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Algebraic structures on typed decorated rooted trees

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Then $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, equipped with all these products, is a \mathcal{T} -multiple pre-Lie algebra (Definition 3.1), also called matching pre-Lie algebras in [25]: for any types t and t' , for any $x, y, z \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}$,

$$x \bullet_{t'} (y \bullet_t z) - (x \bullet_{t'} y) \bullet_t z = x \bullet_t (z \bullet_{t'} y) - (x \bullet_t z) \bullet_{t'} y.$$

This relation appears in [4]. We prove in Corollary 3.11 that it is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} , generalizing the result of [9], first mentioned in [4, Proposition 4.21]. Consequently, we obtain a combinatorial description of the operad of \mathcal{T} -multiple pre-Lie algebras in terms of \mathcal{T} -typed trees with indexed vertices (Theorem 3.17): for example,

$$\begin{array}{c} \downarrow_1^2 \circ_1 \downarrow_1^2 \\ \downarrow_1^2 \end{array} = \begin{array}{c} \downarrow_1^2 \quad \downarrow_1^3 \\ \downarrow_1^2 \end{array} + \begin{array}{c} \downarrow_1^3 \\ \downarrow_1^2 \end{array}, \quad \begin{array}{c} \downarrow_1^2 \circ_2 \downarrow_1^2 \\ \downarrow_1^2 \end{array} = \begin{array}{c} \downarrow_1^3 \\ \downarrow_1^2 \end{array}.$$

We also give a description of the Koszul dual operad and of its free algebras in Propositions 3.19 and 3.20, generalizing a result of [8].

For any family $\lambda = (\lambda_t)_{t \in \mathcal{T}}$ with a finite support, the product $\bullet_\lambda = \sum \lambda_t \bullet_t$ is pre-Lie: using the Guin–Oudom construction [21, 22], we obtain a Hopf algebraic structure $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda} = (S(\mathfrak{g}_{\mathcal{D},\mathcal{T}}), \star_\lambda, \Delta)$ on the symmetric algebra generated by \mathcal{T} -typed and \mathcal{D} -decorated trees, that is to say on the space of \mathcal{T} -typed and \mathcal{D} -decorated forests. The coproduct Δ is given by partitions of forests into two forests and the \star_λ product is given by grafting. For example,

$$\begin{array}{c} \downarrow_a^b \\ \downarrow_a^c \end{array} \star_\lambda \cdot^c = \begin{array}{c} \downarrow_a^b \\ \downarrow_a^c \end{array} \cdot^c + \lambda \mid \begin{array}{c} \downarrow_a^b \quad \downarrow_a^c \\ \downarrow_a^c \end{array} + \lambda \mid \begin{array}{c} \downarrow_a^c \\ \downarrow_a^b \end{array} + \lambda \mid \begin{array}{c} \downarrow_a^c \\ \downarrow_a^c \end{array}.$$

In the non-typed case, we get back the Grossman–Larson Hopf algebra of trees [14]. Dually, we obtain Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$, generalizing the Connes–Kreimer Hopf algebra [10] of rooted trees. For example,

$$\begin{aligned} \Delta^{\text{CK}\lambda}(\downarrow_a^b) &= \downarrow_a^b \otimes 1 + 1 \otimes \downarrow_a^b + \lambda \mid \cdot^a \otimes \cdot^b, \\ \Delta^{\text{CK}\lambda}(\downarrow_a^b \downarrow_a^c) &= \downarrow_a^b \downarrow_a^c \otimes 1 + 1 \otimes \downarrow_a^b \downarrow_a^c + \lambda \mid \downarrow_a^b \otimes \cdot^c + \lambda \mid \downarrow_a^c \otimes \cdot^b + \lambda^2 \mid \cdot^a \otimes \cdot^b \cdot^c, \\ \Delta^{\text{CK}\lambda}(\downarrow_a^b \downarrow_a^c \downarrow_a^d) &= \downarrow_a^b \downarrow_a^c \downarrow_a^d \otimes 1 + 1 \otimes \downarrow_a^b \downarrow_a^c \downarrow_a^d + \lambda \mid \downarrow_a^b \otimes \cdot^c + \lambda \mid \downarrow_a^c \otimes \cdot^b + \lambda \mid \lambda \mid \cdot^a \otimes \cdot^b \cdot^c. \end{aligned}$$

This Hopf algebra satisfies a universal property in Hochschild cohomology, as does the Connes–Kreimer’s Hopf algebra. We describe it in the simpler case where \mathcal{T} is finite (Theorem 5.5). We finally give a second coproduct δ on $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$, such that $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a Hopf algebra in the category of $(S(\mathfrak{g}_{\mathcal{D},\mathcal{T}}), m, \delta)$ -right comodules, generalizing the result of [7]. This coproduct δ is given by a contraction-extraction process. For example, in the non-decorated case,

$$\begin{aligned} \delta(\cdot) &= \cdot \otimes \cdot, \\ \delta(\downarrow) &= \downarrow \otimes \cdot + \cdot \otimes \downarrow, \\ \delta(\downarrow \downarrow) &= \downarrow \downarrow \otimes \cdot + 2 \downarrow \otimes \downarrow + \cdot \otimes \downarrow \downarrow, \\ \delta(\downarrow \downarrow \downarrow) &= \downarrow \downarrow \downarrow \otimes \cdot + \downarrow \otimes \downarrow \downarrow + \downarrow \otimes \downarrow \downarrow + \cdot \otimes \downarrow \downarrow \downarrow. \end{aligned}$$

We are also interested in morphisms between these objects. We prove that if λ and μ are both nonzero, then the pre-Lie algebras $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ and $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\mu)$ are isomorphic (Corollary 4.8).

Consequently, if λ and μ are both nonzero, the Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\mu}$ are isomorphic; dually, the Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\mu}$ are isomorphic (Corollary 5.7). Using Livernet's rigidity theorem [16] and a nonassociative permutative coproduct defined in Proposition 4.1, we prove that if $\lambda \neq 0$, then $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is, as a pre-Lie algebra, freely generated by a family of typed trees $\mathcal{D}' = \mathcal{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ satisfying a condition on the type of edges born from the root (Corollary 4.2). As a consequence, the Hopf algebra $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ of typed and decorated trees is isomorphic to a Connes–Kreimer Hopf algebra of non typed and decorated trees $\mathcal{H}_{\mathcal{D}'}$, and an explicit isomorphism is described with the help of contraction in Proposition 5.9.

This paper is organized as follows: the first section gives the basic definition of typed rooted trees and enumeration results, when the number of types and decorations are finite. The second section is about the \mathcal{T} -multiple pre-Lie algebra structures on these trees and the underlying operads. The freeness of the pre-Lie structures on typed decorated trees and its consequences are studied in the third section. In the last section, the dual Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ are defined, studied and related to the constructions of Bruned, Hairer and Zambotti [3, 5]: forgetting the labels, the two coproducts they use on a family of typed and partially decorated trees are a subquotient of a the construction presented here.

Since the first version of this paper, written in 2018, multiple pre-Lie algebras have appeared in [4, 6] in a context of SPDEs. When the set of types has more structures (for example a product making it a commutative semigroup), then other possibilities for grafting products of typed trees are studied in [13, 18, 25], giving other interesting structures on these objects.

Notation 1.1.

- We denote by \mathbb{K} a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, pre-Lie algebras, ...) in this text will be taken over \mathbb{K} .
- For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \dots, n\}$.
- For any set \mathcal{T} , we denote by $\mathbb{K}^{\mathcal{T}}$ the set of family $\lambda = (\lambda_t)_{t \in \mathcal{T}}$ of elements of \mathbb{K} indexed by \mathcal{T} , and we denote by $\mathbb{K}^{(\mathcal{T})}$ the set of elements $\lambda \in \mathbb{K}^{\mathcal{T}}$ with a finite support. Note that if \mathcal{T} is finite, then $\mathbb{K}^{\mathcal{T}} = \mathbb{K}^{(\mathcal{T})}$.

2 Typed decorated trees

2.1 Definition

Definition 2.1. Let \mathcal{D} and \mathcal{T} be two nonempty sets.

1. A \mathcal{D} -decorated \mathcal{T} -typed forest is a triple $(F, \text{dec}, \text{type})$, where
 - F is a rooted forest. The set of its vertices is denoted by $V(F)$ and the set of its edges by $E(F)$.
 - $\text{dec}: V(F) \longrightarrow \mathcal{D}$ is a map.
 - $\text{type}: E(F) \longrightarrow \mathcal{T}$ is a map.

If the underlying rooted forest of F is connected, we shall say that F is a \mathcal{D} -decorated \mathcal{T} -typed tree.

2. If $(F, \text{dec}_F, \text{type}_F)$ and $(G, \text{dec}_G, \text{type}_G)$ are two \mathcal{D} -decorated \mathcal{T} -typed forests, they are isomorphic if there exists a rooted forest isomorphism f from F to G such that for any vertex v of F , $\text{dec}_G(f(v)) = \text{dec}_F(v)$ and for any edge e of F , $\text{type}_G(f(e)) = \text{type}_F(e)$
3. For any finite set A , we denote by $\mathbb{T}_{\mathcal{T}}(A)$ the set of A -decorated \mathcal{T} -typed trees T such that $V(T) = A$ and $\text{dec} = \text{Id}_A$, and by $\mathbb{F}_{\mathcal{T}}(A)$ the set of A -decorated \mathcal{T} -typed forests F such that $V(F) = A$ and $\text{dec} = \text{Id}_A$.

As any element of $\mathbb{F}_{\mathcal{T},\mathcal{D}}$ can be uniquely decomposed as the disjoint union of its connected components, which are elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, we obtain

$$F_{D,T}(X) = \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)}}. \quad (2.1)$$

We put $\mathcal{T} = \{t_1, \dots, t_T\}$. For any $d \in \mathcal{D}$, we consider

$$B_d: \begin{cases} (\mathbb{F}_{\mathcal{D},\mathcal{T}})^T \longrightarrow \mathbb{T}_{\mathcal{D},\mathcal{T}}, \\ (F_1, \dots, F_T) \longmapsto B_d(F_1, \dots, F_T), \end{cases}$$

where $B_d(F_1, \dots, F_T)$ is the tree obtained by grafting the forests F_1, \dots, F_n on a common root decorated by d ; the edges from this root to the roots of F_i are of type t_i for any $1 \leq i \leq T$. Then B_d is injective, homogeneous of degree 1, and moreover $\mathbb{T}_{\mathcal{D},\mathcal{T}}$ is the disjoint union of the $B_d((\mathbb{F}_{\mathcal{D},\mathcal{T}})^T)$, $d \in \mathcal{D}$. Hence,

$$T_{D,T}(X) = DX(F_{D,T})^T = DX \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)T}}. \quad (2.2)$$

Note that (2.2) allows to compute $t_{D,T}(n)$ by induction on n , and (2.1) allows to deduce $f_{D,T}(n)$.

