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# A SURJECTION THEOREM FOR SINGULAR PERTURBATIONS WITH LOSS OF DERIVATIVES

IVAR EKELAND AND ÉRIC SÉRÉ

ABSTRACT. In this paper we introduce a new algorithm for solving nonlinear functional equations which admit a right-invertible linearization, but such that the inverse loses derivatives. The main difference with the by now classical Nash-Moser algorithm is that, instead of using a regularized Newton scheme, we solve a sequence of Galerkin problems thanks to a topological argument. As a consequence, in our estimates there are *no quadratic terms*. We apply our method to a singular perturbation problem with loss of derivatives as in Texier-Zumbrun [26]. We will compare the two results and we will show that ours improves significantly on theirs, when applied, in particular, to a nonlinear Schrödinger Cauchy problem with highly oscillatory initial data.

## 1. INTRODUCTION

The basic idea of the inverse function theorem (henceforth IFT) is that, if a map  $F$  is differentiable at a point  $u_0$  and the derivative  $DF(u_0)$  is invertible, then the map itself is invertible in some neighbourhood of  $u_0$ . It has a long and distinguished history (see [20] for instance), going back to the inversion of power series in the seventeenth century, and has been extended since to maps between infinite-dimensional spaces. If the underlying space is Banach, and if one is only interested in the local surjectivity of  $F$ , that is, the existence, near  $u_0$ , of a solution  $u$  to the equation  $F(u) = v$  for  $v$  close to  $F(u_0)$ , one just needs to assume that  $F$  is of class  $C^1$  and that  $DF(u_0)$  has a right-inverse  $L(u_0)$ . The standard proof is based on the Picard scheme

$$u_n = u_{n-1} - L(u_0)(F(u_{n-1}) - v)$$

which converges geometrically to a solution of  $F(u) = v$  provided  $\|F(u_0) - v\|$  is small enough. In the  $C^2$  case, the Newton algorithm

$$u_n = u_{n-1} - L(u_{n-1})(F(u_{n-1}) - v)$$

uses the right-invertibility of  $DF(u)$  for  $u$  close to  $u_0$ , and provides local quadratic convergence.

In functional analysis,  $u$  will typically be a function. In many situations the IFT on Banach spaces will be enough, but in the study of Hamiltonian systems and PDEs, one encounters cases when the right-inverse  $L(u)$  of  $DF(u)$  loses derivatives, i.e. when  $L(u)F(w)$  has less derivatives than  $u$  and  $w$ . In such a case, the Picard and Newton schemes lose derivatives at each step. The first solutions to this problem are due, on the one hand, to Kolmogorov [19] and Arnol'd [2], [3], [4] who investigated

perturbations of completely integrable Hamiltonian systems in the analytic class, and showed that invariant tori persist under small perturbations, and, on the other hand, to Nash [23], who showed that any smooth compact Riemannian manifold can be imbedded isometrically into an Euclidian space of sufficiently high dimension<sup>1</sup>.

In both cases, the fast convergence of Newton's scheme was used to outscore the loss of regularity. Since Nash was considering functions with finitely many derivatives, he had to introduce a sequence of smoothing operators  $\mathcal{S}_n$ , in order to regularize  $L(u_{n-1})(F(u_{n-1}) - v)$ , and the new scheme was

$$u_n = u_{n-1} - \mathcal{S}_n L(u_{n-1})(F(u_{n-1}) - v).$$

An early presentation of Nash's method can be found in Schwartz' notes [24]. It was further improved by Moser [22], who used it to extend the Kolmogorov-Arnol'd results to  $C^k$  Hamiltonians. The Nash-Moser method has been the source of a considerable amount of work in many different situations, giving rise in each case to a so-called "hard" IFT. We will not attempt to review this line of work in the present paper. A survey up to 1982 will be found in [15]. In [17], Hörmander introduced a refined version of the Nash-Moser scheme providing optimal estimates on the regularity loss. We refer to [1] for a pedagogical account of this work, and to [5] for recent improvements. We also gained much insight into the Nash-Moser scheme from the papers [7], [8], [9], [10], [26].

The question we want to address here is the following. The IFT implies that the range of  $F$  contains a neighborhood  $\mathcal{V}$  of  $v_0 = F(u_0)$ . What is the size of  $\mathcal{V}$ ?

In general, when one tries to apply directly the abstract Nash-Moser theorem, the estimates which can be derived from its proof are unreasonably small, many orders of magnitude away from what can be observed in numerical simulations or physical experiments. Moreover, precise estimates for the Nash-Moser method are difficult to compute, and most theoretical papers simply do not address the question.

So we shall address instead a "hard" singular perturbation problem with loss of derivatives. The same issue appears in such problems, as we shall explain in a moment, but it takes a simpler form: one tries to find a good estimate on the size of  $\mathcal{V}$  as a power of the perturbation parameter  $\varepsilon$ . Such an asymptotic analysis has been carefully done in the paper of Texier and Zumbun [26] which has been an important source of inspiration to us, and we will be able to compare our results with theirs. As noted by these authors, the use of Newton's scheme implies an intrinsic limit to the size of  $\mathcal{V}$ .

Let us explain this in the "soft" case, without loss of derivatives. Suppose that for every  $0 < \varepsilon \leq 1$  we have a  $C^2$  map  $F_\varepsilon$  between two Banach spaces  $X$  and  $Y$ , such that  $F_\varepsilon(0) = 0$ , and, for all  $\|u\| \leq R$ ,

$$\begin{aligned} \|D_u F_\varepsilon(u)^{-1}\| &\leq \varepsilon^{-1} M \\ \|D_{uu}^2 F_\varepsilon(u)\| &\leq K \end{aligned}$$

Then the Newton-Kantorovich Theorem (see [11], section 7.7 for a comprehensive discussion) tells us that the solution  $u_\varepsilon$  of  $F_\varepsilon(u) = v$  exists for  $\|v\| < \frac{\varepsilon^2}{2KM^2}$ , and this is essentially the best result one can hope for using Newton's algorithm, as

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<sup>1</sup>Nash's theorem on isometric embeddings was later proved by Gunther [14], who found a different formulation of the problem and was able to use the classical IFT in Banach spaces.

mentioned by Texier and Zumbrun in [26], Remark 2.22. Note that the use of a Picard iteration would give a similar condition.

However, in this simple situation where no derivatives are lost, it is possible, using topological arguments instead of Newton's method, to find a solution  $u$  provided  $\|v\| \leq \varepsilon R/M$ : one order of magnitude in  $\varepsilon$  has been gained. The first result of this kind, when  $F$  is  $C^1$  and  $\dim X = \dim Y < \infty$ , is due to Wazewski [27] who used a continuation method. See also [18] and [25] and the references in these papers, for more general results in this direction. In [12] (Theorem 2), using Ekeland's variational principle, Wazewski's result is proved in Banach spaces, assuming only that  $F$  is continuous and Gâteaux differentiable, the differential having a uniformly bounded right-inverse (in §2 below, we recall this result, as Theorem 5).

Our goal is to extend such a topological approach to "hard" problems with loss of derivatives, which up to now have been tackled by the Nash-Moser algorithm. A first attempt in this direction was made in [12] (Theorem 1), in the case when the estimates on the right-inverse do not depend on the base point, but it is very hard to find examples of such situations. The present paper fulfills the program in the general case, where estimates on the inverse do depend on the base point with loss of derivatives. Note, however, that our smoothing operators are necessarily projectors  $\Pi_n$  acting on the domain  $\Omega$ , and  $\Pi'_n$  acting on the target space.

