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Algebraic structures on typed decorated rooted trees

Loïc Foissy

► **To cite this version:**

Loïc Foissy. Algebraic structures on typed decorated rooted trees. SIGMA (Symetry, Integrability and Geometry: Methods and Applications), 17 (086), pp.28, 2021, Algebraic Structures in Perturbative Quantum Field Theory in honor of Dirk Kreimer for his 60th birthday, 10.3842/SIGMA.2021.086 . hal-01924416v3

HAL Id: hal-01924416

<https://hal.science/hal-01924416v3>

Submitted on 21 Sep 2021

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Consequently, if λ and μ are both nonzero, the Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\mu}$ are isomorphic; dually, the Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\mu}$ are isomorphic (Corollary 5.7). Using Livernet's rigidity theorem [16] and a nonassociative permutative coproduct defined in Proposition 4.1, we prove that if $\lambda \neq 0$, then $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is, as a pre-Lie algebra, freely generated by a family of typed trees $\mathcal{D}' = \mathcal{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ satisfying a condition on the type of edges born from the root (Corollary 4.2). As a consequence, the Hopf algebra $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ of typed and decorated trees is isomorphic to a Connes–Kreimer Hopf algebra of non typed and decorated trees $\mathcal{H}_{\mathcal{D}'}$, and an explicit isomorphism is described with the help of contraction in Proposition 5.9.

This paper is organized as follows: the first section gives the basic definition of typed rooted trees and enumeration results, when the number of types and decorations are finite. The second section is about the \mathcal{T} -multiple pre-Lie algebra structures on these trees and the underlying operads. The freeness of the pre-Lie structures on typed decorated trees and its consequences are studied in the third section. In the last section, the dual Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ are defined, studied and related to the constructions of Bruned, Hairer and Zambotti [3, 5]: forgetting the labels, the two coproducts they use on a family of typed and partially decorated trees are a subquotient of a the construction presented here.

Since the first version of this paper, written in 2018, multiple pre-Lie algebras have appeared in [4, 6] in a context of SPDEs. When the set of types has more structures (for example a product making it a commutative semigroup), then other possibilities for grafting products of typed trees are studied in [13, 18, 25], giving other interesting structures on these objects.

Notation 1.1.

- We denote by \mathbb{K} a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, pre-Lie algebras, ...) in this text will be taken over \mathbb{K} .
- For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \dots, n\}$.
- For any set \mathcal{T} , we denote by $\mathbb{K}^{\mathcal{T}}$ the set of family $\lambda = (\lambda_t)_{t \in \mathcal{T}}$ of elements of \mathbb{K} indexed by \mathcal{T} , and we denote by $\mathbb{K}^{(\mathcal{T})}$ the set of elements $\lambda \in \mathbb{K}^{\mathcal{T}}$ with a finite support. Note that if \mathcal{T} is finite, then $\mathbb{K}^{\mathcal{T}} = \mathbb{K}^{(\mathcal{T})}$.

2 Typed decorated trees

2.1 Definition

Definition 2.1. Let \mathcal{D} and \mathcal{T} be two nonempty sets.

1. A \mathcal{D} -decorated \mathcal{T} -typed forest is a triple $(F, \text{dec}, \text{type})$, where
 - F is a rooted forest. The set of its vertices is denoted by $V(F)$ and the set of its edges by $E(F)$.
 - $\text{dec}: V(F) \longrightarrow \mathcal{D}$ is a map.
 - $\text{type}: E(F) \longrightarrow \mathcal{T}$ is a map.

If the underlying rooted forest of F is connected, we shall say that F is a \mathcal{D} -decorated \mathcal{T} -typed tree.

2. If $(F, \text{dec}_F, \text{type}_F)$ and $(G, \text{dec}_G, \text{type}_G)$ are two \mathcal{D} -decorated \mathcal{T} -typed forests, they are isomorphic if there exists a rooted forest isomorphism f from F to G such that for any vertex v of F , $\text{dec}_G(f(v)) = \text{dec}_F(v)$ and for any edge e of F , $\text{type}_G(f(e)) = \text{type}_F(e)$
3. For any finite set A , we denote by $\mathbb{T}_{\mathcal{T}}(A)$ the set of A -decorated \mathcal{T} -typed trees T such that $V(T) = A$ and $\text{dec} = \text{Id}_A$, and by $\mathbb{F}_{\mathcal{T}}(A)$ the set of A -decorated \mathcal{T} -typed forests F such that $V(F) = A$ and $\text{dec} = \text{Id}_A$.

As any element of $\mathbb{F}_{\mathcal{T},\mathcal{D}}$ can be uniquely decomposed as the disjoint union of its connected components, which are elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, we obtain

$$F_{D,T}(X) = \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)}}. \quad (2.1)$$

We put $\mathcal{T} = \{t_1, \dots, t_T\}$. For any $d \in \mathcal{D}$, we consider

$$B_d: \begin{cases} (\mathbb{F}_{\mathcal{D},\mathcal{T}})^T \longrightarrow \mathbb{T}_{\mathcal{D},\mathcal{T}}, \\ (F_1, \dots, F_T) \longmapsto B_d(F_1, \dots, F_T), \end{cases}$$

where $B_d(F_1, \dots, F_T)$ is the tree obtained by grafting the forests F_1, \dots, F_n on a common root decorated by d ; the edges from this root to the roots of F_i are of type t_i for any $1 \leq i \leq T$. Then B_d is injective, homogeneous of degree 1, and moreover $\mathbb{T}_{\mathcal{D},\mathcal{T}}$ is the disjoint union of the $B_d((\mathbb{F}_{\mathcal{D},\mathcal{T}})^T)$, $d \in \mathcal{D}$. Hence,

$$T_{D,T}(X) = DX(F_{D,T})^T = DX \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)T}}. \quad (2.2)$$

Note that (2.2) allows to compute $t_{D,T}(n)$ by induction on n , and (2.1) allows to deduce $f_{D,T}(n)$.

Lemma 2.4. *For any $n \in \mathbb{N}$,*

$$t_{D,T}(n) = \frac{t_{TD,1}(n)}{T}.$$

Proof. By induction on n . If $n = 1$, $t_{D,T}(1) = D$ and $t_{TD,1}(1) = TD$, which gives the result. Let us assume the result at all ranks $k < n$. Then $t_{D,T}(n)T$ is the coefficient of X^n in

$$TDX \prod_{k=1}^{n-1} \frac{1}{(1 - X^k)^{t_{D,T}(k)T}} = TDX \prod_{k=1}^{n-1} \frac{1}{(1 - X^k)^{t_{TD,1}(k)}},$$

which is precisely $t_{TD,1}(n)$. ■

Example 2.5. We obtain

$$\begin{aligned} t_{D,T}(1) &= D, \\ t_{D,T}(2) &= D^2T, \\ t_{D,T}(3) &= \frac{D^2T(3DT + 1)}{2}, \\ t_{D,T}(4) &= \frac{D^2T(8D^2T^2 + 3DT + 1)}{3}, \\ t_{D,T}(5) &= \frac{D^2T(125D^3T^3 + 54D^2T^2 + 31DT + 6)}{24}, \\ t_{D,T}(6) &= \frac{D^2T(162D^4T^4 + 80D^3T^3 + 45D^2T^2 + 10DT + 3)}{15}, \\ t_{D,T}(7) &= \frac{D^2T(16807D^5T^5 + 9375D^4T^4 + 5395D^3T^3 + 2025D^2T^2 + 838DT + 120)}{720}. \end{aligned}$$

We shall give tables of values of $t_{D,T}(k)$ in the appendix.

Lemma 3.6. *If for any $t \in \mathcal{T}$, \bullet_t is a bilinear product on a vector space V , we define $\bullet : (V^{\oplus \mathcal{T}})^{\otimes 2} \rightarrow V^{\oplus \mathcal{T}}$ by*

$$x \partial_t \bullet x' \partial_{t'} = (x \bullet_{t'} y) \partial_t.$$

Then $(V, (\bullet_t)_{t \in \mathcal{T}})$ is a \mathcal{T} -multiple pre-Lie algebra if, and only if, $(V^{\oplus \mathcal{T}}, \bullet)$ is a pre-Lie algebra.

Proof. Let $x, x', x'' \in V$, $t, t', t'' \in \mathcal{T}$. Then

$$x \partial_t \bullet (x' \partial_{t'} \bullet x'' \partial_{t''}) - (x \partial_t \bullet x' \partial_{t'}) \bullet x'' \partial_{t''} = ((x \bullet_{t'} x') \bullet_{t''} x'' - x \bullet_{t'} (x' \bullet_{t''} x'')) \partial_t,$$

which implies the result. ■

Notation 3.7. The symmetric algebra $S(V)$ is given with its usual coproduct Δ , making it a bialgebra:

$$\forall x \in V, \quad \Delta(x) = x \otimes 1 + 1 \otimes x.$$

We shall use Sweedler's notation: for any $w \in S(V)$, $\Delta(w) = \sum w^{(1)} \otimes w^{(2)}$. The counit of this coproduct is denoted by ε : this is the unique algebra morphism from $S(V)$ to \mathbb{K} sending any $v \in V$ to 0.

Theorem 3.8. *Let V be a \mathcal{T} -multiple pre-Lie algebra. One can define a product*

$$\bullet : S(V) \otimes S(V^{\oplus \mathcal{T}}) \rightarrow S(V)$$

in the following way: for any $u, v \in S(V)$, $w \in S(V^{\oplus \mathcal{T}})$, $x \in V$, $t \in \mathcal{T}$,

$$\begin{aligned} 1 \bullet w &= \varepsilon(w), \\ u \bullet 1 &= u, \\ uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}), \\ u \bullet w(x \partial_t) &= (u \bullet w) \bullet_t x - x \bullet (w \bullet_t x), \end{aligned}$$

where \bullet_t is extended to $S(V) \otimes V$ and $S(V^{\oplus \mathcal{T}}) \otimes V$ by the following: for any $x_1, \dots, x_k, x \in V$, for any $t_1, \dots, t_k \in \mathcal{T}$,

$$\begin{aligned} x_1 \cdots x_k \bullet_t x &= \sum_{i=1}^k x_1 \cdots (x_i \bullet_t x) \cdots x_k, \\ (x_1 \partial_{t_1}) \cdots (x_k \partial_{t_k}) \bullet_t x &= \sum_{i=1}^k (x_1 \partial_{t_1}) \cdots ((x_i \bullet_t x) \partial_{t_i}) \cdots (x_k \partial_{t_k}). \end{aligned}$$

Proof. Uniqueness. The last formula allows to compute $x \bullet w$ for any $x \in V$ and $w \in S(V^{\oplus \mathcal{T}})$ by induction on the length of w ; the other ones allow to compute $u \bullet w$ for any $u \in S(V)$ by induction on the length on u . So this product \bullet is unique.

