\ v_m^n(s,t) &:= \sup_{v\in\mathbb{R}^d} \int \mu(dx) \int |\partial_v^n[\partial_\mu p_m(\mu,s,t,x,y)](v)|\, dy\end{aligned}$$

and prove by induction on m the following key inequalities:

$$u_m^n(s,t) \le C_{m,n}(s,t)(t-s)^{-(\frac{1+n}{2}-\eta)}, \qquad v_m^n(s,t) \le C_{m,n}(s,t)(t-s)^{-\frac{(1+n-\eta)}{2}},$$

with $C_{m,n}(s,t) := \sum_{k=1}^m C^k \prod_{i=1}^k B\left(\frac{\eta}{2}, \frac{1-n+\eta}{2} + (i-1)\frac{\eta}{2}\right)(t-s)^{(k-1)\frac{\eta}{2}}$, $C := C(T,a,b,[\delta a/\delta m],[\delta b/\delta m]b,\eta)$ being a positive constant independent of m. This result being straightforward for $m = 1$, we assume that it holds at step m. We first remark that from (5.37) and (5.29) there exist positive constant $K := K(T,a,[\delta a/\delta m],\eta)$, $c := c(\lambda)$, which may vary from line to line, such that for all $m \ge 1$

$$\begin{aligned}&|\partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](v)|\\ &\quad\le K\Big\{\frac{1}{(t-s)^{\frac{1+n-\eta}{2}}} + \frac{1}{t-s}\int_s^t \int\int (|y'-x'|^\eta \wedge 1)|\partial_v^n[\partial_\mu p_m(\mu,s,r,x',y')](v)|\, dy'\, \mu(dx')\, dr\Big\}\\ &\qquad\times g(c(t-s), z-x),\end{aligned}$$

so that by the induction hypothesis

$$
\begin{aligned}
&|\partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](v)| \\
&\quad \le K\left\{\frac{1}{(t-s)^{\frac{1+n-\eta}{2}}}+\frac{1}{(t-s)}\int_s^t \frac{C_{m,n}(s,r)}{(r-s)^{\frac{1+n}{2}-\eta}}\,dr\right\} g(c(t-s),z-x) \\
&\quad \le K\left\{\frac{1}{(t-s)^{\frac{1+n-\eta}{2}}}+\frac{1}{(t-s)^{\frac{\eta}{2}}}\int_s^t \frac{C_{m,n}(s,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}-\eta}}\,dr\right\} g(c(t-s),z-x) \\
&\quad \le K\left\{\frac{1}{(t-s)^{\frac{1+n-\eta}{2}}}+\frac{1}{(t-s)^{\frac{1+n-\eta}{2}}}\sum_{k=1}^m C^k\prod_{i=1}^{k+1} B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\
&\qquad \times g(c(t-s),z-x). \qquad (5.41)
\end{aligned}
$$

Hence, by induction on r, there exists a positive constant $K := K(T,a,b,[\delta a/\delta m],\eta)$ which may change from line to line but is *independent of* m such that

$$
\begin{aligned}
|(\partial_v^n[\partial_\mu \widehat{p}_{m+1}]\otimes \mathcal{H}_{m+1}^{(r)})(\mu,s,t,x,z)(v)| &\le K^r\left\{1+\sum_{k=1}^m C^k\prod_{i=1}^{k+1} B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\
&\quad \times (t-s)^{-\frac{(1+n-\eta)}{2}+r\frac{\eta}{2}}\prod_{i=1}^r B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right) \\
&\quad \times g(c(t-s),z-x),
\end{aligned}
$$

which in turn implies

$$
\begin{aligned}
&\sum_{r\ge 0}|(\partial_v^n[\partial_\mu \widehat{p}_{m+1}]\otimes \mathcal{H}_{m+1}^{(r)})(\mu,s,t,x,z)(v)| \\
&\quad \le \frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{1+\sum_{k=1}^m C^k\prod_{i=1}^{k+1} B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} g(c(t-s),z-x) \\
&\quad \le \frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1} B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\
&\qquad \times g(c(t-s),z-x). \qquad (5.42)
\end{aligned}
$$

We now come back to the decomposition (5.38) of $\partial_v^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,x,z)](y)$. From (5.29) and the induction hypothesis, we directly get

$$
\begin{aligned}
|\mathrm{I}_1^n(y)| &= \left|-\sum_{i=1}^d H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv,z-x\right)\partial_y^n\left[\partial_\mu b_i(r,x,[X_r^{s,\xi,(m)}])\right](y)\right|\widehat{p}_{m+1}(\mu,r,t,x,z) \\
&\le K\frac{(1+(r-s)^{\frac{1+n-\eta}{2}}u_m^n(s,r))}{(t-r)^{\frac12}(r-s)^{\frac{1+n-\eta}{2}}}g(c(t-r),z-x) \\
&\le K\frac{(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}})}{(t-r)^{\frac12}(r-s)^{\frac{1+n-\eta}{2}}}\,g(c(t-r),z-x).
\end{aligned}
$$

Next again from (5.29), the induction hypothesis and the space-time inequality (1.4)

$$
\begin{aligned}
|\mathrm{I}_2^n(y)| &= \left|-\sum_{i=1}^d b_i(r,x,[X_r^{s,\xi,(m)}])\partial_y^n\left[\partial_\mu H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv,z-x\right)\right](y)\right|\widehat{p}_{m+1}(\mu,r,t,x,z) \\
&\le K\frac{|z-x|}{(t-r)^2}\left(\int_r^t\frac{(1+(v-s)^{\frac{1+n-\eta}{2}}u_m^n(s,v))}{(v-s)^{\frac{1+n-\eta}{2}}}dv\right)g(c(t-r),z-x) \\
&\le K\frac{(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)}{(t-r)^{\frac12}(r-s)^{\frac{1+n-\eta}{2}}}\,g(c(t-r),z-x).
\end{aligned}
$$

Hence, one concludes that for all y in $\mathbb{R}^d$,

$$
\begin{aligned}
|\mathrm{I}^n(y)| &\le K\frac{(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)}{(t-r)^{\frac12}(r-s)^{\frac{1+n-\eta}{2}}}\,g(c(t-r),z-x)\\
&\le K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)}\wedge\frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right)\\
&\qquad\times\left(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv\right)g(c(t-r),z-x).
\end{aligned}
$$

We now estimate II^n which is the tricky part of our computations. From Proposition 2.2, $\mathrm{II}_1^n(y)$ can be written as

$$
\begin{aligned}
\mathrm{II}_1^n(y) &= \frac12\Bigg\{\sum_{i,j=1}^d H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv,z-x\right)\\
&\quad\times\partial_y^n\Big[\partial_\nu\Big[\int\Big\{\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y')\Big\}p_m(\nu,s,r,y')dy'\Big]_{|\nu=\mu}\Big](y)\Bigg\}\\
&\quad\times\widehat{p}_m(\mu,r,t,x,z)\\
&=:\Big\{\frac12\sum_{i,j=1}^d H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv,z-x\right)\times\mathrm{J}_{i,j}(y)\Big\}\,\widehat{p}_m(\mu,r,t,x,z).
\end{aligned}
$$

On the one hand, using the induction hypothesis and the fact that $x\mapsto[\delta a_{i,j}/\delta m](r,x,\mu)(y)$ is η-Hölder, one has

$$
\begin{aligned}
|\mathrm{J}_{i,j}(y)| &= \left|\partial_y^n\Big[\partial_\nu\Big[\int\Big\{\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y')\Big\}p_m(\nu,s,r,y')\,dy'\Big]_{|\nu=\mu}\Big](y)\right|\\
&\le K\Big\{\int(|z-x|^\eta\wedge1)|\partial_x^{1+n}p_m(\mu,s,r,y,y')|\,dy'\\
&\quad+\int\int(|z-x|^\eta\wedge1)|\partial_y^n[\partial_\mu p_m(\mu,s,r,x',y')](y)|\,dy'\mu(dx')\Big\}\\
&\le K\frac{|z-x|^\eta}{(r-s)^{\frac{1+n}{2}}}(1+(r-s)^{\frac{1+n}{2}}v_m^n(s,r)). \qquad (5.43)
\end{aligned}
$$

On the other hand, from Proposition 2.2, one gets the following decomposition

$$
\begin{aligned}
\mathrm{J}_{i,j}(y) &= \Big[\int\int\Big\{\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(x')\big)\\
&\quad-\big(\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(x')\big)\Big\}\partial_y^n\Big[\partial_\mu p_m(\mu,s,r,x',y')\Big](y)\,\mu(dx')\,dy'\Big]\\
&\quad+\Big[\int\Big\{\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y)\big)\\
&\quad-\big(\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y)\big)\Big\}\partial_x^{1+n}p_m(\mu,s,r,y,y')\,dy'\Big] \qquad (5.44)
\end{aligned}
$$

which in turn, by the η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](r,x,m)(y)$ and the induction hypothesis, yields

$$
|\mathrm{J}_{i,j}(y)| \le K\frac{(1+(r-s)^{\frac{1+n-\eta}{2}}u_m^n(s,r))}{(r-s)^{\frac{1+n-\eta}{2}}}. \qquad (5.45)
$$

Consequently, combining the estimates (5.43) and (5.45), we obtain

$$
|\mathrm{J}_{i,j}(y)| \le K\left\{\frac{|z-x|^\eta}{(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(r-s)^{\frac{1+n-\eta}{2}}}\right\}(1+(r-s)^{\frac{1+n-\eta}{2}}u_m^n(s,r)+(r-s)^{\frac{1+n}{2}}v_m^n(s,r)). \qquad (5.46)
$$

Hence, from the previous estimate and the space-time inequality (1.4), we deduce

$$
\begin{aligned}
|\mathrm{II}_1^n(y)| &\leq \frac{K}{t-r}\left(\frac{|z-x|^\eta}{(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(r-s)^{\frac{1+n-\eta}{2}}}\right)(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}})\,g(c(t-r),z-x)\\
&\leq K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right)(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}})\,g(c(t-r),z-x).
\end{aligned}
$$

We now turn to II_2^n. From the very definition of $H_2^{i,j}$, (5.29) and noticing that for any differentiable map $\mathcal{P}_2(\mathbb{R}^d)\ni\nu\mapsto\Sigma(\nu)$ taking values in the set of positive definite matrix one has $\partial_y[\partial_\mu(\Sigma^{-1}(\mu))_{i,j}](y)=-(\Sigma^{-1}(\mu)\partial_y[\partial_\mu\Sigma(\mu)](y)\Sigma^{-1}(\mu))_{i,j}=-\sum_{k_1,k_2}(\Sigma^{-1}(\mu))_{i,k_1}\partial_y[\partial_\mu(\Sigma(\mu))_{k_1,k_2}](y)(\Sigma^{-1}(\mu))_{k_2,j}$, we get

$$
\begin{aligned}
&\left|\partial_y^n\partial_\mu\left[H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv,z-x\right)\right](y)\right|\\
&\quad\leq K\left(\frac{|z-x|^2}{(t-r)^3}+\frac{1}{(t-r)^2}\right)\max_{i,j}\left|\int_r^t\partial_y^n\partial_\mu[a_{i,j}(v,z,[X_v^{s,\xi,(m)}])](y)dv\right|\\
&\quad\leq K\left(\frac{|z-x|^2}{(t-r)^2}+\frac{1}{t-r}\right)\frac{(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)}{(r-s)^{\frac{1+n-\eta}{2}}},
\end{aligned}
$$

so that, from the space-time inequality (1.4), we clearly deduce

$$
\begin{aligned}
|\mathrm{II}_2^n(y)| &\leq K\left(\frac{|z-x|^{2+\eta}}{(t-r)^2}+\frac{|z-x|^\eta}{(t-r)}\right)\frac{(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)}{(r-s)^{\frac{1+n-\eta}{2}}}\,g(c(t-r),z-x)\\
&\leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n-\eta}{2}}}(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)\,g(c(t-r),z-x)\\
&\leq K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right)(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)\\
&\quad\times g(c(t-r),z-x).
\end{aligned}
$$

Hence, gathering estimates on II_1^n and II_2^n, we get for all $y\in\mathbb{R}^d$

$$
\begin{aligned}
|\mathrm{II}^n(\mathrm{y})| &\leq K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right)(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)\\
&\quad\times g(c(t-r),z-x).
\end{aligned}
$$

Finally, for $r\neq s$, from the relation (5.37) and the estimate (5.29), we get

$$
\left|\partial_y^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,x,z)](y)\right|\leq\frac{K}{(r-s)^{\frac{1+n-\eta}{2}}}(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)\,g(c(t-r),z-x),
$$

so that

$$
|\mathrm{III}^n(y)|\leq K\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n-\eta}{2}}}(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)g(c(t-r),z-x).
$$

Gathering the previous estimates together, we finally obtain

$$
\begin{aligned}
|\partial_y^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s,r,t,x,z)](y)| &\leq K\left(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\right)\\
&\quad\times\left[1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv\right]\\
&\quad\times g(c(t-r),z-x).
\end{aligned}
\tag{5.47}
$$

Now, our aim is to establish an upper-bound of the quantity $p_{m+1}\otimes\partial_y^n[\partial_\mu\mathcal{H}_{m+1}](\mu,r,t,x,z)$. The estimate (5.47) allows to balance the singularity in time induced by $\partial_y^n[\partial_\mu\mathcal{H}_{m+1}]$. Indeed, assuming first

that $r \in [s,(t+s)/2]$, one has $t-r \geq (t-s)/2$ which directly implies

$$\begin{aligned}\int |p_{m+1}(\mu,s,r,x,y')||\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,y',z)](y)|dy' \\ \leq \frac{K}{(t-s)(r-s)^{\frac{1+n-\eta}{2}}}\left(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}}+(t-s)^{-1}\int_s^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv\right) \\ \times g(c(t-s),z-x),\end{aligned}$$

so that

$$\begin{aligned}&\int_s^{\frac{t+s}{2}}\int |p_{m+1}(\mu,s,r,x,y')||\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,y',z)](y)|dy'\,dr \\ &\quad\leq K\left\{\frac{B(1,\frac{1-n+\eta}{2})}{(t-s)^{\frac{1+n-\eta}{2}}}+\frac{1}{t-s}\int_s^t \frac{C_{m,n}(s,r)}{(r-s)^{\frac{1+n}{2}-\eta}}\,dr\right\} g(c(t-s),z-x) \\ &\quad\leq K\left\{\frac{B(\frac{\eta}{2},\frac{1-n+\eta}{2})}{(t-s)^{\frac{1+n-\eta}{2}}}+\frac{1}{(t-s)^{\frac{\eta}{2}}}\int_s^t \frac{C_{m,n}(s,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n}{2}-\eta}}\,dr\right\} \\ &\qquad\times g(c(t-s),z-x) \\ &\quad\leq \frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\ &\qquad\times g(c(t-s),z-x).\end{aligned}$$

Then, assuming that $r \in [(t+s)/2,t]$, one has $r-s \geq (t-s)/2$ so that

$$\begin{aligned}\int |p_{m+1}(\mu,s,r,x,y')||\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,y',z)](y)|dy' \\ \leq \frac{K}{(t-s)^{\frac{1+n}{2}}(t-r)^{1-\frac{\eta}{2}}}(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv) \\ \times g(c(t-s),z-x),\end{aligned}$$

which in turn, by Fubini's theorem, directly yields

$$\begin{aligned}&\int_{\frac{t+s}{2}}^t\int |p_{m+1}(\mu,s,r,x,y')||\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,y',z)](y)|\,dy'\,dr \\ &\quad\leq \frac{K}{(t-s)^{\frac{1+n}{2}}}\int_{\frac{t+s}{2}}^t \frac{1}{(t-r)^{1-\frac{\eta}{2}}}(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv)\,dr \\ &\qquad\times g(c(t-s),z-x) \\ &\quad\leq K\left\{\frac{B(1,\frac{\eta}{2})}{(t-s)^{\frac{1+n-\eta}{2}}}+\frac{1}{(t-s)^{\frac{\eta}{2}}}\int_s^t \frac{C_{m,n}(s,v)}{(t-v)^{1-\frac{\eta}{2}}(v-s)^{\frac{1+n}{2}-\eta}}\,dv\right\} g(c(t-s),z-x) \\ &\quad\leq \frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\ &\qquad\times g(c(t-s),z-x).\end{aligned}$$

Gathering the two previous cases, we clearly obtain

$$\begin{aligned}&|p_{m+1}\otimes\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,t,x,z)](y)| \\ &\leq \frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left(B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right) \\ &\quad\times g(c(t-s),z-x),\end{aligned} \tag{5.48}$$

so that

$$\sum_{r\geq 0}|(p_{m+1}\otimes\partial_y^n[\partial_\mu\mathcal{H}_{m+1}])\otimes\mathcal{H}_{m+1}^{(r)}(\mu,s,t,x,z)(y)|$$
$$\leq\frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^{m}C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}$$
$$\times g(c(t-s),z-x). \tag{5.49}$$

The estimates (5.42) and (5.49) together with the representation formula (5.40) imply that there exist two positive constants K, c such that

$$|\partial_y^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](y)|$$
$$\leq\frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^{m}C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \tag{5.50}$$
$$\times g(c(t-s),z-x),$$

so that

$$u_{m+1}^n(s,t)\leq\frac{K}{(t-s)^{\frac{1+n}{2}-\eta}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^{m}C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}$$

and similarly,

$$v_{m+1}^n(s,t)\leq\frac{K}{(t-s)^{\frac{1+n-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}\right)+\sum_{k=1}^{m}C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}.$$

Since the constant K does not depend either on the constant C appearing in the definition of $C_{m,n}(s,t)$ or m, one may change C once for all and derive the induction hypothesis at step $m+1$ for u_{m+1}^n and v_{m+1}^n. From (5.50) and the asymptotics of the Beta function, we also conclude that (5.12) holds at step $m+1$.

The estimate (5.13) at step $m+1$ follows by combining the relations (5.34), (5.31), (5.36) with estimates analogous to the one established above while the estimate (5.14) follows from the representation formula (5.40). The remaining technical details are omitted.

In order to derive (5.15) at step $m+1$, we proceed similarly. For a map $h\in\mathcal{C}^2(\mathcal{P}_2(\mathbb{R}^d))$ and for any fixed $y,y'\in\mathbb{R}^d$, we will use the notation $\Delta_{y,y'}\partial_y[\partial_\mu h(\mu)](y)=\partial_y[\partial_\mu h(\mu)](y)-\partial_y[\partial_\mu h(\mu)](y')$. We introduce the quantities

$$u_m(s,t):=\sup_{(y,y')\in(\mathbb{R}^d)^2,\,y\neq y'}\int\mu(dx')\int(|y''-x'|^\eta\wedge 1)\frac{|\Delta_{y,y'}\partial_y[\partial_\mu p_m(\mu,s,t,x',y'')](y)|}{|y-y'|^\beta}\,dy'',$$
$$v_m(s,t):=\sup_{(y,y')\in(\mathbb{R}^d)^2,\,y\neq y'}\int\mu(dx')\int\frac{|\Delta_{y,y'}\partial_y[\partial_\mu p_m(\mu,s,t,x',y'')](y)|}{|y-y'|^\beta}\,dy'',$$

for any $\beta\in[0,\eta)$ and $m\geq 1$. We prove by induction the following key inequalities:

$$u_m(s,t)\leq C_m(s,t)(t-s)^{-(1+\frac{\beta}{2}-\eta)}\quad\text{ and }\quad v_m(s,t)\leq C_m(s,t)(t-s)^{-(1+\frac{\beta-\eta}{2})},$$

with $C_m(s,t):=\sum_{k=1}^m C^k\prod_{i=1}^k B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{(k-1)\frac{\eta}{2}}$, $C:=C(T,a,b,[\delta a/\delta m],[\delta b/\delta m],\eta)$ being a positive constant independent of m. The result being straightforward for $m=1$, we assume that it holds at step m. By direct computations, we obtain the following decomposition

$$\Delta_{y,y'}\partial_y[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,x,z)](y)=Df_{z-x}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv\right)$$
$$\cdot\int_r^t\left\{\int\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'')[\partial_x^2 p_m(\mu,s,v,y,y'')-\partial_x^2 p_m(\mu,s,v,y',y'')]dy''\right. \tag{5.51}$$
$$\left.+\int\int[\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'')-\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(x')]\Delta_{y,y'}\partial_y[\partial_\mu p_m(\mu,s,v,x',y'')](y)\Big)\,dy''\,\mu(dx')\right\}dv.$$