Lemma 2.4. *For any $n \in \mathbb{N}$,*

$$t_{D,T}(n) = \frac{t_{TD,1}(n)}{T}.$$

Proof. By induction on n . If $n = 1$, $t_{D,T}(1) = D$ and $t_{TD,1}(1) = TD$, which gives the result. Let us assume the result at all ranks $k < n$. Then $t_{D,T}(n)T$ is the coefficient of X^n in

$$TDX \prod_{k=1}^{n-1} \frac{1}{(1 - X^k)^{t_{D,T}(k)T}} = TDX \prod_{k=1}^{n-1} \frac{1}{(1 - X^k)^{t_{TD,1}(k)}},$$

which is precisely $t_{TD,1}(n)$. ■

Example 2.5. We obtain

$$\begin{aligned} t_{D,T}(1) &= D, \\ t_{D,T}(2) &= D^2T, \\ t_{D,T}(3) &= \frac{D^2T(3DT + 1)}{2}, \\ t_{D,T}(4) &= \frac{D^2T(8D^2T^2 + 3DT + 1)}{3}, \\ t_{D,T}(5) &= \frac{D^2T(125D^3T^3 + 54D^2T^2 + 31DT + 6)}{24}, \\ t_{D,T}(6) &= \frac{D^2T(162D^4T^4 + 80D^3T^3 + 45D^2T^2 + 10DT + 3)}{15}, \\ t_{D,T}(7) &= \frac{D^2T(16807D^5T^5 + 9375D^4T^4 + 5395D^3T^3 + 2025D^2T^2 + 838DT + 120)}{720}. \end{aligned}$$

We shall give tables of values of $t_{D,T}(k)$ in the appendix.

3 Multiple pre-Lie algebras

We here fix a nonempty set \mathcal{T} of edges types.

3.1 Definition

Definition 3.1. A \mathcal{T} -multiple pre-Lie algebra is a family $(V, (\bullet_t)_{t \in \mathcal{T}})$, where V is a vector space and for all $t \in \mathcal{T}$, \bullet_t is a bilinear product on V such that

$$\forall t, t' \in \mathcal{T}, \quad \forall x, y, z \in V, \quad x \bullet_{t'} (y \bullet_t z) - (x \bullet_{t'} y) \bullet_t z = x \bullet_t (z \bullet_{t'} y) - (x \bullet_t z) \bullet_{t'} y.$$

Remark 3.2. For any $t \in \mathcal{T}$, (V, \bullet_t) is a pre-Lie algebra. More generally, for any family $\lambda = (\lambda_t)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, putting $\bullet_\lambda = \sum \lambda_t \bullet_t$, (V, \bullet_λ) is a pre-Lie algebra.

Proposition 3.3. Let \mathcal{D} be any set; we denote by $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ the vector space generated by $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$. For any $T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $v \in V(T)$ and $t \in \mathcal{T}$, we denote by $T \bullet_t^{(v)} T'$ the \mathcal{D} -decorated \mathcal{T} -typed tree obtained by grafting T' on v (that is to say adding an edge between v and the root of T'), the created edge being of type t . We then define a product \bullet_t on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ by

$$\forall T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad T \bullet_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T'.$$

Then $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, (\bullet_t)_{t \in \mathcal{T}})$ is a \mathcal{T} -multiple pre-Lie algebra.

Proof. Let T, T', T'' be elements of $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $t', t'' \in \mathcal{T}$.

$$\begin{aligned} & (T \bullet_{t'} T') \bullet_{t''} T'' - T \bullet_{t'} (T' \bullet_{t''} T'') \\ &= \sum_{v \in V(T), v' \in V(T) \sqcup V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' - \sum_{v \in V(T), v' \in V(T')} T \bullet_{t'}^{(v)} (T' \bullet_{t''}^{(v')} T'') \\ &= \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' + \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' - T \bullet_{t'}^{(v)} (T' \bullet_{t''}^{(v')} T'') \\ &= \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' \\ &= \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t''}^{(v')} T'') \bullet_{t'}^{(v)} T' \\ &= (T \bullet_{t''} T'') \bullet_{t'} T' - T \bullet_{t''} (T' \bullet_{t'} T'). \end{aligned}$$

So $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ is indeed a \mathcal{T} -multiple pre-Lie algebra. ■

Example 3.4. If $a, b, c \in \mathcal{D}$ and $|, \vdash \in \mathcal{T}$,

$$\begin{array}{c} \begin{array}{c} \overset{b}{\downarrow} \\ \downarrow_a \bullet | \end{array} \cdot \overset{c}{\downarrow} = \begin{array}{c} \overset{b}{\downarrow} \quad \overset{c}{\downarrow} \\ \downarrow_a \end{array} + \begin{array}{c} \overset{c}{\downarrow} \\ \downarrow_b \\ \downarrow_a \end{array}, \quad \begin{array}{c} \overset{b}{\downarrow} \\ \downarrow_a \bullet \vdash \end{array} \cdot \overset{c}{\downarrow} = \begin{array}{c} \overset{b}{\downarrow} \quad \overset{c}{\downarrow} \\ \downarrow_a \end{array} + \begin{array}{c} \overset{c}{\downarrow} \\ \downarrow_b \\ \downarrow_a \end{array}. \end{array}$$

3.2 Guin–Oudom extension of the pre-Lie products

Notation 3.5. Let $(\partial_t)_{t \in \mathcal{T}}$ be a family of formal symbols indexed by \mathcal{T} . For any vector space V , we denote

$$V^{\oplus \mathcal{T}} = V \otimes \text{Vect}(\partial_t, t \in \mathcal{T}).$$

Then

$$V^{\oplus \mathcal{T}} = \bigoplus_{t \in \mathcal{T}} V \otimes \partial_t.$$

In order to enlighten the notations, we write $v\partial_t$ instead of $v \otimes \partial_t$ for any $v \in V$ and for any $t \in \mathcal{T}$.

Lemma 3.6. *If for any $t \in \mathcal{T}$, \bullet_t is a bilinear product on a vector space V , we define $\bullet : (V^{\oplus \mathcal{T}})^{\otimes 2} \rightarrow V^{\oplus \mathcal{T}}$ by*

$$x \partial_t \bullet x' \partial_{t'} = (x \bullet_{t'} y) \partial_t.$$

Then $(V, (\bullet_t)_{t \in \mathcal{T}})$ is a \mathcal{T} -multiple pre-Lie algebra if, and only if, $(V^{\oplus \mathcal{T}}, \bullet)$ is a pre-Lie algebra.

Proof. Let $x, x', x'' \in V$, $t, t', t'' \in \mathcal{T}$. Then

$$x \partial_t \bullet (x' \partial_{t'} \bullet x'' \partial_{t''}) - (x \partial_t \bullet x' \partial_{t'}) \bullet x'' \partial_{t''} = ((x \bullet_{t'} x') \bullet_{t''} x'' - x \bullet_{t'} (x' \bullet_{t''} x'')) \partial_t,$$

which implies the result. ■

Notation 3.7. The symmetric algebra $S(V)$ is given with its usual coproduct Δ , making it a bialgebra:

$$\forall x \in V, \quad \Delta(x) = x \otimes 1 + 1 \otimes x.$$

We shall use Sweedler's notation: for any $w \in S(V)$, $\Delta(w) = \sum w^{(1)} \otimes w^{(2)}$. The counit of this coproduct is denoted by ε : this is the unique algebra morphism from $S(V)$ to \mathbb{K} sending any $v \in V$ to 0.

Theorem 3.8. *Let V be a \mathcal{T} -multiple pre-Lie algebra. One can define a product*

$$\bullet : S(V) \otimes S(V^{\oplus \mathcal{T}}) \rightarrow S(V)$$

in the following way: for any $u, v \in S(V)$, $w \in S(V^{\oplus \mathcal{T}})$, $x \in V$, $t \in \mathcal{T}$,

$$\begin{aligned} 1 \bullet w &= \varepsilon(w), \\ u \bullet 1 &= u, \\ uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}), \\ u \bullet w(x \partial_t) &= (u \bullet w) \bullet_t x - x \bullet (w \bullet_t x), \end{aligned}$$

where \bullet_t is extended to $S(V) \otimes V$ and $S(V^{\oplus \mathcal{T}}) \otimes V$ by the following: for any $x_1, \dots, x_k, x \in V$, for any $t_1, \dots, t_k \in \mathcal{T}$,

$$\begin{aligned} x_1 \cdots x_k \bullet_t x &= \sum_{i=1}^k x_1 \cdots (x_i \bullet_t x) \cdots x_k, \\ (x_1 \partial_{t_1}) \cdots (x_k \partial_{t_k}) \bullet_t x &= \sum_{i=1}^k (x_1 \partial_{t_1}) \cdots ((x_i \bullet_t x) \partial_{t_i}) \cdots (x_k \partial_{t_k}). \end{aligned}$$

Proof. Uniqueness. The last formula allows to compute $x \bullet w$ for any $x \in V$ and $w \in S(V^{\oplus \mathcal{T}})$ by induction on the length of w ; the other ones allow to compute $u \bullet w$ for any $u \in S(V)$ by induction on the length on u . So this product \bullet is unique.

Existence. Let us use the Guin–Oudom construction [21, 22] on the pre-Lie algebra $V^{\otimes \mathcal{T}}$. We obtain a product \bullet defined on $S(V^{\oplus \mathcal{T}})$ such that for any $u, v, w \in S(V^{\oplus \mathcal{T}})$, $x \in V^{\oplus \mathcal{T}}$:

$$\begin{aligned} 1 \bullet w &= \varepsilon(w), \\ u \bullet 1 &= u, \\ uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}), \\ u \bullet wx &= (u \bullet w) \bullet x - x \bullet (w \bullet x). \end{aligned}$$

Let $f: \mathcal{T} \rightarrow \mathbb{K}$ be any nonzero map. We consider the surjective algebra morphism $F: S(V^{\oplus \mathcal{T}}) \rightarrow S(V)$, sending $x\partial_t$ to $f(t)x$ for any $x \in V$, $t \in \mathcal{T}$. Its kernel is generated by the elements $X_{t,t'}x = (f(t')\partial_t - f(t)\partial_{t'})x$, where $x \in V$ and $t, t' \in \mathcal{T}$. We denote by J the vector space generated by the elements $X_{t,t'}x$. Let us prove that for any $w \in S(V^{\oplus \mathcal{T}})$, $J \bullet w \subseteq J$ by induction on the length n of w . If $n = 0$, we can assume that $w = 1$ and this is obvious. If $n = 1$, we can assume that $w = x'\partial_{t'}$. Then

$$X_{t,t'}x \bullet w = (f(t')\partial_t - f(t)\partial_{t'})x \bullet_{t''} x' = X_{t,t''}x \bullet_{t''} x' \in J.$$

Let us assume the result at rank $n - 1$. We can assume that $w = w'x'\partial_t$, the length of w' being $n - 1$. For any $x \in J$,

$$x \bullet w = (x \bullet w') \bullet x' - x \bullet (w' \bullet x').$$

The length of w' and $w' \bullet x'$ is $n - 1$, so $x \bullet w'$ and $x \bullet (w' \bullet x')$ belong to J . From the case $n = 1$, $(x \bullet w') \bullet x' \in J$, so $x \bullet w \in J$.

For any $x \in J$, $u, v \in S(V^{\oplus \mathcal{T}})$,

$$xu \bullet v = \underbrace{(x \bullet v^{(1)})}_{\in J} (u \bullet v^{(2)}) \in \text{Ker}(F).$$

This proves that $\text{Ker}(F) \bullet S(V^{\oplus \mathcal{T}}) \subseteq \text{Ker}(F)$. Hence, \bullet induces a product also denoted by \bullet , defined from $S(V) \otimes S(V^{\oplus \mathcal{T}})$ to $S(V)$. It is not difficult to show that it does not depend on the choice of f and satisfies the required properties. \blacksquare

Definition 3.9. Let $d \in \mathcal{D}$, $T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $t_1, \dots, t_k \in \mathcal{T}$. We denote by

$$B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right)$$

the \mathcal{T} -typed \mathcal{D} -decorated tree obtained by grafting T_1, \dots, T_k on a common root decorated by d , the edge between this root and the root of T_i being of type t_i for any i . This defines a map $B_d: S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}})^{\oplus \mathcal{T}}) \rightarrow S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}}))$.