Existence. Let us use the Guin–Oudom construction [21, 22] on the pre-Lie algebra $V^{\otimes \mathcal{T}}$. We obtain a product \bullet defined on $S(V^{\oplus \mathcal{T}})$ such that for any $u, v, w \in S(V^{\oplus \mathcal{T}})$, $x \in V^{\oplus \mathcal{T}}$:

$$\begin{aligned} 1 \bullet w &= \varepsilon(w), \\ u \bullet 1 &= u, \\ uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}), \\ u \bullet wx &= (u \bullet w) \bullet x - x \bullet (w \bullet x). \end{aligned}$$

Let $f: \mathcal{T} \longrightarrow \mathbb{K}$ be any nonzero map. We consider the surjective algebra morphism $F: S(V^{\oplus \mathcal{T}}) \longrightarrow S(V)$, sending $x\partial_t$ to $f(t)x$ for any $x \in V$, $t \in \mathcal{T}$. Its kernel is generated by the elements $X_{t,t'}x = (f(t')\partial_t - f(t)\partial_{t'})x$, where $x \in V$ and $t, t' \in \mathcal{T}$. We denote by J the vector space generated by the elements $X_{t,t'}x$. Let us prove that for any $w \in S(V^{\oplus \mathcal{T}})$, $J \bullet w \subseteq J$ by induction on the length n of w . If $n = 0$, we can assume that $w = 1$ and this is obvious. If $n = 1$, we can assume that $w = x'\partial_{t'}$. Then

$$X_{t,t'}x \bullet w = (f(t')\partial_t - f(t)\partial_{t'})x \bullet_{t'} x' = X_{t,t'}x \bullet_{t'} x' \in J.$$

Let us assume the result at rank $n - 1$. We can assume that $w = w'x'\partial_t$, the length of w' being $n - 1$. For any $x \in J$,

$$x \bullet w = (x \bullet w') \bullet x' - x \bullet (w' \bullet x').$$

The length of w' and $w' \bullet x'$ is $n - 1$, so $x \bullet w'$ and $x \bullet (w' \bullet x')$ belong to J . From the case $n = 1$, $(x \bullet w') \bullet x' \in J$, so $x \bullet w \in J$.

For any $x \in J$, $u, v \in S(V^{\oplus \mathcal{T}})$,

$$xu \bullet v = \underbrace{(x \bullet v^{(1)})}_{\in J} (u \bullet v^{(2)}) \in \text{Ker}(F).$$

This proves that $\text{Ker}(F) \bullet S(V^{\oplus \mathcal{T}}) \subseteq \text{Ker}(F)$. Hence, \bullet induces a product also denoted by \bullet , defined from $S(V) \otimes S(V^{\oplus \mathcal{T}})$ to $S(V)$. It is not difficult to show that it does not depend on the choice of f and satisfies the required properties. \blacksquare

Definition 3.9. Let $d \in \mathcal{D}$, $T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $t_1, \dots, t_k \in \mathcal{T}$. We denote by

$$B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right)$$

the \mathcal{T} -typed \mathcal{D} -decorated tree obtained by grafting T_1, \dots, T_k on a common root decorated by d , the edge between this root and the root of T_i being of type t_i for any i . This defines a map $B_d: S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}})^{\oplus \mathcal{T}}) \longrightarrow S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}}))$.

Lemma 3.10. For any $d \in \mathcal{D}$, $T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $t_1, \dots, t_k \in \mathcal{T}$,

$$B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) = \bullet^d \bullet \prod_{i \in [k]} T_i \partial_{t_i}.$$

Proof. We write $F = \prod_{i \in [k]} T_i \partial_{t_i}$: the integer k is unique and denoted by $\ell(F)$. We proceed by induction on $\ell(F)$. If $\ell(F) = 0$, then $F = 1$ and $\bullet^d \bullet 1 = \bullet^d = B_d(1)$. let us assume the result for any forest F' with $\ell(F') = \ell(F) - 1$. Putting $k = \ell(F)$, we can write $F = F'T\partial_t$, with $\ell(F') = \ell(F) - 1$, $T = T_k$ and $t = t_k$. Then

$$\begin{aligned} \bullet^d \bullet F &= (\bullet^d \bullet F') \bullet T \partial_t - \bullet^d \bullet (F' \bullet T \partial_t) \\ &= B_d(F') \bullet_t T - B_d(F' \bullet_t T) \\ &= B_d(F'T\partial_t) + B_d(F' \bullet_t T) - B_d(F' \bullet_t T) \\ &= B_d(F). \end{aligned}$$

So the result holds for all forests. \blacksquare

Corollary 3.11. *Let A be a \mathcal{T} -multiple pre-Lie algebra and, for any $d \in \mathcal{D}$, $a_d \in A$. There exists a unique \mathcal{T} -multiple algebra morphism $\phi: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \rightarrow A$, such that for any $d \in \mathcal{D}$, $\phi(\bullet^d) = a_d$. In other words, $\mathfrak{g}_{\mathcal{T}, \mathcal{D}}$ is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} .*

Proof. *Uniqueness.* Using the Guin–Oudom product and lemma 3.10, ϕ is the unique linear map inductively defined by

$$\phi\left(B_d\left(\prod_{i \in [k]} T_i \partial_{t_i}\right)\right) = a_d \bullet \prod_{i \in [k]} \phi(T_i) \partial_{t_i}.$$

Existence. Let $T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $t \in \mathcal{T}$. Let us prove that $\phi(T \bullet_t T') = \phi(T) \bullet_t \phi(T')$ by induction on $n = |T|$. If $n = 1$, we assume that $T = \bullet^d$. Then $T \bullet_t T' = B_d(T' \partial_t)$, so

$$\phi(T \bullet_t T') = a_d \bullet (\phi(T') \partial_t) = a_d \bullet_t \phi(T') = \phi(T) \bullet_t \phi(T').$$

Let us assume the result at all ranks $< |T|$. We put

$$T = B_d\left(\prod_{i=1}^k T_i \partial_{t_i}\right).$$

By definition of the pre-Lie product of $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ in terms of grafting,

$$\begin{aligned} T \bullet T' &= B_d\left(\prod_{i=1}^k T_i \partial_{t_i} T' \partial_t\right) + \sum_{j=1}^k B_d\left(\prod_{i \neq j} T_i \partial_{t_i} (T_j \bullet_t T') \partial_{t_j}\right), \\ \phi(T \bullet T') &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + \sum_{j=1}^k a_d \bullet \prod_{i \neq j} \phi(T_i) \partial_{t_i} (\phi(T_j) \bullet_t \phi(T')) \partial_{t_j} \\ &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + \sum_{j=1}^k a_d \bullet \prod_{i \neq j} \phi(T_i) \partial_{t_i} (\phi(T_j) \bullet_t \phi(T')) \partial_{t_j} \\ &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + a_d \bullet \left(\left(\prod_{i=1}^k \phi(T_i) \partial_{t_i}\right) \bullet \phi(T') \partial_t\right) \\ &= \left(a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i}\right) \bullet \phi(T') \partial_t \\ &= \phi(T) \bullet_t \phi(T'). \end{aligned}$$

So ϕ is a \mathcal{T} -multiple pre-Lie algebra morphism. ■

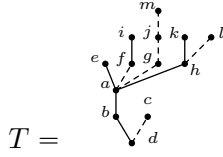
3.3 Operad of typed trees

We now describe an operad of typed trees, in the category of species. We refer to [2, 17, 19] for notations and definitions on operads.

Notation 3.12. Let A be a finite set. If $T \in \mathbb{T}_{\mathcal{T}}(A)$ and $a \in T$:

1. The subtrees formed by the connected components of the set of vertices, descendants of a (a excluded) are denoted by $T_1^{(a)}, \dots, T_{n_a}^{(a)}$. The type of the edge from a to the root of $T_i^{(a)}$ is denoted by t_i .
2. The tree formed by the vertices of T which are not in $T_1^{(a)}, \dots, T_{n_a}^{(a)}$, at the exception of a , is denoted by $T_0^{(a)}$.

Example 3.13. Let us consider the following tree:



with $a, \dots, m \in \mathcal{D}$, then

$$T_0^{(a)} = \begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ d \end{array}, \quad \{T_1^{(a)}, \dots, T_n^{(a)}\} = \left\{ \begin{array}{c} \bullet \cdot e, \downarrow f, \downarrow g, \downarrow h \\ \bullet \cdot i, \downarrow j, \downarrow k, \downarrow l \\ \bullet \cdot m \end{array} \right\}.$$

Proposition 3.14. For any nonempty finite set A , we denote by $\mathcal{P}_{\mathcal{T}}(A)$ the vector space generated by $\mathbb{T}_{\mathcal{T}}(A)$. We define a composition \circ on $\mathcal{P}_{\mathcal{T}}$ in the following way: for any $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$ and $a \in A$,

$$T \circ_a T' = \sum_{v_1, \dots, v_{n_a} \in V(T')} (\dots ((T_0^{(a)} \bullet_{a'}^{(t_0)} T') \bullet_{v_1}^{(t_1)} T_1^{(a)}) \dots) \bullet_{v_{n_a}}^{(t_{n_a})} T_{n_a}^{(a)},$$

where a' is the direct ascendant of a in T and t_0 is the type of the edge between a' and a . If a is the root of T , by convention $T_0^{(a)} \bullet_{a'}^{(t_0)} T' = T'$. With this composition, $\mathcal{P}_{\mathcal{T}}$ is an operad in the category of species.

Proof. Note that the tree $(\dots ((T_0^{(a)} \bullet_{a'}^{(t_0)} T') \bullet_{v_1}^{(t_1)} T_1^{(a)}) \dots) \bullet_{v_{n_a}}^{(t_{n_a})} T_{n_a}^{(a)}$, which is shortly denoted by $T \bullet_{\lambda}^{(v)} T'$, is obtained in the following process:

1. Delete the branches $T_1^{(a)}, \dots, T_{n_a}^{(a)}$ coming from a in T . One obtains a tree T'' , and a is a leaf of T'' .
2. Identify $a \in V(T'')$ with the root of T' . One obtains a tree T''' .
3. Graft $T_1^{(a)}$ on the vertex v_1 of T''' with an edge of type t_1, \dots , graft $T_{n_a}^{(a)}$ on the vertex v_{n_a} of T''' with an edge of type t_n . The obtained tree is $T \bullet_{\lambda}^{(v)} T'$.

This obviously does not depend on the choice of the indexation of $T_1^{(a)}, \dots, T_{n_a}^{(a)}$.