Such smoothing projectors are used in the paper of Berti, Bolle and Procesi [10] who prove a new version of the Nash-Moser theorem by solving a sequence of Galerkin problems  $\Pi'_n F(u_n) = \Pi'_n v$ ,  $u_n \in E_n$ , where  $E_n$  is the range of  $\Pi_n$ . They find the solution of each projected equation thanks to a Picard iteration:

$$u_n = \lim_{k \rightarrow \infty} w^k \quad \text{with} \quad w^0 = u_{n-1} \quad \text{and} \quad w^{k+1} = w^k - L_n(u_{n-1})(F(w^k) - v),$$

where  $L_n(u_{n-1})$  is a right inverse of  $D(\Pi'_n F|_{E_n})(u_{n-1})$ . So, in [10] the regularized Newton step is not really absent: it is essentially the first step in each Picard iteration. As a consequence, the proof in [10] involves quadratic estimates similar to the ones of more standard Nash-Moser schemes. Moreover, Berti, Bolle and Procesi assume the right-invertibility of  $D(\Pi'_n F|_{E_n})(u_{n-1})$ . This assumption is perfectly suitable for the applications they consider (periodic solutions of a nonlinear wave equation), but in general it is not a consequence of the right-invertibility of  $DF(u_{n-1})$ , and this restricts the generality of their method as compared with the standard Nash-Moser scheme.

As in [10], we work with projectors and solve a sequence of Galerkin problems. But in contrast with [10], the Newton steps are completely absent in our new algorithm, they are replaced by the topological argument from [12] (Theorem 2), ensuring the solvability of each projected equation. Incidentally, this allows us to work with functionals  $F$  that are only continuous and Gâteaux-differentiable, while the standard Nash-Moser scheme requires twice-differentiable functionals. Our regularity assumption on  $v$  also seems to be optimal, and weaker than in [17]. Moreover, our method works assuming either the right-invertibility of  $D(\Pi'_n F|_{E_n})(u)$  as in [10], or the right-invertibility of  $DF(u)$  (in the second case, our proof is more complicated). But in our opinion, the main advantage of our approach is the following: there are *no more quadratic terms* in our estimates, as a consequence we can deal with larger  $v$ 's, and this advantage is particularly obvious in the case of singular perturbations.

To emphasize this, we will give an abstract existence theorem with a precise estimate of the range of  $F$  in a singular perturbation situation: this is Theorem 3 below. Comparing our result with the abstract theorem of [26], one can see that we have weaker assumptions and a stronger conclusion. Then we will apply Theorem 3 to an example given by Texier and Zumbrun, namely a Cauchy problem for a quasilinear Schrödinger system first studied by Métivier and Rauch [21]. Texier and Zumbrun use their abstract Nash-Moser theorem to prove the existence of solutions of this system on a fixed time interval, for highly oscillatory initial data. Our abstract theorem allows a significant improvement in this situation: we are able to increase the order of magnitude of the oscillations in the initial data. After reading our paper, Baldi and Haus [6] communicated to us that they are able to recover the same order of magnitude, using their own version [5] of the Newton scheme for Nash-Moser. They can do this thanks to a clever modification of the norms considered by Texier-Zumbrun, which leads to an improved estimate on the second derivative of their functional. In contrast, our proof follows directly from our abstract theorem, taking exactly the same norms and estimates as in Texier-Zumbrun, and without even considering the second derivative of the functional.

The paper is constructed as follows. In Section 2, we present the general framework: we are trying to solve the equation  $F_\varepsilon(u) = v$ , when  $F_\varepsilon$  maps a scale of Banach spaces of functions into another, admits a right-invertible Gâteaux differential, with "tame estimates" involving losses of derivatives and negative powers of  $\varepsilon$ . After giving our precise assumptions, we state our main theorem. Section 3 is devoted to its proof. In Section 4, we apply it to the example taken from Texier and Zumbrun [26], and we compare our results with theirs.

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## 2. MAIN ASSUMPTIONS AND RESULTS.

**2.1. Two tame scales of Banach spaces.** Let  $(V_s, \|\cdot\|_s)_{0 \leq s \leq S}$  be a scale of Banach spaces, namely:

$$0 \leq s_1 \leq s_2 \leq S \implies [V_{s_2} \subset V_{s_1} \text{ and } \|\cdot\|_{s_1} \leq \|\cdot\|_{s_2}]$$

We shall assume that  $V_S$  contains an nondecreasing family of subspaces  $E(\Lambda)$ ,  $\Lambda \in [1, \infty)$ . We shall also assume that there exists a nondecreasing family of continuous projectors  $\Pi(\Lambda) : V_0 \rightarrow E(\Lambda)$  enjoying the following properties:

**Polynomial growth:** *There are constants  $A_1, A_2 \geq 1$  such that, for all numbers  $0 \leq s, t \leq S$ , all  $\Lambda \in [1, \infty)$  and all  $u \in V_s$ , we have:*

$$(2.1) \quad \|\Pi(\Lambda)u\|_t \leq A_1 \Lambda^{(t-s)^+} \|u\|_s$$

$$(2.2) \quad \|(1 - \Pi(\Lambda))u\|_t \leq A_2 \Lambda^{-(s-t)^+} \|u\|_s$$

When the above properties are met, we shall say that  $(V_s, \|\cdot\|_s)_{0 \leq s \leq S}$  endowed with the family of projectors  $\Pi(\Lambda)$ ,  $\Lambda \geq 1$ , is a *tame* Banach scale.

It is well-known (see e.g. [10]) that (2.1,2.2) imply:

**Interpolation inequality:** For  $0 \leq t_1 \leq s \leq t_2 \leq S$ ,

$$(2.3) \quad \|u\|_s \leq A_3 \|u\|_{t_1}^{\frac{t_2-s}{t_2-t_1}} \|u\|_{t_2}^{\frac{s-t_1}{t_2-t_1}}.$$

Let  $W_s$ ,  $0 \leq s \leq S$  be another tame scale of Banach spaces. We shall denote by  $\|\cdot\|'_s$  the associated norms, by  $\Pi'(\Lambda) : W_0 \rightarrow E'(\Lambda) \subset W_S$  the corresponding projections, and by  $A'_i$  ( $i = 1, 2, 3$ ) the corresponding constants in (2.1), (2.2) and (2.3).

**Remark.** In many practical situations, the projectors rather form a discrete family as, for instance,  $\Pi(N)$ ,  $N \in \mathbb{N}^*$ , or  $\Pi(2^j)$ ,  $j \in \mathbb{Z}$ . The first case occurs when  $\Pi(N)$  acts on periodic functions by truncating their Fourier series, keeping only frequencies of size less or equal to  $N$ , as in [10]. The second case occurs when truncating orthogonal wavelet expansions as in an earlier version of the present work [13]. Our choice of notation and assumptions covers these cases, taking  $\Pi(\Lambda) = \Pi(\lfloor \Lambda \rfloor)$  or  $\Pi(\Lambda) = \Pi(2^{\lfloor \log_2(\Lambda) \rfloor})$ , where  $\lfloor \cdot \rfloor$  denotes the integer part. This unified notation was suggested to us by Pietro Baldi.

**2.2. Main theorem.** We state our result in the framework of singular perturbations, in the spirit of Texier and Zumbrun [26]. The norms  $\|\cdot\|_s$ ,  $\|\cdot\|'_s$  on the tame scales  $(V_s)$ ,  $(W_s)$  may depend on a parameter  $\varepsilon \in (0, 1]$ , as well as the projectors  $\Pi(\Lambda)$ ,  $\Pi'(\Lambda)$  and the corresponding subspaces  $E(\Lambda)$ ,  $E'(\Lambda)$ . But we impose that  $S$  and the constants  $A_i$ ,  $A'_i$  appearing in estimates (2.1, 2.2, 2.3) be independent of  $\varepsilon$ . In order to avoid burdensome notations, the dependence of the norms, projectors and subspaces on  $\varepsilon$  will not be explicit in the sequel.