Lemma 3.10. For any $d \in \mathcal{D}$, $T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $t_1, \dots, t_k \in \mathcal{T}$,

$$B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) = \bullet^d \bullet \prod_{i \in [k]} T_i \partial_{t_i}.$$

Proof. We write $F = \prod_{i \in [k]} T_i \partial_{t_i}$: the integer k is unique and denoted by $\ell(F)$. We proceed by induction on $\ell(F)$. If $\ell(F) = 0$, then $F = 1$ and $\bullet^d \bullet 1 = \bullet^d = B_d(1)$. let us assume the result for any forest F' with $\ell(F') = \ell(F) - 1$. Putting $k = \ell(F)$, we can write $F = F'T\partial_t$, with $\ell(F') = \ell(F) - 1$, $T = T_k$ and $t = t_k$. Then

$$\begin{aligned} \bullet^d \bullet F &= (\bullet^d \bullet F') \bullet T \partial_t - \bullet^d \bullet (F' \bullet T \partial_t) \\ &= B_d(F') \bullet_t T - B_d(F' \bullet_t T) \\ &= B_d(F'T\partial_t) + B_d(F' \bullet_t T) - B_d(F' \bullet_t T) \\ &= B_d(F). \end{aligned}$$

So the result holds for all forests. \blacksquare

Corollary 3.11. *Let A be a \mathcal{T} -multiple pre-Lie algebra and, for any $d \in \mathcal{D}$, $a_d \in A$. There exists a unique \mathcal{T} -multiple algebra morphism $\phi: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \rightarrow A$, such that for any $d \in \mathcal{D}$, $\phi(\bullet^d) = a_d$. In other words, $\mathfrak{g}_{\mathcal{T}, \mathcal{D}}$ is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} .*

Proof. *Uniqueness.* Using the Guin–Oudom product and lemma 3.10, ϕ is the unique linear map inductively defined by

$$\phi\left(B_d\left(\prod_{i \in [k]} T_i \partial_{t_i}\right)\right) = a_d \bullet \prod_{i \in [k]} \phi(T_i) \partial_{t_i}.$$

Existence. Let $T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $t \in \mathcal{T}$. Let us prove that $\phi(T \bullet_t T') = \phi(T) \bullet_t \phi(T')$ by induction on $n = |T|$. If $n = 1$, we assume that $T = \bullet^d$. Then $T \bullet_t T' = B_d(T' \partial_t)$, so

$$\phi(T \bullet_t T') = a_d \bullet (\phi(T') \partial_t) = a_d \bullet_t \phi(T') = \phi(T) \bullet_t \phi(T').$$

Let us assume the result at all ranks $< |T|$. We put

$$T = B_d\left(\prod_{i=1}^k T_i \partial_{t_i}\right).$$

By definition of the pre-Lie product of $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ in terms of grafting,

$$\begin{aligned} T \bullet T' &= B_d\left(\prod_{i=1}^k T_i \partial_{t_i} T' \partial_t\right) + \sum_{j=1}^k B_d\left(\prod_{i \neq j} T_i \partial_{t_i} (T_j \bullet_t T') \partial_{t_j}\right), \\ \phi(T \bullet T') &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + \sum_{j=1}^k a_d \bullet \prod_{i \neq j} \phi(T_i) \partial_{t_i} (\phi(T_j) \bullet_t \phi(T')) \partial_{t_j} \\ &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + \sum_{j=1}^k a_d \bullet \prod_{i \neq j} \phi(T_i) \partial_{t_i} (\phi(T_j) \bullet_t \phi(T')) \partial_{t_j} \\ &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + a_d \bullet \left(\left(\prod_{i=1}^k \phi(T_i) \partial_{t_i}\right) \bullet \phi(T') \partial_t\right) \\ &= \left(a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i}\right) \bullet \phi(T') \partial_t \\ &= \phi(T) \bullet_t \phi(T'). \end{aligned}$$

So ϕ is a \mathcal{T} -multiple pre-Lie algebra morphism. ■

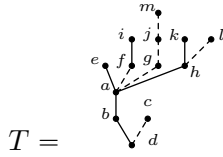
3.3 Operad of typed trees

We now describe an operad of typed trees, in the category of species. We refer to [2, 17, 19] for notations and definitions on operads.

Notation 3.12. Let A be a finite set. If $T \in \mathbb{T}_{\mathcal{T}}(A)$ and $a \in T$:

1. The subtrees formed by the connected components of the set of vertices, descendants of a (a excluded) are denoted by $T_1^{(a)}, \dots, T_{n_a}^{(a)}$. The type of the edge from a to the root of $T_i^{(a)}$ is denoted by t_i .
2. The tree formed by the vertices of T which are not in $T_1^{(a)}, \dots, T_{n_a}^{(a)}$, at the exception of a , is denoted by $T_0^{(a)}$.

Example 3.13. Let us consider the following tree:



with $a, \dots, m \in \mathcal{D}$, then

$$T_0^{(a)} = \begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ d \end{array}, \quad \{T_1^{(a)}, \dots, T_n^{(a)}\} = \left\{ \begin{array}{c} \bullet \cdot e, \downarrow f, \downarrow g, \downarrow h \\ \bullet \cdot i, \downarrow j, \downarrow k, \downarrow l \\ \bullet \cdot m \end{array} \right\}.$$

Proposition 3.14. For any nonempty finite set A , we denote by $\mathcal{P}_{\mathcal{T}}(A)$ the vector space generated by $\mathbb{T}_{\mathcal{T}}(A)$. We define a composition \circ on $\mathcal{P}_{\mathcal{T}}$ in the following way: for any $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$ and $a \in A$,

$$T \circ_a T' = \sum_{v_1, \dots, v_{n_a} \in V(T')} (\dots ((T_0^{(a)} \bullet_{a'}^{(t_0)} T') \bullet_{v_1}^{(t_1)} T_1^{(a)}) \dots) \bullet_{v_{n_a}}^{(t_{n_a})} T_{n_a}^{(a)},$$

where a' is the direct ascendant of a in T and t_0 is the type of the edge between a' and a . If a is the root of T , by convention $T_0^{(a)} \bullet_{a'}^{(t_0)} T' = T'$. With this composition, $\mathcal{P}_{\mathcal{T}}$ is an operad in the category of species.

Proof. Note that the tree $(\dots ((T_0^{(a)} \bullet_{a'}^{(t_0)} T') \bullet_{v_1}^{(t_1)} T_1^{(a)}) \dots) \bullet_{v_{n_a}}^{(t_{n_a})} T_{n_a}^{(a)}$, which is shortly denoted by $T \bullet_{\lambda}^{(v)} T'$, is obtained in the following process:

1. Delete the branches $T_1^{(a)}, \dots, T_{n_a}^{(a)}$ coming from a in T . One obtains a tree T'' , and a is a leaf of T'' .
2. Identify $a \in V(T'')$ with the root of T' . One obtains a tree T''' .
3. Graft $T_1^{(a)}$ on the vertex v_1 of T''' with an edge of type t_1, \dots , graft $T_{n_a}^{(a)}$ on the vertex v_{n_a} of T''' with an edge of type t_n . The obtained tree is $T \bullet_{\lambda}^{(v)} T'$.

This obviously does not depend on the choice of the indexation of $T_1^{(a)}, \dots, T_{n_a}^{(a)}$.

Let $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$, $T'' \in \mathbb{T}_{\mathcal{T}}(C)$.

- If $a', a'' \in A$, with $a' \neq a''$, then

$$\begin{aligned} (T \circ_{a'} T') \circ_{a''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a'}^{(v')} T') \bullet_{a''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a''}^{(v'')} T'') \bullet_{a'}^{(v')} T' \\ &= (T \circ_{a''} T'') \circ_{a'} T'. \end{aligned}$$

- If $a' \in A$ and $b'' \in B$, then

$$\begin{aligned} (T \circ_{a'} T') \circ_{b''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} (T \bullet_{a'}^{(v')} T') \bullet_{b''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} T \bullet_{a'}^{(v')} (T' \bullet_{b''}^{(v'')} T'') \\ &= T \circ_{a'} (T' \circ_{b''} T''). \end{aligned}$$

Moreover, $\bullet^a \bullet_\lambda T = T$ for any tree T , and if $a \in V(T)$, $T \bullet_\lambda \bullet^a T$. So $\mathcal{P}_\mathbb{T}$ is indeed an operad in the category of species. ■

Consequently, the family $(\mathcal{P}_\mathcal{T}(n))_{n \geq 0}$ is an operad in the category of vector spaces, which we denote by $\mathcal{P}_\mathcal{T}$.

Example 3.15.

$$\begin{array}{c} \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\circ}_1 \end{array} \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} = \begin{array}{c} \color{red}{\downarrow}^2 \color{red}{\downarrow}^3 \\ \color{red}{\vee}_1 \end{array} + \begin{array}{c} \color{red}{\downarrow}^3 \\ \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array}, \quad \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} \color{red}{\circ}_2 \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} = \begin{array}{c} \color{red}{\downarrow}^3 \\ \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array}. \end{array}$$

Remark 3.16. Another operad on typed trees is introduced in [11]. It is a typed version of the operad of nonassociative, permutative operad of [16], and is different from ours.

In the non-typed case, this theorem is proved in [9]:

Theorem 3.17. *The operad of \mathcal{T} -multiple pre-Lie algebras is isomorphic to $\mathcal{P}_\mathcal{T}$, via the isomorphism Φ sending, for any $t \in \mathcal{T}$, \bullet_t to the tree $\color{red}{\downarrow}^2_1$, where the edge is of type t .*

Proof. The operad of \mathcal{T} -multiple pre-Lie algebras is generated by the binary elements \bullet_t , $t \in \mathcal{T}$, with the relations

$$\forall t, t' \in \mathcal{T}, \quad \bullet_{t'} \circ_2 \bullet_t - \bullet_t \circ_1 \bullet_{t'} = (\bullet_t \circ_2 \bullet_{t'} - \bullet_{t'} \circ_1 \bullet_t)^{(23)},$$

where in the right side we used the action of the symmetric group \mathfrak{S}_3 on $\mathcal{T}(3)$, and more specifically the action of the transposition (23). Firstly, if t and t' are elements of \mathcal{T} , symbolized by $\color{red}{\downarrow}$ and $\color{red}{\downarrow}$, by the preceding example:

$$\begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} \color{red}{\circ}_1 \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} - \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} \color{red}{\circ}_2 \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} = \begin{array}{c} \color{red}{\downarrow}^2 \color{red}{\downarrow}^3 \\ \color{red}{\vee}_1 \end{array} = \left(\begin{array}{c} \color{red}{\downarrow}^2 \color{red}{\downarrow}^3 \\ \color{red}{\vee}_1 \end{array} \right)^{(23)} = \left(\begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} \color{red}{\circ}_1 \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} - \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} \color{red}{\circ}_2 \begin{array}{c} \color{red}{\downarrow}^2 \\ \color{red}{\downarrow}^1 \end{array} \right)^{(23)}.$$

So the morphism ϕ exists. Let us prove that it is surjective: let $T \in \mathbb{T}_\mathcal{T}(n)$, we show that it belongs to $\text{Im}(\Phi)$ by induction on n . It is obvious if $n = 1$ or $n = 2$. Let us assume the result at all ranks $< n$. Up to a reindexation we assume that

$$T = B_1(T_1 \partial_{t_1} \cdots T_k \partial_{t_k}),$$

where for any $1 \leq i < j \leq k$, if $x \in V(T_i)$ and $y \in V(T_j)$, then $x < y$. We denote by T'_i the standardization of T_i . By the induction hypothesis on n , $T'_i \in \text{Im}(\Phi)$ for all i . We proceed by induction on k . The type t_k will be represented in red. If $k = 1$, then

$$T = \color{red}{\downarrow}^2_1 \circ_2 T_1 \in \text{Im}(\Phi).$$

Let us assume the result at rank $k - 1$. We put $T' = B_1(T_1 \partial_{t_1} \cdots T_{k-1} \partial_{t_{k-1}})$. By the induction hypothesis on n , $T' \in \text{Im}(\Phi)$. Then

$$\color{red}{\downarrow}^2_1 \circ_1 T' = T + x,$$

where x is a sum of trees with n vertices, such that the fertility of the root is $k - 1$. Hence, $x \in \text{Im}(\Phi)$, so $T \in \text{Im}(\Phi)$.