Let $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$, $T'' \in \mathbb{T}_{\mathcal{T}}(C)$.

- If $a', a'' \in A$, with $a' \neq a''$, then

$$\begin{aligned} (T \circ_{a'} T') \circ_{a''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a'}^{(v')} T') \bullet_{a''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a''}^{(v'')} T'') \bullet_{a'}^{(v')} T' \\ &= (T \circ_{a''} T'') \circ_{a'} T'. \end{aligned}$$

- If $a' \in A$ and $b'' \in B$, then

$$\begin{aligned} (T \circ_{a'} T') \circ_{b''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} (T \bullet_{a'}^{(v')} T') \bullet_{b''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} T \bullet_{a'}^{(v')} (T' \bullet_{b''}^{(v'')} T'') \\ &= T \circ_{a'} (T' \circ_{b''} T''). \end{aligned}$$

Let \mathcal{D} be a set. The morphism ϕ implies that the free $\mathcal{P}_{\mathbb{T}}$ -algebra generated by \mathcal{D} , that is to say $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, inherits a \mathcal{T} -multiple pre-Lie algebra structure defined by

$$\forall x, y \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}, \quad \forall \dagger \in \mathcal{T}, \quad x \circ_{\dagger} y = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}^2 \cdot (x \otimes y),$$

where \cdot is the $\mathcal{P}_{\mathcal{T}}$ -algebra structure of $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$. For any trees T, T' in $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, by definition of the operadic composition of $\mathcal{P}_{\mathcal{T}}$,

$$T \circ_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T',$$

so $\circ_t = \bullet_t$ for any t . As $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, (\bullet_t)_{t \in \mathcal{T}})$ is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} , Φ is an isomorphism. \blacksquare

Remark 3.18. Let us assume that \mathcal{T} is finite, of cardinality T . Then the components of $\mathcal{P}_{\mathcal{T}}$ are finite-dimensional. As the number of rooted trees which vertices are the elements of $[n]$ is n^{n-1} , for any $n \geq 0$ the dimension of $\mathcal{P}_{\mathcal{T}}(n)$ is $T^{n-1}n^{n-1}$, and the formal series of $\mathcal{P}_{\mathcal{T}}$ is

$$f_{\mathcal{T}}(X) = \sum_{n \geq 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}(n))}{n!} X^n = \sum_{n \geq 1} \frac{(Tn)^{n-1}}{n!} X^n = \frac{f_1(TX)}{T}.$$

3.4 Koszul dual operad

If \mathcal{T} is finite, then $\mathcal{P}_{\mathcal{T}}$ is a quadratic operad. Its Koszul dual can be directly computed:

Proposition 3.19. *The Koszul dual operad $\mathcal{P}_{\mathcal{T}}^{\perp}$ of $\mathcal{P}_{\mathcal{T}}$ is generated by \diamond_t , $t \in \mathcal{T}$, with the relations*

$$\forall t, t' \in \mathcal{T}, \quad \diamond_{t'} \circ_1 \diamond_t = \diamond_t \circ_2 \diamond_{t'}, \quad \diamond_{t'} \circ_1 \diamond_t = (\diamond_t \circ_1 \diamond_{t'})^{(23)}.$$

The algebras on $\mathcal{P}_{\mathcal{T}}^{\perp}$ are called \mathcal{T} -multiple permutative algebras. Such an algebra A is given with the bilinear products \diamond_t , $t \in \mathcal{T}$, such that

$$\forall x, y, z \in A, \quad (x \diamond_t y) \diamond_{t'} z = x \diamond_t (y \diamond_{t'} z), \quad (x \diamond_t y) \diamond_{t'} z = (x \diamond_{t'} z) \diamond_t y.$$

In particular, for any t , \diamond_t is a permutative product.

Of course, the definition of \mathcal{T} -multiple permutative algebras makes sense even if \mathcal{T} is infinite. Permutative algebras are introduced in [8]. If A is a \mathcal{T} -multiple permutative algebra, then for any $(\lambda_t)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, $\diamond_a = \sum \lambda_t \diamond_t$ is a permutative product on A .

Proposition 3.20. *Let V be a vector space. Then $V \otimes S(V^{\oplus \mathcal{T}})$ is given with a \mathcal{T} -multiple permutative algebra structure:*

$$\forall t \in \mathcal{T}, \quad v, v' \in V, \quad w, w' \in S(V^{\oplus \mathcal{T}}), \quad (v \otimes w) \diamond_t (v' \otimes w') = v \otimes w w' (v' \partial_t).$$

This \mathcal{T} -multiple permutative algebra is denoted by $P_{\mathcal{T}}(V)$. For any \mathcal{T} -multiple permutative algebra V and any linear map $\phi: V \rightarrow A$, there exists a unique morphism $\Phi: P_{\mathcal{T}}(V) \rightarrow A$ such that for any $v \in V$, $\Phi(v \otimes 1) = \phi(v)$.

Proof. Let $t, t' \in \mathcal{T}$, $v, v', v'' \in V$, $w, w', w'' \in S(V^{\oplus \mathcal{T}})$.

$$\begin{aligned} (v \otimes w \diamond_t v' \otimes w') \diamond_{t'} v'' \otimes w'' &= v \otimes w \diamond_t (v' \otimes w' \diamond_{t'} v'' \otimes w'') \\ &= (v \otimes w \diamond_t v' \diamond_{t'} v'' \otimes w'') \otimes w' \\ &= v \otimes w w' w'' (v' \partial_t) (v'' \partial_{t'}), \end{aligned}$$

so $P_{\mathcal{T}}(V)$ is \mathcal{T} -multiple permutative.

Existence of Φ . Let $t_1, \dots, t_k \in \mathcal{T}$, $v, v_1, \dots, v_k \in V$. We inductively define $\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}))$ by

$$\begin{aligned} \Phi(v \otimes 1) &= \phi(v), \\ \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k})) &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \quad \text{if } k \geq 1. \end{aligned}$$

Let us prove that this does not depend on the order chosen on the factors $v_i \partial_{t_i}$ by induction on k . If $k = 0$ or 1 , there is nothing to prove. Otherwise, if $i < k$,

$$\begin{aligned} \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_k \partial_{t_k})) \diamond_{t_i} \phi(v_i) \\ &= (\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k)) \diamond_{t_i} \phi(v_i) \\ &= (\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_i} \phi(v_i)) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k})). \end{aligned}$$

So Φ is well-defined. Let us prove that Φ is a \mathcal{T} -multiple permutative algebra morphism. Let $v, v' \in V$, $w, w' = (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}) \in S(V^{\oplus \mathcal{T}})$, and $t \in \mathcal{T}$. Let us prove that $\Phi(v \otimes w \diamond_t v' \otimes w'') = \Phi(v \otimes w) \diamond_t \Phi(v' \otimes w')$ by induction on k . If $k = 0$,

$$\begin{aligned} \Phi(v \otimes w \diamond_t v' \otimes 1) &= \Phi(v \otimes w (v' \partial_t)) \\ &= \Phi(v \otimes w) \diamond_t \phi(v') \\ &= \Phi(v \otimes w) \diamond_t \Phi(v' \otimes 1). \end{aligned}$$

Otherwise, we put $w'' = (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})$. Then

$$\begin{aligned} \Phi(v \otimes w \diamond_t v' \otimes w') &= \Phi(v \otimes w w'' (v' \partial_t) (v_k \partial_{t_k})) \\ &= \Phi(v \otimes w w'' (v' \partial_t)) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes w \diamond_t v' \otimes w'') \diamond_{t_k} \phi(v_k) \\ &= (\Phi(v \otimes w) \diamond_t \Phi(v' \otimes w'')) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes w) \diamond_t (\Phi(v' \otimes w'') \diamond_{t_k} \phi(v_k)) \\ &= \Phi(v \otimes w') \diamond_t \Phi(v' \otimes w'). \end{aligned}$$

So Φ is a \mathcal{T} -multiple permutative algebra morphism.

Uniqueness. For any $v, v_1, \dots, v_k \in V$, $t_1, \dots, t_k \in \mathcal{T}$,

$$v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}) = (v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} v_k.$$

It is then easy to prove that $P_{\mathcal{T}}(V)$ is generated by $V \otimes 1$ as a \mathcal{T} -multiple permutative algebra. Consequently, Φ is unique. ■

Remark 3.21.

1. We proved that $P_{\mathcal{T}}(V)$ is freely generated by V , identified with $V \otimes 1$. Consequently, $\mathcal{P}_{\mathcal{T}}^!(n)$ has the same dimension as the multilinear component of $V \otimes S(V^{\oplus \mathcal{T}})$ with $V = \text{Vect}(X_1, \dots, X_n)$, that is to say

$$\text{Vect}(X_i \otimes (X_1 \partial_{t_1}) \cdots (X_{i-1} \partial_{t_{i-1}}) (X_{i+1} \partial_{t_{i+1}}) \cdots (X_n \partial_{t_n}), 1 \leq i \leq n, t_j \in \mathcal{T}),$$

so

$$\dim(\mathcal{P}_{\mathcal{T}}^!(n)) = nT^{n-1}.$$

The formal series of $\mathcal{P}_{\mathcal{T}}^!$ is

$$f_{\mathcal{T}}^!(X) = \sum_{n \geq 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}^!(n))}{n!} X^n = \sum_{n \geq 1} \frac{T^{n-1}}{(n-1)!} X^n = X \exp(TX) = \frac{f_1^!(TX)}{T}.$$

2. It is possible to prove that $\mathcal{P}_{\mathcal{T}}^!$ is a Koszul operad (and, hence, $\mathcal{P}_{\mathcal{T}}$ too) using the rewriting method of [17].

4 Structure of the pre-Lie products

4.1 A nonassociative permutative coproduct

For all $t \in \mathcal{T}$, we define a coproduct $\rho_t: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 2}$ by

$$\forall T = B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad \rho_t(T) = \sum_{j \in [k]} \delta_{t, t_j} B_d \left(\prod_{i \in [k], i \neq j} T_i \partial_{t_i} \right) \otimes T_j \partial_t.$$

Proposition 4.1.