Our first aim is to bound the integrand in time appearing in the first integral of the right-hand side of (5.51), namely $\int[\delta a/\delta m](v,z,[X_v^{s,\xi,(m)}])(y'')[\partial_x^2 p_m(\mu,s,v,y,y'')-\partial_x^2 p_m(\mu,s,v,y',y'')]dy''$. We split the

computations into the two following cases: $|y-y'|^2 \le v-s$ and $|y-y'|^2 \ge v-s$. In the first case, we remark that

$$\begin{aligned}&\int \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'')[\partial_x^2 p_m(\mu,s,v,y,y'') - \partial_x^2 p_m(\mu,s,v,y',y'')]dy''\\ &\quad = \int [\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y)][\partial_x^2 p_m(\mu,s,v,y,y'') - \partial_x^2 p_m(\mu,s,v,y',y'')]dy''\end{aligned}$$

so that, from (5.7) with $n=2$, $\beta \in [0,\eta)$, the η-Hölder regularity of the map $y \mapsto [\delta a/\delta m](v,z,[X_v^{s,\xi,(m)}])(y)$ and the space-time inequality (1.4), we get

$$\begin{aligned}&\Big| \int [\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_r^{s,\xi,(m)}])(y)][\partial_x^2 p_m(\mu,s,v,y,y'') - \partial_x^2 p_m(\mu,s,v,y',y'')]dy''\Big|\\ &\quad \le K|y-y'|^\beta \int \frac{|y''-y|^\eta}{(v-s)^{1+\frac{\beta}{2}}} \left\{g(c(v-s),y''-y) + g(c(v-s),y''-y')\right\} dy''\\ &\quad \le K|y-y'|^\beta \int \frac{1}{(v-s)^{1+\frac{\beta}{2}}} \Big\{|y''-y|^\eta g(c(v-s),y''-y) + (|y''-y'|^\eta + |v-s|^{\frac{\eta}{2}}) g(c(v-s),y''-y')\Big\} dy''\end{aligned} \tag{5.52}$$

$$\le K\frac{|y-y'|^\beta}{(v-s)^{1+\frac{\beta-\eta}{2}}}.$$

Otherwise, if $|y-y'|^2 \ge v-s$ using the decomposition

$$\begin{aligned}&\int \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'')[\partial_x^2 p_m(\mu,s,v,y,y'') - \partial_x^2 p_m(\mu,s,v,y',y'')]dy''\\ &\quad = \int [\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y)]\partial_x^2 p_m(\mu,s,v,y,y'')dy''\\ &\qquad - \int [\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y')]\partial_x^2 p_m(\mu,s,v,y',y'')]dy''\end{aligned}$$

the η-Hölder regularity of $y' \mapsto [\delta a/\delta m](t,z,\mu)(y')$, (5.6) for $n=2$ and the space-time inequality (1.4), we also obtain (5.52). Plugging the previous bound in (5.51) yields

$$\begin{aligned}|\Delta_{y,y'}\partial_y[\partial_\mu \widehat{p}_{m+1}(\mu,s,r,t,x,z)](y)| &\le K\frac{|y-y'|^\beta}{t-r}\Big\{\int_r^t \Big[\frac{1}{(v-s)^{1+\frac{(\beta-\eta)}{2}}}\\ &+ \int\int (|y''-x'|^\eta \wedge 1)\frac{|\Delta_{y,y'}\partial_v[\partial_\mu p_m(\mu,s,v,x',y'')](y)|}{|y-y'|^\beta}\, dy''\,\mu(dx')\Big]dv\Big\}\\ &\times g(c(t-r),z-x),\end{aligned} \tag{5.53}$$

for any $\beta \in [0,\eta)$ so that, from the previous inequality with $r=s$ and the induction hypothesis,

$$\begin{aligned}&|\Delta_{y,y'}\partial_y[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](y)|\\ &\quad \le K|y-y'|^\beta \left\{\frac{1}{(t-s)^{1+\frac{(\beta-\eta)}{2}}} + \frac{1}{t-s}\int_s^t \frac{C_m(s,r)}{(r-s)^{1+\frac{\beta}{2}-\eta}}\,dr\right\} g(c(t-s),z-x)\\ &\quad \le K|y-y'|^\beta \left\{\frac{1}{(t-s)^{1+\frac{(\beta-\eta)}{2}}} + \frac{1}{(t-s)^{\frac{\eta}{2}}}\int_s^t \frac{C_m(s,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\beta}{2}-\eta}}\,dr\right\} g(c(t-s),z-x)\\ &\quad = K\frac{|y-y'|^\beta}{(t-s)^{1-\frac{(\eta-\beta)}{2}}}\left\{1+\sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} g(c(t-s),z-x).\end{aligned} \tag{5.54}$$

Introducing the notation $\Delta_y\partial_y[\partial_\mu \widehat{p}_m(\mu,s,t,x,z)](y) := \partial_y[\partial_\mu \widehat{p}_m(\mu,s,t,x,z)](y) - \partial_y[\partial_\mu \hat{p}_m(\mu,s,t,x,z)](y')$, by induction on r, there exists a positive constant $K := K(T,a,b)$ (which may change from line to line but is independent of m and C) such that

$$\begin{aligned}&|(\Delta_{y,y'}\partial_y\partial_\mu \widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(r)})(\mu,s,t,x,z)(y)|\\ &\quad \le K^r|y-y'|^\beta \left\{1+\sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}(t-s)^{-1+\frac{(\eta-\beta)}{2}+r\frac{\eta}{2}}\\ &\qquad \times \prod_{i=1}^r B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\right) g(c(t-s),z-x),\end{aligned}$$

which in turn implies

$$\begin{aligned}\sum_{r\geq 0}&|(\Delta_{y,y'}\partial_y\partial_\mu\widehat{p}_{m+1}\otimes\mathcal{H}^{(r)}_{m+1})(\mu,s,t,x,z)(y)|\\&\leq K\frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}}\left\{1+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}\\&\quad\times g(c(t-s),z-x)\\&\leq K\frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}}\left\{B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}\\&\quad\times g(c(t-s),z-x).\end{aligned}\tag{5.55}$$

Then, from (5.38), we write

$$\begin{aligned}\Delta_{y,y'}\partial_y[\partial_\mu\mathcal{H}_m(\mu,s,r,t,x,z)](y)&=\mathrm{I}^1(y)-\mathrm{I}^1(y')+\mathrm{II}^1(y)-\mathrm{II}^1(y')+\mathrm{III}^1(y)-\mathrm{III}^1(y')\\&=\mathrm{I}^1_1(y)-\mathrm{I}^1_1(y')+\mathrm{I}^1_2(y)-\mathrm{I}^1_2(y')+\mathrm{II}^1_1(y)-\mathrm{II}^1_1(y')+\mathrm{II}^1_2(y)-\mathrm{II}^1_2(y')\\&\quad+\mathrm{III}^1(y)-\mathrm{III}^1(y')\end{aligned}$$

and prove appropriate estimates for each terms. With our notations, one has

$$\mathrm{I}^1_1(y)-\mathrm{I}^1_1(y')=\sum_{i=1}^d H^i_1\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv,z-x\right)\cdot\Delta_{y,y'}\partial_y\left[\partial_\mu b_i(r,x,[X^{s,\xi,(m)}_r])\right](y)\widehat{p}_{m+1}(\mu,r,t,x,z).$$

Moreover, from (5.27), one gets

$$\begin{aligned}\Delta_{y,y'}\partial_y\left[\partial_\mu b_i(r,x,[X^{s,\xi,(m)}_r])\right](y)&=\int(\frac{\delta b_i}{\delta m}(r,x,[X^{s,\xi,(m)}_r])(z).(\partial^2_x p_m(\mu,s,r,y,z)-\partial^2_x p_m(\mu,s,r,y',z))dz\\&\quad+\int\int(\frac{\delta b_i}{\delta m}(r,x,[X^{s,\xi,(m)}_r])(z)-\frac{\delta b_i}{\delta m}(r,x,[X^{s,\xi,(m)}_r])(x'))\\&\quad.\Delta_{y,y'}\partial_v[\partial_\mu p_m(\mu,s,t,x',z)](y)\,dz\,\mu(dx').\end{aligned}\tag{5.56}$$

Then, using (5.7), the uniform η-Hölder regularity of $z\mapsto[\delta b_i/\delta m](r,x,\mu)(z)$, as well as similar arguments as those employed in order to establish (5.52) for the first part appearing in the right-hand side of the previous equality, we get

$$\forall\beta\in[0,\eta),\quad|\Delta_{y,y'}\partial_y\left[\partial_\mu b_i(r,x,[X^{s,\xi,(m)}_r])\right](y)|\leq K\frac{|y-y'|^\beta}{(r-s)^{1+\frac{(\beta-\eta)}{2}}}\left[1+(r-s)^{1+\frac{(\beta-\eta)}{2}}u_m(s,r)\right],\tag{5.57}$$

which in turn by the induction hypothesis implies

$$\begin{aligned}|\mathrm{I}^1_1(y)-\mathrm{I}^1_1(y')|&\leq K\frac{|y-y'|^\beta}{(t-r)^{\frac12}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\left[1+(r-s)^{1+\frac{(\beta-\eta)}{2}}u_m(s,r)\right]g(c(t-r),z-x)\\&\leq K\frac{|y-y'|^\beta}{(t-r)^{\frac12}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\left[1+C_m(s,r)(r-s)^{\frac{\eta}{2}}\right]g(c(t-r),z-x).\end{aligned}$$

Next, we proceed similarly

$$\begin{aligned}|\mathrm{I}^1_2(y)-\mathrm{I}^1_2(y')|&:=\left|\sum_{i=1}^d b_i(r,x,[X^{s,\xi,(m)}_r])\Delta_{y,y'}\partial_y\left[\partial_\mu H^i_1\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv,z-x\right)\right](y)\right|\widehat{p}_m(\mu,r,t,x,z)\\&\leq\frac{K}{(t-r)^{\frac32}}\int_r^t\max_{i,j}\left|\Delta_{y,y'}\partial_y[\partial_\mu a_{i,j}(v,z,[X^{s,\xi,(m)}_v])](y)\right|dv\\&\quad\times g(c(t-r),z-x).\end{aligned}$$

From (5.26) and similar arguments as those employed to establish (5.57), that is, the decomposition (5.56) with the maps $a_{i,j}$ and $[\delta a_{i,j}/\delta m]$ instead of b_i and $[\delta b_i/\delta m]$, in a completely analogous manner, one gets

$$\left|\Delta_{y,y'}\partial_y\left[\partial_\mu a_{i,j}(v,z,[X^{s,\xi,(m)}_v])\right](y)\right|\leq K\frac{|y-y'|^\beta}{(v-s)^{1+\frac{(\beta-\eta)}{2}}}(1+C_m(s,v)(v-s)^{\frac{\eta}{2}}),\tag{5.58}$$

for any $\beta \in [0,\eta)$, so that

$$|\mathrm{I}_2^1(y)-\mathrm{I}_2^1(y')| \le K\frac{|y-y'|^\beta}{(t-r)^{\frac12}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\Big(1+(t-r)^{-1}\int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}}\,dv\Big)\,g(c(t-r),z-x).$$

Gathering the two previous estimates, we conclude

$$|\mathrm{I}^1(y)-\mathrm{I}^1(y')| \le K\frac{|y-y'|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\Big(1+C_m(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}}\Big)\,g(c(t-r),z-x).$$

Still using our notations, we have

$$\mathrm{II}_1^1(y)-\mathrm{II}_1^1(y') \;=\; \sum_{i,j=1}^d\Big\{H_2^{i,j}\Big(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv,z-x\Big)\times(\mathrm{J}_{i,j}(y)-\mathrm{J}_{i,j}(y'))\Big\}\,\widehat{p}_{m+1}(\mu,r,t,x,z).$$

On the one hand, using (5.44) (with $n=1$), the induction hypothesis and the fact that $x\mapsto[\delta a_{i,j}/\delta m](r,x,\mu)(y)$ is η-Hölder uniformly with respect to the other variables, we obtain

$$|\mathrm{J}_{i,j}(y)-\mathrm{J}_{i,j}(y')| \;\le\; K\frac{|z-x|^\eta|y-y'|^\beta}{(r-s)^{1+\frac{\beta}{2}}}(1+(r-s)^{1+\frac{\beta}{2}}v_m(s,r)).$$

On the other hand, the following decomposition holds

$$\begin{aligned}\mathrm{J}_{i,j}(y)-\mathrm{J}_{i,j}(y') &= \int\int\Big\{\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(x')\big)\\ &\quad-\Big(\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(x')\Big)\Big\}\\ &\quad.\Big[\partial_y[\partial_\mu p_m(\mu,s,r,x',y'')](y)-\partial_y[\partial_\mu p_m(\mu,s,r,x',y'')](y')\Big]\,dy''\,\mu(dx')\\ &\quad+\int\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')\big)\\ &\quad.[\partial_x^2 p_m(\mu,s,r,y,y'')-\partial_x^2 p_m(\mu,s,r,y',y'')]\,dy''\end{aligned}$$

so that using the fact that $y\mapsto[\delta a_{i,j}/\delta m](t,z,\mu)(y)$ is η-Hölder, we get that the first term appearing in the right-hand side of the above decomposition is bounded by $K|y-y'|^\beta u_m(s,r)$. For the second term, one has to consider the two disjoint cases: $|y-y'|\le(r-s)^{1/2}$ and $|y-y'|>(r-s)^{1/2}$. In the first case, we write

$$\begin{aligned}&\int\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')\big).[\partial_x^2 p_m(\mu,s,r,y,y'')-\partial_x^2 p_m(\mu,s,r,y',y'')]\,dy''\\ &=\int\Big[\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y)\\ &\quad-\big(\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y)\big)\Big].[\partial_x^2 p_m(\mu,s,r,y,y'')-\partial_x^2 p_m(\mu,s,r,y',y'')]\,dy''\end{aligned}$$

and uses the estimate (5.7) (with $n=2$), the η-Hölder regularity of $y''\mapsto[\delta a_{i,j}/\delta m](r,x,\mu)(y'')$ as well as the inequality $|y''-y|^\eta\le|y''-y'|^\eta+(r-s)^{\eta/2}$. In the second case, that is, when $|y-y'|\ge(r-s)^{1/2}$, we write

$$\begin{aligned}&\int\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')\big).[\partial_x^2 p_m(\mu,s,r,y,y'')-\partial_x^2 p_m(\mu,s,r,y',y'')]\,dy''\\ &=\int\big(\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y)\big).\partial_x^2 p_m(\mu,s,r,y,y'')dy''\\ &\quad-\int\big(\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y')\big).\partial_x^2 p_m(\mu,s,r,y',y'')]\,dy''\end{aligned}$$

and bound each term using directly (5.6) together with the η-Hölder regularity of $y''\mapsto[\delta a_{i,j}/\delta m](r,x,\mu)(y'')$ and the space-time inequality (1.4).

In both cases, one concludes that the second term is bounded by $K|y-y'|^\beta(r-s)^{-1-\frac{\beta-\eta}{2}}$. We thus finally obtain

$$|\mathrm{J}_{i,j}(y)-\mathrm{J}_{i,j}(y')| \le K\frac{|y-y'|^\beta}{(r-s)^{1+\frac{(\beta-\eta)}{2}}}(1+(r-s)^{1+\frac{(\beta-\eta)}{2}}u_m(s,r)).$$

Combining the two previous estimates yields

$$\forall (y,y')\in(\mathbb{R}^d)^2, |\mathrm{II}_1^1(y)-\mathrm{II}_1^1(y')| \le K\left\{\frac{|y-y'|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\beta}{2}}}\wedge\frac{|y-y'|^\beta}{(t-r)(r-s)^{1+\frac{(\beta-\eta)}{2}}}\right\}\\ \times\Big[1+C_m(s,r)(r-s)^{\frac{\eta}{2}}\Big]\,g(c(t-r),z-x)$$

for any $\beta\in[0,\eta)$.

Using (5.58), for all $\beta\in[0,\eta)$ and for all $(y,y')\in(\mathbb{R}^d)^2$, one gets

$$|\mathrm{II}_2^1(y)-\mathrm{II}_2^1(y')| \le K\left\{\frac{|z-x|^{2+\eta}}{(t-r)^3}+\frac{|z-x|^\eta}{(t-r)^2}\right\}\int_r^t \max_{i,j}\Big|\Delta_{y,y'}\partial_y[\partial_\mu a_{i,j}(v,z,[X_v^{s,\xi,(m)}])](y)\Big|\,dv\\ \times g(c(t-r),z-x)\\ \le K\frac{|y-y'|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\Big[1+(t-r)^{-1}\int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}}\,dv\Big]\,g(c(t-r),z-x).$$

Finally, for the last term, from (5.53) and the induction hypothesis, we obtain

$$\forall\beta\in[0,\eta),\ \forall(y,y')\in(\mathbb{R}^d)^2,\quad |\mathrm{III}^1(y)-\mathrm{III}^1(y')|\\ \le K\frac{1}{(t-r)^{1-\frac{\eta}{2}}}|\Delta_{y,y'}\partial_y[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,x,z)](y)|\\ \le K\frac{|y-y'|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\Big[1+\int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}}\,dv\Big]\,g(c(t-r),z-x).$$

Gathering all the previous computations, we finally conclude

$$\Big|\Delta_{y,y'}\partial_y\Big[\partial_\mu\mathcal{H}_{m+1}(\mu,s,r,t,x,z)\Big](y)\Big| \le K\left\{\frac{|y-y'|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\beta}{2}}}\wedge\frac{|y-y'|^\beta}{(t-r)(r-s)^{1-\frac{(\eta-\beta)}{2}}}\right\}\\ \times\Big[1+C_m(s,r)(r-s)^{\frac{\eta}{2}}+(t-r)^{-1}\int_r^t C_m(s,v)(v-s)^{\frac{\eta}{2}}\,dv\Big]\\ \times g(c(t-r),z-x),$$

for any $(y,y')\in(\mathbb{R}^d)^2$ and for any $\beta\in[0,\eta)$. We again separate the space-time convolution into the two disjoint cases: $r\in[s,(t+s)/2]$ and $r\in[(t+s)/2,t]$. Skipping technical details, we obtain

$$|p_{m+1}\otimes\Delta_{y,y'}\partial_y[\partial_\mu\mathcal{H}_{m+1}(\mu,s,t,x,z)](y)|\\ \le K|y-y'|^\beta\left(\frac{1}{(t-s)^{1+\frac{(\beta-\eta)}{2}}}B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}\Big)+\frac{1}{(t-s)^{\frac{\eta}{2}}}\int_s^t\frac{C_m(s,r)(r-s)^{\frac{\eta}{2}}}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{(\beta-\eta)}{2}}}\,dr\right)\\ \times g(c(t-s),z-x)\\ \le K\frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}}\left(B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}\Big)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\Big)(t-s)^{k\frac{\eta}{2}}\right)\\ \times g(c(t-s),z-x),$$

so that

$$\sum_{r\ge0}|(p_{m+1}\otimes\Delta_{y,y'}\partial_y\partial_\mu\mathcal{H}_{m+1})\otimes\mathcal{H}_{m+1}^{(r)}(\mu,s,t,x,z)(y)|\\ \le K\frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}}\Big(B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}\Big)\\ +\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\Big)(t-s)^{k\frac{\eta}{2}}\Big)g(c(t-s),z-x). \tag{5.59}$$

Now, combining (5.55) and (5.59) with the following representation

$$\Delta_{y,y'}\partial_y[\partial_\mu p_{m+1}(\mu,s,t,x,z)](y)=\sum_{r\ge0}[\Delta_{y,y'}\partial_y\partial_\mu\widehat{p}_{m+1}+p_{m+1}\otimes\Delta_{y,y'}\partial_y\partial_\mu\mathcal{H}_{m+1}]\otimes\mathcal{H}_{m+1}^{(r)}(\mu,s,t,x,z)(y),$$

we deduce that there exist two constants K, c (independent of C and m) such that

$$\begin{aligned}|\Delta_{y,y'}\partial_y[\partial_\mu p_{m+1}(\mu,s,t,x,z)](y)| \le K\frac{|y-y'|^\beta}{(t-s)^{1+\frac{(\beta-\eta)}{2}}}\\ \times\Big\{B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}\Big)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\Big)(t-s)^{k\frac{\eta}{2}}\Big\}\\ \times g(c(t-s),z-x),\end{aligned}$$

so that

$$u_{m+1}(s,t)\le\frac{K}{(t-s)^{1+\frac{\beta}{2}-\eta}}\left\{B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}\Big)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\Big)(t-s)^{k\frac{\eta}{2}}\right\}$$

and similarly,

$$v_{m+1}(s,t)\le\frac{K}{(t-s)^{1+\frac{\beta-\eta}{2}}}\left\{B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}\Big)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\Big(\frac{\eta}{2},\frac{\eta-\beta}{2}+(i-1)\frac{\eta}{2}\Big)(t-s)^{k\frac{\eta}{2}}\right\}.$$

Since the constant K does not depend either on the constant C appearing in the definition of $C_m(s,t)$ or m, one may change C once for all and derive the induction hypothesis at step m for $u_{m+1}(s,t)$ and $v_{m+1}(s,t)$. This completes the proof of (5.15) at step $m+1$. We thus conclude that for all $m\ge1$, $(s,x,\mu)\mapsto p_m(\mu,s,t,x,z)\in\mathcal{C}^{1,2,2}([0,t)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ and that the estimates (5.12) to (5.15) are valid.