Denote by  $B_s$  the unit ball in  $V_s$ :

$$B_s = \{u \mid \|u\|_s \leq 1\}$$

In the sequel we fix nonnegative constants  $s_0, m, \ell, \ell'$  and  $g$ , independent of  $\varepsilon$ . We will assume that  $S$  is large enough.

We first recall the definition of Gâteaux-differentiability, in a form adapted to our framework:

**Definition 1.** We shall say that a function  $F : B_{s_0+m} \rightarrow W_{s_0}$  is Gâteaux-differentiable (henceforth *G-differentiable*) if for every  $u \in B_{s_0+m}$ , there exists a linear map  $DF(u) : V_{s_0+m} \rightarrow W_{s_0}$  such that for every  $s \in [s_0, S - m]$ , if  $u \in B_{s_0+m} \cap V_{s+m}$ , then  $DF(u)$  maps continuously  $V_{s+m}$  into  $W_s$ , and

$$\forall h \in V_{s+m}, \quad \lim_{t \rightarrow 0} \left\| \frac{1}{t} [F(u+th) - F(u)] - DF(u)h \right\|'_s = 0$$

Note that in general, a G-differentiable map need not be  $C^1$ , or even continuous. However, if  $DF : V_{s+m} \rightarrow \mathcal{L}(V_{s+m}, W_s)$  is locally bounded, then  $F : V_{s+m} \rightarrow W_s$  is locally Lipschitz, hence continuous. In the present paper, we will always be in such a situation.

We now consider a family of maps  $(F_\varepsilon)_{0 < \varepsilon \leq 1}$  with  $F_\varepsilon : B_{s_0+m} \rightarrow W_{s_0}$ . We are ready to state our assumptions on the family  $(F_\varepsilon)$ :

**Definition 2.**

- We shall say that the maps  $F_\varepsilon : B_{s_0+m} \rightarrow W_{s_0}$  ( $0 < \varepsilon \leq 1$ ) form an  $S$ -tame differentiable family if, for every  $0 < \varepsilon \leq 1$ , the map  $F_\varepsilon$  is  $G$ -differentiable with respect to  $u$ , and there are constants  $a$ ,  $b$  and  $g > 0$  such that for all  $0 < \varepsilon \leq 1$ ,  $s_0 \leq s \leq S - m$  and  $u \in B_{s_0+m} \cap V_{s+m}$ , we have

$$(2.4) \quad \forall h \in V_{s+m}, \|DF_\varepsilon(u)h\|'_s \leq a (\|h\|_{s+m} + \|u\|_{s+m} \|h\|_{s_0+m}) .$$

- Then we shall say that  $DF_\varepsilon$  is tame right-invertible if there are  $\ell, \ell' \geq 0$  such that for all  $0 < \varepsilon \leq 1$  and  $u \in B_{s_0+\max\{\ell, m\}}$ , there is a linear map  $L_\varepsilon(u) : W_{s_0+\ell'} \rightarrow V_{s_0}$  satisfying

$$(2.5) \quad \forall k \in W_{s_0+\ell'}, \quad DF_\varepsilon(u) L_\varepsilon(u) k = k$$

and for all  $s_0 \leq s \leq S - \max\{\ell, \ell'\}$ , if  $u \in B_{s_0+\max\{\ell, m\}} \cap V_{s+\ell}$  then

$$(2.6) \quad \forall k \in W_{s+\ell'}, \|L_\varepsilon(u)k\|_s \leq b\varepsilon^{-g} (\|k\|'_{s+\ell'} + \|k\|'_{s_0+\ell'} \|u\|_{s+\ell}) .$$

- Alternatively, we shall say that  $DF_\varepsilon$  is tame Galerkin right-invertible if there are  $\ell, \ell', \underline{\Lambda} \geq 0$  such that for all  $\Lambda \geq \underline{\Lambda}$ ,  $0 < \varepsilon \leq 1$  and any  $u \in B_{s_0+\ell} \cap E(\Lambda)$ , there is a linear map  $L_{\Lambda, \varepsilon}(u) : E'(\Lambda) \rightarrow E(\Lambda)$  satisfying

$$(2.7) \quad \forall k \in E'(\Lambda), \quad \Pi'(\Lambda)DF_\varepsilon(u) L_{\Lambda, \varepsilon}(u) k = k$$

and for all  $s_0 \leq s \leq S - \max\{\ell, \ell'\}$ ,

$$(2.8) \quad \forall k \in E'(\Lambda), \|L_{\Lambda, \varepsilon}(u)k\|_s \leq b\varepsilon^{-g} (\|k\|'_{s+\ell'} + \|k\|'_{s_0+\ell'} \|u\|_{s+\ell}) .$$

As we will see, the proof of Theorem 3 below is much shorter under the assumptions that  $DF_\varepsilon$  is Galerkin right-invertible. But in many applications, it is much easier to check that  $DF_\varepsilon$  is tame right-invertible than tame Galerkin right-invertible. See [10], however, where an assumption similar to (2.7, 2.8) is used.

**Theorem 3.** *Assume that the maps  $F_\varepsilon$  ( $0 < \varepsilon \leq 1$ ) form an  $S$ -tame differentiable family between the tame scales  $(V_s)_{0 \leq s \leq S}$  and  $(W_s)_{0 \leq s \leq S}$ , with  $F_\varepsilon(0) = 0$  for all  $0 < \varepsilon \leq 1$ . Assume, in addition, that  $DF_\varepsilon$  is either tame right-invertible or tame Galerkin right-invertible. Let  $s_0, m, g, \ell, \ell'$  be the associated parameters.*

*Let  $s_1 \geq s_0 + \max\{m, \ell\}$ ,  $\delta > s_1 + \ell'$  and  $g' > g$ .*

*Then, for  $S$  large enough, there are  $r > 0$  and  $q > 0$  such that, whenever  $0 < \varepsilon \leq 1$  and  $\|v\|'_\delta \leq r\varepsilon^{g'}$ , there exists some  $u_\varepsilon \in V_{s_1}$  such that:*

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= v \\ \|u_\varepsilon\|_{s_0+\max\{m, \ell\}} &\leq 1 \\ \|u_\varepsilon\|_{s_1} &\leq q\varepsilon^{-g'} \|v\|'_\delta \end{aligned}$$

Note that  $s_0 + \max\{m, \ell\} + \ell'$  is the minimal regularity requirement that one should make on  $v$ , so our result is nearly optimal from the point of view of the regularity of  $v$  and we haven't found the same assumption in the literature. In [17] for instance, a stronger assumption is made, of the form  $\delta > s_0 + \max\{2m + \ell', \ell\} + \ell'$ . Moreover, the condition on  $S$  is of the form  $S \geq S_0$  where  $S_0$  depends only on the parameters  $s_0, m, g, \ell, \ell'$  and  $g', s, \delta$ . Then  $r$  and  $q$  depend only on these parameters and the constants  $A_i, A'_i$  associated with the tame scales. In principle, all these constants could be made explicit, but we will not do it here. Let us just mention that one can take  $S_0 = \mathcal{O}\left(\frac{1}{g'-g}\right)$  as  $g' \rightarrow g$ , all other parameters remaining fixed. This follows from the inequality  $\sigma < \zeta g/\eta$  in Lemma 1.