1. For all $t, t' \in \mathcal{T}$, $(\rho_t \otimes \text{Id}) \circ \rho_{t'} = ((\rho_{t'} \otimes \text{Id}) \circ \rho_t)^{(23)}$, where in the right side we use the classical action of the symmetric group \mathfrak{S}_3 on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 3}$ by permutations of the tensors, and more specifically the action of the transposition (23). With Sweedler's notation for ρ_t , this can be written, for any $x \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, as

$$\sum \sum (x^{(1)t'})^{(1)t} \otimes (x^{(1)t'})^{(2)t} \otimes x^{(2)t'} = \sum \sum (x^{(1)t})^{(1)t'} \otimes x^{(2)t} \otimes (x^{(1)t})^{(2)t'}.$$

2. For any $x, y \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, $t, t' \in \mathcal{T}$, with Sweedler's notations $\rho_t(x) = \sum x^{(1)t} \otimes x^{(2)t}$,

$$\rho_t(x \bullet_{t'} y) = \delta_{t, t'} x \otimes y + \sum x^{(1)t} \bullet_{t'} y \otimes x^{(2)t} + \sum x^{(1)t} \otimes x^{(2)t} \bullet_{t'} y.$$

3. For any $\mu = (\mu_t)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, we put

$$\rho_{\mu} = \sum_{t \in \mathcal{T}} \mu_t \rho_t: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 2}.$$

This makes sense, as any tree in $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$ does not vanish only under a finite number of ρ_t . Then ρ_{μ} is a nonassociative permutative (NAP) coproduct; for any $x, y \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, by the second point, using Sweedler's notation for ρ_{μ} ,

$$\rho_{\mu}(x \bullet_{\lambda} y) = \left(\sum_{t \in \mathcal{T}} \lambda_t \mu_t \right) x \otimes y + \sum x^{(1)\mu} \bullet_{\lambda} y \otimes x^{(2)\mu} + \sum x^{(1)\mu} \otimes x^{(2)\mu} \bullet_{\lambda} y.$$

In particular, if $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda}, \rho_{\mu})$ is a NAP pre-Lie bialgebra in the sense of [16].

Proof. 1. For any tree T ,

$$(\rho_t \otimes \text{Id}) \circ \rho_{t'}(T) = \sum_{p,q \in [k], p \neq q} \delta_{t_p, t} \delta_{t_q, t'} B_d \left(\prod_{i \in [k], i \neq p, q} T_i \partial_{t_i} \right) \otimes T_p \partial_t \otimes T_q \partial_{t'},$$

which implies the result.

2. For any tree T, T' ,

$$\begin{aligned} \rho_t(T \bullet_{t'} T') &= \rho_t \left(B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} T' \partial_{t'} \right) + \sum_{i \in [k]} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} (T_i \bullet_{t'} T') \partial_{t_i} \right) \right) \\ &= \delta_{t, t'} B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) \otimes T' + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} T' \partial_{t'} \right) \otimes T_i \partial_t \\ &\quad + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \otimes (T_i \bullet_{t'} T') \partial_t \\ &\quad + \sum_{i \neq j \in [k]} \delta_{t_i, t} B_d \left(\prod_{p \in [k], p \neq i, j} T_p \partial_{t_p} (T_j \bullet_{t'} T') \partial_{t_j} \right) \otimes T_i \partial_t \\ &= \delta_{t, t'} T \otimes T' + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \bullet_{t'} T' \otimes T_i \partial_t \\ &\quad + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \otimes T_i \bullet_{t'} T' \partial_t \\ &= \delta_{t, t'} T \otimes T' + T^{(1)t} \bullet_{t'} T' \otimes T^{(2)t} + T^{(1)t} \otimes T^{(2)t} \bullet_{t'} T'. \end{aligned}$$

3. Obtained by summation. ■

Corollary 4.2. *If $\lambda \in \mathbb{K}^{(\mathcal{T})}$ is nonzero, let us choose $t_0 \in \mathcal{T}$ such that $\lambda_{t_0} \neq 0$. The pre-Lie algebra $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by the set $\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$ of \mathcal{T} -typed \mathcal{D} -decorated trees T such that there is no edge outgoing the root of T of type t_0 .*

Proof. For any tree T , we denote by α_T the number of edges outgoing the root of T of type T_0 . Our aim is to prove that $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by the trees T such that $\alpha_T = 0$. We define a family of scalars $(\mu_t)_{t \in \mathcal{T}}$ by

$$\forall t \in \mathcal{T}, \quad \mu_t = \begin{cases} 0 & \text{if } t \neq t_0, \\ \frac{1}{\lambda_{t_0}} & \text{if } t = t_0. \end{cases}$$

Note that $\rho_\mu = \frac{1}{\lambda_{t_0}} \rho_{t_0}$. By Proposition 4.1, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda, \rho_\mu)$ is a NAP pre-Lie bialgebra, so by Livernet's rigidity theorem [16], it is freely generated by $\text{Ker}(\rho_\mu) = \text{Ker}(\rho_{t_0})$. Obviously, if $\alpha_T = 0$, $T \in \text{Ker}(\rho_{t_0})$. Let us consider $x = \sum_{T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}} x_T T \in \text{Ker}(\rho_{t_0})$. We consider the map

$$\Upsilon: \begin{cases} \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \otimes \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \\ T \otimes T' \longmapsto T \bullet_{t_0}^{\text{root}(T)} T'. \end{cases}$$

By definition of ρ_{t_0} , for any tree T , $\Upsilon \circ \rho_{t_0}(T) = \alpha_T T$. Consequently,

$$0 = \Upsilon \circ \rho_{t_0}(x) = \sum_{T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}} x_T \alpha_T T.$$

So if $\alpha_T \neq 0$, $x_T = 0$, and x is a linear span of trees such that $\alpha_T = 0$: the set of trees T such that $\alpha_T = 0$ is a basis of $\text{Ker}(\rho_{t_0})$. ■

If $|\mathcal{D}| = D$ and $|\mathcal{T}| = T$, the number of elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ of degree n is denoted by $t'_{D,T}(n)$; it does not depend on t_0 . By direct computations,

$$\begin{aligned} t'_{D,T}(1) &= D, \\ t'_{D,T}(2) &= D^2(T-1), \\ t'_{D,T}(3) &= \frac{D^2(T-1)(3DT-D+1)}{2}, \\ t'_{D,T}(4) &= \frac{D^2(T-1)(16D^2T^2-8D^2T+D^2+6DT-3D+2)}{6}. \end{aligned}$$

In the particular case $D = 1$, $T = 2$, we recover sequence A005750 of the OEIS.

4.2 Pre-Lie algebra morphisms

Notation 4.3. Let \mathcal{T} and \mathcal{T}' be two sets of types. We denote by $\mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ the space of matrices $M = (m_{t,t'})_{(t,t') \in \mathcal{T} \times \mathcal{T}'}$, such that for any $t' \in \mathcal{T}'$, $(m_{t,t'})_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$. If $\mathcal{T} = \mathcal{T}'$, we shall simply write $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$. If $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ and $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, then

$$MM' = \left(\sum_{t' \in \mathcal{T}'} m_{t,t'} m'_{t',t''} \right)_{(t,t'') \in \mathcal{T} \times \mathcal{T}''} \in \mathcal{M}_{\mathcal{T},\mathcal{T}''}(\mathbb{K}).$$

If $\lambda \in \mathbb{K}^{(\mathcal{T}'})$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, then

$$M\lambda = \left(\sum_{t' \in \mathcal{T}'} m_{t,t'} \lambda_{t'} \right)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}, \quad M^\top \mu = \left(\sum_{t \in \mathcal{T}} m_{t,t'} \mu_t \right)_{t' \in \mathcal{T}'}. \in \mathbb{K}^{\mathcal{T}'}$$

In particular, $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$ is an algebra, acting on $\mathbb{K}^{(\mathcal{T})}$ on the left and on $\mathbb{K}^{\mathcal{T}}$ on the right.

Definition 4.4. Let $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. We define a map $\Phi_M: \mathcal{H}_{\mathcal{D},\mathcal{T}'} \rightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}$, sending $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}'}$ to the forest obtained by replacing $\text{type}(e)$ by $\sum_{t \in \mathcal{T}} m_{t,\text{type}(e)} t$ for any $e \in E(F)$, F being considered as linear in any of its edges. The restriction of Φ_M to $\mathfrak{g}_{\mathcal{D},\mathcal{T}'}$ is denoted by $\phi_M: \mathfrak{g}_{\mathcal{D},\mathcal{T}'} \rightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}$.

Example 4.5. If \mathcal{T} contains two elements, the first one represented by $|$ and the second one by $\dot{!}$, if $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, for any $x, y, z \in \mathcal{D}$,

$$\begin{aligned} \phi_M(\dot{!}^y_x) &= \alpha \dot{!}^y_x + \gamma \dot{!}^y_x, \\ \phi_M(\dot{!}^y_x) &= \beta \dot{!}^y_x + \delta \dot{!}^y_x, \\ \phi_M(\begin{smallmatrix} y & z \\ \diagdown & \diagup \\ & x \end{smallmatrix}) &= \alpha\beta \begin{smallmatrix} y & z \\ \diagdown & \diagup \\ & x \end{smallmatrix} + \alpha\delta \begin{smallmatrix} y & z \\ \diagdown & \diagup \\ & x \end{smallmatrix} + \beta\gamma \begin{smallmatrix} y & z \\ \diagdown & \diagup \\ & x \end{smallmatrix} + \gamma\delta \begin{smallmatrix} y & z \\ \diagdown & \diagup \\ & x \end{smallmatrix}. \end{aligned}$$

Remark 4.6. For any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, $\Phi_M \circ \Phi_{M'} = \Phi_{MM'}$.

Proposition 4.7. Let $\lambda \in \mathbb{K}^{(\mathcal{T}'})$, $\mu \in \mathbb{K}^{\mathcal{T}}$ and $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. Then ϕ_M is a pre-Lie morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$ and a NAP coalgebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \rho_{M^\top \mu})$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \rho_\mu)$.

Proof. Let $T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$. For any $t \in \mathcal{T}$, for any $v \in V(T)$,

$$\phi_M(T \bullet_t^{(v)} T') = \sum_{t' \in \mathcal{T}'} m_{t',t} \phi_M(T) \bullet_{t'} \phi_M(T'),$$

so

$$\phi_M(T \bullet_\lambda T') = \sum_{t,t' \in \mathcal{T}} m_{t',t} \lambda_t \phi_M(T) \bullet_{t'} \phi_M(T') = \phi_M(T) \bullet_{M\lambda} \phi_M(T').$$

We proved that ϕ_M is a pre-Lie algebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$.