Let \mathcal{D} be a set. The morphism ϕ implies that the free $\mathcal{P}_{\mathbb{T}}$ -algebra generated by \mathcal{D} , that is to say $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, inherits a \mathcal{T} -multiple pre-Lie algebra structure defined by

$$\forall x, y \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}, \quad \forall \dagger \in \mathcal{T}, \quad x \circ_{\dagger} y = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}^2 \cdot (x \otimes y),$$

where \cdot is the $\mathcal{P}_{\mathcal{T}}$ -algebra structure of $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$. For any trees T, T' in $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, by definition of the operadic composition of $\mathcal{P}_{\mathcal{T}}$,

$$T \circ_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T',$$

so $\circ_t = \bullet_t$ for any t . As $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, (\bullet_t)_{t \in \mathcal{T}})$ is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} , Φ is an isomorphism. ■

Remark 3.18. Let us assume that \mathcal{T} is finite, of cardinality T . Then the components of $\mathcal{P}_{\mathcal{T}}$ are finite-dimensional. As the number of rooted trees which vertices are the elements of $[n]$ is n^{n-1} , for any $n \geq 0$ the dimension of $\mathcal{P}_{\mathcal{T}}(n)$ is $T^{n-1}n^{n-1}$, and the formal series of $\mathcal{P}_{\mathcal{T}}$ is

$$f_{\mathcal{T}}(X) = \sum_{n \geq 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}(n))}{n!} X^n = \sum_{n \geq 1} \frac{(Tn)^{n-1}}{n!} X^n = \frac{f_1(TX)}{T}.$$

3.4 Koszul dual operad

If \mathcal{T} is finite, then $\mathcal{P}_{\mathcal{T}}$ is a quadratic operad. Its Koszul dual can be directly computed:

Proposition 3.19. *The Koszul dual operad $\mathcal{P}_{\mathcal{T}}^!$ of $\mathcal{P}_{\mathcal{T}}$ is generated by \diamond_t , $t \in \mathcal{T}$, with the relations*

$$\forall t, t' \in \mathcal{T}, \quad \diamond_{t'} \circ_1 \diamond_t = \diamond_t \circ_2 \diamond_{t'}, \quad \diamond_{t'} \circ_1 \diamond_t = (\diamond_t \circ_1 \diamond_{t'})^{(23)}.$$

The algebras on $\mathcal{P}_{\mathcal{T}}^!$ are called \mathcal{T} -multiple permutative algebras. Such an algebra A is given with the bilinear products \diamond_t , $t \in \mathcal{T}$, such that

$$\forall x, y, z \in A, \quad (x \diamond_t y) \diamond_{t'} z = x \diamond_t (y \diamond_{t'} z), \quad (x \diamond_t y) \diamond_{t'} z = (x \diamond_{t'} z) \diamond_t y.$$

In particular, for any t , \diamond_t is a permutative product.

Of course, the definition of \mathcal{T} -multiple permutative algebras makes sense even if \mathcal{T} is infinite. Permutative algebras are introduced in [8]. If A is a \mathcal{T} -multiple permutative algebra, then for any $(\lambda_t)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, $\diamond_a = \sum \lambda_t \diamond_t$ is a permutative product on A .

Proposition 3.20. *Let V be a vector space. Then $V \otimes S(V^{\oplus \mathcal{T}})$ is given with a \mathcal{T} -multiple permutative algebra structure:*

$$\forall t \in \mathcal{T}, \quad v, v' \in V, \quad w, w' \in S(V^{\oplus \mathcal{T}}), \quad (v \otimes w) \diamond_t (v' \otimes w') = v \otimes w w' (v' \diamond_t).$$

This \mathcal{T} -multiple permutative algebra is denoted by $P_{\mathcal{T}}(V)$. For any \mathcal{T} -multiple permutative algebra V and any linear map $\phi: V \rightarrow A$, there exists a unique morphism $\Phi: P_{\mathcal{T}}(V) \rightarrow A$ such that for any $v \in V$, $\Phi(v \otimes 1) = \phi(v)$.

Proof. Let $t, t' \in \mathcal{T}$, $v, v', v'' \in V$, $w, w', w'' \in S(V^{\oplus \mathcal{T}})$.

$$\begin{aligned} (v \otimes w \diamond_t v' \otimes w') \diamond_{t'} v'' \otimes w'' &= v \otimes w \diamond_t (v' \otimes w' \diamond_{t'} v'' \otimes w'') \\ &= (v \otimes w \diamond_t v' \diamond_{t'} v'' \otimes w'') \otimes w' \\ &= v \otimes w w' w'' (v' \partial_t) (v'' \partial_{t'}), \end{aligned}$$

so $P_{\mathcal{T}}(V)$ is \mathcal{T} -multiple permutative.

Existence of Φ . Let $t_1, \dots, t_k \in \mathcal{T}$, $v, v_1, \dots, v_k \in V$. We inductively define $\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}))$ by

$$\begin{aligned} \Phi(v \otimes 1) &= \phi(v), \\ \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k})) &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \quad \text{if } k \geq 1. \end{aligned}$$

Let us prove that this does not depend on the order chosen on the factors $v_i \partial_{t_i}$ by induction on k . If $k = 0$ or 1 , there is nothing to prove. Otherwise, if $i < k$,

$$\begin{aligned} \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_k \partial_{t_k})) \diamond_{t_i} \phi(v_i) \\ &= (\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k)) \diamond_{t_i} \phi(v_i) \\ &= (\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_i} \phi(v_i)) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k})). \end{aligned}$$

So Φ is well-defined. Let us prove that Φ is a \mathcal{T} -multiple permutative algebra morphism. Let $v, v' \in V$, $w, w' = (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}) \in S(V^{\oplus \mathcal{T}})$, and $t \in \mathcal{T}$. Let us prove that $\Phi(v \otimes w \diamond_t v' \otimes w'') = \Phi(v \otimes w) \diamond_t \Phi(v' \otimes w')$ by induction on k . If $k = 0$,

$$\begin{aligned} \Phi(v \otimes w \diamond_t v' \otimes 1) &= \Phi(v \otimes w (v' \partial_t)) \\ &= \Phi(v \otimes w) \diamond_t \phi(v') \\ &= \Phi(v \otimes w) \diamond_t \Phi(v' \otimes 1). \end{aligned}$$

Otherwise, we put $w'' = (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})$. Then

$$\begin{aligned} \Phi(v \otimes w \diamond_t v' \otimes w') &= \Phi(v \otimes w w'' (v' \partial_t) (v_k \partial_{t_k})) \\ &= \Phi(v \otimes w w'' (v' \partial_t)) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes w \diamond_t v' \otimes w'') \diamond_{t_k} \phi(v_k) \\ &= (\Phi(v \otimes w) \diamond_t \Phi(v' \otimes w'')) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes w) \diamond_t (\Phi(v' \otimes w'') \diamond_{t_k} \phi(v_k)) \\ &= \Phi(v \otimes w') \diamond_t \Phi(v' \otimes w'). \end{aligned}$$

So Φ is a \mathcal{T} -multiple permutative algebra morphism.

Uniqueness. For any $v, v_1, \dots, v_k \in V$, $t_1, \dots, t_k \in \mathcal{T}$,

$$v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}) = (v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} v_k.$$

It is then easy to prove that $P_{\mathcal{T}}(V)$ is generated by $V \otimes 1$ as a \mathcal{T} -multiple permutative algebra. Consequently, Φ is unique. ■

Remark 3.21.

1. We proved that $P_{\mathcal{T}}(V)$ is freely generated by V , identified with $V \otimes 1$. Consequently, $\mathcal{P}_{\mathcal{T}}^!(n)$ has the same dimension as the multilinear component of $V \otimes S(V^{\oplus \mathcal{T}})$ with $V = \text{Vect}(X_1, \dots, X_n)$, that is to say

$$\text{Vect}(X_i \otimes (X_1 \partial_{t_1}) \cdots (X_{i-1} \partial_{t_{i-1}}) (X_{i+1} \partial_{t_{i+1}}) \cdots (X_n \partial_{t_n}), 1 \leq i \leq n, t_j \in \mathcal{T}),$$

so

$$\dim(\mathcal{P}_{\mathcal{T}}^!(n)) = nT^{n-1}.$$

The formal series of $\mathcal{P}_{\mathcal{T}}^!$ is

$$f_{\mathcal{T}}^!(X) = \sum_{n \geq 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}^!(n))}{n!} X^n = \sum_{n \geq 1} \frac{T^{n-1}}{(n-1)!} X^n = X \exp(TX) = \frac{f_1^!(TX)}{T}.$$

2. It is possible to prove that $\mathcal{P}_{\mathcal{T}}^!$ is a Koszul operad (and, hence, $\mathcal{P}_{\mathcal{T}}$ too) using the rewriting method of [17].

4 Structure of the pre-Lie products

4.1 A nonassociative permutative coproduct

For all $t \in \mathcal{T}$, we define a coproduct $\rho_t: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 2}$ by

$$\forall T = B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad \rho_t(T) = \sum_{j \in [k]} \delta_{t, t_j} B_d \left(\prod_{i \in [k], i \neq j} T_i \partial_{t_i} \right) \otimes T_j \partial_t.$$

Proposition 4.1.

1. For all $t, t' \in \mathcal{T}$, $(\rho_t \otimes \text{Id}) \circ \rho_{t'} = ((\rho_{t'} \otimes \text{Id}) \circ \rho_t)^{(23)}$, where in the right side we use the classical action of the symmetric group \mathfrak{S}_3 on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 3}$ by permutations of the tensors, and more specifically the action of the transposition (23). With Sweedler's notation for ρ_t , this can be written, for any $x \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, as

$$\sum \sum (x^{(1)t'})^{(1)t} \otimes (x^{(1)t'})^{(2)t} \otimes x^{(2)t'} = \sum \sum (x^{(1)t})^{(1)t'} \otimes x^{(2)t} \otimes (x^{(1)t})^{(2)t'}.$$

2. For any $x, y \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, $t, t' \in \mathcal{T}$, with Sweedler's notations $\rho_t(x) = \sum x^{(1)t} \otimes x^{(2)t}$,

$$\rho_t(x \bullet_{t'} y) = \delta_{t, t'} x \otimes y + \sum x^{(1)t} \bullet_{t'} y \otimes x^{(2)t} + \sum x^{(1)t} \otimes x^{(2)t} \bullet_{t'} y.$$

3. For any $\mu = (\mu_t)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, we put

$$\rho_{\mu} = \sum_{t \in \mathcal{T}} \mu_t \rho_t: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 2}.$$

This makes sense, as any tree in $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$ does not vanish only under a finite number of ρ_t . Then ρ_{μ} is a nonassociative permutative (NAP) coproduct; for any $x, y \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, by the second point, using Sweedler's notation for ρ_{μ} ,

$$\rho_{\mu}(x \bullet_{\lambda} y) = \left(\sum_{t \in \mathcal{T}} \lambda_t \mu_t \right) x \otimes y + \sum x^{(1)\mu} \bullet_{\lambda} y \otimes x^{(2)\mu} + \sum x^{(1)\mu} \otimes x^{(2)\mu} \bullet_{\lambda} y.$$

In particular, if $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda}, \rho_{\mu})$ is a NAP pre-Lie bialgebra in the sense of [16].