Multiplying the norms  $\|\cdot\|_s$  and  $\|\cdot\|'_s$  by  $\varepsilon^\gamma$  for some  $\gamma > 0$ , and replacing  $F$  by  $\Phi_\varepsilon(u) = F_\varepsilon(u) - v$ , we can restate our assumptions as follows (here we only consider the case of a tame right-invertible differential, which is more adapted to the applications we have in mind):

For some  $\gamma > 0$  and any  $0 < \varepsilon \leq 1$ , the map  $\Phi_\varepsilon : B_{s_0+m}(\varepsilon^\gamma) \rightarrow W_{s_0}$  is  $G$ -differentiable with respect to  $u$ , and there are constants  $a, b$  and  $g > 0$  such that:

- for all  $0 < \varepsilon \leq 1$ ,  $s_0 \leq s \leq S - m$  and  $u \in B_{s_0+m}(\varepsilon^\gamma) \cap V_{s+m}$ , we have:

$$(2.9) \quad \forall h \in V_{s+m}, \quad \|D\Phi_\varepsilon(u)h\|'_s \leq a (\|h\|_{s+m} + \varepsilon^{-\gamma} \|u\|_{s+m} \|h\|_{s_0+m})$$

- for all  $0 < \varepsilon \leq 1$  and  $u \in B_{s_0+\ell}(\varepsilon^\gamma)$ , there is  $L_\varepsilon(u) : W_{s_0+\ell'} \rightarrow V_{s_0}$  linear, satisfying:

$$(2.10) \quad \forall k \in W_{s_0+\ell'}, \quad D\Phi_\varepsilon(u) L_\varepsilon(u) k = k$$

and for all  $s_0 \leq s \leq S - \max\{\ell, \ell'\}$ , if  $u \in B_{s_0+\ell}(\varepsilon^\gamma) \cap V_{s+\ell}$ , then

$$(2.11) \quad \forall k \in W_{s+\ell'}, \quad \|L_\varepsilon(u)k\|_s \leq b\varepsilon^{-g} (\|k\|'_{s+\ell'} + \varepsilon^{-\gamma} \|k\|'_{s_0+\ell'} \|u\|_{s+\ell})$$

So our main theorem can be restated in a form that allows direct comparison with [26]:

**Corollary 4.** Consider two Banach tame scales  $(V_s)_{0 \leq s \leq S}$ ,  $(W_s)_{0 \leq s \leq S}$ , nonnegative constants  $s_0, m, \ell, \ell', g, \gamma$ , and two positive constants  $a, b$ . Take any  $g' > g$ ,  $s_1 \geq s_0 + \max\{m, \ell\}$  and  $\delta > s_1 + \ell'$ . Then, for  $S$  large enough, there are  $r$  and  $q$  positive such that, for any family of  $G$ -differentiable maps  $\Phi_\varepsilon : B_{s_0+m}(\varepsilon^\gamma) \rightarrow W_{s_0}$  ( $0 < \varepsilon \leq 1$ ) satisfying (2.9, 2.10, 2.11), if, in addition,  $\|\Phi_\varepsilon(0)\|'_\delta \leq r\varepsilon^{\gamma+g'}$ , then there exists some  $u_\varepsilon$  satisfying:

$$\begin{aligned} \Phi_\varepsilon(u_\varepsilon) &= 0 \\ \|u_\varepsilon\|_{s_0+\max\{m, \ell\}} &\leq \varepsilon^\gamma \\ \|u_\varepsilon\|_{s_1} &\leq q\varepsilon^{-g'} \|\Phi_\varepsilon(0)\|'_\delta \end{aligned}$$

In [26], the assumptions are stronger, since they involve the second derivative of  $\Phi_\varepsilon$ . More importantly, we only need the norm of  $\Phi_\varepsilon(0)$  to be controlled by  $\varepsilon^{\gamma+g'}$  with  $g' > g$ , provided  $S \geq S_0$  with  $S_0 = \mathcal{O}\left(\frac{1}{g'-g}\right)$ , while in [26] (Assumption 2.15 and Remark 2.23), due to quadratic estimates, one needs  $g' > 2g$  with the faster growth  $S_0 = \mathcal{O}\left(\frac{1}{(g'-2g)^2}\right)$ .

### 3. PROOF OF THEOREM 3

The proof consists in constructing a sequence  $u_n, n \geq 1$ , which converges to a solution  $u$  of  $F(u) = v$ . This sequence depends on a number of parameters  $\eta, \alpha, \beta, \theta$  and  $\sigma$  satisfying various conditions: in the first section we prove that these conditions are compatible. We then construct an initial point  $u_1$  depending on  $\eta, \alpha$  and  $\theta$ . In the third section we construct, by induction, the remaining points  $u_n$  which also depend on  $\beta$  and  $\sigma$ . Finally we prove that the sequence  $(u_n)$  converges to a solution  $u$  of the problem, satisfying the desired estimates.

At each step, in order to find  $u_n$ , we solve a nonlinear equation in a Banach space, using Theorem 2 in [12], which we restate here for the reader's convenience:



**Theorem 5.** *Let  $X$  and  $Y$  be Banach spaces. Let  $f : X \rightarrow Y$  be continuous and Gâteaux-differentiable, with  $f(0) = 0$ . Assume that the derivative  $Df(u)$  has a right-inverse  $L(u)$ , uniformly bounded in a neighbourhood of 0:*

$$\begin{aligned} \forall h \in Y, \quad Df(u) L(u)v &= v \\ \sup \{ \|L(u)\| \mid \|u\| \leq R \} &< M \end{aligned}$$

*Then, for every  $v \in Y$  with  $\|v\| \leq RM^{-1}$  there is some  $u \in X$  satisfying  $f(u) = v$  and:*

$$(3.1) \quad \|u\| \leq M \|v\| \leq R$$

Note first that this is a local surjection theorem, not an inverse function theorem: with respect to the IFT, we lose uniqueness. On the other hand, the regularity requirement on  $f$  and the smallness condition on  $v$  are much weaker. As mentioned in the Introduction, for a  $C^1$  functional in finite dimensions, this theorem has been proved a long time ago by Wazewski [27] by a continuation argument. For a comparison of the existence and uniqueness domains in the  $C^2$  case with  $\dim X = \dim Y$ , see [16], chapter II, exercise 2.3.

It turns out that the proof of Theorem 3 is much easier if one assumes that the family  $(DF_\varepsilon)$  is tame Galerkin right-invertible. But most applications require that  $(DF_\varepsilon)$  be tame right-invertible. Let us explain why the proof is longer in this case. In our algorithm, we will use two sequences of projectors  $\Pi_n := \Pi(\Lambda_n)$  and  $\Pi'_n := \Pi'(M_n)$  with associated ranges  $E_n = E(\Lambda_n)$  and  $E'_n = E'(M_n)$ , where  $\Lambda^0 \approx \varepsilon^{-\eta}$  for some small  $\eta > 0$ ,  $\Lambda_n = \Lambda_0^{\alpha^n}$  for some  $\alpha > 1$  close to 1, and  $M_n = \Lambda_n^\vartheta$  for some  $\vartheta \leq 1$  such that  $\vartheta\alpha > 1$ . The algorithm consists in finding, by induction on  $n$  and using Theorem 5 a each step, a solution  $u_n \in E_n$  of the problem  $\Pi'_n DF_\varepsilon(u_n) = \Pi'_{n-1}v$ . For this, we need the right-invertibility of  $\Pi'_n DF_\varepsilon(u)|_{E_n}$  for  $u$  in a certain ball  $\mathcal{B}_n$ , with estimates on the right inverse for a certain norm  $\|\cdot\|_{\mathcal{N}_n}$ .

When the family  $(DF_\varepsilon)$  is tame Galerkin right-invertible, we can take  $\vartheta = 1$  so that  $M_n = \Lambda_n$ , instead of assuming  $\vartheta < 1$ . Then the right-invertibility of  $\Pi'_n DF_\varepsilon(u)|_{E_n}$  is immediate.

But when  $(DF_\varepsilon)$  is only tame right-invertible, it is crucial to take  $\vartheta < 1$ . The intuitive idea is the following. One can think of  $DF_\varepsilon(u)$  as very large right-invertible matrix. The topological argument we use requires  $\Pi'_n DF_\varepsilon(u)|_{E_n}$  to have a right-inverse for  $u$  in a suitable ball. If we take  $M_n = \Lambda_n$ , this is like asking that the square submatrix of a right-invertible matrix be invertible. In general this is not true. But a rectangular submatrix, with more columns than lines, will be right-invertible if the full matrix is and if there are enough columns in the submatrix. This is why we impose  $M_n < \Lambda_n$  when we do not assume the tame Galerkin right-invertibility.