For any $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$,

$$\rho_t \circ \phi_M(T) = \sum_{t' \in \mathcal{T}} m_{t,t'}(\phi_M \otimes \phi_M) \circ \rho_{t'}(T),$$

so

$$\rho_\mu \circ \phi_M(T) = \sum_{t,t' \in \mathcal{T}} m_{t,t'} \mu_t(\phi_M \otimes \phi_M) \circ \rho_{t'}(T) = (\phi_M \otimes \phi_M) \circ \rho_{M^\top \mu}(T).$$

So $\phi_M: (\mathfrak{g}_{\mathcal{D}, \mathcal{T}'}, \rho_{M^\top \mu}) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \rho_\mu)$ is a NAP coalgebra morphism. \blacksquare

Corollary 4.8. *For any $\lambda \in \mathbb{K}^{\mathcal{T}}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, such that $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, for any $t_0 \in \mathcal{T}$, the NAP pre-Lie bialgebras $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda, \rho_\mu)$ and $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{t_0}, \rho_{t_0})$ are isomorphic.*

Proof. Let us denote by $\lambda^{(0)}$ the element of $\mathbb{K}^{\mathcal{T}}$ defined by

$$\lambda_t^{(0)} = \delta_{t,t_0}.$$

Note that for any $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, $\phi_M: (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda^{(0)}}) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{M\lambda^{(0)}})$ is an isomorphism. In particular, for a well-chosen M , $M\lambda^{(0)} = \lambda$; we can assume that $\lambda = \lambda^{(0)}$ without loss of generality. Then, by hypothesis, $\mu_{t_0} = 1$. We define a matrix $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$ in the following way:

$$m_{t,t'} = \begin{cases} \delta_{t,t_0} & \text{if } t' = t_0, \\ \delta_{t,t'} - \mu_{t'} \delta_{t,t_0} & \text{otherwise.} \end{cases}$$

Then M is invertible. Moreover, $M\lambda^{(0)} = \lambda^{(0)}$ and $M^\top \mu = \lambda^{(0)}$. So ϕ_M is an isomorphism from $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda^{(0)}})$ to $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$. \blacksquare

Proposition 4.9. *Let $\lambda \in \mathbb{K}^{\mathcal{T}}$, and $t_0 \in \mathcal{T}$. We define a pre-Lie algebra morphism $\psi_{t_0}: (\mathfrak{g}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}, \bullet) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$, sending \cdot^x to T for any $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$. Then ψ_{t_0} is a pre-Lie algebra isomorphism if, and only if, $\lambda_{t_0} \neq 0$.*

Proof. If $\lambda_{t_0} \neq 0$, then by corollary 4.2, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by $\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$, so ψ_{t_0} is an isomorphism. If $\lambda_{t_0} = 0$, then it is not difficult to show that any tree T with two vertices, with its unique edge of type t_0 , does not belong to $\text{Im}(\psi_{t_0})$. \blacksquare

5 Hopf algebraic structures

We here fix $\lambda \in \mathbb{K}^{\mathcal{T}}$.

5.1 Enveloping algebra of $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$

Using again the Guin–Oudom construction, we obtain the enveloping algebra of $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$. We first identify the symmetric coalgebra $S(\mathfrak{g}_{\mathcal{D}, \mathcal{T}})$ with the vector space generated by $\mathbb{F}_{\mathcal{D}, \mathcal{T}}$, which we denote by $\mathcal{H}_{\mathcal{D}, \mathcal{T}}$. Its product m is given by disjoint union of forests, its coproduct by

$$\forall T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad \Delta(T_1 \cdots T_n) = \sum_{I \subseteq [n]} \prod_{i \in I} T_i \otimes \prod_{i \notin I} T_i.$$

We denote by \bullet_λ the Guin–Oudom extension of \bullet_λ to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}$ and \star_λ the associated associative product.

Theorem 5.1. For any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, $T_1, \dots, T_n \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$,

$$F \bullet_{\lambda} T_1 \cdots T_n = \sum_{\substack{v_1, \dots, v_n \in V(F), \\ t_1, \dots, t_n \in \mathcal{T}}} \left(\prod_{i \in [n]} \lambda_{t_i} \right) (\cdots (F \bullet_{t_1}^{(v_1)} T_1) \cdots) \bullet_{t_n}^{(v_n)} T_n,$$

$$F \star_{\lambda} T_1 \cdots T_n = \sum_{I \subseteq [n]} \left(F \bullet_{\lambda} \prod_{i \in I} T_i \right) \prod_{i \notin I} T_i.$$

The Hopf algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, \star_{\lambda}, \Delta)$ is denoted by $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$. Moreover, for any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, for any $\lambda \in \mathbb{K}^{(\mathcal{T}')}$, Φ_M is a Hopf algebra morphism from $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}M\lambda}$. The extension of ψ_{t_0} as a Hopf algebra morphism from $\mathcal{H}_{\mathbb{T}_{\mathcal{D},\mathcal{T}}(t_0)}^{\text{GL}\lambda}$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ is denoted by Ψ_{t_0} ; it is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

In particular, if $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the Grossman–Larson Hopf algebra [14].

5.2 Dual construction

Proposition 5.2. Let $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$.

1. A cut c of T is a nonempty subset of $E(T)$; it is said to be admissible if any path in the tree from the root to a leaf meets at most one edge in c . The set of admissible cuts of T is denoted by $\text{Adm}(T)$.
2. If c is admissible, one of the connected components of $T \setminus c$ contains the root of c : we denote it by $R^c(T)$. The product of the other connected components of $T \setminus c$ is denoted by $P^c(T)$.

Let $\lambda \in \mathbb{K}^{\mathcal{T}}$. We define a multiplicative coproduct $\Delta^{\text{CK}\lambda}$ on the algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m)$ by

$$\forall T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \quad \Delta^{\text{CK}\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in \text{Adm}(T)} \left(\prod_{e \in c} \lambda_{\text{type}(e)} \right) R^c(T) \otimes P^c(T).$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \Delta^{\text{CK}\lambda})$ is a Hopf algebra, which we denote by $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$.

Proof. We first assume that $\lambda \in \mathbb{K}^{(\mathcal{T})}$. Let us define a nondegenerate pairing $\langle -, - \rangle$ on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by

$$\forall F, F' \in \mathbb{F}_{\mathcal{D},\mathcal{T}}, \quad \langle F, F' \rangle = \delta_{F, F'} s_F,$$

where s_F is the number of symmetries of F . Let us consider three forests F, F', F'' . We put

$$F = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\lambda_t}, \quad F' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a'_T}, \quad F'' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a''_T}.$$

Then

$$\begin{aligned} \langle \Delta(F), F' \otimes F'' \rangle &= \sum_{a=b+c} \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} \frac{\lambda_t!}{\mu_t! c_T!} \left\langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\mu_t}, F' \right\rangle \left\langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{c_T}, F'' \right\rangle \\ &= \sum_{a=b+c} \delta_{b, a'} \delta_{c, a''} \frac{\lambda_t!}{a'_T! a''_T!} s_{F'} s_{F''} = \delta_{a, a' + a''} \frac{\lambda_t!}{a'_T! a''_T!} a'_T! a''_T! s_T^{a'_T + a''_T} \\ &= \delta_{a, a' + a''} \lambda_t! s_T^{\lambda_t} = \delta_{F, F' F''} s_F \\ &= \langle F, F' F'' \rangle. \end{aligned}$$

Therefore,

$$\forall x, y, z \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}, \quad \langle \Delta(x), y \otimes z \rangle = \langle x, yz \rangle.$$

Let F, G be two forests and T be a tree. Observe that if F is a forest with at least two trees, then $F \star_\lambda G$ does not contain any tree, so $\langle F \star_\lambda G, T \rangle = 0$. If $F = 1$, then $\langle F \star_\lambda G, T \rangle \neq 0$ if, and only if, $G = T$; moreover, $\langle 1 \star_\lambda T, T \rangle = 1$. If F is a tree, then

$$\langle F \star_\lambda G, T \rangle = \langle F \bullet_\lambda G, T \rangle.$$

Moreover, if $F = B_d(F')$ and $G = T_1 \cdots T_k$,

$$F \bullet_\lambda G = \sum_{I \subseteq [k]} \sum_{(t_i) \in \mathcal{T}^k} \left(\prod_{i \in [k]} \lambda_{t_i} \right) B_d \left(\prod_{i \in I} T_i \partial_{t_i} F' \bullet \prod_{i \notin I} T_i \partial_{t_i} \right),$$

where \bullet is the pre-Lie product on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\mathcal{T}}$ induced by the \mathcal{T} -multiple pre-Lie structure. Consequently, we can inductively define a coproduct $\Delta^{\text{CK}\lambda}: \mathcal{H}_{\mathcal{D}, \mathcal{T}} \rightarrow \mathcal{H}_{\mathcal{D}, \mathcal{T}} \otimes \mathcal{H}_{\mathcal{D}, \mathcal{T}}$, multiplicative for the product m , such that, if we denote for any tree T , $\bar{\Delta}_{\text{CK}}(T) = \Delta(T) - 1 \otimes T$, for any tree $T = B_d(T_1 \partial_{t_1} \cdots T_k \partial_{t_k})$,

$$\bar{\Delta}_\lambda^{\text{CK}}(T) = (B_d \otimes \text{Id}) \left(\prod_{i \in [k]} \left(\bar{\Delta}_\lambda^{\text{CK}}(T_i) \partial_{t_i} \otimes 1 + \lambda_{t_i} 1 \otimes T_i \right) \right).$$

Then, for any $x, y, z \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$,

$$\langle x \star_\lambda y, z \rangle = \langle x \otimes y, \Delta^{\text{CK}\lambda}(z) \rangle.$$

A quite easy induction on the number of vertices of trees proves that this coproduct is indeed the one we define in the statement of the proposition. As $\langle -, - \rangle$ is nondegenerate, $(\mathcal{H}_{\mathcal{D}, \mathcal{T}}, m, \Delta^{\text{CK}\lambda})$ is a Hopf algebra, dual to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}\lambda}$.

In the general case, for any $x \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$, there exists a finite subset \mathcal{T}' of \mathcal{T} such that $x \in \mathcal{H}_{\mathcal{D}, \mathcal{T}'}$. Putting $\lambda' = \lambda|_{\mathcal{T}'}$, $\lambda' \in \mathbb{K}^{\mathcal{T}'} = \mathbb{K}^{(\mathcal{T}'})$, so

$$\begin{aligned} (\Delta^{\text{CK}\lambda} \otimes \text{Id}) \circ \Delta^{\text{CK}\lambda}(x) &= (\Delta^{\text{CK}\lambda'} \otimes \text{Id}) \circ \Delta^{\text{CK}\lambda'}(x) \\ &= (\text{Id} \otimes \Delta^{\text{CK}\lambda'}) \circ \Delta^{\text{CK}\lambda'}(x) \\ &= (\text{Id} \otimes \Delta^{\text{CK}\lambda}) \circ \Delta^{\text{CK}\lambda}(x). \end{aligned}$$

Hence, Δ_λ is coassociative, and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$ is a Hopf algebra. ■

Example 5.3. Let us fix a subset \mathcal{T}' of \mathcal{T} and choose $(\lambda_t)_{t \in \mathcal{T}}$ such that

$$\lambda_t = \begin{cases} 1 & \text{if } t \in \mathcal{T}', \\ 0 & \text{otherwise.} \end{cases}$$

For any tree $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, let us denote by $\text{Adm}_{\mathcal{T}'}(T)$ the set of admissible cuts c of T such that the type of any edge in c belongs to \mathcal{T}' . Then

$$\Delta^{\text{CK}\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in \text{Adm}_{\mathcal{T}'}(T)} R^c(T) \otimes P^c(T).$$

Remark 5.4.