Proof. 1. For any tree T ,

$$(\rho_t \otimes \text{Id}) \circ \rho_{t'}(T) = \sum_{p,q \in [k], p \neq q} \delta_{t_p, t} \delta_{t_q, t'} B_d \left(\prod_{i \in [k], i \neq p, q} T_i \partial_{t_i} \right) \otimes T_p \partial_t \otimes T_q \partial_{t'},$$

which implies the result.

2. For any tree T, T' ,

$$\begin{aligned} \rho_t(T \bullet_{t'} T') &= \rho_t \left(B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} T' \partial_{t'} \right) + \sum_{i \in [k]} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} (T_i \bullet_{t'} T') \partial_{t_i} \right) \right) \\ &= \delta_{t, t'} B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) \otimes T' + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} T' \partial_{t'} \right) \otimes T_i \partial_t \\ &\quad + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \otimes (T_i \bullet_{t'} T') \partial_t \\ &\quad + \sum_{i \neq j \in [k]} \delta_{t_i, t} B_d \left(\prod_{p \in [k], p \neq i, j} T_p \partial_{t_p} (T_j \bullet_{t'} T') \partial_{t_j} \right) \otimes T_i \partial_t \\ &= \delta_{t, t'} T \otimes T' + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \bullet_{t'} T' \otimes T_i \partial_t \\ &\quad + \sum_{i \in [k]} \delta_{t_i, t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \otimes T_i \bullet_{t'} T' \partial_t \\ &= \delta_{t, t'} T \otimes T' + T^{(1)t} \bullet_{t'} T' \otimes T^{(2)t} + T^{(1)t} \otimes T^{(2)t} \bullet_{t'} T'. \end{aligned}$$

3. Obtained by summation. ■

Corollary 4.2. *If $\lambda \in \mathbb{K}^{(\mathcal{T})}$ is nonzero, let us choose $t_0 \in \mathcal{T}$ such that $\lambda_{t_0} \neq 0$. The pre-Lie algebra $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by the set $\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$ of \mathcal{T} -typed \mathcal{D} -decorated trees T such that there is no edge outgoing the root of T of type t_0 .*

Proof. For any tree T , we denote by α_T the number of edges outgoing the root of T of type T_0 . Our aim is to prove that $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by the trees T such that $\alpha_T = 0$. We define a family of scalars $(\mu_t)_{t \in \mathcal{T}}$ by

$$\forall t \in \mathcal{T}, \quad \mu_t = \begin{cases} 0 & \text{if } t \neq t_0, \\ \frac{1}{\lambda_{t_0}} & \text{if } t = t_0. \end{cases}$$

Note that $\rho_\mu = \frac{1}{\lambda_{t_0}} \rho_{t_0}$. By Proposition 4.1, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda, \rho_\mu)$ is a NAP pre-Lie bialgebra, so by Livernet's rigidity theorem [16], it is freely generated by $\text{Ker}(\rho_\mu) = \text{Ker}(\rho_{t_0})$. Obviously, if $\alpha_T = 0$, $T \in \text{Ker}(\rho_{t_0})$. Let us consider $x = \sum_{T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}} x_T T \in \text{Ker}(\rho_{t_0})$. We consider the map

$$\Upsilon: \begin{cases} \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \otimes \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \\ T \otimes T' \longmapsto T \bullet_{t_0}^{\text{root}(T)} T'. \end{cases}$$

By definition of ρ_{t_0} , for any tree T , $\Upsilon \circ \rho_{t_0}(T) = \alpha_T T$. Consequently,

$$0 = \Upsilon \circ \rho_{t_0}(x) = \sum_{T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}} x_T \alpha_T T.$$

So if $\alpha_T \neq 0$, $x_T = 0$, and x is a linear span of trees such that $\alpha_T = 0$: the set of trees T such that $\alpha_T = 0$ is a basis of $\text{Ker}(\rho_{t_0})$. ■

If $|\mathcal{D}| = D$ and $|\mathcal{T}| = T$, the number of elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ of degree n is denoted by $t'_{D,T}(n)$; it does not depend on t_0 . By direct computations,

$$\begin{aligned} t'_{D,T}(1) &= D, \\ t'_{D,T}(2) &= D^2(T-1), \\ t'_{D,T}(3) &= \frac{D^2(T-1)(3DT-D+1)}{2}, \\ t'_{D,T}(4) &= \frac{D^2(T-1)(16D^2T^2-8D^2T+D^2+6DT-3D+2)}{6}. \end{aligned}$$

In the particular case $D = 1$, $T = 2$, we recover sequence A005750 of the OEIS.

4.2 Pre-Lie algebra morphisms

Notation 4.3. Let \mathcal{T} and \mathcal{T}' be two sets of types. We denote by $\mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ the space of matrices $M = (m_{t,t'})_{(t,t') \in \mathcal{T} \times \mathcal{T}'}$, such that for any $t' \in \mathcal{T}'$, $(m_{t,t'})_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$. If $\mathcal{T} = \mathcal{T}'$, we shall simply write $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$. If $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ and $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, then

$$MM' = \left(\sum_{t' \in \mathcal{T}'} m_{t,t'} m'_{t',t''} \right)_{(t,t'') \in \mathcal{T} \times \mathcal{T}''} \in \mathcal{M}_{\mathcal{T},\mathcal{T}''}(\mathbb{K}).$$

If $\lambda \in \mathbb{K}^{(\mathcal{T}')}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, then

$$M\lambda = \left(\sum_{t' \in \mathcal{T}'} m_{t,t'} \lambda_{t'} \right)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}, \quad M^\top \mu = \left(\sum_{t \in \mathcal{T}} m_{t,t'} \mu_t \right)_{t' \in \mathcal{T}'} \in \mathbb{K}^{\mathcal{T}'}$$

In particular, $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$ is an algebra, acting on $\mathbb{K}^{(\mathcal{T})}$ on the left and on $\mathbb{K}^{\mathcal{T}}$ on the right.

Definition 4.4. Let $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. We define a map $\Phi_M: \mathcal{H}_{\mathcal{D},\mathcal{T}'} \rightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}$, sending $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}'}$ to the forest obtained by replacing $\text{type}(e)$ by $\sum_{t \in \mathcal{T}} m_{t,\text{type}(e)} t$ for any $e \in E(F)$, F being considered as linear in any of its edges. The restriction of Φ_M to $\mathfrak{g}_{\mathcal{D},\mathcal{T}'}$ is denoted by $\phi_M: \mathfrak{g}_{\mathcal{D},\mathcal{T}'} \rightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}$.

Example 4.5. If \mathcal{T} contains two elements, the first one represented by $|$ and the second one by $\dot{|}$, if $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, for any $x, y, z \in \mathcal{D}$,

$$\begin{aligned} \phi_M(\dot{|}^y_x) &= \alpha \dot{|}^y_x + \gamma |^y_x, \\ \phi_M(|^y_x) &= \beta \dot{|}^y_x + \delta |^y_x, \\ \phi_M(\begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array}) &= \alpha\beta \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} + \alpha\delta \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} + \beta\gamma \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} + \gamma\delta \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array}. \end{aligned}$$

Remark 4.6. For any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, $\Phi_M \circ \Phi_{M'} = \Phi_{MM'}$.

Proposition 4.7. Let $\lambda \in \mathbb{K}^{(\mathcal{T})}$, $\mu \in \mathbb{K}^{\mathcal{T}}$ and $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. Then ϕ_M is a pre-Lie morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$ and a NAP coalgebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \rho_{M^\top \mu})$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \rho_\mu)$.

Proof. Let $T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$. For any $t \in \mathcal{T}$, for any $v \in V(T)$,

$$\phi_M(T \bullet_t^{(v)} T') = \sum_{t' \in \mathcal{T}'} m_{t',t} \phi_M(T) \bullet_{t'} \phi_M(T'),$$

so

$$\phi_M(T \bullet_\lambda T') = \sum_{t,t' \in \mathcal{T}} m_{t',t} \lambda_t \phi_M(T) \bullet_{t'} \phi_M(T') = \phi_M(T) \bullet_{M\lambda} \phi_M(T').$$

We proved that ϕ_M is a pre-Lie algebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$.

For any $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$,

$$\rho_t \circ \phi_M(T) = \sum_{t' \in \mathcal{T}} m_{t,t'}(\phi_M \otimes \phi_M) \circ \rho_{t'}(T),$$

so

$$\rho_\mu \circ \phi_M(T) = \sum_{t,t' \in \mathcal{T}} m_{t,t'} \mu_t(\phi_M \otimes \phi_M) \circ \rho_{t'}(T) = (\phi_M \otimes \phi_M) \circ \rho_{M^\top \mu}(T).$$

So $\phi_M: (\mathfrak{g}_{\mathcal{D}, \mathcal{T}'}, \rho_{M^\top \mu}) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \rho_\mu)$ is a NAP coalgebra morphism. \blacksquare

Corollary 4.8. *For any $\lambda \in \mathbb{K}^{\mathcal{T}}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, such that $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, for any $t_0 \in \mathcal{T}$, the NAP pre-Lie bialgebras $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda, \rho_\mu)$ and $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{t_0}, \rho_{t_0})$ are isomorphic.*

Proof. Let us denote by $\lambda^{(0)}$ the element of $\mathbb{K}^{\mathcal{T}}$ defined by

$$\lambda_t^{(0)} = \delta_{t,t_0}.$$

Note that for any $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, $\phi_M: (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda^{(0)}}) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{M\lambda^{(0)}})$ is an isomorphism. In particular, for a well-chosen M , $M\lambda^{(0)} = \lambda$; we can assume that $\lambda = \lambda^{(0)}$ without loss of generality. Then, by hypothesis, $\mu_{t_0} = 1$. We define a matrix $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$ in the following way:

$$m_{t,t'} = \begin{cases} \delta_{t,t_0} & \text{if } t' = t_0, \\ \delta_{t,t'} - \mu_{t'} \delta_{t,t_0} & \text{otherwise.} \end{cases}$$

Then M is invertible. Moreover, $M\lambda^{(0)} = \lambda^{(0)}$ and $M^\top \mu = \lambda^{(0)}$. So ϕ_M is an isomorphism from $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda^{(0)}})$ to $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$. \blacksquare

Proposition 4.9. *Let $\lambda \in \mathbb{K}^{\mathcal{T}}$, and $t_0 \in \mathcal{T}$. We define a pre-Lie algebra morphism $\psi_{t_0}: (\mathfrak{g}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}, \bullet) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$, sending \cdot^x to T for any $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$. Then ψ_{t_0} is a pre-Lie algebra isomorphism if, and only if, $\lambda_{t_0} \neq 0$.*

Proof. If $\lambda_{t_0} \neq 0$, then by corollary 4.2, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by $\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$, so ψ_{t_0} is an isomorphism. If $\lambda_{t_0} = 0$, then it is not difficult to show that any tree T with two vertices, with its unique edge of type t_0 , does not belong to $\text{Im}(\psi_{t_0})$. \blacksquare

5 Hopf algebraic structures

We here fix $\lambda \in \mathbb{K}^{\mathcal{T}}$.

5.1 Enveloping algebra of $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$

Using again the Guin–Oudom construction, we obtain the enveloping algebra of $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$. We first identify the symmetric coalgebra $S(\mathfrak{g}_{\mathcal{D}, \mathcal{T}})$ with the vector space generated by $\mathbb{F}_{\mathcal{D}, \mathcal{T}}$, which we denote by $\mathcal{H}_{\mathcal{D}, \mathcal{T}}$. Its product m is given by disjoint union of forests, its coproduct by

$$\forall T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad \Delta(T_1 \cdots T_n) = \sum_{I \subseteq [n]} \prod_{i \in I} T_i \otimes \prod_{i \notin I} T_i.$$

We denote by \bullet_λ the Guin–Oudom extension of \bullet_λ to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}$ and \star_λ the associated associative product.