We now establish the estimates (5.16) and (5.17) at step $m+1$. For a map h defined on $\mathcal{P}_2(\mathbb{R}^d)$ and two fixed measures $\mu,\mu'\in\mathcal{P}_2(\mathbb{R}^d)$ we will sometimes write $\Delta_{\mu,\mu'}h(\mu)=h(\mu)-h(\mu')$. We first prove that there exist some positive constants $K:=K(T,a,b,[\delta a/\delta m],[\delta b/\delta m])$, $c:=c(\lambda)$ such that for all $m\ge1$, for all $\mu,\mu'\in\mathcal{P}_2(\mathbb{R}^d)$

$$\forall\alpha\in[0,1],\ |\Delta_{\mu,\mu'}p_m(\mu,s,t,x,z)|\le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{\alpha-\eta}{2}}}\,g(c(t-s),z-x).\tag{5.60}$$

We first remark that if $W_2(\mu,\mu')\le(t-s)^{1/2}$, then for any $\xi,\xi'\in\mathbb{L}^2$ such that $[\xi]=\mu$ and $[\xi']=\mu'$, one has

$$\Delta_{\mu,\mu'}p_m(\mu,s,t,x,z)=\int_0^1\partial_\lambda p_m(\mu_\lambda,s,r,x,z)\,d\lambda=\int_0^1\mathbb{E}\Big[\partial_\mu p_m(\mu_\lambda,s,r,x,z)(\xi_\lambda)(\xi-\xi')\Big]d\lambda,$$

where we introduced the notations $\xi_\lambda:=\xi+\lambda(\xi-\xi')$, $\mu_\lambda:=\mu'+\lambda(\mu-\mu')$, so that, using (5.12) (note that the constant C does not depend on m) and optimising over joint distributions with μ as first marginal and μ' as second marginal

$$|\Delta_{\mu,\mu'}p_m(\mu,s,r,x,z)|\le K\frac{W_2(\mu,\mu')}{(t-s)^{\frac{1-\eta}{2}}}g(c(t-s),z-x)\le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{\alpha-\eta}{2}}}g(c(t-s),z-x).$$

Otherwise, if $W_2(\mu,\mu')>(t-s)^{1/2}$, we directly get

$$\forall m\ge1,\ |\Delta_{\mu,\mu'}p_m(\mu,s,r,x,z)|\le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{\alpha}{2}}}g(c(t-s),z-x)$$

so that, from Proposition 2.2 applied to the map $\mu\mapsto a_{i,j}(r,z,\mu)$, for any $m\ge1$ and any $\alpha\in[0,1]$

$$\begin{aligned}&|a_{i,j}(r,z,[X_r^{s,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s,\xi',(m)}])|\\ &\quad\le\Big|\int_0^1\int_{(\mathbb{R}^d)^2}\frac{\delta a_{i,j}}{\delta m}(r,z,\Theta_{\lambda,r}^{(m)})(y')p_m(\mu,s,r,x',y')\,dy'\,(\mu-\mu')(dx')\,d\lambda\Big|\\ &\quad+\Big|\int_0^1\int_{(\mathbb{R}^d)^2}\Big(\frac{\delta a_{i,j}}{\delta m}(r,z,\Theta_{\lambda,r}^{(m)})(y')-\frac{\delta a_{i,j}}{\delta m}(r,z,\Theta_{\lambda,r}^{(m)})(x')\Big)\\ &\qquad\times\Delta_{\mu,\mu'}p_m(\mu,s,r,x',y')\,dy'\mu'(dx')\,d\lambda\Big|\\ &\quad\le K\left\{\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}+\int(|y'-x'|^\eta\wedge1)|\Delta_{\mu,\mu'}p_m(\mu,s,r,x',y')|\,dy'\mu'(dx')\right\}\\ &\quad\le K\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha-\eta}{2}}},\end{aligned}\tag{5.61}$$

where we used the notation $\Theta^{(m)}_{\lambda,r} := (1-\lambda)[X^{s,\xi,(m)}_r] + \lambda[X^{s,\xi',(m)}_r]$ and the fact that the map $x' \mapsto \int_0^1 \int_{\mathbb{R}^d}[\delta a_{i,j}/\delta m](r,z,\Theta^{(m)}_{\lambda,r})(y')p_m(\mu,s,r,x',y')\,dy'\,d\lambda$ is α-Hölder with modulus bounded by $K(r-s)^{(\eta-\alpha)/2}$. The previous bound and the mean-value theorem thus yield

$$\begin{aligned}
&|\Delta_{\mu,\mu'}\widehat{p}_m(\mu,s,t,x,z)| \\
&\leq \frac{K}{t-s}\int_s^t \max_{i,j}|a_{i,j}(r,z,[X^{s,\xi,(m-1)}_r]) - a_{i,j}(r,z,[X^{s,\xi',(m-1)}_r])|\,dr\, g(c(t-s),z-x) \\
&\leq K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-s),z-x).
\end{aligned}$$

More generally, differentiating n-times the map $x \mapsto \widehat{p}_m(\mu,s,t,x,z)$, from the mean-value theorem and (5.61) and similar arguments, we also obtain

$$\forall \alpha \in [0,1], \quad |\Delta_{\mu,\mu'}\partial_x^n \widehat{p}_m(\mu,s,t,x,z)| \leq K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{n+\alpha-\eta}{2}}}\, g(c(t-s),z-x), \tag{5.62}$$

for $n = 0,1,2$. Now, using the decomposition $p_m(\mu,s,t,x,z) = \widehat{p}_m(\mu,s,t,x,z) + R_m(\mu,s,t,x,z)$ with $R_m(\mu,s,t,x,z) := \sum_{k\geq 1}\widehat{p}_m \otimes \mathcal{H}^{(k)}_m(\mu,s,t,x,z)$ satisfying $|R_m(\mu,s,t,x,z)| \leq K(t-s)^{\eta/2}g(c(t-s),z-x)$, still in the case $W_2(\mu,\mu') > (t-s)^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
|\Delta_{\mu,\mu'}p_m(\mu,s,t,x,z)| &\leq |\Delta_{\mu,\mu'}\widehat{p}_m(\mu,s,t,x,z)| + K(t-s)^{\frac{\eta}{2}}g(c(t-s),z-x) \\
&\leq K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-s),z-x).
\end{aligned}$$

This last estimate concludes the proof of (5.60). We now make use of the following decomposition

$$\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,r,t,y,z) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

with

$$\begin{aligned}
\mathrm{I} &:= -\sum_{i=1}^d [b_i(r,y,[X^{s,\xi,(m)}_r]) - b_i(r,y,[X^{s,\xi',(m)}_r])]H_1^i\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv, z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z), \\
\mathrm{II} &:= -\sum_{i=1}^d b_i(r,y,[X^{s,\xi',(m)}_r])\Big[H_1^i\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv, z-y\right) - H_1^i\left(\int_r^t a(v,z,[X^{s,\xi',(m)}_v])dv, z-y\right)\Big] \\
&\quad \times \widehat{p}_{m+1}(\mu,s,r,t,y,z), \\
\mathrm{III} &:= \frac{1}{2}\sum_{i,j=1}^d \Big[(a_{i,j}(r,y,[X^{s,\xi,(m)}_r]) - a_{i,j}(r,z,[X^{s,\xi,(m)}_r])) \\
&\quad - (a_{i,j}(r,y,[X^{s,\xi',(m)}_r]) - a_{i,j}(r,z,[X^{s,\xi',(m)}_r]))\Big]H_2^{i,j}\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv, z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z), \\
\mathrm{IV} &:= \frac{1}{2}\sum_{i,j=1}^d (a_{i,j}(r,y,[X^{s,\xi',(m)}_r]) - a_{i,j}(r,z,[X^{s,\xi',(m)}_r])) \\
&\quad \times \Big[H_2^{i,j}\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv, z-y\right) - H_2^{i,j}\left(\int_r^t a(v,z,[X^{s,\xi',(m)}_v])dv, z-y\right)\Big]\widehat{p}_{m+1}(\mu,s,r,t,y,z), \\
\mathrm{V} &:= -\sum_{i=1}^d b_i(r,y,[X^{s,\xi',(m)}_r])H_1^i\left(\int_r^t a(v,z,[X^{s,\xi,(m)}_v])dv, z-y\right)\Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^d (a_{i,j}(r,y,[X^{s,\xi',(m)}_r]) - a_{i,j}(r,z,[X^{s,\xi',(m)}_r]))H_2^{i,j}\left(\int_r^t a(v,z,[X^{s,\xi',(m)}_v])dv, z-y\right) \\
&\quad \times \Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z).
\end{aligned}$$

From similar arguments as those previously used, for all $r \neq s$ and for all $\alpha \in [0,1]$, we get

$$
\begin{aligned}
|\mathrm{I}| &\leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r), z-y),\\
|\mathrm{II}| &\leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r), z-y),\\
|\mathrm{III}| &\leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\alpha}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu,\mu')\, g(c(t-r), z-x),\\
|\mathrm{IV}| &\leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r), z-y),\\
|\mathrm{V}| &\leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r), z-y).
\end{aligned}
$$

We only prove the estimates on I and III. The estimates on II, IV and V are obtained similarly and we the remaining technical details are omitted. From Proposition 2.2 applied to the map $\mu \mapsto b_i(r,z,\mu)$, still with our notations $\Theta^{(m)}_{\lambda,r} := (1-\lambda)[X_r^{s,\xi,(m)}] + \lambda[X_r^{s,\xi',(m)}]$, we get

$$
\begin{aligned}
|\mathrm{I}| &\leq \frac{K}{(t-r)^{\frac{1}{2}}} \sup_i \Big| \int_0^1 \int_{\mathbb{R}^d} \frac{\delta b_i}{\delta m}(r,y,\Theta^{(m)}_{\lambda,r})(z')(p_m(\mu,s,r,z') - p_m(\mu',s,r,z')dz'd\lambda \Big|\, g(c(t-r), z-y)\\
&\leq \frac{K}{(t-r)^{\frac{1}{2}}} \sup_i \Big| \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta b_i}{\delta m}(r,y,\Theta^{(m)}_{\lambda,r})(z') p_m(\mu,s,r,x',z')\, dz'\, (\mu-\mu')(dx')d\lambda \Big|\, g(c(t-r), z-y)\\
&+ \frac{K}{(t-r)^{\frac{1}{2}}} \sup_i \Big| \int_0^1 \int_{(\mathbb{R}^d)^2} \frac{\delta b_i}{\delta m}(r,y,\Theta^{(m)}_{\lambda,r})(z') \Delta_{\mu,\mu'} p_m(\mu,s,r,x',z')\, dz'\, \mu'(dx')d\lambda \Big|\\
&\quad \times g(c(t-r), z-y)\\
&\leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r), z-y),
\end{aligned}
$$

where we used the estimate (5.60) and also the fact that, since $z' \mapsto [\delta b_i/\delta m](r,y,\mu)(z')$ is η-Hölder uniformly with respect to the other variables, the map $x \mapsto \int_0^1 \int_{\mathbb{R}^d} [\delta b_i/\delta m](r,y,\Theta^{(m)}_{r,\lambda})(z') p_m(\mu,s,r,x,z')\, dz'\, d\lambda$ is α-Hölder, for any $\alpha \in [0,1]$ with a modulus bounded by $K(r-s)^{(\eta-\alpha)/2}$, $K := K(T,a,b,[\delta b/\delta m])$ being a positive constant independent of m. From the following identity

$$
\begin{aligned}
h(y) &:= a_{i,j}(r,y,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,y,[X_r^{s,\xi',(m)}])\\
&= -\int_0^1 \partial_\lambda a_{i,j}(r,y,\Theta^{(m)}_{\lambda,r})\, d\lambda\\
&= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta a_{i,j}}{\delta m}(r,y,\Theta^{(m)}_{r,\lambda})(y')(p_m(\mu,s,r,y') - p_m(\mu',s,r,y'))\, dy'\, d\lambda
\end{aligned}
$$

and the η-Hölder regularity of $y \mapsto [\delta a_{i,j}/\delta m](r,y,\mu)(z)$, (5.60) and (5.6), we deduce

$$
|h(y) - h(z)| \leq K \frac{|y-z|^\eta}{(r-s)^{\frac{1}{2}}} \mathbb{E}[|\xi - \xi'|]. \tag{5.63}
$$

Taking infimum over all joint distributions of the random variables ξ and ξ' with marginals μ and μ' respectively and plugging the corresponding bound in III, we get

$$
|\mathrm{III}| \leq K \frac{W_2(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1}{2}}}\, g(c(t-r), z-y) \leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\alpha}{2}}}\, g(c(t-r), z-y),
$$

when $W_2(\mu,\mu') \leq (r-s)^{1/2}$. If $W_2(\mu,\mu') > (r-s)^{1/2}$, we directly get

$$
|\mathrm{III}| \leq K \frac{1}{(t-r)^{1-\frac{\eta}{2}}}\, g(c(t-r), z-y) \leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\alpha}{2}}}\, g(c(t-r), z-y).
$$

Then, from (5.61), we get

$$
|\mathrm{III}| \leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r), z-y).
$$

This last bound completes the proof of the third estimate. Gathering all the previous estimates together, we thus obtain

(5.64)
$$\forall \alpha \in [0,1],\ |\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,r,t,y,z)| \leq K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{\alpha}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{\alpha-\eta}{2}}}\right\} W_2^{\alpha}(\mu,\mu')\, g(c(t-r),z-y).$$

For $r = s$, using a similar decomposition as the one employed above and omitting the remaining technical details, we also obtain

$$\forall \alpha \in [0,\eta),\quad |\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,t,y,z)| \leq K\frac{W_2^{\alpha}(\mu,\mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}}\, g(c(t-s),z-y). \tag{5.65}$$

With the previous computations, notably (5.62) and (5.64), by induction on k, it follows that for any $\alpha \in [0,1]$

$$\begin{aligned}|\Delta_{\mu,\mu'}\widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)| &\leq K^k W_2^{\alpha}(\mu,\mu')(t-s)^{\frac{(\eta-\alpha)}{2}+k\frac{\eta}{2}} \prod_{i=1}^{k} B\left(\frac{\eta}{2}, 1+\frac{\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)\\ &\quad \times g(c(t-s),z-x),\\ |(p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)| &\leq K^k W_2^{\alpha}(\mu,\mu')(t-s)^{\frac{(\eta-\alpha)}{2}+k\frac{\eta}{2}} \prod_{i=1}^{k} B\left(\frac{\eta}{2}, 1+\frac{\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)\\ &\quad \times g(c(t-s),z-x).\end{aligned}$$

From the representation in infinite series of p_{m+1}, the following relation holds

$$\Delta_{\mu,\mu'}p_{m+1}(\mu,s,t,x,z) = \Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,t,x,z)+p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,t,x,z)+\Delta_{\mu,\mu'}p_{m+1}\otimes\mathcal{H}_{m+1}(\mu,s,t,x,z),$$

which in turn by iteration yields

$$\Delta_{\mu,\mu'}p_{m+1}(\mu,s,t,x,z) = \sum_{k\geq 0}\{\Delta_{\mu,\mu'}\widehat{p}_{m+1} + p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}\}\otimes\mathcal{H}_m^{(k)}(\mu,s,t,x,z). \tag{5.66}$$

Moreover, one may differentiate the infinite series (5.66) with respect to x so that for $n = 0,1,2$

$$\partial_x^n\Delta_{\mu,\mu'}p_{m+1}(\mu,s,t,x,z) = \sum_{k\geq 0}\{\partial_x^n\Delta_{\mu,\mu'}\widehat{p}_{m+1} + \partial_x^n p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}\}\otimes\mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z). \tag{5.67}$$

In order to make the previous formula totally rigorous, one has to study the convergence of the above series. From the estimates (5.62) and (5.64) we get

$$\begin{aligned}&|\partial_x\Delta_{\mu,\mu'}\widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)|\\ &\qquad \leq K^k W_2^{\alpha}(\mu,\mu')(t-s)^{-\frac{1}{2}+\frac{(\eta-\alpha)}{2}+k\frac{\eta}{2}} \prod_{i=1}^{k} B\left(\frac{\eta}{2}, \frac{1}{2}+\frac{\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right) g(c(t-s),z-x),\\ &|(\partial_x p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)|\\ &\qquad \leq K^k W_2^{\alpha}(\mu,\mu')(t-s)^{-\frac{1}{2}+\frac{(\eta-\alpha)}{2}+k\frac{\eta}{2}} \prod_{i=1}^{k} B\left(\frac{\eta}{2}, \frac{1}{2}+\frac{\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right) g(c(t-s),z-x),\end{aligned}$$

for any $\alpha \in [0,1]$ and

$$\begin{aligned}&|\partial_x^2\Delta_{\mu,\mu'}\widehat{p}_{m+1} \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)|\\ &\qquad \leq K^k W_2^{\alpha}(\mu,\mu')(t-s)^{-1+\frac{(\eta-\alpha)}{2}+k\frac{\eta}{2}} \prod_{i=1}^{k} B\left(\frac{\eta}{2}, \frac{\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right) g(c(t-s),z-x),\end{aligned}$$

for any $\alpha \in [0,\eta)$. On the one hand, if $r \in [s,(t+s)/2]$, from (5.64), we get

$$\begin{aligned}\int_{\mathbb{R}^d}|\partial_x^2 p_m(\mu,s,r,x,y)|\Delta_{\mu,\mu'}\mathcal{H}_m(\mu,s,r,t,y,z)|dy &\leq K\frac{W_2^{\alpha}(\mu,\mu')}{(t-r)(r-s)^{1+\frac{\alpha-\eta}{2}}}g(c(t-s),z-x)\\ &\leq K\frac{W_2^{\alpha}(\mu,\mu')}{(t-s)(r-s)^{1+\frac{\alpha-\eta}{2}}}g(c(t-s),z-x),\end{aligned}$$

so that for any $\alpha \in [0,\eta)$

$$\int_s^{\frac{t+s}{2}} \int_{\mathbb{R}^d} |\partial_x^2 p_m(\mu,s,r,x,y)||\Delta_{\mu,\mu'}\mathcal{H}_m(\mu,s,r,t,y,z)|\,dy\,dr \le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}} g(c(t-s),z-x). \tag{5.68}$$

On the other hand, if $r \in [(t+s)/2,t]$, again from (5.64), we get

$$\begin{aligned}\int_{\mathbb{R}^d} |\partial_x^2 p_m(\mu,s,r,x,y)||\Delta_{\mu,\mu'}\mathcal{H}_m(\mu,s,r,t,y,z)|dy &\le K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{1+\frac{\alpha}{2}}} g(c(t-s),z-x)\\ &\le K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(t-s)^{1+\frac{\alpha}{2}}} g(c(t-s),z-x),\end{aligned}$$

so that

$$\int_{\frac{t+s}{2}}^t \int_{\mathbb{R}^d} |\partial_x^2 p_m(\mu,s,r,x,y)||\Delta_{\mu,\mu'}\mathcal{H}_m(\mu,s,r,t,y,z)|\,dy\,dr \le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}} g(c(t-s),z-x). \tag{5.69}$$

Combining (5.68) with (5.69), we finally conclude

$$\forall \alpha \in [0,\eta), \quad |(\partial_x^2 p_m \otimes \Delta_{\mu,\mu'}\mathcal{H}_m)(\mu,s,t,x,z)| \le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{1+\frac{\alpha-\eta}{2}}} g(c(t-s),z-x),$$

so that, by induction on k, we obtain

$$\begin{aligned}|(\partial_x^2 p_m \otimes \Delta_{\mu,\mu'}\mathcal{H}_m)\otimes \mathcal{H}_m^{(k)}(\mu,s,t,x,z)| \le K^k W_2^\alpha(\mu,\mu')(t-s)^{-1+\frac{(\eta-\alpha)}{2}+k\frac{\eta}{2}} \prod_{i=1}^k B\left(\frac{\eta}{2},\frac{\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)\\ \times g(c(t-s),z-x),\end{aligned}$$

for any $\alpha \in [0,\eta)$. From the asymptotics of the Beta function, we conclude that the infinite series appearing in the right-hand side of (5.67) is absolutely and uniformly convergent in the two following cases: $n = 0$, 1 for any $\alpha \in [0,1]$ and $n = 2$ for any $\alpha \in [0,\eta)$. Moreover, there exist two constants $K := K(T,\eta,a,b,[\delta a/\delta m],[\delta b/\delta m])$, $c := c(\lambda)$ such that for any $\mu,\mu' \in \mathcal{P}_2(\mathbb{R}^d)$

$$|\Delta_{\mu,\mu'}\partial_x^n p_m(\mu,s,t,x,z)| \le K\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{n+\alpha-\eta}{2}}}\, g(c(t-s),z-x),\ n=0,1,2. \tag{5.70}$$

This completes the proof of (5.16).

In order to obtain (5.17) we proceed as for the previous estimates. To lighten the notations, we introduce the quantities

$$\begin{aligned}u_m^n(s,t) &:= \sup_{(y,\mu,\mu')\in\mathbb{R}^d\times(\mathcal{P}_2(\mathbb{R}^d))^2,\mu\neq\mu'} \int\int (|y''-x'|^\eta \wedge 1)\frac{|\Delta_{\mu,\mu'}\partial_y^n[\partial_\mu p_m(\mu,s,t,x,y'')](y)|}{W_2^\alpha(\mu,\mu')}\,dy''\,\mu'(dx'),\\ v_m^n(s,t) &:= \sup_{(y,\mu,\mu')\in\mathbb{R}^d\times(\mathcal{P}_2(\mathbb{R}^d))^2,\mu\neq\mu'} \int\int \frac{|\Delta_{\mu,\mu'}\partial_y^n[\partial_\mu p_m(\mu,s,t,x',y'')](y)|}{W_2^\alpha(\mu,\mu')}\,dy''\,\mu'(dx'),\end{aligned}$$

for a fixed $n = 0$, 1, $\alpha \in [0,1]$ for $n = 0$ and $\alpha \in [0,\eta)$ if $n = 1$. We prove by induction the following key inequalities:

$$u_m^n(s,t) \le C_{m,n}(s,t)(t-s)^{-(\frac{1+n+\alpha}{2}-\eta)} \quad \text{and} \quad v_m^n(s,t) \le C_{m,n}(s,t)(t-s)^{-(\frac{1+n+\alpha-\eta}{2})},$$

with $C_{m,n}(s,t) := \sum_{k=1}^m C^k \prod_{i=1}^k B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{(k-1)\frac{\eta}{2}}$. The result being straightforward for $m = 1$, we assume that it holds at step m. We first claim

$$\begin{aligned}|\Delta_{\mu,\mu'}\partial_y^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,r,t,x,z)](y)| \le KW_2^\alpha(\mu,\mu')\left(\left\{\frac{1}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}1_{\{r=s\}}+\frac{1}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}}1_{\{r>s\}}\right\}\right.\\ \left.+\frac{1}{t-r}\int_r^t u_m^n(s,v)dv\right)\, g(c(t-r),z-x),\end{aligned} \tag{5.71}$$

where $\alpha \in [0,1]$ if $n = 0$ and $\alpha \in [0,\eta)$ if $n = 1$.