In the sequel, we assume that the family  $(DF_\varepsilon)$  is tame right-invertible, so we take  $\vartheta < 1$ , and we point out the specific places where the arguments would be easier assuming, instead, that  $(DF_\varepsilon)$  is tame Galerkin right-invertible.

**3.1. Choosing the values of the parameters.** We are given  $s_1 \geq s_0 + \max\{m, \ell\}$ ,  $\delta > s_1 + \ell'$  and  $g' > g$ . These are fixed throughout the proof.

We introduce positive parameters  $\eta, \alpha, \beta, \vartheta$  and  $\sigma$  satisfying the following conditions:

$$(3.2) \quad \eta < \frac{g' - g}{\max\{m, \ell\}}$$

$$(3.3) \quad \frac{1}{\alpha} < \vartheta < 1$$

$$(3.4) \quad (1 - \vartheta)(\sigma - \delta) > \vartheta m + \max\{\ell, \vartheta \ell'\} + \frac{g}{\eta}$$

$$(3.5) \quad \sigma > \alpha\beta + s_1$$

$$(3.6) \quad (1 + \alpha - \vartheta\alpha)(\sigma - s_0) > \alpha\beta + \alpha(m + \ell) + \ell' + \frac{g}{\eta}$$

$$(3.7) \quad (1 - \vartheta)(\sigma - s_0) > m + \vartheta \ell' + \frac{g}{\alpha\eta}$$

$$(3.8) \quad \delta > s_0 + \frac{\alpha}{\vartheta}(\sigma - s_0 - \alpha\beta + \ell'')$$

$$(3.9) \quad (\alpha - 1)\beta > (1 - \vartheta)(\sigma - s_0) + \vartheta m + \ell'' + \frac{g}{\eta}$$

$$(3.10) \quad \ell'' = \max\{(\alpha - 1)\ell + \ell', \alpha\vartheta \ell'\}$$

Note that condition (3.8) may be rewritten as

$$\beta > \frac{1}{\alpha}(\sigma - \delta) + \left(1 - \frac{\vartheta}{\alpha}\right) \frac{\delta - s_0}{\alpha} + \frac{\ell''}{\alpha}$$

which implies the simpler inequality

$$(3.11) \quad \beta > \frac{1}{\alpha}(\sigma - \delta)$$

Inequality (3.11) will also be used in the proof.

As already mentioned, if we assume tame Galerkin right-invertibility instead of tame right-invertibility, we do not need conditions (3.4), (3.6), (3.7) and we can take  $\vartheta = 1$  instead of  $\vartheta < 1$ .

**Lemma 1.** *The set of parameters  $(\eta, \alpha, \beta, \vartheta, \sigma)$  satisfying the above conditions is non-empty. More precisely, there are some  $\alpha > 1$  and  $\zeta > 0$  depending only on  $(s_0, m, \ell, \ell', s, \delta)$ , such that, for  $\vartheta = \alpha^{-1/2}$  and for every  $\eta > 0$ , there exist  $\beta, \sigma$  with  $\sigma < \zeta g/\eta$  such that the constraints (3.4) to (3.10) are satisfied.*

*Proof.* Since  $\delta > s + \ell'$ , and  $\ell'' \rightarrow \ell'$  when  $\alpha$  and  $\vartheta \rightarrow 1$ , it is possible to choose  $\vartheta$  and  $\alpha = \vartheta^{-2}$  close enough to 1 so that  $\delta > s_0 + \frac{\alpha}{\vartheta}(s_1 - s_0 + \ell'')$ . Take some  $\tau$  with  $0 < \tau < \frac{\vartheta}{\alpha}(\delta - s_0) - s_1 + s_0 - \ell''$ , and set:

$$(3.12) \quad \beta = \frac{\sigma}{\alpha} - \frac{s_1 + \tau}{\alpha}$$

Then conditions (3.3), (3.5) and (3.8) are satisfied. Note that  $\alpha, \vartheta$  and  $\tau$  depend only on  $\delta$ .

The remaining constraints can be rewritten as follows:

$$(3.13) \quad \sigma > \delta + \frac{1}{1-\vartheta} \left[ \vartheta m + \max\{\ell, \vartheta \ell'\} + \frac{g}{\eta} \right]$$

$$(3.14) \quad \beta < \left( \frac{1}{\alpha} + 1 - \vartheta \right) \sigma - m - \ell - \frac{\ell'}{\alpha} - \left( \frac{1}{\alpha} + 1 - \vartheta \right) s_0 - \frac{g}{\alpha \eta}$$

$$(3.15) \quad \sigma > s_0 + \frac{1}{(1-\vartheta)} \left( m + \vartheta \ell' + \frac{g}{\alpha \eta} \right)$$

$$(3.16) \quad \beta > \frac{1-\vartheta}{\alpha-1} \sigma + \frac{1}{\alpha-1} \left( \vartheta m + \ell'' + \frac{g}{\eta} - (1-\vartheta) s_0 \right)$$

Since  $\alpha\vartheta > 1$ , the slopes in (3.12), (3.14) and (3.16) are ordered as follows:

$$0 < \frac{1-\vartheta}{\alpha-1} < \frac{1}{\alpha} < \frac{1}{\alpha} + 1 - \vartheta < 1$$

It follows that, for the chosen values of  $\alpha, \vartheta$  and  $\tau$ , the domain defined by these three conditions in the  $(\beta, \sigma)$ -plane is an infinite half-line stretching to the North-West. The remaining two, (3.13) and (3.15), just tell us that  $\sigma$  should be large enough. So the set of solutions is of the form  $\sigma > \bar{\sigma}$ ,  $\beta = \frac{\sigma}{\alpha} - \frac{s_0 + \tau}{\alpha}$  and  $\bar{\sigma}$  is clearly a piecewise affine function of  $g/\eta$ .  $\square$

**Remark.** *As already mentioned, if we assume that  $(DF_\varepsilon)$  is tame Galerkin right-invertible, we do not need conditions (3.4), (3.6) and (3.7). The remaining conditions can be satisfied by taking  $\vartheta = 1$  and for a larger set of the other parameters. The corresponding variant of Lemma 1 has a simpler proof. We can choose  $\alpha > 1$  such that  $\delta > s_0 + \alpha(s_1 - s_0 + \ell'')$  and  $\tau$  such that  $0 < \tau < \frac{1}{\alpha}(\delta - s_0) - s_1 + s_0 - \ell''$ , and we may impose condition (3.12). Then conditions (3.13), (3.14) and (3.15) are no longer required, and the only remaining condition (3.16) is easily satisfied by taking  $\sigma$  large enough.*

The values  $(\delta, \alpha, \beta, \vartheta, \sigma)$  are now fixed. For the remainder of the proof we introduce an important notation. By

$$x \lesssim y$$

we mean that there is some constant  $C$ , depending on our scales of spaces, our families of projectors, our family of functions  $(F_\varepsilon)_{0 < \varepsilon \leq 1}$  and our additional parameters  $(\eta, \alpha, \beta, \vartheta, \sigma)$ , but NOT on  $\varepsilon$ , nor on the regularity index  $s \in [0, S]$  or the rank  $n$  in any of the sequences which will be introduced in the sequel, such that  $x \leq Cy$ . For instance, the tame inequalities become:

$$\begin{aligned} \|DF_\varepsilon(u)h\|_s &\lesssim (\|u\|_{s+m} \|h\|_{s_0+m} + \|h\|_{s+m}) \\ \|L_\varepsilon(u)k\|_s &\lesssim \varepsilon^{-g} (\|u\|_{s+l} \|k\|_{s_0+l'} + \|k\|_{s+l'}) \end{aligned}$$