Theorem 5.1. For any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, $T_1, \dots, T_n \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$,

$$F \bullet_{\lambda} T_1 \cdots T_n = \sum_{\substack{v_1, \dots, v_n \in V(F), \\ t_1, \dots, t_n \in \mathcal{T}}} \left(\prod_{i \in [n]} \lambda_{t_i} \right) (\cdots (F \bullet_{t_1}^{(v_1)} T_1) \cdots) \bullet_{t_n}^{(v_n)} T_n,$$

$$F \star_{\lambda} T_1 \cdots T_n = \sum_{I \subseteq [n]} \left(F \bullet_{\lambda} \prod_{i \in I} T_i \right) \prod_{i \notin I} T_i.$$

The Hopf algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, \star_{\lambda}, \Delta)$ is denoted by $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$. Moreover, for any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, for any $\lambda \in \mathbb{K}^{(\mathcal{T}')}$, Φ_M is a Hopf algebra morphism from $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}M\lambda}$. The extension of ψ_{t_0} as a Hopf algebra morphism from $\mathcal{H}_{\mathbb{T}_{\mathcal{D},\mathcal{T}}(t_0)}^{\text{GL}\lambda}$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ is denoted by Ψ_{t_0} ; it is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

In particular, if $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the Grossman–Larson Hopf algebra [14].

5.2 Dual construction

Proposition 5.2. Let $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$.

1. A cut c of T is a nonempty subset of $E(T)$; it is said to be admissible if any path in the tree from the root to a leaf meets at most one edge in c . The set of admissible cuts of T is denoted by $\text{Adm}(T)$.
2. If c is admissible, one of the connected components of $T \setminus c$ contains the root of c : we denote it by $R^c(T)$. The product of the other connected components of $T \setminus c$ is denoted by $P^c(T)$.

Let $\lambda \in \mathbb{K}^{\mathcal{T}}$. We define a multiplicative coproduct $\Delta^{\text{CK}\lambda}$ on the algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m)$ by

$$\forall T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \quad \Delta^{\text{CK}\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in \text{Adm}(T)} \left(\prod_{e \in c} \lambda_{\text{type}(e)} \right) R^c(T) \otimes P^c(T).$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \Delta^{\text{CK}\lambda})$ is a Hopf algebra, which we denote by $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$.

Proof. We first assume that $\lambda \in \mathbb{K}^{(\mathcal{T})}$. Let us define a nondegenerate pairing $\langle -, - \rangle$ on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by

$$\forall F, F' \in \mathbb{F}_{\mathcal{D},\mathcal{T}}, \quad \langle F, F' \rangle = \delta_{F, F'} s_F,$$

where s_F is the number of symmetries of F . Let us consider three forests F, F', F'' . We put

$$F = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\lambda_t}, \quad F' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a'_T}, \quad F'' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a''_T}.$$

Then

$$\begin{aligned} \langle \Delta(F), F' \otimes F'' \rangle &= \sum_{a=b+c} \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} \frac{\lambda_t!}{\mu_t! c_T!} \left\langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\mu_t}, F' \right\rangle \left\langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{c_T}, F'' \right\rangle \\ &= \sum_{a=b+c} \delta_{b, a'} \delta_{c, a''} \frac{\lambda_t!}{a'_T! a''_T!} s_{F'} s_{F''} = \delta_{a, a' + a''} \frac{\lambda_t!}{a'_T! a''_T!} a'_T! a''_T! s_T^{a'_T + a''_T} \\ &= \delta_{a, a' + a''} \lambda_t! s_T^{\lambda_t} = \delta_{F, F' F''} s_F \\ &= \langle F, F' F'' \rangle. \end{aligned}$$

Therefore,

$$\forall x, y, z \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}, \quad \langle \Delta(x), y \otimes z \rangle = \langle x, yz \rangle.$$

Let F, G be two forests and T be a tree. Observe that if F is a forest with at least two trees, then $F \star_\lambda G$ does not contain any tree, so $\langle F \star_\lambda G, T \rangle = 0$. If $F = 1$, then $\langle F \star_\lambda G, T \rangle \neq 0$ if, and only if, $G = T$; moreover, $\langle 1 \star_\lambda T, T \rangle = 1$. If F is a tree, then

$$\langle F \star_\lambda G, T \rangle = \langle F \bullet_\lambda G, T \rangle.$$

Moreover, if $F = B_d(F')$ and $G = T_1 \cdots T_k$,

$$F \bullet_\lambda G = \sum_{I \subseteq [k]} \sum_{(t_i) \in \mathcal{T}^k} \left(\prod_{i \in [k]} \lambda_{t_i} \right) B_d \left(\prod_{i \in I} T_i \partial_{t_i} F' \bullet \prod_{i \notin I} T_i \partial_{t_i} \right),$$

where \bullet is the pre-Lie product on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\mathcal{T}}$ induced by the \mathcal{T} -multiple pre-Lie structure. Consequently, we can inductively define a coproduct $\Delta^{\text{CK}\lambda}: \mathcal{H}_{\mathcal{D}, \mathcal{T}} \rightarrow \mathcal{H}_{\mathcal{D}, \mathcal{T}} \otimes \mathcal{H}_{\mathcal{D}, \mathcal{T}}$, multiplicative for the product m , such that, if we denote for any tree T , $\bar{\Delta}_{\text{CK}}(T) = \Delta(T) - 1 \otimes T$, for any tree $T = B_d(T_1 \partial_{t_1} \cdots T_k \partial_{t_k})$,

$$\bar{\Delta}_\lambda^{\text{CK}}(T) = (B_d \otimes \text{Id}) \left(\prod_{i \in [k]} \left(\bar{\Delta}_\lambda^{\text{CK}}(T_i) \partial_{t_i} \otimes 1 + \lambda_{t_i} 1 \otimes T_i \right) \right).$$

Then, for any $x, y, z \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$,

$$\langle x \star_\lambda y, z \rangle = \langle x \otimes y, \Delta^{\text{CK}\lambda}(z) \rangle.$$

A quite easy induction on the number of vertices of trees proves that this coproduct is indeed the one we define in the statement of the proposition. As $\langle -, - \rangle$ is nondegenerate, $(\mathcal{H}_{\mathcal{D}, \mathcal{T}}, m, \Delta^{\text{CK}\lambda})$ is a Hopf algebra, dual to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}\lambda}$.

In the general case, for any $x \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$, there exists a finite subset \mathcal{T}' of \mathcal{T} such that $x \in \mathcal{H}_{\mathcal{D}, \mathcal{T}'}$. Putting $\lambda' = \lambda|_{\mathcal{T}'}$, $\lambda' \in \mathbb{K}^{\mathcal{T}'} = \mathbb{K}^{(\mathcal{T}'})$, so

$$\begin{aligned} (\Delta^{\text{CK}\lambda} \otimes \text{Id}) \circ \Delta^{\text{CK}\lambda}(x) &= (\Delta^{\text{CK}\lambda'} \otimes \text{Id}) \circ \Delta^{\text{CK}\lambda'}(x) \\ &= (\text{Id} \otimes \Delta^{\text{CK}\lambda'}) \circ \Delta^{\text{CK}\lambda'}(x) \\ &= (\text{Id} \otimes \Delta^{\text{CK}\lambda}) \circ \Delta^{\text{CK}\lambda}(x). \end{aligned}$$

Hence, Δ_λ is coassociative, and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$ is a Hopf algebra. ■

Example 5.3. Let us fix a subset \mathcal{T}' of \mathcal{T} and choose $(\lambda_t)_{t \in \mathcal{T}}$ such that

$$\lambda_t = \begin{cases} 1 & \text{if } t \in \mathcal{T}', \\ 0 & \text{otherwise.} \end{cases}$$

For any tree $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, let us denote by $\text{Adm}_{\mathcal{T}'}(T)$ the set of admissible cuts c of T such that the type of any edge in c belongs to \mathcal{T}' . Then

$$\Delta^{\text{CK}\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in \text{Adm}_{\mathcal{T}'}(T)} R^c(T) \otimes P^c(T).$$

Remark 5.4.

1. If $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the usual Connes–Kreimer Hopf algebra of \mathcal{D} -decorated rooted trees, which we denote by $\mathcal{H}_{\mathcal{D}}^{\text{CK}}$, and its duality with the Grossman–Larson Hopf algebra [10, 15, 23].
2. If \mathcal{T} and \mathcal{D} are finite, for any $\lambda \in \mathbb{K}^{\mathcal{T}}$, both $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}\lambda}$ are graded Hopf algebras (by the number of vertices), and their homogeneous components are finite-dimensional. Via the pairing $\langle -, - \rangle$, each one is the graded dual of the other.

5.3 Hochschild cohomology of coalgebras

For the sake of simplicity, we assume that the set of types \mathcal{T} is finite and we put $\mathcal{T} = \{t_1, \dots, t_N\}$.

Let (C, Δ) be a coalgebra and let (M, δ_L, δ_R) be a C -bicomodule. We define a complex, dual to the Hochschild complex for algebras, in the following way:

1. For any $n \geq 0$, $H_n = \mathcal{L}(M, C^{\otimes n})$.
2. For any $L \in H_n$,

$$b_n(L) = (\text{Id} \otimes L) \circ \delta_L + \sum_{i=1}^n (-1)^i (\text{Id}^{\otimes(i-1)} \otimes \Delta \otimes \text{Id}^{\otimes(n-i)}) \circ L \\ + (-1)^{n+1} (L \otimes \text{Id}) \circ \delta_R.$$

In particular, one-cocycles are maps $L: M \rightarrow C$ such that

$$\Delta \circ L = (\text{Id} \otimes L) \circ \delta_L + (L \otimes \text{Id}) \circ \delta_R.$$

We shall consider in particular the bicomodule (M, δ_L, δ_R) such that

$$\forall x \in C, \quad \begin{cases} \delta_L(x) = 1 \otimes x, \\ \delta_R(x) = \Delta(x). \end{cases}$$

If C is a bialgebra, then $M^{\otimes N}$ is also a bicomodule:

$$\forall x_t \in C, \quad \begin{cases} \delta_L \left(\bigotimes_{1 \leq i \leq N} x_i \right) = 1 \otimes \bigotimes_{1 \leq i \leq N} x_i, \\ \delta_R \left(\bigotimes_{1 \leq i \leq N} x_i \right) = \bigotimes_{1 \leq i \leq N} x_i^{(1)} \otimes \prod_{1 \leq i \leq N} x_i^{(2)}. \end{cases}$$

We denote by $\underline{1} = (1)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, and we take $C = \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_1}$. One can identify $S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}})^{\oplus \mathcal{T}})$ and $C^{\otimes N}$, $x \delta_{T_i}$ being identified with $1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$ for any $x \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $1 \leq i \leq N$. Then for any d , $B_d: C^{\otimes N} \rightarrow C$ is a 1-cocycle. Moreover, there is a universal property, proved in the same way as for the Connes–Kreimer's one [10]:

Theorem 5.5. *Let B be a commutative bialgebra and, for any $d \in \mathcal{D}$, let $L_d: C^{\otimes N} \rightarrow C$ be a 1-cocycle:*

$$\forall d \in \mathcal{D}, \quad \forall x_t \in B, \quad \Delta \circ L_d \left(\bigotimes_{1 \leq i \leq N} x_i \right) = 1 \otimes \bigotimes_{1 \leq i \leq N} x_i + L_d \left(\bigotimes_{1 \leq i \leq N} x_i^{(1)} \right) \otimes \prod_{1 \leq i \leq N} x_i^{(2)}.$$

There exists a unique bialgebra morphism $\phi: \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_1} \rightarrow C$ such that for any $d \in \mathcal{D}$, $\phi \circ L_d = B_d \circ \phi^{\otimes N}$.