In order to prove the previous inequality, we make use of the following decomposition:

$$\Delta_{\mu,\mu'}\partial_y^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,r,t,x,z)](y) = \mathrm{I}(y)+\mathrm{II}(y)+\mathrm{III}(y)+\mathrm{IV}(y),$$

with

$$\mathrm{I}(y) := \left\{ Df_{z-x}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])dv\right) - Df_{z-x}\left(\int_r^t a(v,z,[X_v^{s,\xi',(m)}])dv\right)\right\}$$
$$\cdot \int_r^t \left\{ \int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y))\partial_x^{1+n} p_m(\mu,s,v,y,y')dy' \right.$$
$$\left. + \int\int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(x'))\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y')](y)dy'\mu(dx')\right\} dv,$$

$$\mathrm{II}(y) := Df_{z-x}\left(\int_r^t a(v,z,[X_r^{s,\xi',(m)}])dr\right)$$
$$\cdot \int_r^t \left\{ \int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y'))\partial_x^{1+n} p_m(\mu,s,v,y,y')\, dy' \right.$$
$$\left. + \int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y))\Delta_{\mu,\mu'}\partial_x^{1+n} p_m(\mu,s,v,y,y')dy'\right\} dv,$$

$$\mathrm{III}(y) := Df_{z-x}\left(\int_r^t a(v,z,[X_v^{s,\xi',(m)}])dv\right)$$
$$\cdot \int_r^t \left\{ \int\int \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y')\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y')](y)\, dy'(\mu-\mu')(dx') \right.$$
$$\left. + \int\int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y'))\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y')](y)\, dy'\mu'(dx') \right\} dv,$$

$$\mathrm{IV}(y) := Df_{z-x}\left(\int_r^t a(v,z,[X_v^{s,\xi',(m)}])dv\right) \cdot \int_r^t \int\int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(x'))$$
$$\cdot \Delta_{\mu,\mu'}\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y')](y)dy'dv\mu'(dx').$$

From the mean-value theorem and the estimates (5.61), (5.6) and (5.12), there exists a constant $K := K(T,a,b)$ independent of m such that

$$|\mathrm{I}(y)| \le \frac{K}{(t-r)^2}\int_r^t \max_{i,j} |a_{i,j}(v,z,[X_v^{s,\xi,(m)}]) - a_{i,j}(v,z,[X_v^{s,\xi',(m)}])|\, dv$$
$$\times \left(\int_r^t \left\{(v-s)^{-\frac{(1+n-\eta)}{2}} + (v-s)^{-\frac{(1+n-\eta)}{2}-\frac{\eta}{2}}\right\} dv\right) g(c(t-r),z-x)$$
$$\le \frac{K}{(t-r)^2}\int_r^t \frac{W_2^\alpha(\mu,\mu')}{(v-s)^{\frac{\alpha-\eta}{2}}}\, dv \times [(t-s)^{1-\frac{(1+n-\eta)}{2}}1_{\{r=s\}} + (t-r)(r-s)^{-\frac{(1+n-\eta)}{2}}1_{\{r>s\}}]g(c(t-r),z-x)$$
$$\le KW_2^\alpha(\mu,\mu')\left[\frac{1}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}}1_{\{r=s\}} + \frac{1}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}1_{\{r>s\}}\right] g(c(t-r),z-x)$$

for any $\alpha \in [0,1]$. Similarly to (5.61) with the map $[\delta a_{i,j}/\delta m]$ instead of $a_{i,j}$, one gets

$$\forall \alpha \in [0,1],\ |\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y'')| \le K\, W_2^\alpha(\mu,\mu')(v-s)^{\frac{\eta-\alpha}{2}} \tag{5.72}$$

so that by (5.6)

$$\left|\int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi,(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y''))\partial_x^{1+n} p_m(\mu,s,v,y,y'')\, dy''\right| \le K\frac{W_2^\alpha(\mu,\mu')}{(v-s)^{\frac{1+n+\alpha-\eta}{2}}}$$

and, from (5.70), the η-Hölder regularity of $y \mapsto [\delta a/\delta m](v,z,\mu)(y)$ and the space-time inequality (1.4), one has

$$\left|\int (\frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y'') - \frac{\delta a}{\delta m}(v,z,[X_v^{s,\xi',(m)}])(y))(\partial_x^{1+n} p_m(\mu,s,v,y,y'') - \partial_x^{1+n} p_m(\mu',s,v,y,y''))dy''\right|$$
$$\le K\frac{W_2^\alpha(\mu,\mu')}{(v-s)^{\frac{1+n+\alpha}{2}-\eta}},$$

for any $\alpha \in [0,1]$ if $n=0$ and for any $\alpha \in [0,\eta)$ if $n=1$. Combining the two previous estimates, we finally obtain

$$|\mathrm{II}(y)| \le KW_2^\alpha(\mu,\mu')\left[\frac{1}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}1_{\{r=s\}} + \frac{1}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}1_{\{r>s\}}\right] g(c(t-r),z-x),$$

with $\alpha \in [0,1]$ if $n=0$ and $\alpha \in [0,\eta)$ if $n=1$. From the representation in infinite series (5.40), we derive that $x \mapsto \partial_v^n[\partial_\mu p_m(\mu,s,t,x,z)(y)$ is α-Hölder continuous, with $\alpha \in [0,1]$ if $n=0$ and $\alpha \in [0,\eta)$ if $n=1$. More precisely, for any $x,x' \in \mathbb{R}^d$ and for any $m \geq 1$, one has

$$\begin{aligned}\Big|\partial_v^n[\partial_\mu p_m(\mu,s,t,x,z)](y) - \partial_v^n[\partial_\mu p_m(\mu,s,t,x',z)](y)\Big| \\ \leq K\frac{|x-x'|^\alpha}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}\left\{g(c(t-s),z-x)+g(c(t-s),z-x')\right\}. \end{aligned} \tag{5.73}$$

where $K := K(T,a,,b,[\delta a/\delta m],[\delta b/\delta m],\eta)$ is a positive constant independent of m. We omit its proof. The previous estimate combined with the η-Hölder continuity of $y \mapsto [\delta a/\delta m](v,z,m)(y)$ implies that $x \mapsto \int [\delta a/\delta m]a(v,z,[X_v^{s,\xi,(m)}])(y'')\partial_y^n[\partial_\mu p_m(\mu,s,v,x,y'')](y)dy''$ is α-Hölder, with a modulus bounded by $K(v-s)^{-\frac{(1+n+\alpha)}{2}+\eta}$, K being a positive constant independent of m, where $\alpha \in [0,1]$ of $n=0$ and $\alpha \in [0,\eta)$ if $n=1$. We thus deduce

$$\Big|\int_{(\mathbb{R}^d)^2}[\delta a/\delta m](v,y,[X_v^{s,\xi,(m)}])(y')\,\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y')](y)\,dy'\,(\mu-\mu')(dx')\Big| \leq K\frac{W_2^\alpha(\mu,\mu')}{(v-s)^{\frac{1+n+\alpha}{2}-\eta}}.$$

From (5.72) and (5.12), we also obtain

$$\begin{aligned}&\Big|\int_{(\mathbb{R}^d)^2}\Big(\frac{\delta}{\delta m}a(v,z,[X_v^{s,\xi,(m)}])(y') - \frac{\delta}{\delta m}a(v,z,[X_v^{s,\xi',(m)}])(y')\Big)\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y')](y)\,dy'\,\mu'(dx')\Big| \\ &\leq K\frac{W_2^\alpha(\mu,\mu')}{(v-s)^{\frac{1+n+\alpha}{2}-\eta}},\end{aligned}$$

for any $\alpha \in [0,1]$. Consequently, combining the two previous estimates, we conclude

$$|\mathrm{III}(y)| \leq KW_2^\alpha(\mu,\mu')\left\{\frac{1}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}}1_{\{r=s\}} + \frac{1}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}1_{\{r>s\}}\right\}g(c(t-r),z-x),$$

for any $\alpha \in [0,1]$ if $n=0$ and for any $\alpha \in [0,\eta)$ if $n=1$. Finally, one has

$$|\mathrm{IV}(y)| \leq \frac{K}{t-r}\int_r^t\int_{(\mathbb{R}^d)^2}(|y''-x'|^\eta \wedge 1)|\Delta_{\mu,\mu'}\partial_v^n\partial_\mu[p_m(\mu,s,v,x',y'')](v)|\,dy''\,\mu'(dx')\,dv\;g(c(t-r),z-x).$$

Gathering the estimates on $\mathrm{I}(y)$, $\mathrm{II}(y)$, $\mathrm{III}(y)$ and $\mathrm{IV}(y)$, we obtain

$$\begin{aligned}|\Delta_{\mu,\mu'}\partial_y^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,x,z)](y)| \leq K\Big(&W_2^\alpha(\mu,\mu')\big[\frac{1}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}1_{\{r=s\}} + \frac{1}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}}1_{\{r>s\}}\big] \\ &+ \frac{1}{t-r}\int_r^t\int_{(\mathbb{R}^d)^2}(|y''-x'|^\eta\wedge 1)|\Delta_{\mu,\mu'}\partial_y^n[\partial_\mu p_m(\mu,s,v,x',y'')](y)|\,dy''\,\mu'(dx')\,dv\Big) \\ &\times g(c(t-r),z-x),\end{aligned} \tag{5.74}$$

for any $\alpha \in [0,1]$ if $n=0$ and for any $\alpha \in [0,\eta)$ if $n=1$. This completes the proof of (5.71). As a consequence, from the induction hypothesis, we directly get

$$\begin{aligned}&|\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,t,x,z)](v)| \\ &\leq K\left\{\frac{1}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}} + \frac{1}{t-s}\int_s^t\frac{C_{m,n}(s,r)}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}dr\right\}W_2^\alpha(\mu,\mu')\,g(c(t-s),z-x) \\ &\leq \frac{K}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\alpha-\eta}{2}\right) + \sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\ &\quad\times W_2^\alpha(\mu,\mu')\,g(c(t-s),z-x),\end{aligned}$$

which in turn yields

$$\begin{aligned}&\sum_{r\geq 0}|\Delta_{\mu,\mu'}\partial_v^n\partial_\mu\widehat{p}_{m+1}\otimes\mathcal{H}_{m+1}^{(r)}(\mu,s,t,x,z)(v)| \\ &\leq K\left\{B\left(\frac{\eta}{2},\frac{1-n+\alpha-\eta}{2}\right) + \sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\} \\ &\quad\times\frac{W_2^\alpha(\mu,\mu')}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}g(c(t-s),z-x).\end{aligned} \tag{5.75}$$

From (5.38), we easily obtain the following decomposition

$$\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,y,z)](v) = \mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}$$

with

$$\begin{aligned}
\mathrm{A} &:= \mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3,\\
\mathrm{A}_1 &:= -\sum_{i=1}^d \Delta_{\mu,\mu'}\partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi,(m)}])](v)\, H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{A}_2 &:= -\sum_{i=1}^d \partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi',(m)}])](v)\,\Delta_{\mu,\mu'}H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{A}_3 &:= -\sum_{i=1}^d \partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi',(m)}])](v)\, H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi',(m)}])dv', z-y\right)\Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{B} &:= \mathrm{B}_1+\mathrm{B}_2+\mathrm{B}_3,\\
\mathrm{B}_1 &:= \frac{1}{2}\sum_{i,j=1}^d \Delta_{\mu,\mu'}\partial_v^n[\partial_\mu[a_{i,j}(r,y,[X_r^{s,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s,\xi,(m)}])]](v)\\
&\qquad\times H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{B}_2 &:= \frac{1}{2}\sum_{i,j=1}^d \partial_v^n[\partial_\mu[a_{i,j}(r,y,[X_r^{s,\xi',(m)}])-a_{i,j}(r,z,[X_r^{s,\xi',(m)}])]](v)\\
&\qquad\times \Delta_{\mu,\mu'}H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{B}_3 &:= \frac{1}{2}\sum_{i,j=1}^d \partial_v^n[\partial_\mu[a_{i,j}(r,y,[X_r^{s,\xi',(m)}])-a_{i,j}(r,z,[X_r^{s,\xi',(m)}])]](v)\\
&\qquad\times H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi',(m)}])dv', z-y\right)\Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{C} &:= \mathrm{C}_1+\mathrm{C}_2+\mathrm{C}_3,\\
\mathrm{C}_1 &:= -\sum_{i=1}^d \Delta_{\mu,\mu'}b_i(r,y,[X_r^{s,\xi,(m)}])\,\partial_v^n\left[\partial_\mu H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\right](v)\,\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{C}_2 &:= -\sum_{i=1}^d b_i(r,y,[X_r^{s,\xi',(m)}])\,\Delta_{\mu,\mu'}\partial_v^n\left[\partial_\mu H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\right](v)\,\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{C}_3 &:= -\sum_{i=1}^d b_i(r,y,[X_r^{s,\xi',(m)}])\,\partial_v^n\left[\partial_\mu H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi',(m)}])dv', z-y\right)\right](v)\,\Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{D} &:= \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3,\\
\mathrm{D}_1 &:= \frac{1}{2}\sum_{i,j=1}^{d} \Delta_{\mu,\mu'}\Big(a_{i,j}(r,y,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])\Big)\\
&\qquad \times \partial_v^n\left[\partial_\mu H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\right](v)\,\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{D}_2 &:= \frac{1}{2}\sum_{i,j=1}^{d} (a_{i,j}(r,y,[X_r^{s,\xi',(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi',(m)}]))\\
&\qquad \times \Delta_{\mu,\mu'}\partial_v^n\left[\partial_\mu H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\right](v)\,\widehat{p}_{m+1}(\mu,s,r,t,y,z),\\
\mathrm{D}_3 &:= \frac{1}{2}\sum_{i,j=1}^{d} (a_{i,j}(r,y,[X_r^{s,\xi',(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi',(m)}]))\\
&\qquad \times \partial_v^n\left[\partial_\mu H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi',(m)}])dv', z-y\right)\right](v)\,\Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E} &:= \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3\\
\mathrm{E}_1 &:= -\sum_{i=1}^{d} \Delta_{\mu,\mu'}\Big[b_i(r,y,[X_r^{s,\xi,(m)}])H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\Big]\,\partial_v^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,y,z)](v),\\
\mathrm{E}_2 &:= \frac{1}{2}\sum_{i,j=1}^{d} \Delta_{\mu,\mu'}\Big[\Big(a_{i,j}(r,y,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])\Big)H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\Big]\\
&\qquad \times \partial_v^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,y,z)](v),\\
\mathrm{E}_3 &:= \Bigg\{-\sum_{i=1}^{d} b_i(r,y,[X_r^{s,\xi,(m)}])H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\\
&\qquad +\frac{1}{2}\sum_{i,j=1}^{d}\Big(a_{i,j}(r,y,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])\Big)H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\Bigg\}\\
&\qquad \times \Delta_{\mu,\mu'}\partial_v^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,r,t,y,z)](v).
\end{aligned}
$$

• **Estimates on** A**:**
In order to deal with A_1, we use the following decomposition

$$\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi,(m)}])](v) = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V},$$

with

$$
\begin{aligned}
\mathrm{I} &:= \int\Big(\frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z) - \frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z)\Big)\,\partial_x^{1+n}p_m(\mu,s,r,v,z)\,dz,\\
\mathrm{II} &:= \int\Big(\frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z) - \frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(v)\Big)\,\Delta_{\mu,\mu'}\partial_x^{1+n}p_m(\mu,s,r,v,z)\,dz,\\
\mathrm{III} &:= \int\int \frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z)\,\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z)](v)\,dz\,(\mu-\mu')(dx'),\\
\mathrm{IV} &:= \int\int\Big(\frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z) - \frac{\delta b_i}{\delta m}(r,y,[X^{s,\xi',(m)}])(z)\Big)\,\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z)](v)\,dz\,\mu'(dx'),\\
\mathrm{V} &:= \int\int\Big(\frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z) - \frac{\delta b_i}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(x')\Big)\\
&\qquad .\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z)](v)\,dz\,\mu'(dx').
\end{aligned}
$$

We now need to quantify the contribution of each term in the above decomposition. Using (5.70) (with $n=0$) and computations similar to those employed for (5.61) with the map $[\delta b_i/\delta m]$ instead of $a_{i,j}$, we directly get $|[\delta b_i/\delta m](r,y,[X_r^{s,\xi,(m)}])(z) - [\delta b_i/\delta m](r,y,[X_r^{s,\xi',(m)}])(z)| \le K W_2^\alpha(\mu,\mu')(r-s)^{-(\alpha-\eta)/2}$, for

any $\alpha \in [0,1]$, so that

$$|\mathrm{I}| \leq K \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}}.$$

From (5.70) and the η-Hölder regularity of $z \mapsto [\delta b_i/\delta m](r,x,\mu)(z)$, we also obtain

$$|\mathrm{II}| \leq K \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}.$$

Using the fact that $x \mapsto \int [\delta b_i/\delta m](r,y,[X_r^{s,\xi,(m)}])(z)\partial_v^n[\partial_\mu p_m(\mu,s,r,x,z)](v)\,dz$ is α-Hölder, (with $\alpha \in [0,1]$ if $n=0$ and $\alpha \in [0,\eta)$ if $n=1$), with modulus bounded by $K(r-s)^{-((1+n+\alpha)/2-\eta)}$, we get

$$|\mathrm{III}| \leq K \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}.$$

Similarly to I, from (5.70) (with $n=0$) and (5.12), one has

$$|\mathrm{IV}| \leq K \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha}{2}-\eta}}.$$

Finally, for the last term, from the boundedness and the η-Hölder regularity of $z \mapsto [\delta b_i/\delta m](r,x,\mu)(z)$, one has

$$|\mathrm{V}| \leq K \int\int (|z-x'|^\eta \wedge 1)|\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z)](v)|\,dz\,\mu'(dx').$$

Gathering the previous estimates and using the induction hypothesis, we finally obtain

$$|\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi,(m)}])](v)| \leq K\Big(\frac{1}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}} + u_m^n(s,r)\Big) W_2^\alpha(\mu,\mu'), \tag{5.76}$$

so that

$$|\mathrm{A}_1| \leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}})\, W_2^\alpha(\mu,\mu')\, g(c(t-r),z-y).$$

For A_2, from (5.29) and (5.12) one gets (5.30) which in turn yields

$$|\partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi',(m)}])](v)| \leq \frac{K}{(r-s)^{\frac{1+n-\eta}{2}}} \tag{5.77}$$

and, from the mean-value theorem and (5.61), for any $\alpha \in [0,1]$

$$\begin{aligned}
&\Big|\Delta_{\mu,\mu'} H_1^i\Big(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\Big)\widehat{p}_{m+1}(\mu,s,r,t,y,z)\Big| \\
&\qquad \leq \frac{K}{(t-r)^{\frac{3}{2}}}\int_r^t \max_{i,j}|a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}]) - a_{i,j}(v',z,[X_{v'}^{s,\xi',(m)}])|dv'\, g(c(t-r),z-y) \\
&\qquad \leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{\alpha-\eta}{2}}} g(c(t-r),z-y),
\end{aligned}$$

so that

$$\forall \alpha \in [0,1], \quad |\mathrm{A}_2| \leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}} g(c(t-r),z-y).$$

For A_3, from (5.61) and the mean value theorem

$$\Big|\Delta_{\mu,\mu'}\widehat{p}_{m+1}(\mu,s,r,t,y,z)\Big| \leq K\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}\, g(c(t-r),z-y), \tag{5.78}$$

which in turn, with (5.77), imply

$$\forall \alpha \in [0,1], \quad |\mathrm{A}_3| \leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\, g(c(t-r),z-y).$$

Combining the previous estimates, we finally obtain

$$|\mathrm{A}| \leq \frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}})\, W_2^\alpha(\mu,\mu')\, g(c(t-r),z-y).$$

- **Estimates on** B**:**

For B_1, we employ a similar decomposition as for $\Delta_{\mu,\mu'}\partial_v[\partial_\mu b_i(r,y,[X_r^{s,\xi,(m)}])](v)$, namely

$$\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu a_{i,j}(r,y,[X_r^{s,\xi,(m)}]) - \partial_\mu a_{i,j}(r,z,[X_r^{s,\xi,(m)}])](v) = \mathrm{I}_{i,j} + \mathrm{II}_{i,j} + \mathrm{III}_{i,j} + \mathrm{IV}_{i,j} + \mathrm{V}_{i,j},$$

with

$$
\begin{aligned}
\mathrm{I}_{i,j} &:= \int \Big[\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(z') \\
&\quad - (\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi',(m)}])(z'))\Big].\partial_x^{1+n} p_m(\mu,s,r,v,z')\,dz', \\
\mathrm{II}_{i,j} &:= \int \Big[\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi',(m)}])(z') \\
&\quad - (\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(v) - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi',(m)}])(v))\Big].\Delta_{\mu,\mu'}\partial_x^{1+n} p_m(\mu,s,r,v,z')\,dz', \\
\mathrm{III}_{i,j} &:= \int_{(\mathbb{R}^d)^2} \Big[\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(z')\Big]\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z')](v)\,dz'\,(\mu-\mu')(dx'), \\
\mathrm{IV}_{i,j} &:= \int_{(\mathbb{R}^d)^2} \Big[\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(z') \\
&\quad - (\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi',(m)}])(z'))\Big].\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z')](v)\,dz'\,\mu'(dx'), \\
\mathrm{V}_{i,j} &:= \int_{(\mathbb{R}^d)^2} \Big[\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi',(m)}])(z') \\
&\quad - (\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(x') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi',(m)}])(x'))\Big].\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z')](v)\,dz'\mu'(dx').
\end{aligned}
$$

As previously done, we quantify the contribution of each term in the above decomposition. We establish a bound similar to (5.63) but with the map $[\delta a_{i,j}/\delta m]$ instead of $a_{i,j}$. To be more specific, let $\Theta_{\lambda,r}^{(m)} := (1-\lambda)[X_r^{s,\xi,(m)}] + \lambda[X_r^{s,\xi',(m)}]$ and then write

$$
\begin{aligned}
h(y) &:= \frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z') - \frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi',(m)}])(z') \\
&= -\int_0^1 \partial_\lambda \frac{\delta a_{i,j}}{\delta m}(r,y,\Theta_{\lambda,r}^{(m)})(z')\,d\lambda \\
&= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta^2 a_{i,j}}{\delta m^2}(r,y,\Theta_{r,\lambda})(z',y')(p_m(\mu,s,r,y') - p_m(\mu',s,r,y'))\,dy'\,d\lambda.
\end{aligned}
$$