In the iteration process, we will need the following result:

**Lemma 2.** *If the maps  $F_\varepsilon$  form an  $S$ -tame differentiable family and  $F_\varepsilon(0) = 0$ , then, for  $u \in B_{s_0+m} \cap V_{s+m}$  and  $s_0 \leq s \leq S - m$ , we have:*

$$\|F_\varepsilon(u)\|_s \lesssim \|u\|_{s+m}$$

*Proof.* Consider the function  $\varphi(t) = \|F_\varepsilon(tu)\|_s$ . Since  $F_\varepsilon$  is G-differentiable, we have:

$$\begin{aligned}\varphi'(t) &= \left\langle DF_\varepsilon(tu)u, \frac{F_\varepsilon(tu)}{\|F_\varepsilon(tu)\|_s} \right\rangle_s \\ &\leq a(t\|u\|_{s_0+m}\|u\|_{s+m} + \|u\|_{s+m})\end{aligned}$$

and since  $\varphi(0) = 0$ , we get the result.  $\square$

### 3.2. Initialization.

3.2.1. *Some subspaces.* This subsection uses condition (3.3), and  $s + \ell' < \delta < \sigma$ .

We are given  $(\alpha, \vartheta, \delta, \varepsilon, \eta)$ . We fix a large constant  $K > 1$ , to be chosen later independently of  $0 < \varepsilon \leq 1$ . We set  $\Lambda_0 = (K\varepsilon^{-\eta})^{1/\alpha}$ ,  $\Lambda_1 := \Lambda_0^\alpha = K\varepsilon^{-\eta}$ ,  $M_0 := \Lambda_0^\vartheta = (K\varepsilon^{-\eta})^{\vartheta/\alpha}$  and  $M_1 := \Lambda_1^\vartheta = (K\varepsilon^{-\eta})^\vartheta$ .

We then have the inequalities  $M_0 < \Lambda_0 < M_1 < \Lambda_1$ .

Let  $E_1 := E(\Lambda_1)$ ,  $\Pi_1 := \Pi(\Lambda_1)$ ,  $E'_1 = E(M_1)$  and  $\Pi'_i := \Pi'(M_i)$  for  $i = 0, 1$ .

On  $E_1$ ,  $E'_1$ , we introduce the norms:

$$\begin{aligned}\|h\|_{\mathcal{N}_1} &:= \|h\|_\delta + \Lambda_1^{-\frac{\vartheta}{\alpha}(\sigma-\delta)} \|h\|_\sigma \\ \|k\|'_{\mathcal{N}_1} &:= \|k\|'_\delta + \Lambda_1^{-\frac{\vartheta}{\alpha}(\sigma-\delta)} \|k\|'_\sigma\end{aligned}$$

The map  $F_\varepsilon$  induces a map  $f_1 : B_{s_0+m} \cap E_1 \rightarrow E'_1$  defined by  $f_1(u) := \Pi'_1 F_\varepsilon(u)$ ,  $u \in B_{s_0+m} \cap E_1$ . Note that  $f_1(0) = 0$ . We will use the local surjection theorem to show that the range of  $f_1$  covers a neighbourhood of 0 in  $E'_1$ . We begin by showing that  $Df_1$  has a right inverse.

*Note that, if we assume that  $DF$  is tame Galerkin right-invertible, we can take  $M_1 = \Lambda_1 \geq \underline{\Lambda}$ , and  $Df_1$  is automatically right-invertible, with the tame estimate (2.8). So the next subsection is only necessary if we assume that  $DF$  is tame right-invertible.*

3.2.2.  *$Df_1$  has a right inverse.* This section uses condition (3.4). We recall it here for the reader's convenience:

$$(1 - \vartheta)(\sigma - \delta) > \vartheta m + \max\{\ell, \vartheta \ell'\} + \frac{g}{\eta}$$

**Lemma 3.** *For  $K$  large enough and for all  $u \in E_1$  with  $\|u\|_{\mathcal{N}_1} \leq 1$ :*

$$\|\Pi'_1 DF_\varepsilon(u)(1 - \Pi_1)L_\varepsilon(u)\|_{\mathcal{N}_1} \leq \frac{1}{2}$$

*Proof.* From  $\|u\|_{\mathcal{N}_1} \leq 1$ , it follows that  $\|u\|_\delta \leq 1$ , and since  $\delta > s_0 + \max\{\ell, m\} + \ell'$ , the tame estimates hold for  $u$ .

Take any  $k \in E'_1$  and set  $h = (1 - \Pi_1)L_\varepsilon(u)k$ .

We have  $\|h\|_\delta \lesssim \Lambda_1^{\delta-\sigma} \|L_\varepsilon(u)k\|_\sigma$ , and:

$$\begin{aligned}\|\Pi'_1 DF_\varepsilon(u)h\|_{\delta-m} &\lesssim \|h\|_{s_0+m}\|u\|_\delta + \|h\|_\delta \lesssim \|h\|_\delta \\ \|\Pi'_1 DF_\varepsilon(u)h\|_\delta &\lesssim M_1^m \|\Pi'_1 DF_\varepsilon(u)h\|_{\delta-m} \lesssim M_1^m \|h\|_\delta.\end{aligned}$$

Hence:

$$\|\Pi'_1 DF_\varepsilon(u)h\|_\delta \lesssim M_1^m \Lambda_1^{\delta-\sigma} \|L_\varepsilon(u)k\|_\sigma.$$

Writing  $\|\Pi'_1 DF_\varepsilon(u)h\|_\sigma \lesssim M_1^{\sigma-\delta} \|\Pi'_1 DF_\varepsilon(u)h\|_\delta$  we finally get:

$$(3.17) \quad \|\Pi'_1 DF_\varepsilon(u)h\|_{\mathcal{N}_1} \lesssim M_1^m \Lambda_1^{\delta-\sigma} \left(1 + \Lambda_1^{-\frac{\vartheta}{\alpha}(\sigma-\delta)} M_1^{\sigma-\delta}\right) \|L_\varepsilon(u)k\|_\sigma.$$

We now have to estimate  $\|L_\varepsilon(u)k\|_\sigma$ . By the tame estimates, we have:

$$\begin{aligned} \|L_\varepsilon(u)k\|_\sigma &\lesssim \varepsilon^{-g} (\|k\|'_{\sigma+\ell'} + \|u\|_{\sigma+\ell} \|k\|'_{s_0+\ell'}) \\ &\lesssim \varepsilon^{-g} (M_1^{\ell'} \|k\|'_\sigma + \Lambda_1^\ell \|u\|_\sigma \|k\|'_\delta) \end{aligned}$$

Since  $\|u\|_{\mathcal{N}_1} \leq 1$ , we have  $\|u\|_\sigma \leq \Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)}$ . Substituting, we get:

$$\begin{aligned} \|L_\varepsilon(u)k\|_\sigma &\lesssim \varepsilon^{-g} (M_1^{\ell'} \|k\|'_\sigma + \Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)+\ell} \|k\|'_\delta) \\ (3.18) \quad &\lesssim \varepsilon^{-g} \Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)} \max\{M_1^{\ell'}, \Lambda_1^\ell\} \|k\|'_{\mathcal{N}_1} \end{aligned}$$

Putting (3.17) and (3.18) together, we get:

$$\|\Pi'_1 DF_\varepsilon(u)h\|_{\mathcal{N}_1} \lesssim \varepsilon^{-g} M_1^m \Lambda_1^{\delta-\sigma} (\Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)} + M_1^{\sigma-\delta}) \max\{M_1^{\ell'}, \Lambda_1^\ell\} \|k\|'_{\mathcal{N}_1}$$

Since  $\alpha > 1$ , we have  $\Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)} \leq \Lambda_1^{\vartheta(\sigma-\delta)} = M_1^{\sigma-\delta}$ , so that:

$$\begin{aligned} \|\Pi'_1 DF_\varepsilon(u)h\|_{\mathcal{N}_1} &\lesssim \varepsilon^{-g} M_1^m \Lambda_1^{\delta-\sigma} M_1^{\sigma-\delta} \max\{M_1^{\ell'}, \Lambda_1^\ell\} \|k\|'_{\mathcal{N}_1} \\ &\lesssim \varepsilon^{-g} \Lambda_1^{\vartheta m - (1-\vartheta)(\sigma-\delta) + \max\{\ell, \vartheta\ell'\}} \|k\|'_{\mathcal{N}_1} \end{aligned}$$

Since  $\Lambda_1 = K\varepsilon^{-\eta}$ , the inequality becomes:

$$\|\Pi'_1 DF_\varepsilon(u)h\|_{\mathcal{N}_1} \lesssim K^{-C_0} \varepsilon^{-g+\eta C_0} \|k\|'_{\mathcal{N}_1}$$

with  $C_0 := (1-\vartheta)(\sigma-\delta) - \vartheta m - \max\{\ell, \vartheta\ell'\}$ .