5.4 Hopf algebra morphisms

Our aim is, firstly, to construct Hopf algebra morphisms between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$; secondly, to construct Hopf algebra isomorphisms between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}', \mathcal{T}}^{\text{CK}}$ for a well-chosen \mathcal{D}' .

Proposition 5.6. *Let $M \in \mathcal{M}_{\mathcal{T}, \mathcal{T}'}(\mathbb{K})$, $\lambda \in \mathbb{K}^{\mathcal{T}}$. Then $\Phi_M: \mathcal{H}_{\mathcal{D}, \mathcal{T}'}^{\text{CK}_{M^{\top \lambda}}} \rightarrow \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ is a Hopf algebra morphism.*

Proof. Φ_M is obviously an algebra morphism. Let $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$.

$$\begin{aligned} \Delta_\lambda \circ \Phi_M(T) &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T) \\ &+ \sum_{c \in \text{Adm}(T)} \prod_{e \in c} \left(\sum_{t \in \mathcal{T}} m_{t, \text{type}(e)} \lambda_t \right) \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T)) \\ &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T) \\ &+ \sum_{c \in \text{Adm}(T)} \prod_{e \in c} (M^\top \lambda)_{\text{type}(e)} \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T)) \\ &= (\Phi_M \otimes \Phi_M) \circ \Delta_{M^\top a}(T). \end{aligned}$$

So Φ_M is a coalgebra morphism from $\mathcal{H}_{\mathcal{D}, \mathcal{T}'}^{\text{CK}_{M^\top \lambda}}$ to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$. \blacksquare

Corollary 5.7. Let $\lambda, \mu \in \mathbb{K}^{\mathcal{T}}$, both nonzero. Then $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$ are isomorphic Hopf algebras.

Proof. There exists $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, such that $M^\top \lambda = \mu$. Then Φ_M is an isomorphism between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$. \blacksquare

Definition 5.8. Let us fix $t_0 \in \mathcal{T}$. For any $F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, we shall say that $\{T_1, \dots, T_k\} \triangleleft_{t_0} F$ if the following conditions hold:

- $\{T_1, \dots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, by restriction.
- For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$.

If $\{T_1, \dots, T_k\} \triangleleft_{t_0} F$, we denote by $F/\{T_1, \dots, T_k\}$ the forest obtained by contracting T_i to a single vertex for any $i \in [k]$, decorating this vertex by T_i , and forgetting the type of the remaining edges. Then $F/\{T_1, \dots, T_k\}$ is a $\mathcal{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$ -decorated forest.

Proposition 5.9. Let $\lambda \in \mathbb{K}^{\mathcal{T}}$, $t_0 \in \mathcal{T}$. Let us consider the map

$$\Psi_{t_0}^* : \begin{cases} \mathcal{H}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}, \\ F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}} \longmapsto \sum_{\{T_1, \dots, T_k\} \triangleleft_{t_0} F} \left(\prod_{e \in E(F) \setminus \sqcup E(T_i)} \lambda_{\text{type}(e)} \right) F/\{T_1, \dots, T_k\}. \end{cases}$$

Then $\Psi_{t_0}^*$ is a Hopf algebra morphism from $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ to $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{CK}}$. It is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

Proof. *First case.* We first assume that \mathcal{D} and \mathcal{T} are finite. In this case, $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ is the graded dual of $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}_\lambda}$, with the Hopf pairing $\langle -, - \rangle$; grading $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}$ by the number of vertices of the decorations, $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{CK}}$ is the graded dual of $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{GL}}$. Moreover, $\Psi_{t_0}^*$ is the transpose of Ψ_{t_0} of Proposition 4.9, so is a Hopf algebra morphism. If $\lambda_{t_0} \neq 0$, Ψ_{t_0} is an isomorphism, so $\Psi_{t_0}^*$ also is.

General case. Let $x, y \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$. There exist finite $\mathcal{D}', \mathcal{T}'$, such that $x, y \in \mathcal{H}_{\mathcal{D}', \mathcal{T}'}$; we can assume that $t_0 \in \mathcal{T}'$. We denote by $\lambda' = \lambda|_{\mathcal{T}'}$. Then, by the preceding case, denoting by Ψ'_{t_0} the restriction of $\Psi_{t_0}^*$ to $\mathcal{H}_{\mathcal{D}', \mathcal{T}'}$,

$$\begin{aligned} \Psi_{t_0}^*(xy) &= \Psi'_{t_0}(xy) = \Psi'_{t_0}(x)\Psi'_{t_0}(y) = \Psi_{t_0}^*(x)\Psi_{t_0}^*(y), \\ \Delta^{\text{CK}_\lambda} \circ \Psi_{t_0}^*(x) &= \Delta^{\text{CK}_{\lambda'}} \circ \Psi'_{t_0}(x) = (\Psi'_{t_0} \otimes \Psi'_{t_0}) \circ \Delta^{\text{CK}_{\lambda'}}(x) = (\Psi_{t_0}^* \otimes \Psi_{t_0}^*) \circ \Delta^{\text{CK}_\lambda}(x), \end{aligned}$$

so Ψ is a Hopf algebra morphism.

Let us assume that $\lambda_{t_0} \neq 0$. If $\Psi_{t_0}^*(x) = 0$, then $\Psi'_{t_0}(x) = 0$. As $a'_{t_0} \neq 0$, by the first case, $x = 0$, so $\Psi_{t_0}^*$ is injective. Moreover, there exists $z \in \mathcal{H}_{\mathcal{D}', \mathcal{T}'}$, such that $\Psi'_{t_0}(z) = y$; so $\Psi_{t_0}^*(z) = y$, and $\Psi_{t_0}^*$ is surjective.

Let us assume that $\lambda_{t_0} = 0$. Let T be a tree with two vertices, such that its unique edge is of type t_0 . As $T \notin \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$, $\Phi_{t_0}(T)$ has a unique term, given by the partition $X = \{\{x_1\}, \{x_2\}\}$, where x_1 and x_2 are the vertices of T . Hence,

$$\Psi_{t_0}^*(T) = \lambda_{t_0} T' = 0,$$

so $\Psi_{t_0}^*$ is not injective. ■

Example 5.10. Here, \mathcal{T} contains two elements, $|$ and $\dot{|}$. In order to simplify, we omit the decorations of vertices. We put

$$x = \bullet, \quad y = \dot{|}, \quad z = \vee, \quad u = \dot{|}, \quad v = \ddot{|}.$$

Applying $\Psi_{|}^*$:

$$\begin{aligned} \Psi_{|}^*(\bullet) &= \bullet^x, & \Psi_{|}^*(\vee) &= \lambda^2 | \vee^x, \\ \Psi_{|}^*(\dot{|}) &= \lambda | \dot{|}^x, & \Psi_{|}^*(\dot{|}) &= \lambda^2 | \dot{|}^x, \\ \Psi_{|}^*(\ddot{|}) &= \lambda \dot{|} \dot{|}^x + \bullet^y, & \Psi_{|}^*(\ddot{|}) &= \lambda | \lambda \dot{|} \dot{|}^x + \lambda | \dot{|}^y, \\ \Psi_{|}^*(\vee) &= \lambda | \lambda \dot{|} \vee^x + \lambda | \dot{|}^y, & \Psi_{|}^*(\dot{|}) &= \lambda | \lambda \dot{|} \dot{|}^x + \lambda | \dot{|}^y + \bullet^u, \\ \Psi_{|}^*(\vee) &= \lambda^2 | \vee^x + 2\lambda \dot{|} \dot{|}^x + \bullet^z, & \Psi_{|}^*(\ddot{|}) &= \lambda^2 | \dot{|} \dot{|}^x + \lambda \dot{|} \dot{|}^x + \lambda \dot{|} \dot{|}^y + \bullet^v. \end{aligned}$$

Remark 5.11. Although it is not indicated, Ψ_{t_0} and $\Psi_{t_0}^*$ depend on λ .

5.5 Bialgebras in cointeraction

By [12], for any $\lambda \in \mathbb{K}^{(\mathcal{T})}$, the operad morphism $\theta_\lambda: \mathbf{PreLie} \rightarrow \mathcal{P}_{\mathcal{T}}$, which send \bullet to \bullet_λ , where \mathbf{PreLie} is the operad of pre-Lie algebras, induces a pair of cointeracting bialgebras for any finite set \mathcal{D} . By construction, the first bialgebra of the pair is $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$. Let us describe the second one.

Definition 5.12. Let $F \in \mathbb{F}_{\mathcal{T}, \mathcal{D}}$. We shall say that $\{T_1, \dots, T_k\} \triangleleft F$ if

1. $\{T_1, \dots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, by restriction.
2. For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$.

If $\{T_1, \dots, T_k\} \triangleleft F$ and $\text{dec}: [k] \rightarrow \mathcal{D}$, we denote by $(F/\{T_1, \dots, T_k\}, \text{dec})$ the forest obtained by contracting T_i to a single vertex, and decorating this vertex by $\text{dec}(i)$, for all $i \in [k]$. This is an element of $\mathbb{F}_{\mathcal{D}, \mathcal{T}}$.

Proposition 5.13. If \mathcal{D} is finite, $\mathcal{H}'_{\mathcal{D}, \mathcal{T}}$ is the free commutative algebra generated by pairs (T, d) , where $T \in \mathbb{T}_{\mathcal{T}, \mathcal{D}}$ and $d \in \mathcal{D}$. The coproduct is given, for any $F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, $d \in \mathcal{D}$, by

$$\delta(F, d) = \sum_{\{T_1, \dots, T_k\} \triangleleft F} \sum_{\text{dec}: [k] \rightarrow \mathcal{D}} ((F/\{T_1, \dots, T_k\}, \text{dec}), d) \otimes (T_1, \text{dec}(1)) \cdots (T_k, \text{dec}(k)).$$

Then $(\mathcal{H}'_{\mathcal{D},\mathcal{T}}, m, \delta)$ is a bialgebra, and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a coalgebra in the category of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ -comodules via the coaction given, for any $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, by

$$\bar{\delta}(T) = \sum_{\{T_1, \dots, T_k\} \triangleleft T} \sum_{\text{dec}: [k] \rightarrow \mathcal{D}} ((T/\{T_1, \dots, T_k\}, \text{dec}) \otimes (T_1, \text{dec}(1)) \cdots (T_k, \text{dec}(k))).$$

Corollary 5.14. *Let us assume that \mathcal{D} is given a semigroup law denoted by $+$. If $F \in \mathbb{F}_{\mathcal{T},\mathcal{D}}$, and $\{T_1, \dots, T_k\} \triangleleft F$, then naturally $T_i \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$ for any i and the \mathcal{T} -typed forest $F/\{T_1, \dots, T_k\}$ is given a \mathcal{D} -decoration, decorating the vertex obtained in the contradiction of T_i by the sum of the decorations of the vertices of T_i . Then $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ is given a second coproduct δ such that for any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$,*

$$\delta(F) = \sum_{\{T_1, \dots, T_k\} \triangleleft F} F/\{T_1, \dots, T_k\} \otimes T_1 \cdots T_k.$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \delta)$ is a bialgebra and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a coalgebra in the category of $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ -comodules via the coaction δ .