1. If $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the usual Connes–Kreimer Hopf algebra of \mathcal{D} -decorated rooted trees, which we denote by $\mathcal{H}_{\mathcal{D}}^{\text{CK}}$, and its duality with the Grossman–Larson Hopf algebra [10, 15, 23].
2. If \mathcal{T} and \mathcal{D} are finite, for any $\lambda \in \mathbb{K}^{\mathcal{T}}$, both $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}\lambda}$ are graded Hopf algebras (by the number of vertices), and their homogeneous components are finite-dimensional. Via the pairing $\langle -, - \rangle$, each one is the graded dual of the other.

5.3 Hochschild cohomology of coalgebras

For the sake of simplicity, we assume that the set of types \mathcal{T} is finite and we put $\mathcal{T} = \{t_1, \dots, t_N\}$.

Let (C, Δ) be a coalgebra and let (M, δ_L, δ_R) be a C -bicomodule. We define a complex, dual to the Hochschild complex for algebras, in the following way:

1. For any $n \geq 0$, $H_n = \mathcal{L}(M, C^{\otimes n})$.
2. For any $L \in H_n$,

$$b_n(L) = (\text{Id} \otimes L) \circ \delta_L + \sum_{i=1}^n (-1)^i (\text{Id}^{\otimes(i-1)} \otimes \Delta \otimes \text{Id}^{\otimes(n-i)}) \circ L \\ + (-1)^{n+1} (L \otimes \text{Id}) \circ \delta_R.$$

In particular, one-cocycles are maps $L: M \rightarrow C$ such that

$$\Delta \circ L = (\text{Id} \otimes L) \circ \delta_L + (L \otimes \text{Id}) \circ \delta_R.$$

We shall consider in particular the bicomodule (M, δ_L, δ_R) such that

$$\forall x \in C, \quad \begin{cases} \delta_L(x) = 1 \otimes x, \\ \delta_R(x) = \Delta(x). \end{cases}$$

If C is a bialgebra, then $M^{\otimes N}$ is also a bicomodule:

$$\forall x_t \in C, \quad \begin{cases} \delta_L \left(\bigotimes_{1 \leq i \leq N} x_i \right) = 1 \otimes \bigotimes_{1 \leq i \leq N} x_i, \\ \delta_R \left(\bigotimes_{1 \leq i \leq N} x_i \right) = \bigotimes_{1 \leq i \leq N} x_i^{(1)} \otimes \prod_{1 \leq i \leq N} x_i^{(2)}. \end{cases}$$

We denote by $\underline{1} = (1)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, and we take $C = \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_1}$. One can identify $S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}})^{\oplus \mathcal{T}})$ and $C^{\otimes N}$, $x \delta_{T_i}$ being identified with $1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$ for any $x \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $1 \leq i \leq N$. Then for any d , $B_d: C^{\otimes N} \rightarrow C$ is a 1-cocycle. Moreover, there is a universal property, proved in the same way as for the Connes–Kreimer's one [10]:

Theorem 5.5. *Let B be a commutative bialgebra and, for any $d \in \mathcal{D}$, let $L_d: C^{\otimes N} \rightarrow C$ be a 1-cocycle:*

$$\forall d \in \mathcal{D}, \quad \forall x_t \in B, \quad \Delta \circ L_d \left(\bigotimes_{1 \leq i \leq N} x_i \right) = 1 \otimes \bigotimes_{1 \leq i \leq N} x_i + L_d \left(\bigotimes_{1 \leq i \leq N} x_i^{(1)} \right) \otimes \prod_{1 \leq i \leq N} x_i^{(2)}.$$

There exists a unique bialgebra morphism $\phi: \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_1} \rightarrow C$ such that for any $d \in \mathcal{D}$, $\phi \circ L_d = B_d \circ \phi^{\otimes N}$.

5.4 Hopf algebra morphisms

Our aim is, firstly, to construct Hopf algebra morphisms between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$; secondly, to construct Hopf algebra isomorphisms between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}', \mathcal{T}}^{\text{CK}}$ for a well-chosen \mathcal{D}' .

Proposition 5.6. *Let $M \in \mathcal{M}_{\mathcal{T}, \mathcal{T}'}(\mathbb{K})$, $\lambda \in \mathbb{K}^{\mathcal{T}}$. Then $\Phi_M: \mathcal{H}_{\mathcal{D}, \mathcal{T}'}^{\text{CK}_{M^{\top \lambda}}} \rightarrow \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ is a Hopf algebra morphism.*

Proof. Φ_M is obviously an algebra morphism. Let $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$.

$$\begin{aligned} \Delta_\lambda \circ \Phi_M(T) &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T) \\ &+ \sum_{c \in \text{Adm}(T)} \prod_{e \in c} \left(\sum_{t \in \mathcal{T}} m_{t, \text{type}(e)} \lambda_t \right) \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T)) \\ &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T) \\ &+ \sum_{c \in \text{Adm}(T)} \prod_{e \in c} (M^\top \lambda)_{\text{type}(e)} \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T)) \\ &= (\Phi_M \otimes \Phi_M) \circ \Delta_{M^\top a}(T). \end{aligned}$$

So Φ_M is a coalgebra morphism from $\mathcal{H}_{\mathcal{D}, \mathcal{T}'}^{\text{CK}_{M^\top \lambda}}$ to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$. \blacksquare

Corollary 5.7. *Let $\lambda, \mu \in \mathbb{K}^{\mathcal{T}}$, both nonzero. Then $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$ are isomorphic Hopf algebras.*

Proof. There exists $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, such that $M^\top \lambda = \mu$. Then Φ_M is an isomorphism between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$. \blacksquare

Definition 5.8. Let us fix $t_0 \in \mathcal{T}$. For any $F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, we shall say that $\{T_1, \dots, T_k\} \triangleleft_{t_0} F$ if the following conditions hold:

- $\{T_1, \dots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, by restriction.
- For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$.

If $\{T_1, \dots, T_k\} \triangleleft_{t_0} F$, we denote by $F/\{T_1, \dots, T_k\}$ the forest obtained by contracting T_i to a single vertex for any $i \in [k]$, decorating this vertex by T_i , and forgetting the type of the remaining edges. Then $F/\{T_1, \dots, T_k\}$ is a $\mathcal{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$ -decorated forest.

Proposition 5.9. *Let $\lambda \in \mathbb{K}^{\mathcal{T}}$, $t_0 \in \mathcal{T}$. Let us consider the map*

$$\Psi_{t_0}^* : \begin{cases} \mathcal{H}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}, \\ F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}} \longmapsto \sum_{\{T_1, \dots, T_k\} \triangleleft_{t_0} F} \left(\prod_{e \in E(F) \setminus \sqcup E(T_i)} \lambda_{\text{type}(e)} \right) F/\{T_1, \dots, T_k\}. \end{cases}$$

Then $\Psi_{t_0}^*$ is a Hopf algebra morphism from $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ to $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{CK}_{\lambda_{t_0}}}$. It is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

Proof. *First case.* We first assume that \mathcal{D} and \mathcal{T} are finite. In this case, $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ is the graded dual of $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}_\lambda}$, with the Hopf pairing $\langle -, - \rangle$; grading $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}$ by the number of vertices of the decorations, $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{CK}_{\lambda_{t_0}}}$ is the graded dual of $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{GL}_{\lambda_{t_0}}}$. Moreover, $\Psi_{t_0}^*$ is the transpose of Ψ_{t_0} of Proposition 4.9, so is a Hopf algebra morphism. If $\lambda_{t_0} \neq 0$, Ψ_{t_0} is an isomorphism, so $\Psi_{t_0}^*$ also is.

General case. Let $x, y \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$. There exist finite $\mathcal{D}', \mathcal{T}'$, such that $x, y \in \mathcal{H}_{\mathcal{D}', \mathcal{T}'}$; we can assume that $t_0 \in \mathcal{T}'$. We denote by $\lambda' = \lambda|_{\mathcal{T}'}$. Then, by the preceding case, denoting by Ψ'_{t_0} the restriction of $\Psi_{t_0}^*$ to $\mathcal{H}_{\mathcal{D}', \mathcal{T}'}$,

$$\begin{aligned} \Psi_{t_0}^*(xy) &= \Psi'_{t_0}(xy) = \Psi'_{t_0}(x)\Psi'_{t_0}(y) = \Psi_{t_0}^*(x)\Psi_{t_0}^*(y), \\ \Delta^{\text{CK}_\lambda} \circ \Psi_{t_0}^*(x) &= \Delta^{\text{CK}_{\lambda'}} \circ \Psi'_{t_0}(x) = (\Psi'_{t_0} \otimes \Psi'_{t_0}) \circ \Delta^{\text{CK}_{\lambda'}}(x) = (\Psi_{t_0}^* \otimes \Psi_{t_0}^*) \circ \Delta^{\text{CK}_\lambda}(x), \end{aligned}$$

so Ψ is a Hopf algebra morphism.

Let us assume that $\lambda_{t_0} \neq 0$. If $\Psi_{t_0}^*(x) = 0$, then $\Psi'_{t_0}(x) = 0$. As $a'_{t_0} \neq 0$, by the first case, $x = 0$, so $\Psi_{t_0}^*$ is injective. Moreover, there exists $z \in \mathcal{H}_{\mathcal{D}', \mathcal{T}'}$, such that $\Psi'_{t_0}(z) = y$; so $\Psi_{t_0}^*(z) = y$, and $\Psi_{t_0}^*$ is surjective.