The uniform η-Hölder regularity of $y \mapsto [\delta^2 a_{i,j}/\delta m^2](r,y,\mu)(z',y')$, (5.60) and (5.6) yield

$$
|h(y) - h(z)| \le K\frac{|y-z|^\eta}{(r-s)^{\frac{1}{2}}} W_2(\mu,\mu'), \tag{5.79}
$$

so that

$$
|\mathrm{I}_{i,j}| \le K|y-z|^\eta \frac{W_2(\mu,\mu')}{(r-s)^{1+\frac{\eta}{2}}} \le K|y-z|^\eta \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha}{2}}},
$$

if $W_2(\mu,\mu') \le (r-s)^{1/2}$. The previous estimate directly follows from the uniform η-Hölder regularity of $y \mapsto [\delta a_{i,j}/\delta m](r,y,[X_r^{s,\xi,(m)}])(z')$ and (5.6) if $W_2(\mu,\mu') \ge (r-s)^{1/2}$. From computations similar to those employed for (5.61) with the map $[\delta a_{i,j}/\delta m]$ instead of $a_{i,j}$, we directly get $|[\delta a_{i,j}/\delta m](r,y,[X_r^{s,\xi,(m)}])(z) - [\delta a_{i,j}/\delta m](r,y,[X_r^{s,\xi',(m)}])(z)| \le K W_2^\alpha(\mu,\mu')(r-s)^{-(\alpha-\eta)/2}$, for any $\alpha \in [0,1]$, so that

$$
|\mathrm{I}_{i,j}| \le K\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}}.
$$

Hence, combining both estimates with the space-time inequality (1.4), we deduce

$$
\begin{aligned}
&\forall \alpha \in [0,1], \quad \Big|\mathrm{I}_{i,j} \times H_2^{i,j}\Big(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\Big)\,\widehat{p}_{m+1}(\mu,s,r,t,y,z)\Big| \\
&\qquad \le K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\right\} W_2^\alpha(\mu,\mu')\,g(c(t-r),z-y).
\end{aligned} \tag{5.80}
$$

Again, from (5.70) and the η-Hölder regularity of $y \mapsto [\delta a_{i,j}/\delta m](r,y,\mu)(z)$, we obtain

$$
|\mathrm{II}_{i,j}| \le K\frac{|y-z|^\eta \wedge 1}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}} W_2^\alpha(\mu,\mu'),
$$

so that

$$\left|\mathrm{II}_{i,j}\times H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv',z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z)\right|\leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\,g(c(t-r),z-y).$$

From (5.73), the map

$$x'\mapsto\int(\frac{\delta a_{i,j}}{\delta m}(r,y,[X_r^{s,\xi,(m)}])(z')-\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(z'))\partial_v^n[\partial_\mu p_m(\mu,s,r,x',z')](v)\,dz',$$

is α-Hölder with a modulus bounded by $K(|y-z|^\eta\wedge 1)(r-s)^{-(1+n+\alpha-\eta)/2}$ so that

$$|\mathrm{III}_{i,j}|\leq K\frac{|y-z|^\eta\wedge 1}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}}W_2^\alpha(\mu,\mu'),$$

which in turn yields

$$\left|\mathrm{III}_{i,j}\times H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv',z-y\right)\widehat{p}_{m+1}(\mu,s,r,t,y,z)\right|\leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\,g(c(t-r),z-y).$$

Using similar arguments as those employed for $\mathrm{I}_{i,j}$, with (5.12) instead of (5.6), we get

$$|\mathrm{IV}_{i,j}|\leq C|y-z|^\eta\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n+\alpha-\eta}{2}}},$$

so that

$$|\mathrm{IV}_{i,j}\times H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv',z-y\right)\widehat{p}_{m+1}(\mu,r,t,y,z)|\leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\,g(c(t-r),z-y).$$

For the last term, using either the η-Hölder regularity of $x\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(z)$ on the one hand or the η-Hölder regularity of $z\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(z)$ on the other hand, as well as the induction hypothesis, we get

$$|\mathrm{V}_{i,j}|\leq K\left\{u_m^n(s,r)\wedge|y-z|^\eta v_m^n(s,r)\right\}W_2^\alpha(\mu,\mu').$$

Gathering the previous estimates and using the space-time inequality (1.4), we finally obtain

$$|\mathrm{B}_1|\leq K\left\{\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}}}\wedge\frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\right\}+\left\{\frac{u_m^n(s,r)}{t-r}\wedge\frac{v_m^n(s,r)}{(t-r)^{1-\frac{\eta}{2}}}\right\}\right\}W_2^\alpha(\mu,\mu')$$
$$\times\,g(c(t-r),z-y),$$

where $\alpha\in[0,1]$ if $n=0$ and $\alpha\in[0,\eta)$ if $n=1$. For B_2, from (5.46) (bounding $(r-s)^{(1+n-\eta)/2}u_m^n(s,r)$ and $(r-s)^{(1+n)/2}v_m^n(s,r)$ by K independent of m), one gets

$$|\partial_v^n[\partial_\mu[a_{i,j}(r,y,[X_r^{s,\xi',(m)}])-a_{i,j}(r,z,[X_r^{s,\xi',(m)}])]](v)|\leq K\left\{\frac{|z-y|^\eta}{(r-s)^{\frac{1+n}{2}}}\wedge\frac{1}{(r-s)^{\frac{1+n-\eta}{2}}}\right\}\tag{5.81}$$

and, by the mean-value theorem and then (5.61)

$$\begin{aligned}&|\Delta_{\mu,\mu'}H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv',z-y\right)\widehat{p}_{m+1}(\mu,r,t,y,z)|\\&\leq\frac{K}{(t-r)^2}\int_r^t\max_{i,j}|a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])-a_{i,j}(v',z,[X_{v'}^{s,\xi',(m)}])|dv'g(c(t-r),z-y)\\&\leq K\frac{W_2^\alpha(\mu,\mu')}{(t-r)(r-s)^{\frac{\alpha-\eta}{2}}}g(c(t-r),z-y),\end{aligned}\tag{5.82}$$

so that combining (5.81) and (5.82)

$$|\mathrm{B}_2|\leq K\left\{\frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\wedge\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\right\}W_2^\alpha(\mu,\mu')\,g(c(t-r),z-y).$$

For B_3, from (5.78), (5.81) and the space-time inequality (1.4), we get

$$\begin{aligned}|\mathrm{B}_3|&\leq K\left\{\frac{1}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\wedge\frac{|z-y|^\eta}{(t-r)(r-s)^{\frac{1+n}{2}}}\right\}\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}\,g(c(t-r),z-y)\\&\leq K\left\{\frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\wedge\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\right\}W_2^\alpha(\mu,\mu')\,g(c(t-r),z-y).\end{aligned}$$

Gathering the previous estimates on B_1, B_2, B_3 and using the induction hypothesis, we finally deduce

$$\begin{aligned}|\mathrm{B}| &\le K\left\{\left\{\frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\wedge\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}}}\right\}+\left\{\frac{u_m^n(s,r)}{t-r}\wedge\frac{v_m^n(s,r)}{(t-r)^{1-\frac{\eta}{2}}}\right\}\right\}W_2^\alpha(\mu,\mu')\\&\quad\times g(c(t-r),z-y)\\&\le K\left\{\frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\wedge\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}}}\right\}(1+C_{m,n}(s,r)(r-s)^{\frac{\eta}{2}})\\&\quad\times W_2^\alpha(\mu,\mu')\,g(c(t-r),z-y).\end{aligned}$$

• **Estimates on** C**:**

For C_1, similarly to (5.61) with the map b_i instead of $a_{i,j}$, one has

$$\forall\alpha\in[0,1],\quad|\Delta_{\mu,\mu'}b_i(r,y,[X_r^{s,\xi,(m)}])|\le K\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}\tag{5.83}$$

and, from (5.30), so that

$$\begin{aligned}&\Big|\partial_v^n\Big[\partial_\mu H_1^i\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv',z-y\right)\Big](v)\widehat{p}_{m+1}(\mu,r,t,y,z)\Big|\\&\qquad\le\frac{K}{(t-r)^{\frac{3}{2}}}\int_r^t\max_{i,j}|\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])](v)|\,dv'\,g(c(t-r),z-y)\\&\qquad\le\frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n-\eta}{2}}}\,g(c(t-r),z-y).\end{aligned}\tag{5.84}$$

Combining both estimates, we obtain

$$|\mathrm{C}_1|\le K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\,g(c(t-r),z-y).$$

From similar computations as those employed for the term $\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu b_i(r,y,[X_r^{s,\xi,(m)}])](v)$, see (5.76), in a completely analogous manner, we get

$$|\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])](v)|\le K\Big(\frac{1}{(v'-s)^{\frac{1+n+\alpha-\eta}{2}}}+u_m^n(s,v')\Big)\,W_2^\alpha(\mu,\mu')\tag{5.85}$$

so that, from the mean-value theorem and the induction hypothesis

$$\begin{aligned}|\mathrm{C}_2|&\le K\left\{\frac{1}{(t-r)^{\frac{3}{2}}}\int_r^t\max_{i,j}|\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m-1)}])](v)|dv'+\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{n+\alpha}{2}-\eta}}\right\}g(c(t-r),z-y)\\&\le\frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\left(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv\right)W_2^\alpha(\mu,\mu')\,g(c(t-r),z-y).\end{aligned}$$

For C_3, from (5.78) and then (5.84), we obtain

$$\begin{aligned}|\mathrm{C}_3|&\le K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{3}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\int_r^t\max_{i,j}|\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])](v)|\,dv'\,g(c(t-r),z-y)\\&\le K\frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\,g(c(t-r),z-y).\end{aligned}$$

Gathering the previous estimates on C_1, C_2 and C_3, we get

$$|\mathrm{C}|\le\frac{K}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}}\left(1+(t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\,dv\right)W_2^\alpha(\mu,\mu')\,g(c(t-r),z-y).$$

• **Estimates on** D**:**

In order to deal with D_1, we first remark that from (5.63) and the computations shortly after, distinguishing the two cases $W_2(\mu,\mu')\ge(r-s)^{1/2}$ and $W_2(\mu,\mu')\le(r-s)^{1/2}$, we get

$$\forall\alpha\in[0,1],\quad|\Delta_{\mu,\mu'}\Big(a_{i,j}(r,y,[X_r^{s,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s,\xi,(m)}])\Big)|\le K\frac{|z-y|^\eta}{(r-s)^{\frac{\alpha}{2}}}W_2^\alpha(\mu,\mu').$$

From (5.61), we also get

$$\forall \alpha \in [0,1], \quad |\Delta_{\mu,\mu'} a_{i,j}(r,y,[X_r^{s,\xi,(m)}])| + |\Delta_{\mu,\mu'} a_{i,j}(r,z,[X_r^{s,\xi,(m)}])| \leq K \frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{\alpha-\eta}{2}}}.$$

Gathering the two previous bounds, one obtains

(5.86)
$$\forall \alpha \in [0,1], \quad |\Delta_{\mu,\mu'}\Big(a_{i,j}(r,y,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])\Big)| \leq K \left\{ \frac{|z-y|^\eta}{(r-s)^{\frac{\alpha}{2}}} \wedge \frac{1}{(r-s)^{\frac{\alpha-\eta}{2}}} \right\} W_2^\alpha(\mu,\mu').$$

Moreover, similarly to (5.84), one has

$$\begin{aligned} &\left|\partial_v^n \left[\partial_\mu H_2^{i,j}\left(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y\right)\right](v)\, \widehat{p}_{m+1}(\mu,s,r,t,y,z)\right| \\ &\leq \frac{K}{(t-r)^2}\int_r^t \max_{i,j} |\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])](v)|\, dv'\, g(c(t-r),z-y) \\ &\leq \frac{K}{(t-r)(r-s)^{\frac{1+n-\eta}{2}}}\, g(c(t-r),z-y). \end{aligned} \tag{5.87}$$

The two previous estimates allow to conclude

$$|\mathrm{D}_1| \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha}{2}-\eta}} \right\} W_2^\alpha(\mu,\mu')\, g(c(t-r),z-y).$$

For D_2, we handle $\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu H_2^{i,j}(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y)](v)$ in a similar way as we did for $\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu H_1^i(\int_r^t a(v',z,[X_{v'}^{s,\xi,(m)}])dv', z-y)](v)$, that is, from the mean-value theorem, (5.61), (5.30), (5.85), one gets

$$\begin{aligned} |\mathrm{D}_2| &\leq K \left\{ \frac{1}{(t-r)^{2-\frac{\eta}{2}}}\int_r^t \max_{i,j} |\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])](v)|dv' + \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}} \right\} g(c(t-r),z-y) \\ &\leq \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \left(1 + (t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\, dv\right) W_2^\alpha(\mu,\mu')\, g(c(t-r),z-y). \end{aligned}$$

To deal with D_3, we employ (5.78) and (5.87). We get

$$\begin{aligned} |\mathrm{D}_3| &\leq \frac{K}{(t-r)^{2-\frac{\eta}{2}}(r-s)^{\frac{\alpha-\eta}{2}}}\int_r^t \max_{i,j} |\partial_v^n[\partial_\mu a_{i,j}(v',z,[X_{v'}^{s,\xi,(m)}])](v)|\, dv'\, g(c(t-r),z-y) \\ &\leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\, g(c(t-r),z-y). \end{aligned}$$

Gathering the previous estimates on D_1, D_2 and D_3, we thus obtain

$$\begin{aligned} |\mathrm{D}| \leq C &\left\{ \frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \right\} \left(1 + (t-r)^{-1}\int_r^t C_{m,n}(s,v)(v-s)^{\frac{\eta}{2}}\, dv\right) \\ &\times W_2^\alpha(\mu,\mu')\, g(c(t-r),z-y). \end{aligned}$$

• **Estimates on** E**:**

For E_1, we proceed as for the previous terms. To be more specific, from (5.83), the mean-value theorem and (5.61) as well as (5.39) (note that the constant C in the latter estimate does not depend on m), we have

$$|\mathrm{E}_1| \leq K \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{\frac{1}{2}}(r-s)^{\frac{1+n+\alpha}{2}-\eta}}\, g(c(t-r),z-y).$$

For E_2, from (5.86), (5.39) and then (5.82), we get

$$|\mathrm{E}_2| \leq K \left\{ \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha}{2}-\eta}} \right\} W_2^\alpha(\mu,\mu')\, g(c(t-r),z-y).$$

For the last term E_3, from (5.74) and the induction hypothesis, one obtains

$$
\begin{aligned}
|\mathrm{E}_3| &\le K \left\{ \frac{W_2^\alpha(\mu,\mu')}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \right. \\
&\quad \left. + \frac{1}{(t-r)^{2-\frac{\eta}{2}}} \int_r^t \int_{(\mathbb{R}^d)^2} (|y''-x'|^\eta \wedge 1) |\Delta_{\mu,\mu'} \partial_v^n [\partial_\mu p_m(\mu,s,v',x',y'')](v)| \, dy'' \, \mu'(dx') \, dv' \right\} \\
&\quad \times g(c(t-r), z-y) \\
&\le \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \left(1 + (t-r)^{-1} \int_r^t C_{m,n}(s,v')(v'-s)^{\frac{\eta}{2}} \, dv' \right) W_2^\alpha(\mu,\mu') \, g(c(t-r), z-y).
\end{aligned}
$$

Gathering the previous estimates, we finally deduce

$$
|\mathrm{E}| \le \frac{K}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \left(1 + (t-r)^{-1} \int_r^t C_{m,n}(s,v')(v'-s)^{\frac{\eta}{2}} \, dv' \right) W_2^\alpha(\mu,\mu') \, g(c(t-r), z-y).
$$

We now collect all the previous estimates on A, B, C, D and E. We finally obtain the following bound

$$
\begin{aligned}
&|\Delta_{\mu,\mu'} \partial_v^n [\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,y,z)](v)| \\
&\le K \left\{ \frac{1}{(t-r)(r-s)^{\frac{1+n+\alpha-\eta}{2}}} \wedge \frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s)^{\frac{1+n+\alpha}{2}}} \right\} \left(1 + \frac{1}{t-r} \int_r^t C_{m,n}(s,v')(v'-s)^{\frac{\eta}{2}} \, dv' \right) \\
&\quad \times W_2^\alpha(\mu,\mu') g(c(t-r), z-y),
\end{aligned}
$$

which in turn, after a space-time convolution with p_{m+1} (separating the time integral into the two disjoint parts $[s,(t+s)/2]$ and $[(t+s)/2,t]$ as we did before in order to balance the time singularity), implies

$$
\begin{aligned}
&|p_{m+1} \otimes \Delta_{\mu,\mu'} \partial_v^n [\partial_\mu \mathcal{H}_{m+1}(\mu,s,t,x,z)](v)| \\
&\quad \le \frac{K}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}} \left(B\left(\frac{\eta}{2}, \frac{1-n+\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{1-n+\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right) \\
&\qquad \times W_2^\alpha(\mu,\mu') \, g(c(t-s), z-x).
\end{aligned}
$$

From standard computations, we deduce that the series $\sum_{k\ge 0} (p_{m+1} \otimes \Delta_{\mu,\mu'} \partial_v^n \partial_\mu \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(v)$ converges absolutely and uniformly. Moreover, there exist positive constants $K := K(T,a,b,[\delta a/\delta m],[\delta b/\delta m],[\delta^2 a/\delta m^2],[\delta^2 b/\delta m^2])$, $c := c(\lambda)$ such that for any $\alpha \in [0,\eta)$

$$
\begin{aligned}
&\sum_{k\ge 0} \left| \left(p_{m+1} \otimes \Delta_{\mu,\mu'} \partial_v^n \partial_\mu \mathcal{H}_{m+1} \right) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(v) \right| \\
&\quad \le \frac{K}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}} \left\{ B\left(\frac{\eta}{2}, \frac{1-n+\eta-\alpha}{2}\right) + \sum_{k=1}^m C^k \prod_{i=1}^{k+1} B\left(\frac{\eta}{2}, \frac{1-n+\eta-\alpha}{2} + (i-1)\frac{\eta}{2}\right) (t-s)^{k\frac{\eta}{2}} \right\} \\
&\qquad \times W_2^\alpha(\mu,\mu') \, g(c(t-s), z-x).
\end{aligned} \tag{5.88}
$$

From (5.64) and (5.12) (at step $m+1$), separating the computations into the two disjoint intervals $[s,(t+s)/2]$ and $[(t+s)/2,t]$ as in (5.68) and (5.69), we get

$$
|(\partial_v^n \partial_\mu p_{m+1} \otimes \Delta_{\mu,\mu'} \mathcal{H}_{m+1})(\mu,s,t,x,z)(v)| \le \frac{K}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}} W_2^\alpha(\mu,\mu') \, g(c(t-s), z-x),
$$

where $\alpha \in [0,1]$ if $n=0$ and $\alpha \in [0,\eta)$ if $n=1$, so that

$$
\sum_{k\ge 0} |(\partial_v^n \partial_\mu p_{m+1} \otimes \Delta_{\mu,\mu'} \mathcal{H}_{m+1}) \otimes \mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(v)| \le \frac{K}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}} W_2^\alpha(\mu,\mu') \, g(c(t-s), z-x). \tag{5.89}
$$

Similarly, from the estimates (5.60), (5.47) (bounding $C_{m,n}$ by a constant K independent of m), separating the time integral into two disjoint intervals as previously done, after some standard computations, we get

$$
|\Delta_{\mu,\mu'} p_{m+1} \otimes \partial_v^n \partial_\mu \mathcal{H}_{m+1}(\mu,s,t,x,z)(v)| \le \frac{K}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}} W_2^\alpha(\mu,\mu') \, g(c(t-s), z-x),
$$

which in turn implies

(5.90)
$$\sum_{k\geq 0}|(\Delta_{\mu,\mu'}p_{m+1}\otimes\partial_v^n\partial_\mu\mathcal{H}_{m+1})\otimes\mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(v)|\leq\frac{K}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}}W_2^\alpha(\mu,\mu')\,g(c(t-s),z-x).$$

If we differentiate with respect to the measure argument (and then possibly with respect to the variable v) the relation $p_{m+1}=\widehat{p}_{m+1}+p_{m+1}\otimes\mathcal{H}_{m+1}$, we obtain $\partial_v^n\partial_\mu p_{m+1}=\partial_v^n\partial_\mu\widehat{p}_{m+1}+p_{m+1}\otimes\partial_v^n\partial_\mu\mathcal{H}_{m+1}+\partial_v^n\partial_\mu p_{m+1}\otimes\mathcal{H}_{m+1}$ so that

$$\begin{aligned}\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](v)&=\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,t,x,z)](v)+p_{m+1}\otimes\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s,t,x,z)](v)\\&+\Delta_{\mu,\mu'}p_{m+1}\otimes\partial_v^n\partial_\mu\mathcal{H}_{m+1}(\mu,s,t,x,z)(v)+\partial_v^n\partial_\mu p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,t,x,z)(v)\\&+\Delta_{\mu,\mu'}\partial_v^n\partial_\mu p_{m+1}\otimes\mathcal{H}_{m+1}(\mu,s,t,x,z)(v).\end{aligned}$$

Iterating the previous relation, we obtain the following representation

$$\begin{aligned}\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](v)=\sum_{k\geq 0}\Big[&\Delta_{\mu,\mu'}\partial_v^n\partial_\mu\widehat{p}_{m+1}+p_{m+1}\otimes\Delta_{\mu,\mu'}\partial_v^n\partial_\mu\mathcal{H}_{m+1}\\&+\Delta_{\mu,\mu'}p_{m+1}\otimes\partial_v^n\partial_\mu\mathcal{H}_{m+1}+\partial_v^n\partial_\mu p_{m+1}\otimes\Delta_{\mu,\mu'}\mathcal{H}_{m+1}\Big]\otimes\mathcal{H}_{m+1}^{(k)}(\mu,s,t,x,z)(v).\end{aligned}$$

Gathering the estimates (5.75), (5.88), (5.89) and (5.90), we deduce that the above series converges absolutely and satifies

$$\begin{aligned}&|\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](v)|\\&\quad\leq\frac{K}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}}W_2^\alpha(\mu,\mu')\,g(c(t-s),z-x)\\&\qquad+\frac{K}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}\\&\qquad\times W_2^\alpha(\mu,\mu')\,g(c(t-s),z-x)\\&\quad\leq\frac{K}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}\\&\qquad\times W_2^\alpha(\mu,\mu')g(c(t-s),z-x),\end{aligned}$$

so that

$$u_{m+1}^n(s,t)\leq\frac{K}{(t-s)^{\frac{1+n+\alpha}{2}-\eta}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}$$

and similarly,

$$v_{m+1}^n(s,t)\leq\frac{K}{(t-s)^{\frac{1+n+\alpha-\eta}{2}}}\left\{B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}\right)+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta-\alpha}{2}+(i-1)\frac{\eta}{2}\right)(t-s)^{k\frac{\eta}{2}}\right\}.$$

Since the constant K does not depend either on the constant C appearing in the definition of $C_{m,n}(s,t)$ or m, one may change C once for all and derive the induction hypothesis at step $m+1$ for u_m^n and v_m^n. This completes the proof of (5.17) at step $m+1$.