Proof. We denote by I the ideal of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ generated by pairs (T, d) such that $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $d \in \mathcal{D}$, with

$$d \neq \sum_{v \in V(T)} \text{dec}(v).$$

The quotient $\mathcal{H}'_{\mathcal{D},\mathcal{T}}/I$ is identified with $\mathcal{H}_{\mathcal{D},\mathcal{T}}$, through the surjective algebra morphism

$$\varpi: \begin{cases} \mathcal{H}'_{\mathcal{D},\mathcal{T}} \longrightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}, \\ (F, d) \in \mathbb{F}_{\mathcal{D},\mathcal{T}} \times \mathcal{D} \longmapsto \begin{cases} F & \text{if } d = \sum_{v \in V(F)} \text{dec}(v), \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Let us prove that I is a coideal. Let $T \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$, $d \in \mathcal{D}$, $\{T_1, \dots, T_k\} \triangleleft F$, $\text{dec}: [k] \rightarrow \mathcal{D}$ such that $((T/\{T_1, \dots, T_k\}, \text{dec}), d) \notin I$ and for any i , $(T_i, \text{dec}(i)) \notin I$. Then

$$\forall i \in [k], \quad \sum_{v \in V(T_i)} \text{dec}(v) = \text{dec}(i), \quad \sum_{i=1}^k \text{dec}(i) = d.$$

Hence,

$$\sum_{v \in V(T)} \text{dec}(v) = \sum_{i=1}^k \sum_{v \in V(T_i)} \text{dec}(v) = \sum_{i=1}^k \text{dec}(i) = d,$$

so $(T, d) \notin I$. Consequently, if $T \in I$, then $((T/\{T_1, \dots, T_k\}, \text{dec}), d) \in I$ or at least one of the $(T_i, \text{dec}(i))$ belongs to I . Hence,

$$\delta(I) \subseteq I \otimes \mathcal{H}'_{\mathcal{D},\mathcal{T}} + \mathcal{H}'_{\mathcal{D},\mathcal{T}} \otimes I.$$

So I is a coideal. The coproduct induced on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by the morphism ϖ is precisely the one given in the setting of this corollary. \blacksquare

In particular, if \mathcal{D} is reduced to a single element, denoted by $*$, if we give it its unique semigroup structure ($* + * = *$), We obtain again the result of [7].

5.6 The Bruned–Hairer–Zambotti construction

We now consider the coproducts on typed trees in [3, Theorem 2.2.16], the first one with $\overline{\mathfrak{A}}(F) = \mathfrak{A}(F)$ and the second one with $\overline{\mathfrak{A}}(F) = \mathfrak{A}^+(F)$ of [3, Definition 2.4.1]. By definition [3, Definition 2.26] of admissible subtrees, according to the notations we choose in this paper:

- Let \mathcal{L} be a finite set of types. The considered trees are \mathcal{L} -typed and the leaves are \mathcal{L} -decorated. Considering that the internal vertices of such a tree are in fact decorated by an element $0 \notin \mathcal{L}$, these trees form a subset of $\mathbb{T}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ which we denote by $\mathbb{T}'_{\mathcal{L}}$.
- The first coproduct Δ_+ is given on any tree $T \in \mathbb{T}'_{\mathcal{L}}$ by

$$\Delta_+(T) = \sum_{c \in \text{Adm}'(T)} R^c(T) \otimes P^c(T),$$

where $\text{Adm}'(T)$ is the set of admissible cuts of T such that the set of leaves of $R^c(T)$ is a subset of leaves of T (note that automatically, the leaves of $P^c(T)$ are also leaves of T).

- The second coproduct is given on any tree $T \in \mathbb{T}'_{\mathcal{L}}$ by

$$\overline{\Delta}(T) = \sum_{\{T_1, \dots, T_k\} \triangleleft' T} T / \{T_1, \dots, T_k\} \otimes T_1 \cdots T_k,$$

where the sum runs over all $\{T_1, \dots, T_k\} \triangleleft T$ such that the leaves of $T / \{T_1, \dots, T_k\}$ are leaves of T .

We denote by $\mathcal{H}'_{\mathcal{L}}$ the subalgebra of $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ generated by $\mathbb{T}'_{\mathcal{L}}$. Then $(\mathcal{H}'_{\mathcal{L}}, m, \Delta_+)$ and $(\mathcal{H}'_{\mathcal{L}}, m, \delta)$ are bialgebras.

Let us give to \mathcal{L} any product \times making it a commutative associative semigroup. We extend this structure to $\mathcal{L} \sqcup \{0\}$ by

$$\forall t \in \mathcal{L} \sqcup \{0\}, \quad 0 \times t = t \times 0 = 0.$$

We take $\lambda_t = 1$ for any $t \in \mathcal{L} \sqcup \{0\}$, and obtain with this data two coproducts Δ and δ on $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$, the first one given by admissible cuts and the second one by contractions of subtrees. The subalgebra $\mathcal{H}''_{\mathcal{L}}$ of $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ generated by trees such that any internal vertex is decorated by 0 (note that the leaves of such a tree are decorated by $\mathcal{L} \sqcup \{0\}$) is a subalgebra for both coproducts. We denote by I the ideal of $\mathcal{H}''_{\mathcal{L}}$ generated by trees with at least one leaf decorated by 0. Then it is a coideal for both coproducts, so the quotient algebra $\mathcal{H}''_{\mathcal{L}}/I$ inherits two coproducts, still denoted by Δ and δ . This algebra is trivially identified with the algebra generated by $\mathbb{T}'_{\mathcal{L}}$, that is to say with $\mathcal{H}'_{\mathcal{L}}$. By construction of the different coproducts, this identification is an isomorphism from the Hopf algebra $(\mathcal{H}''_{\mathcal{L}}/I, \Delta)$ to $(\mathcal{H}'_{\mathcal{L}}/I, \Delta_+)$ and from the Hopf algebra $(\mathcal{H}''_{\mathcal{L}}/I, \delta)$ to $(\mathcal{H}'_{\mathcal{L}}/I, \overline{\Delta})$. In other words, the Bruned–Hairer–Zambotti construction of cointeracting bialgebras on typed trees is a subquotient of the construction presented here.

6 Appendix

6.1 Values of $t_{D,T}(k)$

For $D = 1$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	1	1	2	4	9	20	48	115
2	1	2	7	26	107	458	2058	9498
3	1	3	15	82	495	3144	20875	142773
4	1	4	26	188	1499	12628	111064	1006840
5	1	5	40	360	3570	37476	410490	4635330
6	1	6	57	614	7284	91566	1200705	16232820
7	1	7	77	966	13342	195384	2984142	46990952
8	1	8	100	1432	22570	377320	6578116	118238600
9	1	9	126	2028	35919	674964	13225632	267188229
10	1	10	155	2770	54465	1136402	24723000	554540590

For $D = 2$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	2	4	14	52	214	916	4116	18996
2	2	8	52	376	2998	25256	222128	2013680
3	2	12	114	1228	14568	183132	2401410	32465640
4	2	16	200	2864	45140	754640	13156232	236477200
5	2	20	310	5540	108930	2272804	49446000	1109081180
6	2	24	444	9512	224154	5606520	146204792	3930863232
7	2	28	602	15036	413028	12043500	366122190	11475005616
8	2	32	784	22368	701768	23373216	811575408	29052861280
9	2	36	990	31764	1120590	41969844	1638712716	65965167108
10	2	40	1220	43480	1703710	70875208	3073688160	137426005200

For $D = 3$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	3	9	45	246	1485	9432	62625	428319
2	3	18	171	1842	21852	274698	3602115	48698460
3	3	27	378	6084	107757	2024892	39676896	801564687
4	3	36	666	14268	336231	8409780	219307188	5896294848
5	3	45	1035	27690	814680	25444584	828506340	27812997990
6	3	54	1485	47646	1680885	62954766	2458069074	98947750662
7	3	63	2016	75432	3103002	135520812	6170116638	289616448690
8	3	72	2628	112344	5279562	263423016	13701398868	734709311208
9	3	81	3321	159678	8439471	473586264	27703353159	1670715963729
10	3	90	4095	218730	12842010	800524818	52018920345	3484841027040

For $D = 4$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	4	16	104	752	5996	50512	444256	4027360
2	4	32	400	5728	90280	1509280	26312464	472954400
3	4	48	888	19024	448308	11213040	292409584	7861726464
4	4	64	1568	44736	1403536	46746432	1623150816	58105722560
5	4	80	2440	86960	3407420	141750416	6147376320	274852010400
6	4	96	3504	149792	7039416	351230688	18268531824	979612414944
7	4	112	4760	237328	13006980	756866096	45910215120	2871018269632
8	4	128	6208	353664	22145568	1472317056	102037088448	7290356719488
9	4	144	7848	502896	35418636	2648533968	206451156768	16590568445280
10	4	160	9680	689120	53917640	4479065632	387863411920	34625886677920

we find the following sequences of the OEIS [24]:

$T \setminus D$	1	2	3	4
1	A0081	A038055	A038059	A136793
2	A00151	A136794		
3	A006964			
4	A052763			
5	A052788			
6	A246235			
7	A246236			
8	A246237			
9	A246238			
10	A246239			

6.2 Values of $t'_{D,T}(k)$

For $D = 1$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	1	1	3	10	39	160	702	3177
3	1	2	9	46	268	1660	10845	73270
4	1	3	18	124	963	7968	69236	621999
5	1	4	30	260	2525	26136	283528	3178696
6	1	5	45	470	5480	68096	885805	11904160
7	1	6	63	770	10479	151956	2304974	36110880
8	1	7	84	1176	18298	303296	5255964	94051770
9	1	8	108	1704	29838	556464	10845732	218239560
10	1	9	135	2370	46125	955872	20696076	462558987

For $D = 2$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	2	0	0	0	0	0	0	0
2	2	4	22	144	1090	8864	76162	678532
3	2	8	68	688	7886	96896	1250780	16713504
4	2	12	138	1888	29004	476736	8213588	146342376
5	2	16	232	4000	77060	1586304	34185344	761389360
6	2	20	350	7280	168670	4171744	107932710	2884827980
7	2	24	492	11984	324450	9370368	282934428	8822987856
8	2	28	658	18368	569016	18793600	648698792	23119514576
9	2	32	848	26688	930984	34609920	1344232416	53898191520
10	2	36	1062	37200	1442970	59627808	2573660298	114661732500

For $D = 3$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	3	0	0	0	0	0	0	0
2	3	9	72	705	7947	96588	1237728	16450389
3	3	18	225	3408	58347	1072224	20685195	413084610
4	3	27	459	9405	216081	5315112	136987407	3650993163
5	3	36	774	19992	576405	17763984	572991726	19100718828
6	3	45	1170	36465	1264950	46852884	1815034140	72635168880
7	3	54	1647	60120	2437722	105455952	4768982442	222723271080
8	3	63	2205	92253	4281102	211832208	10953036318	584744300226
9	3	72	2844	134160	7011846	390570336	22727284344	1365242802048
10	3	81	3564	187137	10877085	673533468	43560017892	2907844041231

For $D = 4$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	4	0	0	0	0	0	0	0
2	4	16	168	2192	32844	531200	9051376	159962784
3	4	32	528	10656	242792	5939968	152518064	4053650976
4	4	48	1080	29488	902100	29551104	1014147872	35989518528
5	4	64	1824	62784	2411024	98976256	4252211232	188790415552
6	4	80	2760	114640	5297820	261422336	13491005840	719200139360
7	4	96	3888	189152	10218744	588999936	35487727184	2208096700896
8	4	112	5208	290416	17958052	1184031744	81574704960	5802692175744
9	4	128	6720	422528	29428000	2184360960	169377005376	13557899008896
10	4	144	8424	589584	45668844	3768659712	324805399344	28894042642464

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