Let us assume that $\lambda_{t_0} = 0$. Let T be a tree with two vertices, such that its unique edge is of type t_0 . As $T \notin \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$, $\Phi_{t_0}(T)$ has a unique term, given by the partition $X = \{\{x_1\}, \{x_2\}\}$, where x_1 and x_2 are the vertices of T . Hence,

$$\Psi_{t_0}^*(T) = \lambda_{t_0} T' = 0,$$

so $\Psi_{t_0}^*$ is not injective. ■

Example 5.10. Here, \mathcal{T} contains two elements, $|$ and $\dot{|}$. In order to simplify, we omit the decorations of vertices. We put

$$x = \bullet, \quad y = \dot{|}, \quad z = \vee, \quad u = \dot{|}, \quad v = \ddot{|}.$$

Applying $\Psi_{|}^*$:

$$\begin{aligned} \Psi_{|}^*(\bullet) &= \bullet^x, & \Psi_{|}^*(\vee) &= \lambda^2 | \vee^x, \\ \Psi_{|}^*(\dot{|}) &= \lambda | \dot{|}^x, & \Psi_{|}^*(\dot{\dot{|}}) &= \lambda^2 | \dot{\dot{|}}^x, \\ \Psi_{|}^*(\ddot{|}) &= \lambda \dot{|} \dot{|}^x + \bullet^y, & \Psi_{|}^*(\dot{\dot{\dot{|}}}) &= \lambda | \lambda \dot{\dot{\dot{|}}}|^x + \lambda | \dot{|}^y, \\ \Psi_{|}^*(\vee\vee) &= \lambda | \lambda \dot{|} \vee^x + \lambda | \dot{|}^y, & \Psi_{|}^*(\dot{\dot{\dot{|}}}) &= \lambda | \lambda \dot{\dot{\dot{|}}}|^x + \lambda | \dot{|}^y + \bullet^u, \\ \Psi_{|}^*(\vee\vee\vee) &= \lambda^2 | \vee\vee^x + 2\lambda \dot{|} \dot{|}^x + \bullet^z, & \Psi_{|}^*(\dot{\dot{\dot{\dot{|}}}}) &= \lambda^2 | \dot{\dot{\dot{\dot{|}}}}|^x + \lambda \dot{|} \dot{|}^x + \lambda \dot{|} \dot{|}^y + \bullet^v. \end{aligned}$$

Remark 5.11. Although it is not indicated, Ψ_{t_0} and $\Psi_{t_0}^*$ depend on λ .

5.5 Bialgebras in cointeraction

By [12], for any $\lambda \in \mathbb{K}^{(\mathcal{T})}$, the operad morphism $\theta_\lambda: \mathbf{PreLie} \rightarrow \mathcal{P}_{\mathcal{T}}$, which send \bullet to \bullet_λ , where \mathbf{PreLie} is the operad of pre-Lie algebras, induces a pair of cointeracting bialgebras for any finite set \mathcal{D} . By construction, the first bialgebra of the pair is $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$. Let us describe the second one.

Definition 5.12. Let $F \in \mathbb{F}_{\mathcal{T}, \mathcal{D}}$. We shall say that $\{T_1, \dots, T_k\} \triangleleft F$ if

1. $\{T_1, \dots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, by restriction.
2. For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$.

If $\{T_1, \dots, T_k\} \triangleleft F$ and $\text{dec}: [k] \rightarrow \mathcal{D}$, we denote by $(F/\{T_1, \dots, T_k\}, \text{dec})$ the forest obtained by contracting T_i to a single vertex, and decorating this vertex by $\text{dec}(i)$, for all $i \in [k]$. This is an element of $\mathbb{F}_{\mathcal{D}, \mathcal{T}}$.

Proposition 5.13. If \mathcal{D} is finite, $\mathcal{H}'_{\mathcal{D}, \mathcal{T}}$ is the free commutative algebra generated by pairs (T, d) , where $T \in \mathbb{T}_{\mathcal{T}, \mathcal{D}}$ and $d \in \mathcal{D}$. The coproduct is given, for any $F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, $d \in \mathcal{D}$, by

$$\delta(F, d) = \sum_{\{T_1, \dots, T_k\} \triangleleft F} \sum_{\text{dec}: [k] \rightarrow \mathcal{D}} ((F/\{T_1, \dots, T_k\}, \text{dec}), d) \otimes (T_1, \text{dec}(1)) \cdots (T_k, \text{dec}(k)).$$

Then $(\mathcal{H}'_{\mathcal{D},\mathcal{T}}, m, \delta)$ is a bialgebra, and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a coalgebra in the category of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ -comodules via the coaction given, for any $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, by

$$\bar{\delta}(T) = \sum_{\{T_1, \dots, T_k\} \triangleleft T} \sum_{\text{dec}: [k] \rightarrow \mathcal{D}} ((T/\{T_1, \dots, T_k\}, \text{dec}) \otimes (T_1, \text{dec}(1)) \cdots (T_k, \text{dec}(k))).$$

Corollary 5.14. *Let us assume that \mathcal{D} is given a semigroup law denoted by $+$. If $F \in \mathbb{F}_{\mathcal{T},\mathcal{D}}$, and $\{T_1, \dots, T_k\} \triangleleft F$, then naturally $T_i \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$ for any i and the \mathcal{T} -typed forest $F/\{T_1, \dots, T_k\}$ is given a \mathcal{D} -decoration, decorating the vertex obtained in the contradiction of T_i by the sum of the decorations of the vertices of T_i . Then $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ is given a second coproduct δ such that for any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$,*

$$\delta(F) = \sum_{\{T_1, \dots, T_k\} \triangleleft F} F/\{T_1, \dots, T_k\} \otimes T_1 \cdots T_k.$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \delta)$ is a bialgebra and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a coalgebra in the category of $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ -comodules via the coaction δ .

Proof. We denote by I the ideal of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ generated by pairs (T, d) such that $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $d \in \mathcal{D}$, with

$$d \neq \sum_{v \in V(T)} \text{dec}(v).$$

The quotient $\mathcal{H}'_{\mathcal{D},\mathcal{T}}/I$ is identified with $\mathcal{H}_{\mathcal{D},\mathcal{T}}$, through the surjective algebra morphism

$$\varpi: \begin{cases} \mathcal{H}'_{\mathcal{D},\mathcal{T}} \longrightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}, \\ (F, d) \in \mathbb{F}_{\mathcal{D},\mathcal{T}} \times \mathcal{D} \longmapsto \begin{cases} F & \text{if } d = \sum_{v \in V(F)} \text{dec}(v), \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Let us prove that I is a coideal. Let $T \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$, $d \in \mathcal{D}$, $\{T_1, \dots, T_k\} \triangleleft F$, $\text{dec}: [k] \rightarrow \mathcal{D}$ such that $((T/\{T_1, \dots, T_k\}, \text{dec}), d) \notin I$ and for any i , $(T_i, \text{dec}(i)) \notin I$. Then

$$\forall i \in [k], \quad \sum_{v \in V(T_i)} \text{dec}(v) = \text{dec}(i), \quad \sum_{i=1}^k \text{dec}(i) = d.$$

Hence,

$$\sum_{v \in V(T)} \text{dec}(v) = \sum_{i=1}^k \sum_{v \in V(T_i)} \text{dec}(v) = \sum_{i=1}^k \text{dec}(i) = d,$$

so $(T, d) \notin I$. Consequently, if $T \in I$, then $((T/\{T_1, \dots, T_k\}, \text{dec}), d) \in I$ or at least one of the $(T_i, \text{dec}(i))$ belongs to I . Hence,

$$\delta(I) \subseteq I \otimes \mathcal{H}'_{\mathcal{D},\mathcal{T}} + \mathcal{H}'_{\mathcal{D},\mathcal{T}} \otimes I.$$

So I is a coideal. The coproduct induced on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by the morphism ϖ is precisely the one given in the setting of this corollary. \blacksquare

In particular, if \mathcal{D} is reduced to a single element, denoted by $*$, if we give it its unique semigroup structure ($* + * = *$), We obtain again the result of [7].

5.6 The Bruned–Hairer–Zambotti construction

We now consider the coproducts on typed trees in [3, Theorem 2.2.16], the first one with $\overline{\mathfrak{A}}(F) = \mathfrak{A}(F)$ and the second one with $\overline{\mathfrak{A}}(F) = \mathfrak{A}^+(F)$ of [3, Definition 2.4.1]. By definition [3, Definition 2.26] of admissible subtrees, according to the notations we choose in this paper:

- Let \mathcal{L} be a finite set of types. The considered trees are \mathcal{L} -typed and the leaves are \mathcal{L} -decorated. Considering that the internal vertices of such a tree are in fact decorated by an element $0 \notin \mathcal{L}$, these trees form a subset of $\mathbb{T}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ which we denote by $\mathbb{T}'_{\mathcal{L}}$.
- The first coproduct Δ_+ is given on any tree $T \in \mathbb{T}'_{\mathcal{L}}$ by

$$\Delta_+(T) = \sum_{c \in \text{Adm}'(T)} R^c(T) \otimes P^c(T),$$

where $\text{Adm}'(T)$ is the set of admissible cuts of T such that the set of leaves of $R^c(T)$ is a subset of leaves of T (note that automatically, the leaves of $P^c(T)$ are also leaves of T).

- The second coproduct is given on any tree $T \in \mathbb{T}'_{\mathcal{L}}$ by

$$\overline{\Delta}(T) = \sum_{\{T_1, \dots, T_k\} \triangleleft' T} T / \{T_1, \dots, T_k\} \otimes T_1 \cdots T_k,$$

where the sum runs over all $\{T_1, \dots, T_k\} \triangleleft T$ such that the leaves of $T / \{T_1, \dots, T_k\}$ are leaves of T .

We denote by $\mathcal{H}'_{\mathcal{L}}$ the subalgebra of $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ generated by $\mathbb{T}'_{\mathcal{L}}$. Then $(\mathcal{H}'_{\mathcal{L}}, m, \Delta_+)$ and $(\mathcal{H}'_{\mathcal{L}}, m, \delta)$ are bialgebras.

Let us give to \mathcal{L} any product \times making it a commutative associative semigroup. We extend this structure to $\mathcal{L} \sqcup \{0\}$ by

$$\forall t \in \mathcal{L} \sqcup \{0\}, \quad 0 \times t = t \times 0 = 0.$$

We take $\lambda_t = 1$ for any $t \in \mathcal{L} \sqcup \{0\}$, and obtain with this data two coproducts Δ and δ on $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$, the first one given by admissible cuts and the second one by contractions of subtrees. The subalgebra $\mathcal{H}''_{\mathcal{L}}$ of $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ generated by trees such that any internal vertex is decorated by 0 (note that the leaves of such a tree are decorated by $\mathcal{L} \sqcup \{0\}$) is a subbialgebra for both coproducts. We denote by I the ideal of $\mathcal{H}''_{\mathcal{L}}$ generated by trees with at least one leaf decorated by 0. Then it is a coideal for both coproducts, so the quotient algebra $\mathcal{H}''_{\mathcal{L}}/I$ inherits two coproducts, still denoted by Δ and δ . This algebra is trivially identified with the algebra generated by $\mathbb{T}'_{\mathcal{L}}$, that is to say with $\mathcal{H}'_{\mathcal{L}}$. By construction of the different coproducts, this identification is an isomorphism from the Hopf algebra $(\mathcal{H}''_{\mathcal{L}}/I, \Delta)$ to $(\mathcal{H}'_{\mathcal{L}}/I, \Delta_+)$ and from the Hopf algebra $(\mathcal{H}''_{\mathcal{L}}/I, \delta)$ to $(\mathcal{H}'_{\mathcal{L}}/I, \overline{\Delta})$. In other words, the Bruned–Hairer–Zambotti construction of cointeracting bialgebras on typed trees is a subquotient of the construction presented here.