We now prove the estimates (5.18) and (5.19). Since the proofs are rather long, technical and use similar arguments as those employed before, we will limit ourself to (5.19) and will omit some technical details. The proof of (5.18) follows from the relation (5.23) and similar arguments as those developed below. We remark that if $|s_1-s_2|\geq t-s_1\vee s_2$, the estimate (5.19) follows directly from (5.12). We thus assume that $|s_1-s_2|\leq t-s_1\vee s_2$ for the rest of the proof. To make the notations simpler, for any fixed $(s_1,s_2)\in[0,t)^2$, we write $\Delta_{s_1,s_2}f(s)=f(s_1\vee s_2)-f(s_1\wedge s_2)$ for a function f defined on $[0,t)$. In particular, $\Delta_{s_1,s_2}p_m(\mu,s,t,x,z)=p_m(\mu,s_1\vee s_2,t,x,z)-p_m(\mu,s_1\wedge s_2,t,x,z)$. We first claim

$$\begin{aligned}&\forall\beta\in[0,1],\forall m\geq 1,\quad|\Delta_{s_1,s_2}p_m(\mu,s,t,x,z)|\\&\qquad\qquad\leq K\left\{\frac{|s_1-s_2|^\beta}{(t-s_1)^\beta}g(c(t-s_1),z-x)+\frac{|s_1-s_2|^\beta}{(t-s_2)^\beta}g(c(t-s_2),z-x)\right\}.\end{aligned}\tag{5.91}$$

In order to prove the above statement, one has to consider the two cases $|s_1-s_2|\geq t-s_1\vee s_2$ and $|s_1-s_2|\leq t-s_1\vee s_2$. In the first case, it directly follows from (5.6) with $n=0$, while in the second case, it follows from the mean-value theorem, (5.13) and the inequality $(t-s_1\vee s_2)^{-1}\leq 2(t-s_1\wedge s_2)^{-1}$.

We now start from the representation in infinite series (5.40) and write the following decomposition

$$\begin{aligned}\Delta_{s_1,s_2}\partial_v^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](v) &= \Delta_{s_1,s_2}\partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](v) + \Delta_{s_1,s_2}(p_{m+1}\otimes \partial_v^n[\partial_\mu \mathcal{H}_{m+1}])(\mu,s,t,x,z)(v)\\ &\quad + \Delta_{s_1,s_2}(\partial_v^n[\partial_\mu \widehat{p}_{m+1}]\otimes \Phi_{m+1})(\mu,s,t,x,z)(v)\\ &\quad + \Delta_{s_1,s_2}((p_{m+1}\otimes \partial_y^n[\partial_\mu \mathcal{H}_{m+1}])\otimes \Phi_{m+1})(\mu,s,t,x,z)(v).\end{aligned} \tag{5.92}$$

We investigate the first term appearing in the right-hand side of the above identity and make use of the following decomposition

$$\Delta_{s_1,s_2}\partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](v) = \mathrm{I}(v)+\mathrm{II}(v)+\mathrm{III}(v)+\mathrm{IV}(v),$$

with

$$\begin{aligned}\mathrm{I}(v) &:= \left\{Df_{z-x}\left(\int_{s_1\vee s_2}^t a(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])dr\right) - Df_{z-x}\left(\int_{s_1\wedge s_2}^t a(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])dr\right)\right\}\\ &\quad\cdot\int_{s_1\vee s_2}^t\left\{\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(v))\partial_x^{1+n}p_m(\mu,s_1\vee s_2,r,v,y')dy'\right.\\ &\quad\left.+\int\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(x'))\partial_v^n\partial_\mu[p_m(\mu,s_1\vee s_2,r,x',y')](v)\,dy'\mu(dx')\right\}dr,\\ \mathrm{II}(v) &:= Df_{z-x}\left(\int_{s_1\wedge s_2}^t a(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])dr\right)\\ &\quad\cdot\int_{s_1\vee s_2}^t\left\{\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'))\partial_x^{1+n}p_m(\mu,s_1\vee s_2,r,v,y')\,dy'\right.\\ &\quad\left.+\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(v)).\Delta_{s_1,s_2}\partial_x^{1+n}p_m(\mu,s,r,v,y')dy'\right\}dr,\\ \mathrm{III}(v) &:= Df_{z-x}\left(\int_{s_1\wedge s_2}^t a(r,y,[X_r^{s_1\wedge s_2,\xi,(m)}])dr\right)\\ &\quad\cdot\int_{s_1\vee s_2}^t\left\{\int\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'))\partial_v^n[\partial_\mu p_m(\mu,s_1\vee s_2,r,x',y')](v)dy'\mu(dx')\right.\\ &\quad\left.+\int\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(x')).\Delta_{s_1,s_2}\partial_v^n[\partial_\mu p_m(\mu,s_1\vee s_2,r,x',y')](v)\,dy'\mu(dx')\right\}\\ \mathrm{IV}(v) &:= -Df_{z-x}\left(\int_{s_1\wedge s_2}^t a(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])dr\right)\\ &\quad\cdot\int_{s_1\wedge s_2}^{s_1\vee s_2}\left\{\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(v))\partial_x^{1+n}p_m(\mu,s_1\wedge s_2,r,v,y')dy'\right.\\ &\quad\left.+\int\int(\frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y') - \frac{\delta a}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(x'))\partial_v^n[\partial_\mu p_m(\mu,s_1\wedge s_2,r,x',y')](v)dy'\mu(dx')\right\}dr.\end{aligned}$$

From the mean-value theorem,

$$\begin{aligned}&a_{i,j}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])\\ &\qquad= \int_0^1\int_{\mathbb{R}^d}\frac{\delta a_{i,j}}{\delta m}(r,z,\Theta_{\lambda,r}^{(m)})(y)\,\Delta_{s_1,s_2}p(\mu,s,t,y)\,dyd\lambda\\ &\qquad= \int_0^1\int_{\mathbb{R}^d}\Big(\frac{\delta a_{i,j}}{\delta m}(r,z,\Theta_{\lambda,r}^{(m)})(y) - \frac{\delta a_{i,j}}{\delta m}(r,z,\Theta_{\lambda,r}^{(m)})(x)\Big)\Delta_{s_1,s_2}p(\mu,s,t,x,y)\,dy\,\mu(dx)d\lambda\end{aligned}$$

where we used the notation $\Theta_{\lambda,r}^{(m)} := \lambda[X_r^{s_1\vee s_2,\xi,(m)}] + (1-\lambda)[X_r^{s_1\wedge s_2,\xi,(m)}]$ which combined with (5.91), the η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](r,z,\mu)(y)$ and the space-time inequality (1.4) yield

$$|a_{i,j}(r,z,[X_r^{s_1\vee s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])| \le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}. \tag{5.93}$$

Then, from the previous bound, the mean-value theorem, (5.12) and (5.6) we get

$$\begin{aligned}\forall \beta\in[0,1],\ |\mathrm{I}(v)| &\le \frac{K}{(t-s_1\vee s_2)^2}\left\{|s_1-s_2|+\int_{s_1\vee s_2}^t \frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\,dr\right\}(t-s_1\vee s_2)^{\frac{1-n+\eta}{2}}\\ &\quad\times g(c(t-s_1\wedge s_2),z-x)\\ &\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\,g(c(t-s_1\wedge s_2),z-x),\end{aligned}$$

where we used the inequality $|s_1-s_2|\le t-s_1\vee s_2$ for the last inequality. From (5.93) with the map $[\delta a_{i,j}/\delta m]$ instead of $a_{i,j}$, using the uniform η-Hölder regularity of $z\mapsto[\delta^2 a_{i,j}/\delta m^2](t,x,\mu)(y,z)$, one gets

$$\Big|\frac{\delta}{\delta m}a(r,z,[X_r^{s_1\vee s_2,\xi,(m)}])(y')-\frac{\delta}{\delta m}a(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y')\Big|\le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}},\tag{5.94}$$

which, together with (5.6), (5.18) and the η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](r,z,\mu)(y)$, imply

$$\begin{aligned}|\mathrm{II}(v)| &\le \frac{K}{t-s_1\vee s_2}\int_{s_1\vee s_2}^t\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\,dr\,g(c(t-s_1\wedge s_2),z-x)\\ &\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\,g(c(t-s_1\wedge s_2),z-x),\end{aligned}$$

where $\beta\in[0,(1+\eta)/2)$ for $n=0$ and $\beta\in[0,\eta/2)$ for $n=1$. From (5.94), (5.12) and the uniform η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(y)$, we get

$$\begin{aligned}|\mathrm{III}(v)|\le K\Big\{&\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}+\frac{1}{t-s_1\wedge s_2}\int_{s_1\vee s_2}^t\int\int(|y''-x'|^\eta\wedge 1)\\ &\times|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu p_m(\mu,s,r,x',y'')](v)|\,dr\,dy''\mu(dx')\Big\}g(c(t-s_1\wedge s_2),z-x),\end{aligned}$$

where $\beta\in[0,(1/2+\eta)\wedge 1)$ for $n=0$ and $\beta\in[0,\eta)$ for $n=1$. From (5.6), (5.12), the η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(y)$ and the space-time inequality (1.4), we finally obtain

$$|\mathrm{IV}(v)|\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}g(c(t-s_1\wedge s_2),z-x),$$

where $\beta\in[0,(1+\eta)/2)$ for $n=0$ and $\beta\in[0\eta/2)$ for $n=1$. Collecting the above estimates, we thus obtain

$$\begin{aligned}&|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu\widehat{p}_{m+1}(\mu,s,t,x,z)](v)|\\ &\quad\le K\Big\{\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{1}{t-s_1\wedge s_2}\int_{s_1\vee s_2}^t\int\int(|y''-x'|^\eta\wedge1)\\ &\qquad\times|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu p_m(\mu,s,r,x',y'')](v)|\,dy''\mu(dx')dr\Big\}g(c(t-s_1\wedge s_2),z-x).\end{aligned}\tag{5.95}$$

As we already did before we introduce the quantities

$$\begin{aligned}u_m^n(s_1,s_2,t)&:=\sup_{(s_1,s_2,v)\in[0,t)^2\times\mathbb{R}^d,s_1\neq s_2}\int\int(|y-x|^\eta\wedge1)\frac{|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu p_m(\mu,s,t,x,y)](v)|}{|s_1-s_2|^\beta}\,dy\,\mu(dx),\\ v_m^n(s_1,s_2,t)&:=\sup_{(s_1,s_2,v)\in[0,t)^2\times\mathbb{R}^d,s_1\neq s_2}\int\int\frac{|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu p_m(\mu,s,t,x,y)](v)|}{|s_1-s_2|^\beta}\,dy\,\mu(dx),\end{aligned}$$

for any fixed $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and any fixed $\beta\in[0,\frac{\eta}{2})$ for $n=0,1$. We prove by induction the following key inequalities:

$$\begin{aligned}u_m^n(s_1,s_2,t)&\le C_{m,n}(s_1\vee s_2,t)(t-s_1\vee s_2)^{-\frac{(1+n)}{2}-\beta+\eta},\\ v_m^n(s_1,s_2,t)&\le C_{m,n}(s_1\vee s_2,t)(t-s_1\vee s_2)^{-\frac{(1+n)}{2}-\beta+\frac{\eta}{2}},\end{aligned}$$

with $C_{m,n}(s,t):=\sum_{k=1}^m C^k\prod_{i=1}^k B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}-\beta+(i-1)\frac{\eta}{2}\right)(t-s)^{(k-1)\frac{\eta}{2}}$. The result for $m=1$ being straightforward, we assume that it holds at step m. With the above induction hypothesis applied to

(5.95), we get

$$|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,t,x,z)](v)| \le K\Big\{\frac{1}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}} + \frac{1}{t-s_1\wedge s_2}\int_{s_1\vee s_2}^t \frac{C_{m,n}(s_1\vee s_2,r)}{(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}dr\Big\}$$
$$\times |s_1-s_2|^\beta g(c(t-s_1\wedge s_2),z-x). \tag{5.96}$$

From $\Phi_{m+1}(\mu,s,r,t,x,y) = \mathcal{H}_{m+1}(\mu,s,r,t,x,y)+\mathcal{H}_{m+1}\otimes\Phi_{m+1}(\mu,s,r,t,x,y)$, we obtain the following relation

$$\begin{aligned}\Delta_{s_1,s_2}\Phi_{m+1}(\mu,s,r,t,x,y) &= \Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)\\ &+\int_r^t\int \Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,s,r,v,x,z)\Phi_{m+1}(\mu,s_1\vee s_2,v,t,z,y)\,dv\,dy\\ &+\int_r^t\int \mathcal{H}_{m+1}(\mu,s_1\wedge s_2,r,v,x,z)\Delta_{s_1,s_2}\Phi_{m+1}(\mu,s,v,t,z,y)\,dv\,dy.\end{aligned} \tag{5.97}$$

We then claim

$$|\Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)| \le K\left\{\frac{|s_1-s_2|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^\beta}\wedge\frac{|s_1-s_2|^\beta}{(t-r)(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\right\} g(c(t-r),y-x). \tag{5.98}$$

In order to prove the above statement, we use a similar decomposition as the one employed for $\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)$, namely

$$\Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,s,r,t,x,y) = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V},$$

with

$$\begin{aligned}
\mathrm{I} &:= -\sum_{i=1}^d \Delta_{s_1,s_2}b_i(r,x,[X_r^{s,\xi,(m)}])H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z),\\
\mathrm{II} &:= -\sum_{i=1}^d b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])\Delta_{s_1,s_2}H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
\mathrm{III} &:= \frac{1}{2}\sum_{i,j=1}^d \Delta_{s_1,s_2}[a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])]H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)\\
&\quad\times\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z),\\
\mathrm{IV} &:= \frac{1}{2}\sum_{i,j=1}^d [a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]\Delta_{s_1,s_2}H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\\
&\quad\times\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z),\\
\mathrm{V} &:= -\sum_{i=1}^d b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z),\\
&\quad+\frac{1}{2}\sum_{i,j=1}^d [a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)\\
&\quad\times\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z).
\end{aligned}$$

From (5.93) (with the maps $a_{i,j}$ and b_i), (5.13), following similar arguments as those employed for $\Delta_{\mu,\mu'}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)$, the following estimates hold: for all $\beta\in[0,1]$,

$$
\begin{aligned}
|\mathrm{I}| &\le K\frac{|s_1-s_2|^\beta}{(t-r)^{\frac12}(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\,g(c(t-r),y-x),\\
|\mathrm{II}| &\le K\frac{|s_1-s_2|^\beta}{(t-r)^{\frac12}(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\,g(c(t-r),y-x),\\
|\mathrm{III}| &\le K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\right\}|s_1-s_2|^\beta\,g(c(t-r),y-x),\\
|\mathrm{IV}| &\le K\frac{|s_1-s_2|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\,g(c(t-r),y-x),\\
|\mathrm{V}| &\le K\frac{|s_1-s_2|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}\,g(c(t-r),y-x).
\end{aligned}
$$

We only prove the estimates on I and III. The estimates on II, IV and V are obtained by following similar lines of reasonings and the remaining technical details are omitted. From (5.93) with the map b_i instead of $a_{i,j}$, we get

$$
|\Delta_{s_1,s_2}b_i(r,x,[X_r^{s,\xi,(m)}])|\le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}},\tag{5.99}
$$

which in turn directly yields the announced estimates on I. In order to deal with III, we consider the two disjoint cases: $|s_1-s_2|\ge(r-s_1\vee s_2)$ and $|s_1-s_2|\le(r-s_1\vee s_2)$. In the first case, from the η-Hölder regularity of $x\mapsto a(t,x,\mu)$ and the space-time inequality (1.4), we directly obtain

$$
\forall\beta\in[0,1],\ |\mathrm{III}|\le K\frac{1}{(t-r)^{1-\frac{\eta}{2}}}g(c(t-r),z-x)\le K\frac{|s_1-s_2|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\beta}}g(c(t-r),z-x).
$$

In order to obtain the other part of the estimate, we directly use (5.93)

$$
\forall\beta\in[0,1],\ |\mathrm{III}|\le K\frac{|s_1-s_2|^\beta}{(t-r)(r-s_1\vee s_2)^{\beta-\frac{\eta}{2}}}g(c(t-r),z-x).\tag{5.100}
$$

We thus derive the announced estimate on III in the case $|s_1-s_2|\ge(r-s_1\vee s_2)$. We importantly observe that (5.100) is still valid in the case $|s_1-s_2|\le(r-s_1\vee s_2)$. Now assume that $|s_1-s_2|\le(r-s_1\vee s_2)$. Then, from the mean-value theorem, one gets

$$
\begin{aligned}
h(x)&:=a_{i,j}(r,x,[X_r^{s_1\vee s_2,\xi,(m)}])-a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])\\
&=\int_0^1\int_{(\mathbb{R}^d)^2}\frac{\delta a_{i,j}}{\delta m}(r,x,\Theta^{(m)}_{\lambda,r})(y'')\Delta_{s_1,s_2}p_m(\mu,s,r,x',y'')\,dy''\mu(dx')\\
&=\int_0^1\int_{(\mathbb{R}^d)^2}\Big[\frac{\delta a_{i,j}}{\delta m}(r,x,\Theta^{(m)}_{\lambda,r})(y'')-\frac{\delta a_{i,j}}{\delta m}(r,x,\Theta^{(m)}_{\lambda,r})(x')\Big]\Delta_{s_1,s_2}p_m(\mu,s,r,x',y'')\,dy''\mu(dx')
\end{aligned}
$$

where we used the notation $\Theta^{(m)}_{\lambda,r}:=\lambda[X_r^{s_1\vee s_2}]+(1-\lambda)[X_r^{s_1\wedge s_2}]$. The previous identity together with the η-Hölder regularity of $x\mapsto[\delta a_{i,j}/\delta m](r,x,\mu)(y)$ and (5.91) yield

$$
\forall\beta\in[0,1],\ |h(x)-h(y)|\le K(|y-x|^\eta\wedge1)\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\beta}},
$$

which, combined with the space-time inequality (1.4) directly imply

$$
\forall\beta\in[0,1],\ |\mathrm{III}|\le K\frac{|s_1-s_2|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\beta}}g(c(t-r),z-x).
$$

This concludes the proof of the announced result on III. Gathering the previous estimates allow to conclude that (5.98) holds. With the previous result at hand, we derive an estimate for the second term appearing in the right-hand side of (5.97). More precisely, from (5.98), one gets $|\Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,s,r,v,x,z)|\le K|s_1-s_2|^\beta(v-r)^{-1+\frac{\eta}{2}}(r-s_1\vee s_2)^{-\beta}g(c(v-r),z-x)$, so that, after some standard computations, we get

$$
\Big|\int_r^t\int\Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,s,r,v,x,z)\Phi_{m+1}(\mu,s_1\vee s_2,v,t,z,y)\,dv\,dy\Big|\ \le K\frac{|s_1-s_2|^\beta}{(t-r)^{1-\eta}(r-s_1\vee s_2)^{\beta}}\,g(c(t-r),y-x),
$$

which in turn, from the identity (5.97), the estimate (5.98) and a direct induction argument, yield

$$\forall \beta \in [0,1], \quad |\Delta_{s_1,s_2}\Phi_{m+1}(\mu,s,r,t,x,y)| \leq K \frac{|s_1-s_2|^\beta}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^\beta}\, g(c(t-r),y-x). \tag{5.101}$$