By condition (3.4), the exponent  $C_0$  is larger than  $g/\eta$ , so, choosing  $K$  large enough independently of  $0 < \varepsilon \leq 1$ , we find the desired estimate.  $\square$

Introduce the map  $L_1(u) = \Pi_1 L_\varepsilon(u)|_{E'_1}$ . Since  $DF_\varepsilon(u)L_\varepsilon(u) = 1$ , it follows from the Lemma that, for  $k \in E'_1$ , we have:

$$\|k - Df_1(u)L_1(u)k\|'_{\mathcal{N}_1} \leq \frac{1}{2} \|k\|'_{\mathcal{N}_1}$$

This implies that the Neumann series  $\sum_{i \geq 0} (I_{E'_1} - Df_1(u)L_1(u))^i$  converges in operator norm. Its sum is  $S_1(u) = (Df_1(u)L_1(u))^{-1}$  and it has operator norm at most 2.

Then  $T_1(u) := L_1(u)S_1(u)$  is a right inverse of  $Df_1(u)$  and  $\|T_1(u)\|_{\mathcal{N}_1} \leq 2\|L_1(u)\|_{\mathcal{N}_1}$ . By the tame estimates, for  $k \in E'_1$ , we have:

$$\begin{aligned} \|L_\varepsilon(u)k\|_\delta &\lesssim \varepsilon^{-g} (\|k\|'_{\delta+\ell'} + \|u\|_{\delta+\ell} \|k\|'_{s_0+\ell'}) \\ &\lesssim \varepsilon^{-g} (M_1^{\ell'} + \Lambda_1^\ell) \|k\|'_\delta \end{aligned}$$

Comparing with (3.18), we find:

$$\|T_1(u)\|_{\mathcal{N}_1} \lesssim \|L_\varepsilon(u)\|_{\mathcal{N}_1} \lesssim \varepsilon^{-g} (M_1^{\ell'} + \Lambda_1^\ell)$$

Hence

$$\sup_{\|u\|_{\mathcal{N}_1} \leq 1} \|T_1(u)\|_{\mathcal{N}_1} \lesssim m_1 := \varepsilon^{-g} (M_1^{\ell'} + \Lambda_1^\ell)$$

### 3.2.3. Local inversion of $f_1$ .

Applying Theorem 5, we find that if  $\|\Pi'_0 v\|'_{\mathcal{N}_1} < 1/m_1$ , then equation  $f_1(u) = \Pi'_0 v$  has a solution  $u_1$  with  $\|u_1\|_{\mathcal{N}_1} \leq 1$  and  $\|u_1\|_{\mathcal{N}_1} \leq m_1 \|\Pi'_0 v\|'_{\mathcal{N}_1}$ .

Note that  $\|\Pi'_0 v\|'_\sigma \lesssim M_0^{\sigma-\delta} \|\Pi'_0 v\|'_\delta \lesssim \Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)} \|\Pi'_0 v\|'_\delta$ . It follows that

$$\|\Pi'_0 v\|'_{\mathcal{N}_1} = \|\Pi'_0 v\|'_\delta + \Lambda_1^{-\frac{\vartheta}{\alpha}(\sigma-\delta)} \|\Pi'_0 v\|'_\sigma \lesssim \|\Pi'_0 v\|'_\delta$$

Assume from now on:

$$(3.19) \quad \|v\|'_\delta \lesssim \varepsilon^g \left( M_1^{\ell'} + \Lambda_1^\ell \right)^{-1}$$

Then  $\|\Pi'_0 v\|'_{\mathcal{N}_1} \lesssim m_1^{-1}$ , and Theorem 5 applies. The estimate on  $u_1$  implies:

$$(3.20) \quad \|u_1\|_\delta \lesssim \varepsilon^{-g} \left( M_1^{\ell'} + \Lambda_1^\ell \right) \|v\|'_\delta \leq 1$$

It also implies an estimate in higher norm:

$$(3.21) \quad \|u_1\|_\sigma \lesssim \varepsilon^{-g} \Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)} \left( M_1^{\ell'} + \Lambda_1^\ell \right) \|v\|'_\delta \lesssim \Lambda_1^{\frac{\vartheta}{\alpha}(\sigma-\delta)}.$$

## 3.3. Induction.

**3.3.1. Some uniform bounds.** In addition to  $(\alpha, \vartheta, \delta, \varepsilon, \eta)$  we are given  $\beta$  satisfying relations (3.5) and (3.11). We recall them here for the reader's convenience. With  $s_1 > s_0 + \max\{m, \ell\}$  and  $\delta > s_1 + \ell'$ ,

$$\begin{aligned} \sigma &> \alpha\beta + s_1 \\ \beta &> \frac{1}{\alpha}(\sigma - \delta) \end{aligned}$$

We also inherit  $\Lambda_1 = K\varepsilon^{-\eta}$  and  $u_1$  from the preceding section. Combining (3.11) and (3.21), we immediately obtain the estimate

$$(3.22) \quad \|u_1\|_\sigma \lesssim \varepsilon^{-g} \Lambda_1^\beta \left( M_1^{\ell'} + \Lambda_1^\ell \right) \|v\|'_\delta \lesssim \Lambda_1^\beta.$$

Consider the sequences of integers  $M_n$  and  $\Lambda_n$ ,  $n \geq 1$ , defined by  $\Lambda_n := \Lambda_1^{\alpha^{n-1}}$  and  $M_n := \Lambda_n^{\vartheta}$ .

Let  $\Pi_n := \Pi(\Lambda_n)$ ,  $\Pi'_n := \Pi'(M_n)$ ,  $E_n := E(\Lambda_n)$ ,  $E'_n := E'(M_n)$ .

We will construct a sequence  $u_n \in E_n$ ,  $n \geq 1$ , starting from the initial point  $u_1$  we found in the preceding section. For all  $n \geq 2$  the remaining points should satisfy the following conditions:

$$(3.23) \quad \Pi'_n F_\varepsilon(u_n) = \Pi'_{n-1} v$$

$$(3.24) \quad \|u_n - u_{n-1}\|_{s_0} \leq \varepsilon^{-g} \Lambda_{n-1}^{\alpha\beta - \sigma + s_0} \|v\|'_\delta$$

$$(3.25) \quad \|u_n - u_{n-1}\|_\sigma \leq \varepsilon^{-g} \Lambda_{n-1}^{\alpha\beta} \|v\|'_\delta$$

We proceed by induction. Suppose we have found  $u_2, \dots, u_{n-1}$  satisfying these conditions. We want to construct  $u_n$ .