6 Appendix

6.1 Values of $t_{D,T}(k)$

For $D = 1$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|----|-----|------|-------|---------|----------|-----------|
| 1 | 1 | 1 | 2 | 4 | 9 | 20 | 48 | 115 |
| 2 | 1 | 2 | 7 | 26 | 107 | 458 | 2058 | 9498 |
| 3 | 1 | 3 | 15 | 82 | 495 | 3144 | 20875 | 142773 |
| 4 | 1 | 4 | 26 | 188 | 1499 | 12628 | 111064 | 1006840 |
| 5 | 1 | 5 | 40 | 360 | 3570 | 37476 | 410490 | 4635330 |
| 6 | 1 | 6 | 57 | 614 | 7284 | 91566 | 1200705 | 16232820 |
| 7 | 1 | 7 | 77 | 966 | 13342 | 195384 | 2984142 | 46990952 |
| 8 | 1 | 8 | 100 | 1432 | 22570 | 377320 | 6578116 | 118238600 |
| 9 | 1 | 9 | 126 | 2028 | 35919 | 674964 | 13225632 | 267188229 |
| 10 | 1 | 10 | 155 | 2770 | 54465 | 1136402 | 24723000 | 554540590 |

For $D = 2$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|----|------|-------|---------|----------|------------|--------------|
| 1 | 2 | 4 | 14 | 52 | 214 | 916 | 4116 | 18996 |
| 2 | 2 | 8 | 52 | 376 | 2998 | 25256 | 222128 | 2013680 |
| 3 | 2 | 12 | 114 | 1228 | 14568 | 183132 | 2401410 | 32465640 |
| 4 | 2 | 16 | 200 | 2864 | 45140 | 754640 | 13156232 | 236477200 |
| 5 | 2 | 20 | 310 | 5540 | 108930 | 2272804 | 49446000 | 1109081180 |
| 6 | 2 | 24 | 444 | 9512 | 224154 | 5606520 | 146204792 | 3930863232 |
| 7 | 2 | 28 | 602 | 15036 | 413028 | 12043500 | 366122190 | 11475005616 |
| 8 | 2 | 32 | 784 | 22368 | 701768 | 23373216 | 811575408 | 29052861280 |
| 9 | 2 | 36 | 990 | 31764 | 1120590 | 41969844 | 1638712716 | 65965167108 |
| 10 | 2 | 40 | 1220 | 43480 | 1703710 | 70875208 | 3073688160 | 137426005200 |

For $D = 3$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|----|------|--------|----------|-----------|-------------|---------------|
| 1 | 3 | 9 | 45 | 246 | 1485 | 9432 | 62625 | 428319 |
| 2 | 3 | 18 | 171 | 1842 | 21852 | 274698 | 3602115 | 48698460 |
| 3 | 3 | 27 | 378 | 6084 | 107757 | 2024892 | 39676896 | 801564687 |
| 4 | 3 | 36 | 666 | 14268 | 336231 | 8409780 | 219307188 | 5896294848 |
| 5 | 3 | 45 | 1035 | 27690 | 814680 | 25444584 | 828506340 | 27812997990 |
| 6 | 3 | 54 | 1485 | 47646 | 1680885 | 62954766 | 2458069074 | 98947750662 |
| 7 | 3 | 63 | 2016 | 75432 | 3103002 | 135520812 | 6170116638 | 289616448690 |
| 8 | 3 | 72 | 2628 | 112344 | 5279562 | 263423016 | 13701398868 | 734709311208 |
| 9 | 3 | 81 | 3321 | 159678 | 8439471 | 473586264 | 27703353159 | 1670715963729 |
| 10 | 3 | 90 | 4095 | 218730 | 12842010 | 800524818 | 52018920345 | 3484841027040 |

For $D = 4$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|-----|------|--------|----------|------------|--------------|----------------|
| 1 | 4 | 16 | 104 | 752 | 5996 | 50512 | 444256 | 4027360 |
| 2 | 4 | 32 | 400 | 5728 | 90280 | 1509280 | 26312464 | 472954400 |
| 3 | 4 | 48 | 888 | 19024 | 448308 | 11213040 | 292409584 | 7861726464 |
| 4 | 4 | 64 | 1568 | 44736 | 1403536 | 46746432 | 1623150816 | 58105722560 |
| 5 | 4 | 80 | 2440 | 86960 | 3407420 | 141750416 | 6147376320 | 274852010400 |
| 6 | 4 | 96 | 3504 | 149792 | 7039416 | 351230688 | 18268531824 | 979612414944 |
| 7 | 4 | 112 | 4760 | 237328 | 13006980 | 756866096 | 45910215120 | 2871018269632 |
| 8 | 4 | 128 | 6208 | 353664 | 22145568 | 1472317056 | 102037088448 | 7290356719488 |
| 9 | 4 | 144 | 7848 | 502896 | 35418636 | 2648533968 | 206451156768 | 16590568445280 |
| 10 | 4 | 160 | 9680 | 689120 | 53917640 | 4479065632 | 387863411920 | 34625886677920 |

we find the following sequences of the OEIS [24]:

| $T \setminus D$ | 1 | 2 | 3 | 4 |
|-----------------|---------|---------|---------|---------|
| 1 | A0081 | A038055 | A038059 | A136793 |
| 2 | A00151 | A136794 | | |
| 3 | A006964 | | | |
| 4 | A052763 | | | |
| 5 | A052788 | | | |
| 6 | A246235 | | | |
| 7 | A246236 | | | |
| 8 | A246237 | | | |
| 9 | A246238 | | | |
| 10 | A246239 | | | |

6.2 Values of $t'_{D,T}(k)$

For $D = 1$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|-----|------|-------|--------|----------|-----------|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 3 | 10 | 39 | 160 | 702 | 3177 |
| 3 | 1 | 2 | 9 | 46 | 268 | 1660 | 10845 | 73270 |
| 4 | 1 | 3 | 18 | 124 | 963 | 7968 | 69236 | 621999 |
| 5 | 1 | 4 | 30 | 260 | 2525 | 26136 | 283528 | 3178696 |
| 6 | 1 | 5 | 45 | 470 | 5480 | 68096 | 885805 | 11904160 |
| 7 | 1 | 6 | 63 | 770 | 10479 | 151956 | 2304974 | 36110880 |
| 8 | 1 | 7 | 84 | 1176 | 18298 | 303296 | 5255964 | 94051770 |
| 9 | 1 | 8 | 108 | 1704 | 29838 | 556464 | 10845732 | 218239560 |
| 10 | 1 | 9 | 135 | 2370 | 46125 | 955872 | 20696076 | 462558987 |

For $D = 2$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|----|------|-------|---------|----------|------------|--------------|
| 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 2 | 4 | 22 | 144 | 1090 | 8864 | 76162 | 678532 |
| 3 | 2 | 8 | 68 | 688 | 7886 | 96896 | 1250780 | 16713504 |
| 4 | 2 | 12 | 138 | 1888 | 29004 | 476736 | 8213588 | 146342376 |
| 5 | 2 | 16 | 232 | 4000 | 77060 | 1586304 | 34185344 | 761389360 |
| 6 | 2 | 20 | 350 | 7280 | 168670 | 4171744 | 107932710 | 2884827980 |
| 7 | 2 | 24 | 492 | 11984 | 324450 | 9370368 | 282934428 | 8822987856 |
| 8 | 2 | 28 | 658 | 18368 | 569016 | 18793600 | 648698792 | 23119514576 |
| 9 | 2 | 32 | 848 | 26688 | 930984 | 34609920 | 1344232416 | 53898191520 |
| 10 | 2 | 36 | 1062 | 37200 | 1442970 | 59627808 | 2573660298 | 114661732500 |

For $D = 3$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|----|------|--------|----------|-----------|-------------|---------------|
| 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 3 | 9 | 72 | 705 | 7947 | 96588 | 1237728 | 16450389 |
| 3 | 3 | 18 | 225 | 3408 | 58347 | 1072224 | 20685195 | 413084610 |
| 4 | 3 | 27 | 459 | 9405 | 216081 | 5315112 | 136987407 | 3650993163 |
| 5 | 3 | 36 | 774 | 19992 | 576405 | 17763984 | 572991726 | 19100718828 |
| 6 | 3 | 45 | 1170 | 36465 | 1264950 | 46852884 | 1815034140 | 72635168880 |
| 7 | 3 | 54 | 1647 | 60120 | 2437722 | 105455952 | 4768982442 | 222723271080 |
| 8 | 3 | 63 | 2205 | 92253 | 4281102 | 211832208 | 10953036318 | 584744300226 |
| 9 | 3 | 72 | 2844 | 134160 | 7011846 | 390570336 | 22727284344 | 1365242802048 |
| 10 | 3 | 81 | 3564 | 187137 | 10877085 | 673533468 | 43560017892 | 2907844041231 |

For $D = 4$,

| $T \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|-----|------|--------|----------|------------|--------------|----------------|
| 1 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 4 | 16 | 168 | 2192 | 32844 | 531200 | 9051376 | 159962784 |
| 3 | 4 | 32 | 528 | 10656 | 242792 | 5939968 | 152518064 | 4053650976 |
| 4 | 4 | 48 | 1080 | 29488 | 902100 | 29551104 | 1014147872 | 35989518528 |
| 5 | 4 | 64 | 1824 | 62784 | 2411024 | 98976256 | 4252211232 | 188790415552 |
| 6 | 4 | 80 | 2760 | 114640 | 5297820 | 261422336 | 13491005840 | 719200139360 |
| 7 | 4 | 96 | 3888 | 189152 | 10218744 | 588999936 | 35487727184 | 2208096700896 |
| 8 | 4 | 112 | 5208 | 290416 | 17958052 | 1184031744 | 81574704960 | 5802692175744 |
| 9 | 4 | 128 | 6720 | 422528 | 29428000 | 2184360960 | 169377005376 | 13557899008896 |
| 10 | 4 | 144 | 8424 | 589584 | 45668844 | 3768659712 | 324805399344 | 28894042642464 |

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