From (5.38) we obtain the following decomposition

$$\Delta_{s_1,s_2}\partial_y^n[\partial_\mu \mathcal{H}_{m+1}(\mu,s,r,t,x,z)](y) = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V},$$

with

$$\begin{aligned}
\mathrm{I} = &-\sum_{i=1}^d \Delta_{s_1,s_2}\partial_y^n[\partial_\mu b_i(r,x,[X_r^{s,\xi,(m)}])](y) H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)})\,dv, z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
&-\sum_{i=1}^d \partial_y^n[\partial_\mu b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])](y)\Delta_{s_1,s_2} H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
&-\sum_{i=1}^d \partial_y^n[\partial_\mu b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])](y) H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z)\\
=:&\ \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3,
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} = &-\sum_{i=1}^d \Delta_{s_1,s_2} b_i(r,x,[X_r^{s,\xi,(m)}])\partial_y^n[\partial_\mu H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)](y)\,\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
&-\sum_{i=1}^d b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])\Delta_{s_1,s_2}\partial_y^n[\partial_\mu H_1^i\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)](y)\,\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
&-\sum_{i=1}^d b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])\partial_y^n[\partial_\mu H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)](y)\,\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z)\\
=:&\ \mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3,
\end{aligned}$$

$$\begin{aligned}
\mathrm{III} = &\frac{1}{2}\sum_{i,j=1}^d \Delta_{s_1,s_2}\partial_y^n[\partial_\mu[a_{i,j}(r,x,[X_r^{s,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s,\xi,(m)}])]](y) H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)\\
&\times\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
&+\frac{1}{2}\sum_{i,j=1}^d \partial_y^n[\partial_\mu[a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]](y)\Delta_{s_1,s_2} H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)\\
&\times\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\\
&+\frac{1}{2}\sum_{i,j=1}^d \partial_y^n[\partial_\mu[a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]](y) H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)\\
&\times\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z)\\
=:&\ \mathrm{III}_1+\mathrm{III}_2+\mathrm{III}_3,
\end{aligned}$$

$$
\begin{aligned}
\mathrm{IV} &= \frac{1}{2}\sum_{i,j=1}^{d}\Delta_{s_1,s_2}[a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])]\partial_y^n[\partial_\mu H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)](y)\\
&\quad\times \widehat{p}_{m+1}(\mu, s_1\vee s_2, r, t, x, z)\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{d}[a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]\Delta_{s_1,s_2}\partial_y^n[\partial_\mu H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)](y)\\
&\quad\times \widehat{p}_{m+1}(\mu, s_1\vee s_2, r, t, x, z)\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{d}[a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]\partial_y^n[\partial_\mu H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)](y)\\
&\quad\times \Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu, s, r, t, x, z)\\
&=: \mathrm{IV}_1+\mathrm{IV}_2+\mathrm{IV}_3,
\end{aligned}
$$

and finally

$$
\begin{aligned}
\mathrm{V} &= -\sum_{i=1}^{d}\Delta_{s_1,s_2}[b_i(r,x,[X_r^{s,\xi,(m)}])H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)]\partial_y^n[\partial_\mu\widehat{p}_{m+1}(\mu, s_1\vee s_2, r, t, x, z)](y)\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{d}\Delta_{s_1,s_2}[a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\right)]\\
&\quad\times\partial_y^n[\partial_\mu\widehat{p}_{m+1}(\mu, s_1\vee s_2, r, t, x, z)](y)\\
&\quad+\left\{-\sum_{i=1}^{d}b_i(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])H_1^i\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)\right.\\
&\quad\left.+\frac{1}{2}\sum_{i,j=1}^{d}[a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\wedge s_2,\xi,(m)}])\,dv, z-x\right)\right\}\\
&\quad\times\Delta_{s_1,s_2}\partial_y^n[\partial_\mu\widehat{p}_{m+1}(\mu, s, r, t, x, z)](y)\\
&=: \mathrm{V}_1+\mathrm{V}_2+\mathrm{V}_3.
\end{aligned}
$$

From similar arguments as those employed for the proofs of the estimates appearing in the decomposition of $\Delta_{\mu,\mu'}\partial_v^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s,t,x,z)](v)$, we obtain the following bounds:

$$
|\mathrm{I}| \le K\left\{\frac{1}{(t-r)^{\frac{1}{2}}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{1}{(t-r)^{\frac{1}{2}}}u_m^n(s_1,s_2,r)\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x),
$$

$$
|\mathrm{II}| \le K\left\{\frac{1}{(t-r)^{\frac{1}{2}}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{1}{(t-r)^{\frac{3}{2}}}\int_r^t u_m^n(s_1,s_2,v)\,dv\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x),
$$

$$
\begin{aligned}
|\mathrm{III}| &\le K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}+\frac{v_m^n(s_1,s_2,r)}{(t-r)^{1-\frac{\eta}{2}}}\wedge\frac{u_m^n(s_1,s_2,r)}{t-r}\right\}\\
&\quad\times|s_1-s_2|^\beta\, g(c(t-r),z-x),
\end{aligned}
$$

$$
|\mathrm{IV}| \le K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{1}{(t-r)^{2-\frac{\eta}{2}}}\int_r^t u_m^n(s_1,s_2,v)\,dv\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x),
$$

and

$$
|\mathrm{V}| \le K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{1}{(t-r)^{2-\frac{\eta}{2}}}\int_r^t u_m^n(s_1,s_2,v)\,dv\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x)
$$

where $\beta\in[0,1]$ except for III where $\beta\in[0,(1+\eta)/2)$ for $n=0$ and $\beta\in[0,\eta/2)$. We only prove the estimates on I and III. The estimates on II, IV and V follow from similar lines of reasonings and

technical details are omitted. In order to prove the announced estimate on I, we proceed as follows. For I_1, we use the decomposition

$$\Delta_{s_1,s_2}\partial_y^n[\partial_\mu b_i(r,x,[X_r^{s,\xi,(m)}])](y) = \mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D},$$

with

$$\mathrm{A} := \int \Delta_{s_1,s_2}\frac{\delta b_i}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')\,\partial_x^{1+n}p_m(\mu,s_1\vee s_2,r,y,y'')\,dy'',$$

$$\mathrm{B} := \int \Big(\frac{\delta b_i}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi',(m)}])(y'') - \frac{\delta b_i}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi',(m)}])(y)\Big).\,\Delta_{s_1,s_2}\partial_x^{1+n}p_m(\mu,s,r,y,y'')\,dy'',$$

$$\mathrm{C} := \int\int \Delta_{s_1,s_2}\frac{\delta b_i}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'')\,\partial_y^n[\partial_\mu p_m(\mu,s_1\vee s_2,r,x',y'')](y)\,dy''\,\mu(dx'),$$

$$\mathrm{D} := \int\int \Big(\frac{\delta b_i}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi',(m)}])(y'') - \frac{\delta b_i}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi',(m)}])(x')\Big).\Delta_{s_1,s_2}\partial_y^n[\partial_\mu p_m(\mu,s,r,x',y'')](y)\,dy''\,\mu(dx').$$

Similarly to (5.93) with the map $[\delta b_i/\delta m]$ instead of $a_{i,j}$ and (5.6), we get

$$|\mathrm{A}| \le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}.$$

For B, we combine (5.18) with the uniform η-Hölder regularity of $z\mapsto[\delta b_i/\delta m](r,x,\mu)(z)$

$$|\mathrm{B}| \le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}.$$

For C, similarly to A, using (5.12) instead of (5.6), we get

$$|\mathrm{C}| \le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}.$$

Finally, for the last term, from the uniform η-Hölder regularity of $z\mapsto[\delta b_i/\delta m](r,x,\mu)(z)$, one has

$$|\mathrm{D}| \le K\int\int(|y''-x'|^\eta\wedge 1)|\Delta_{s_1,s_2}\partial_y^n[\partial_\mu p_m(\mu,s,r,x',y'')](y)|\,dy''\,\mu'(dx').$$

Gathering the previous estimates and using the induction hypothesis, we finally obtain

$$|\Delta_{s_1,s_2}\partial_y^n[\partial_\mu b_i(r,x,[X_r^{s,\xi,(m)}])](y)| \le K\Big(\frac{1}{(r-s_1\vee s_2)^{\frac{1+n+\beta-\eta}{2}}} + u_m^n(s_1,s_2,r)\Big)\,|s_1-s_2|^\beta,$$

so that

$$|\mathrm{I}_1| \le K\left\{\frac{1}{(t-r)^{\frac12}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}} + \frac{1}{(t-r)^{\frac12}}u_m^n(s_1,s_2,r)\right\}|s_1-s_2|^\beta\,g(c(t-r),z-x).$$

From the mean-value theorem and (5.93), we get

$$\Big|\Delta_{s_1,s_2}H_1^i\Big(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv, z-x\Big)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)\Big| \le K|s_1-s_2|^\beta(t-r)^{-\frac12}(r-s_1\vee s_2)^{-\beta+\frac{\eta}{2}}g(c(t-r),z-x),$$

which with (5.30) imply

$$|\mathrm{I}_2| \le K\frac{|s_1-s_2|^\beta}{(t-r)^{\frac12}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,g(c(t-r),z-x).$$

Finally, from the mean value theorem and (5.93), one gets $|\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z)| \le K|s_1-s_2|^\beta(r-s_1\vee s_2)^{-\beta+\eta/2}g(c(t-r),z-x)$ which with (5.30) in turn imply

$$|\mathrm{I}_3| \le K\frac{|s_1-s_2|^\beta}{(t-r)^{\frac12}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,g(c(t-r),z-x).$$

Gathering the previous estimates, we obtain the announced estimate on I. In order to deal with III_1, we make use of the decomposition

$$\Delta_{s_1,s_2}\partial_y^n[\partial_\mu[a_{i,j}(r,x,[X_r^{s,\xi,(m)}]) - a_{i,j}(r,z,[X_r^{s,\xi,(m)}])]](y) = \mathrm{A}_{i,j}+\mathrm{B}_{i,j}+\mathrm{C}_{i,j}+\mathrm{D}_{i,j}$$

with

$$\mathrm{A}_{i,j} := \int \Delta_{s_1,s_2}\Big[\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')\Big]\,\partial_x^{1+n} p_m(\mu, s_1\vee s_2, r, y, y'')\,dy'',$$

$$\begin{aligned}\mathrm{B}_{i,j} := \int \Big([\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'') - \frac{\delta a_{i,j}}{\delta m} - (r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'')]\\ - [\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])(y) - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y)]\Big)\,\Delta_{s_1,s_2}\partial_x^{1+n} p_m(\mu,s,r,y,y'')\,dy'',\end{aligned}$$

$$\mathrm{C}_{i,j} := \int\int \Delta_{s_1,s_2}\Big[\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s,\xi,(m)}])(y'') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s,\xi,(m)}])(y'')\Big]\,\partial_y^n[\partial_\mu p_m(\mu,s_1\vee s_2,r,x',y'')](y)\,dy''\,\mu(dx'),$$

$$\begin{aligned}\mathrm{D}_{i,j} := \int\int \Big[\frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'') - \frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])(x')\\ - (\frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'') - \frac{\delta a_{i,j}}{\delta m}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}]))(x')\Big].\Delta_{s_1,s_2}\partial_y^n[\partial_\mu p_m(\mu,s,r,x',y'')](y)\,dy''\,\mu(dx').\end{aligned}$$

In order to handle $\mathrm{A}_{i,j}$, by the mean-value theorem

$$\begin{aligned}h(x) &:= \frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s_1\vee s_2,\xi,(m)}])(y'') - \frac{\delta a_{i,j}}{\delta m}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])(y'')\\ &= \int_0^1\int_{\mathbb{R}^d} \frac{\delta^2 a_{i,j}}{\delta m^2}(r,x,\Theta_{r,\lambda})(y'',z'')\Delta_{s_1,s_2}p_m(\mu,s,r,z'')\,dz''\,d\lambda.\end{aligned} \tag{5.102}$$

where we introduced the notation $\Theta_{\lambda,r} := \lambda[X_r^{s_1\vee s_2,\xi,(m)}] + (1-\lambda)[X_r^{s_1\wedge s_2,\xi,(m)}]$. We now make use of (5.91) and the uniform η-Hölder regularity of $x\mapsto[\delta^2 a_{i,j}/\delta m^2](r,x,\mu)(y'',z'')$

$$|h(x)-h(z)| \le K|x-z|^\eta \frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^\beta}, \tag{5.103}$$

so that with (5.6)

$$|\mathrm{A}_{i,j}| \le K|x-z|^\eta \frac{|s_1-s_2|^\beta}{(r-s)^{\frac{1+n}{2}+\beta}}.$$

From (5.102),

$$h(x) = \int_0^1\int_{\mathbb{R}^d}[\frac{\delta^2 a_{i,j}}{\delta m^2}(r,x,\Theta_{r,\lambda})(y'',z'') - \frac{\delta^2 a_{i,j}}{\delta m^2}(r,x,\Theta_{r,\lambda})(y'',z'')]\Delta_{s_1,s_2}p_m(\mu,s,r,x''z'')\,dz''\mu(dx'')\,d\lambda$$

so that the η-Hölder regularity of $z''\mapsto[\delta^2 a_{i,j}/\delta m^2](r,x,\Theta_{r,\lambda})(y'',z'')$ and (5.18) imply

$$|\Delta_{s_1,s_2}[\delta a_{i,j}/\delta m](r,y,[X_r^{s,\xi,(m)}])(z)| \le K|s_1-s_2|^\beta (r-s_1\vee s_2)^{-\beta+\eta/2},$$

for any $y,z\in\mathbb{R}^d$, which in turn yields

$$|\mathrm{I}_{i,j}| \le K\frac{W_2^\alpha(\mu,\mu')}{(r-s)^{\frac{1+n-\eta}{2}+\beta}}.$$

From the two previous estimates and the space-time inequality (1.4), we thus obtain

$$\begin{aligned}&\forall\beta\in[0,1],\ |\mathrm{A}_{i,j}H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv, z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)|\\ &\le K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x).\end{aligned} \tag{5.104}$$

From (5.18) and the uniform η-Hölder regularity of $x\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(y)$, we get

$$|\mathrm{B}_{i,j}| \le K(|y-x|^\eta\wedge 1)\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta}},$$

while employing the η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(y)$, we get

$$|\mathrm{B}_{i,j}| \le K\frac{|s_1-s_2|^\beta}{(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}},$$

so that

$$|\mathrm{B}_{i,j}H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv,z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)|$$
$$\leq K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x).$$

We deal with $\mathrm{C}_{i,j}$ similarly to $\mathrm{A}_{i,j}$ except that we use the estimate (5.12) instead of (5.6). Skipping some technical details, we obtain

$$|\mathrm{C}_{i,j}H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv,z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)|$$
$$\leq K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x).$$

For $\mathrm{D}_{i,j}$, on the one hand, from the η-Hölder regularity of $x\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(y)$, one gets $|\mathrm{D}_{i,j}|\leq K|s_1-s_2|^\beta|z-x|^\eta v_m^n(s_1,s_2,r)$ while, on the other hand, from the η-Hölder regularity of $y\mapsto[\delta a_{i,j}/\delta m](t,x,\mu)(y)$, one gets $|\mathrm{D}_{i,j}|\leq K|s_1-s_2|^\beta u_m^n(s_1,s_2,r)$. Hence, from the space-time inequality (1.4), we conclude

$$|\mathrm{D}_{i,j}H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s_1\vee s_2,\xi,(m)}])\,dv,z-x\right)\widehat{p}_{m+1}(\mu,s_1\vee s_2,r,t,x,z)|$$
$$\leq K\left\{\frac{v_m^n(s_1,s_2,r)}{(t-r)^{1-\frac{\eta}{2}}}\wedge\frac{u_m^n(s_1,s_2,r)}{t-r}\right\}|s_1-s_2|^\beta\, g(c(t-r),z-x).$$

Gathering the previous bound, we thus obtain

$$|\mathrm{III}_1|\leq K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{v_m^n(s_1,s_2,r)}{(t-r)^{1-\frac{\eta}{2}}}\wedge\frac{u_m^n(s_1,s_2,r)}{t-r}\right\}$$
$$\times|s_1-s_2|^\beta\, g(c(t-r),z-x),$$

with $\beta\in[0,(1+\eta)/2)$ of $n=0$ and $\beta\in[0,\eta/2)$ if $n=1$. In order to deal with III_2 and III_3, we employ (5.46) (note that we can bound $(r-s)^{(1+n-\eta)/2}u_m^n(s,r)$ and $(r-s)^{(1+n)/2}v_m^n(s,r)$ by a positive constant K independent of m) to bound the quantity $\partial_y^n[\partial_\mu[a_{i,j}(r,x,[X_r^{s_1\wedge s_2,\xi,(m)}])-a_{i,j}(r,z,[X_r^{s_1\wedge s_2,\xi,(m)}])]](y):=\mathrm{J}_{i,j}(y)$ as well as the mean value theorem and (5.93) to bound $\Delta_{s_1,s_2}H_2^{i,j}\left(\int_r^t a(v,z,[X_v^{s,\xi,(m)}])\,dv,z-x\right)$ and $\Delta_{s_1,s_2}\widehat{p}_{m+1}(\mu,s,r,t,x,z)$. For both quantities, we obtain

$$|\mathrm{III}_3|+|\mathrm{III}_2|\leq K\left\{\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\right\}$$
$$\times|s_1-s_2|^\beta\, g(c(t-r),z-x),$$

for all $\beta\in[0,1]$. Gathering the three previous estimates on III_1, III_2 and III_3 completes the proof of the announced estimate on III.

The induction hypothesis then allows to conclude

$$|\Delta_{s_1,s_2}\partial_v^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s,r,t,x,z)](v)|$$
$$\leq K\left\{\Big(\frac{1}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta}}\wedge\frac{1}{(t-r)(r-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\Big)\right.$$
$$\times(1+C_{m,n}(s_1\vee s_2,r))+\frac{1}{(t-r)^{2-\frac{\eta}{2}}}\int_r^t C_{m,n}(s_1\vee s_2,v)(v-s_1\vee s_2)^{-\frac{(1+n)}{2}-\beta+\eta}\,dv\Big\} \tag{5.105}$$
$$\times|s_1-s_2|^\beta\, g(c(t-r),z-x).$$

With the above estimates at hand we can now provide upper-bounds for the different terms appearing in the right-hand side of (5.92). The first estimate is given by (5.96). We thus consider the quantity $\Delta_{s_1,s_2}[\partial_v^n[\partial_\mu\widehat{p}_{m+1}]\otimes\Phi_{m+1}](\mu,s,t,x,z)$ and use the following decomposition

$$\Delta_{s_1,s_2}[\partial_v^n\partial_\mu[\widehat{p}_{m+1}]\otimes\Phi_{m+1}](\mu,s,t,x,z)(v)=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

with

$$\mathrm{I} := \int_{s_1\vee s_2}^t \int \Delta_{s_1,s_2}\partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s,r,x,y)](v)\Phi_{m+1}(\mu,s_1\vee s_2,r,t,y,z)\,dy\,dr,$$

$$\mathrm{II} := \int_{s_1\vee s_2}^t \int \partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s_1\wedge s_2,r,x,y)](v)\Delta_{s_1,s_2}\Phi_{m+1}(\mu,s,r,t,y,z)\,dy\,dr,$$

$$\mathrm{III} := -\int_{s_1\wedge s_2}^{s_1\vee s_2} \int \partial_v^n[\partial_\mu \widehat{p}_{m+1}(\mu,s_1\wedge s_2,r,x,y)](v)\Phi_{m+1}(\mu,s_1\wedge s_2,r,t,y,z)\,dy\,dr.$$

From (5.96) and using the fact that $v\mapsto C_{m,n}(s_1\vee s_2,v)$ is non-decreasing, we derive

$$\begin{aligned}|\mathrm{I}| &\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}g(c(t-s_1\wedge s_2),z-x)\\ &\quad + K|s_1-s_2|^\beta\int_{s_1\vee s_2}^t \frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\,g(c(t-s_1\wedge s_2),z-x).\end{aligned}$$

From (5.101) and (5.41), one gets

$$\forall\beta\in[0,1],\quad |\mathrm{II}|\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}g(c(t-s_1\wedge s_2),z-x).$$

Finally, using (5.41) together with the fact that $\beta\in[0,(1+\eta)/2)$ if $n=0$ and $\beta\in[0,\eta/2)$ if $n=1$, one obtains

$$|\mathrm{III}|\le K\frac{|s_1-s_2|^{\frac{1-n+\eta}{2}}}{(t-s_1\vee s_2)^{1-\frac{\eta}{2}}}g(c(t-s_1\wedge s_2),z-x)\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}g(c(t-s_1\wedge s_2),z-x),$$

where we used the fact that $|s_1-s_2|\le t-s_1\vee s_2$ for the last inequality. Gathering the previous estimates finally yields

$$\begin{aligned}|\Delta_{s_1,s_2}[\partial_v^n\partial_\mu[\widehat{p}_{m+1}]\otimes\Phi_{m+1}](\mu,s,t,x,z)(v)| &\le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}g(c(t-s_1\wedge s_2),z-x)\\ &\quad + K|s_1-s_2|^\beta\int_{s_1\vee s_2}^t \frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\,g(c(t-s_1\wedge s_2),z-x).\end{aligned}\tag{5.106}$$

We now turn our attention to the term $\Delta_{s_1,s_2}(p_{m+1}\otimes\partial_v^n[\partial_\mu\mathcal{H}_{m+1}])(\mu,s,t,x,z)(v)$ and make use of a similar decomposition, namely

$$\Delta_{s_1,s_2}(p_{m+1}\otimes\partial_v^n[\partial_\mu\mathcal{H}_{m+1}])(\mu,s,t,x,z)(v)=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

with

$$\mathrm{I} := \int_{s_1\vee s_2}^t \int \Delta_{s_1,s_2}p_{m+1}(\mu,s,r,x,y)\partial_v^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s_1\vee s_2,r,t,y,z)](v)\,dy\,dr,$$

$$\mathrm{II} := \int_{s_1\vee s_2}^t \int p_{m+1}(\mu,s_1\wedge s_2,r,x,y)\Delta_{s_1,s_2}\partial_v^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s,r,t,y,z)(v)\,dy\,dr,$$

and

$$\mathrm{III} := -\int_{s_1\wedge s_2}^{s_1\vee s_2} \int p_{m+1}(\mu,s_1\wedge s_2,r,x,y)\partial_v^n[\partial_\mu\mathcal{H}_{m+1}(\mu,s_1\wedge s_2,r,t,y,z)](v)\,dy\,dr.$$

From (5.47) (bounding $C_{m,n}$ by a positive constant K independent of m) and (5.91), breaking the time integral into the two intervals $[s_1\vee s_2,(t+s_1\vee s_2)/2)$ and $[(t+s_1\vee s_2)/2,t]$ to balance the time singularity, after some standard computations, we obtain

$$\forall\beta\in[0,1],\ |\mathrm{I}|\le K\left\{\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{1+n-\eta}{2}+\beta}}g(c(t-s_1),z-x)+\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{1+n-\eta}{2}+\beta}}g(c(t-s_2),z-x)\right\}.$$

To deal with II, we employ the estimate (5.105). For the first term appearing in the right-hand side of (5.105), we break the time integral into two intervals similarly to the previous estimate in order to balance the time singularity. For the second term, we bound the minimum of the two terms by the first

one, namely $(t-r)^{-1+\eta/2}(r-s_1\vee s_2)^{-(1+n-\eta)/2-\beta}$ while for the third term we use Fubini's theorem. After some standard computations, we obtain

$$|\mathrm{II}| \le K\left\{\frac{1}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}} + \int_{s_1\vee s_2}^t \frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\right\}$$
$$\times |s_1-s_2|^\beta g(c(t-s_1\wedge s_2),z-x),$$

where $\beta\in[0,(1+\eta)/2)$ if $n=0$ and $\beta\in[0,\eta/2)$ if $n=1$.

Finally, using (5.47) (bounding again $C_{m,n}$ by a positive constant K independent of m), we get

$$|\mathrm{III}| \le K\frac{|s_1-s_2|^{\frac{1-n+\eta}{2}}}{t-s_1\vee s_2}\,g(c(t-s_1\wedge s_2),z-x) \le K\frac{|s_1-s_2|^\beta}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\,g(c(t-s_1\wedge s_2),z-x),$$

where $\beta\in[0,\frac{1+\eta}{2})$ if $n=0$ and $\beta\in[0,\frac{\eta}{2})$ if $n=1$. Gathering the three previous estimates finally yields

$$\begin{aligned}
&|\Delta_{s_1,s_2}(p_{m+1}\otimes\partial_v^n[\partial_\mu\mathcal{H}_{m+1}])(\mu,s,t,x,z)(v)|\\
&\le K\left\{\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{1+n-\eta}{2}+\beta}}g(c(t-s_1),z-x)+\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{1+n-\eta}{2}+\beta}}g(c(t-s_2),z-x)\right\}\\
&+K|s_1-s_2|^\beta\int_{s_1\vee s_2}^t\frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\,g(c(t-s_1\wedge s_2),z-x).
\end{aligned}\tag{5.107}$$

For the last term, namely $\Delta_{s_1,s_2}((p_{m+1}\otimes\partial_y^n[\partial_\mu\mathcal{H}_{m+1}])\otimes\Phi_{m+1})(\mu,s,t,x,z)(v)$, as previously done, we decompose it as the sum of three terms in a completely analogous way as for the previous term. We then make use of (5.107), (5.101) and (5.48). Skipping some technical details, we obtain

$$\begin{aligned}
&|\Delta_{s_1,s_2}((p_{m+1}\otimes\partial_y^n[\partial_\mu\mathcal{H}_{m+1}])\otimes\Phi_{m+1})(\mu,s,t,x,z)(v)|\\
&\le K\left\{\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{1+n}{2}+\beta-\eta}}g(c(t-s_1),z-x)+\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{1+n}{2}+\beta-\eta}}g(c(t-s_2),z-x)\right\}\\
&+K|s_1-s_2|^\beta\int_{s_1\vee s_2}^t\frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\,g(c(t-s_1\wedge s_2),z-x).
\end{aligned}\tag{5.108}$$

Coming back to (5.92) and gathering the estimates (5.96), (5.106), (5.107) and (5.108) finally yield

$$\begin{aligned}
&|\Delta_{s_1,s_2}\partial_y^n[\partial_\mu p_{m+1}(\mu,s,t,x,z)](v)|\\
&\le K\Big(\Big[\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{1+n-\eta}{2}+\beta}}+\frac{|s_1-s_2|^\beta}{(t-s_1)^{\frac{\eta}{2}}}\int_{s_1\vee s_2}^t\frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\Big]g(c(t-s_1),z-x)\\
&+\Big[\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{1+n-\eta}{2}+\beta}}+\frac{|s_1-s_2|^\beta}{(t-s_2)^{\frac{\eta}{2}}}\int_{s_1\vee s_2}^t\frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\Big]g(c(t-s_2),z-x)\Big),
\end{aligned}$$

so that by the space-time inequality (1.4)

$$\begin{aligned}
u_{m+1}^n(s_1,s_2,t) &\le K\Big(\frac{1}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}+\int_{s_1\vee s_2}^t\frac{C_{m,n}(s_1\vee s_2,r)}{(t-r)^{1-\frac{\eta}{2}}(r-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\,dr\Big)\\
&\le \frac{K}{(t-s_1\vee s_2)^{\frac{1+n}{2}+\beta-\eta}}\left\{B\left(\frac{\eta}{2},\frac{1-n}{2}-\beta+\eta\right)\right.\\
&\left.+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}-\beta+(i-1)\frac{\eta}{2}\right)(t-s_1\vee s_2)^{k\frac{\eta}{2}}\right\}
\end{aligned}$$

and similarly

$$\begin{aligned}
v_{m+1}^n(s_1,s_2,t) &\le \frac{K}{(t-s_1\vee s_2)^{\frac{1+n-\eta}{2}+\beta}}\left\{B\left(\frac{\eta}{2},\frac{1-n}{2}-\beta+\eta\right)\right.\\
&\left.+\sum_{k=1}^m C^k\prod_{i=1}^{k+1}B\left(\frac{\eta}{2},\frac{1-n+\eta}{2}-\beta+(i-1)\frac{\eta}{2}\right)(t-s_1\vee s_2)^{k\frac{\eta}{2}}\right\}.
\end{aligned}$$

In a completely analogous manner as for the previous estimates, we thus derive that the induction hypothesis remains valid at step $m+1$. Coming back to (5.92) and bounding each term using our estimates combined with the two previous bounds, from the asymptotics of the Beta function, we deduce that (5.19) is valid at step $m+1$. The proof of the proposition is now complete.

6. Solving the related PDE on the Wasserstein space

This section is devoted to the proof of Theorem 3.11. Thanks to the regularity properties provided by Theorem 3.10, we are able to tackle the Cauchy problem (1.2) on any strip $[0,T]$. We first start with the following Proposition.

Proposition 6.1. Under the assumptions of Theorem 3.11, the mapping $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,x,\mu)\mapsto U(t,x,\mu)$ defined by (3.26) is continuous, belongs to $\mathcal{C}^{1,2,2}([0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, satisfies (3.27) and for any $(t,x,v,\mu)\in[0,T)\times(\mathbb{R}^d)^2\times\mathcal{P}_2(\mathbb{R}^d)$

$$\left|\partial_v^n[\partial_\mu U(t,x,\mu)](v)\right|\leq C(T-t)^{-\frac{(1+n)}{2}}\exp(\frac{k|x|^2}{T})(1+|v|^2+M_2^q(\mu)),\ n=0,1, \tag{6.1}$$

and

$$\left|\partial_x U(t,x,\mu)\right|\leq K(T-t)^{-\frac{1}{2}}\exp(\frac{k|x|^2}{T})(1+M_2^q(\mu)),\ n=0,1, \tag{6.2}$$

where $C:=C(T,b,\delta b/\delta m,a,\delta a/\delta m,\lambda,\eta)$, $K:=K(T,b,a,\lambda,\eta)$ and $k:=k(\lambda,\alpha)$ are positive constants. Moreover, U is a solution to the Cauchy problem (1.2) on the strip $[0,T]$.

Proof. We first remark that if $(\mu_n)_{n\geq1}$ is a sequence of $\mathcal{P}_2(\mathbb{R}^d)$ and if $(t_n)_{n\geq1}$ is a sequence of $[0,T)$ both satisfying $\lim_n|t_n-t|=\lim_n W_2(\mu_n,\mu)=0$, for some $(t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$, then, by weak uniqueness, $([X_T^{t_n,\xi_n}])_{n\geq1}$ weakly converges to $[X_T^{t,\xi}]$, where $[\xi_n]=\mu_n$ and $[\xi]=\mu$, so that, passing to the limit in the parametrix infinite series (3.11) and using the relation (3.15), we deduce that $\int_{\mathbb{R}^d}|z|^2p(\mu_n,t_n,T,z)\,dz\to\int_{\mathbb{R}^d}|z|^2p(\mu,t,T,z)\,dz$ which in turn yields $\lim_n W_2([X_T^{t,\xi_n}],[X_T^{t,\xi}])=0$. By continuity of h, f, we deduce that the two maps $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,\mu)\mapsto h(z,[X_T^{t,\xi}])$, $[0,s]\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,\mu)\mapsto f(s,z,[X_s^{t,\xi}])$ are continuous so that the mapping $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,x,\mu)\mapsto U(t,x,\mu)$ is also continuous.

We now prove that $\mathcal{P}_2(\mathbb{R}^d)\ni(x,\mu)\mapsto U(t,x,\mu)\in\mathcal{C}^{2,2}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$, for $t\in[0,T)$ and that $[0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,x,\mu)\mapsto\mathcal{L}_tU(t,x,\mu)$ is continuous, where the operator $\mathcal{L}_t$ is defined by (1.3).

From Theorem 3.10 and the relation (3.15), the map $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto p(\mu,t,T,z)$ is partially $\mathcal{C}^2(\mathcal{P}_2(\mathbb{R}^d))$ (see Chapter 5 of [CD18] for a definition of partial $\mathcal{C}^2(\mathcal{P}_2(\mathbb{R}^d))$ regularity) with derivatives given by

$$\partial_v^n[\partial_\mu p(\mu,t,T,z)](v)=\partial_x^{1+n}p(\mu,t,T,v,z)+\int_{\mathbb{R}^d}\partial_v^n[\partial_\mu p(\mu,t,T,x,z)](v)\,\mu(dx),\quad n=0,\,1.$$

From Proposition 2.2, we thus deduce that the two maps $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto h(z,[X_T^{t,\xi}])$, $\mathcal{P}_2(\mathbb{R}^d)\ni\mu\mapsto f(s,z,[X_s^{t,\xi}])$ are partially $\mathcal{C}^2(\mathcal{P}_2(\mathbb{R}^d))$ for any fixed $T>0$, $s>t\geq0$ and $z\in\mathbb{R}^d$. Moreover, by Fubini's theorem, their L-derivatives are given by

$$\begin{aligned}\partial_v^n[\partial_\mu h(z,[X_T^{t,\xi}])](v)&=\int_{\mathbb{R}^d}\frac{\delta h}{\delta m}(z,[X_T^{t,\xi}])(y)\,\partial_x^{1+n}p(\mu,t,T,v,y)\,dy\\&\quad+\int_{(\mathbb{R}^d)^2}\frac{\delta h}{\delta m}(z,y,[X_T^{t,\xi}])\,\partial_v^n[\partial_\mu p(\mu,t,T,x,y)](v)\,dy\,\mu(dx)\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\partial_v^n[\partial_\mu f(s,z,[X_s^{t,\xi}])](v)&=\int_{\mathbb{R}^d}\Big[\frac{\delta f}{\delta m}(s,z,[X_s^{t,\xi}])(y)-\frac{\delta f}{\delta m}(s,z,[X_s^{t,\xi}])(v)\Big]\,\partial_x^{1+n}p(\mu,t,s,v,y)\,dy\\&\quad+\int_{(\mathbb{R}^d)^2}\frac{\delta f}{\delta m}(s,z,[X_s^{t,\xi}])(y)\,\partial_v^n[\partial_\mu p(\mu,t,s,x,y)](v)\,dy\,\mu(dx).\end{aligned} \tag{6.4}$$

We may break the integral appearing in the right-hand side of (6.4) into two parts J_1 and J_2 by dividing the domain of integration into two domains. In the first part J_1, the dy-integration is taken over a bounded domain D containing v such that $|y-v|\geq1$ if $y\notin D$. Using the η-Hölder regularity of $y\mapsto\frac{\delta f}{\delta m}(s,z,[X_s^{t,\xi}])(y)$ on D, (3.13) and the space-time inequality (1.4), we get

$$|\mathrm{J}_1|\leq C(s-t)^{\frac{-1-n+\eta}{2}}.$$

As for J_2, for $\alpha<c:=c(\lambda)$, where c is the constant appearing in (3.13), from (3.25), the space-time inequality (1.4) and noting that $M_2([X_T^{t,\xi}])\leq C(1+M_2(\mu))$, we obtain

$$\begin{aligned}|\mathrm{J}_2|&\leq C\exp\left(\alpha\frac{|z|^2}{T}\right)\int_{|y-v|\geq1}(s-t)^{-1}(1+|y|^2+M_2^q(\mu))\,g(c(s-t),y-v)\,dy\\&\leq C\exp\left(\alpha\frac{|z|^2}{T}\right)(s-t)^{\frac{-1-n+\eta}{2}}(1+|v|^2+M_2^q(\mu)).\end{aligned}$$

Also, from (3.16) and (3.25), we derive

$$\Big|\int_{(\mathbb{R}^d)^2}\frac{\delta f}{\delta m}(s,z,[X_s^{t,\xi}])(y)\,\partial_v^n[\partial_\mu p(\mu,t,s,x,y)](v)\,dy\,\mu(dx)\Big|\le C\exp\left(\alpha\frac{|z|^2}{T}\right)(s-t)^{\frac{-1-n+\eta}{2}}(1+M_2^q(\mu)).$$

Gathering the previous estimates, we obtain

$$|\partial_v^n[\partial_\mu f(s,z,[X_s^{t,\xi}])](v)|\le C\exp\left(\alpha\frac{|z|^2}{T}\right)(s-t)^{\frac{-1-n+\eta}{2}}(1+|v|^2+M_2^q(\mu)). \tag{6.5}$$

From (6.3), (3.13), (3.25) and similar computations

$$|\partial_v^n[\partial_\mu h(z,[X_T^{t,\xi}])](v)|\le C\exp\left(\alpha\frac{|z|^2}{T}\right)(T-t)^{-\frac{(1+n)}{2}}(1+|v|^2+M_2^q(\mu)). \tag{6.6}$$

The estimates (3.13), (3.16) and (6.5) allow to conclude that the map $(x,\mu)\mapsto U(t,x,\mu)$ is in $\mathcal{C}^{2,2}(\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ with derivatives given by

$$\begin{aligned}\partial_v^n[\partial_\mu U(t,x,\mu)](v)&=\int_{\mathbb{R}^d}h(z,[X_T^{t,\xi}])\,\partial_v^n[\partial_\mu p(\mu,t,T,x,z)](v)\,dz+\int_{\mathbb{R}^d}\partial_v^n[\partial_\mu h(z,[X_T^{t,\xi}])](v)\,p(\mu,t,T,x,z)\,dz\\&\quad-\int_t^T\int_{\mathbb{R}^d}\partial_v^n[\partial_\mu f(s,z,[X_s^{t,\xi}])](v)\,p(\mu,t,s,x,z)\,dz\,ds\\&\quad-\int_t^T\int_{\mathbb{R}^d}f(s,z,[X_s^{t,\xi}])\,\partial_v^n[\partial_\mu p(\mu,t,s,x,z)](v)\,dz\,ds\end{aligned} \tag{6.7}$$

for $n=0$, 1 and

$$\begin{aligned}\partial_x^n U(t,x,\mu)&=\int_{\mathbb{R}^d}h(z,[X_T^{t,\xi}])\,\partial_x^n p(\mu,t,T,x,z)\,dz\\&\quad-\int_t^T\int_{\mathbb{R}^d}[f(s,z,[X_s^{t,\xi}])-f(s,x,[X_s^{t,\xi}])]\,\partial_x^n p(\mu,t,s,x,z)\,dz\,ds\end{aligned} \tag{6.8}$$

for $n=0$, 1, 2. Note that we may break the last integral appearing in the right-hand side of (6.8) into two parts by dividing the domain of integration into two domains as we did before. Then, using the local Hölder continuity of $z\mapsto f(s,z,\mu)$, (3.24), the estimate (3.13), we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}[f(s,z,[X_s^{t,\xi}])-f(s,x,[X_s^{t,\xi}])]\,\partial_x^{1+n}p(\mu,t,s,x,z)\,dz\Big|\\&\quad\le C(s-t)^{\frac{-1-n+\eta}{2}}\left\{\int_{\mathbb{R}^d}e^{\alpha\frac{|z|^2}{T}}g(c(s-t),z-x)\,dz+e^{\alpha\frac{|x|^2}{T}}\right\}(1+M_2^q(\mu))\\&\quad\le C(s-t)^{\frac{-1-n+\eta}{2}}e^{k\frac{|x|^2}{T}}(1+M_2^q(\mu))\end{aligned}$$

for some positive constant $k:=k(c,\alpha)$, $\alpha\mapsto k(c,\alpha)$ being non-decreasing, where we used the fact that the constant α is sufficiently small, namely $\alpha<1/(2c)$, c being the constant appearing in (3.13) and the inequality: for any positive constants α and c' satisfying $0<\alpha<c'$, there exists a positive constant $C:=C(c',\alpha)$ (take e.g. $C=c'\alpha/(c'-\alpha)$) such that for any $(z,x)\in(\mathbb{R}^d)^2$,

$$\alpha|z|^2-c'|z-x|^2\le C|x|^2. \tag{6.9}$$

The previous estimate as well as (6.5) and (3.16) ensure that the integrals appearing in (6.7) and (6.8) are well defined if α is sufficiently small. We thus conclude from (6.7) and (6.8) that $[0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\ni(t,x,\mu)\mapsto\mathcal{L}_tU(t,x,\mu)$ is continuous.

Finally, from (3.12), (3.24) and (6.9), we get

$$\begin{aligned}|U(t,x,\mu)|&\le C\left\{\int_{\mathbb{R}^d}\exp\Big(\alpha\frac{|z|^2}{T}\Big)g(c(T-t),z-x)\,dz+\int_t^T\int_{\mathbb{R}^d}\exp\Big(\alpha\frac{|z|^2}{T}\Big)g(c(s-t),z-x)\,dz\,ds\right\}\\&\quad\times(1+M_2^q(\mu))\\&\le C\exp\Big(\frac{k|x|^2}{T}\Big)(1+M_2^q(\mu))\end{aligned}$$

and the proof of (6.2) follows similarly from (3.13) (with $n=1$), (3.24) and (6.9). The proof of (6.1) is a consequence from (6.5), (6.6) and (6.7).

Let us now prove that U is in $\mathcal{C}^{1,2,2}([0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$. From the Markov property satisfied by the SDE (1.1) (which is inherited from the well-posedness of the associated martingale problem) we obtain the following identity

$$U(t-h,x,\mu)=\mathbb{E}\left[U(t,X_t^{t-h,x,\mu},[X_t^{t-h,\xi}])-\int_{t-h}^t f(r,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}])dr\right],\quad h>0.$$

The chain rule formula of Proposition 2.1 (with respect to the space and measure variables only) together with the estimate (6.1) yield

$$\begin{aligned}U(t,X_t^{t-h,x,\mu},[X_t^{t-h,\xi}])&=U(t,x,\mu)+\int_{t-h}^t\mathcal{L}_rU(t,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}])dr\\&\quad+\int_{t-h}^t\partial_xU(t,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}]).\sigma(r,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}])dW_r.\end{aligned}$$

From (6.2) and (3.12), it follows that the last term appearing in the right-hand side of the previous equality is a square integrable martingale if α is sufficiently small. Hence,

$$\mathbb{E}\left[U(t,X_t^{t-h,x,\mu},[X_t^{t-h,\xi}])\right]=U(t,x,\mu)+\mathbb{E}\left[\int_{t-h}^t\mathcal{L}_rU(t,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}])dr\right].$$

and

$$\frac{1}{h}\left(U(t-h,x,\mu)-U(t,x,\mu)\right)=\frac{1}{h}\mathbb{E}\left[\int_{t-h}^t\left\{\mathcal{L}_rU(t,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}])-f(r,X_r^{t-h,x,\mu},[X_r^{t-h,\xi}])\right\}dr\right]$$

so that letting $h\downarrow 0$, from the boundedness and continuity of the coefficients, we deduce that U is left differentiable in time at any time $t\in[0,T)$. Still from the continuity of the coefficients and f, we then conclude that it is differentiable in time with

$$\partial_tU(t,x,\mu)=-\mathcal{L}_tU(t,x,\mu)+f(t,x,\mu).$$

Hence, the map U solves the PDE (3.26). □

In order to get the uniqueness result, first fix any $0\le t\le s<T$ and consider any solution V to the Cauchy problem (1.2) satisfying (2.6) on any interval $[0,T']$, with $T'<T$, as well as (3.27). We apply the chain rule formula of Proposition 2.1 to $\left\{V(s,X_s^{t,x,\mu},[X_s^{t,\xi}]),\,t\le s<T\right\}$ and use the fact that $(\partial_t+\mathcal{L}_t)V(t,x,\mu)=f(t,x,\mu)$, for $(t,x,\mu)\in[0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ to get

$$\begin{aligned}V(s,X_s^{t,x,\mu},[X_s^{t,\xi}])&=V(t,x,\mu)+\int_t^s f(r,X_r^{t,x,\mu},[X_r^{t,\xi}])\,dr\\&\quad+\int_t^s\sum_{i=1}^d\sum_{j=1}^q\sigma_{i,j}(r,X_r^{t,x,\mu},[X_r^{t,\xi}])\,\partial_{x_i}V(r,X_r^{t,x,\mu},[X_r^{t,\xi}])dB_r^j.\end{aligned}$$

The local martingale appearing in the right-hand side of the above equality is in fact a true martingale since $V(s,X_s^{t,x,\mu},[X_s^{t,\xi}])$ and $\int_t^s f(r,X_r^{t,x,\mu},[X_r^{t,\xi}])\,dr$ are both square integrable if the constant α and k appearing in the two conditions (3.24) and (3.27) are small enough, that is, α and k strictly less than $1/4c$, $c:=c(\lambda)$ being the constant appearing in (3.12) is sufficient.

Hence, taking expectation in the previous equality, then passing to the limit as $s\uparrow T$ and using the continuity assumption at the boundary, we obtain

$$V(t,x,\mu)=\mathbb{E}[h(X_T^{t,x,\mu},[X_T^{t,\xi}])-\int_t^T f(r,X_r^{t,x,\mu},[X_r^{t,\xi}])\,dr]$$

which completes the proof of Theorem 3.11.

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