**Lemma 4.** *Let us impose  $K \geq 2$ . For all  $t$  with  $s_0 \leq t < \sigma - \alpha\beta$ , and all  $i$  with  $2 \leq i \leq n-1$ , we have:*

$$\sum_{i=2}^{n-1} \|u_i - u_{i-1}\|_t \leq \varepsilon^{-g} \Sigma(t) \|v\|'_\delta$$

where  $\Sigma(t)$  is finite and independent of  $n, \varepsilon$ .

*Proof.* By the interpolation formula, we have  $\|u_i - u_{i-1}\|_t \leq \varepsilon^{-g} \Lambda_{i-1}^{\alpha\beta - \sigma + t} \|v\|'_\delta$ , for all  $2 \leq i \leq n$ . Since  $\Lambda_1 = K\varepsilon^{-\eta} \geq 2$ , we have:

$$\begin{aligned} \sum_{i=2}^{n-1} \|u_i - u_{i-1}\|_t &\leq \varepsilon^{-g} \sum_{i=2}^{\infty} \Lambda_{i-1}^{\alpha\beta - \sigma + t} \|v\|'_\delta \\ &\leq \varepsilon^{-g} \sum_{j=0}^{\infty} 2^{\alpha j(\alpha\beta - \sigma + t)} \|v\|'_\delta \end{aligned}$$

□

By (3.5) we can take  $t = s_0 + \max\{m, \ell\}$ , and we find a uniform bound for  $u_{n-1}$  in the  $(s_0 + \max\{m, \ell\})$ -norm, namely:

$$\begin{aligned} \|u_{n-1}\|_{s_0 + \max\{m, \ell\}} &\leq \|u_1\|_\delta + \sum_{i=2}^{n-1} \|u_i - u_{i-1}\|_{s_0 + \max\{m, \ell\}} \\ &\lesssim \varepsilon^{-g} \|v\|'_\delta \left( M_1^{\ell'} + \Lambda_1^\ell + \Sigma(s_0 + \max\{m, \ell\}) \right) \\ &\lesssim \varepsilon^{-g} \|v\|'_\delta \left( M_1^{\ell'} + \Lambda_1^\ell \right) \end{aligned}$$

In particular, we will have  $\|u_{n-1}\|_{s_0 + \max\{m, \ell\}} \leq 1$  if  $\|v\|'_\delta \lesssim \varepsilon^g \left( M_1^{\ell'} + \Lambda_1^\ell \right)^{-1}$ , so the tame estimates hold at  $u_{n-1}$ .

Similarly, if  $\|v\|'_\delta \lesssim \varepsilon^g \left( M_1^{\ell'} + \Lambda_1^\ell \right)^{-1}$  we find a uniform bound in the  $\sigma$ -norm. We have:

$$\|u_{n-1}\|_\sigma \leq \|u_1\|_\sigma + \sum_{i=2}^{n-1} \|u_i - u_{i-1}\|_\sigma$$

and

$$\sum_{i=2}^{n-1} \|u_i - u_{i-1}\|_\sigma \lesssim \varepsilon^{-g} \sum_{i=1}^{n-1} \Lambda_i^\beta \|v\|'_\delta \lesssim \varepsilon^{-g} \Lambda_{n-1}^\beta \|v\|'_\delta$$

so, combining this with (3.22), we get:

$$(3.26) \quad \|u_{n-1}\|_\sigma \lesssim \varepsilon^{-g} \Lambda_{n-1}^\beta \left( M_1^{\ell'} + \Lambda_1^\ell \right) \|v\|'_\delta \lesssim \Lambda_{n-1}^\beta.$$

**3.3.2. Setting up the induction step.**

Suppose, as above, that  $\|v\|'_\delta \lesssim \varepsilon^g \left( M_1^{\ell'} + \Lambda_1^\ell \right)^{-1}$  and that  $u_2, \dots, u_{n-1}$  have been found. We have seen that  $\|u_{n-1}\|_{s_0+m} \leq 1$ , so that the tame estimates hold at  $u_{n-1}$ , and we also have  $\|u_{n-1}\|_\sigma \lesssim \Lambda_{n-1}^\beta$ . We want to find  $u_n$  satisfying (3.23),

(3.24) and (3.25). Since  $\Pi'_{n-1}F_\varepsilon(u_n) = \Pi'_{n-2}v$ , we rewrite the latter equation as follows:

$$(3.27) \quad \Pi'_n(F_\varepsilon(u_n) - F_\varepsilon(u_{n-1})) + (\Pi'_n - \Pi'_{n-1})F_\varepsilon(u_{n-1}) = (\Pi'_{n-1} - \Pi'_{n-2})v$$

Define a map  $f_n : E_n \rightarrow E'_n$  with  $f_n(0) = 0$  by:

$$f_n(z) = \Pi'_n(F_\varepsilon(u_{n-1} + z) - F_\varepsilon(u_{n-1}))$$

Equation (3.27) can be rewritten as follows:

$$(3.28) \quad f_n(z) = \Delta_n v + e_n$$

$$(3.29) \quad \Delta_n v = \Pi'_{n-1}(1 - \Pi'_{n-2})v$$

$$(3.30) \quad e_n = -\Pi'_n(1 - \Pi'_{n-1})F_\varepsilon(u_{n-1})$$

We introduce on  $E_n$  and  $E'_n$  the norms:

$$\begin{aligned} \|x\|_{\mathcal{N}_n} &= \|x\|_{s_0} + N_{n-1}^{-\sigma+s_0} \|x\|_\sigma \\ \|y\|'_{\mathcal{N}_n} &= \|y\|'_{s_0} + N_{n-1}^{-\sigma+s_0} \|y\|'_\sigma \end{aligned}$$

**Lemma 5.** *If  $0 \leq t \leq \sigma - s_0$ , then:*

$$\begin{aligned} \|x\|_{s_0+t} &\leq \Lambda_{n-1}^t \|x\|_{\mathcal{N}_n} \\ \|y\|'_{s_0+t} &\leq \Lambda_{n-1}^t \|y\|'_{\mathcal{N}_n} \end{aligned}$$

*Proof.* Use the interpolation inequality. □

We will solve the system (3.28), (3.29), (3.30) by applying the local surjection theorem to  $f_n$  on the ball  $B_{\mathcal{N}_n}(0, r_n) \subset E_n$  where:

$$(3.31) \quad r_n = \varepsilon^{-g} \Lambda_{n-1}^{\alpha\beta - \sigma + s_0} \|v\|'_\delta$$

Note that, if  $z \in B_{\mathcal{N}_n}(0, r_n)$ , then  $\|z\|_{s_0} \leq \varepsilon^{-g} \Lambda_{n-1}^{\alpha\beta - \sigma + s_0} \|v\|'_\delta$  and  $\|z\|_\sigma \leq \varepsilon^{-g} \Lambda_{n-1}^{\alpha\beta} \|v\|'_\delta$ . In other words,  $u_n = u_{n-1} + z$  satisfies (3.24) and (3.25). Arguing as above, we find that:

$$(3.32) \quad \|u_{n-1} + z\|_{s_0 + \max\{m, l\}} \leq 1$$

$$(3.33) \quad \|u_{n-1} + z\|_\sigma \lesssim \Lambda_n^\beta$$

By (3.32) the tame estimates hold on  $z \in B_{\mathcal{N}_n}(0, r_n)$ . We begin by showing that  $Df_n(z)$  has a right inverse.

*Note that, if we assume that  $DF_\varepsilon$  is tame Galerkin right-invertible, we can take  $M_n = \Lambda_n$ , and the result of the next subsection is obvious. This subsection is only useful if we assume that  $DF$  is tame right-invertible but not tame Galerkin right-invertible.*

**3.3.3.  $Df_n$  has a right inverse.** In this section, we use conditions (3.6) and (3.7). We recall them for the reader's convenience:

$$\begin{aligned} (1 + \alpha - \vartheta\alpha)(\sigma - s_0) &> \alpha\beta + \alpha(m + \ell) + \ell' + \frac{g}{\alpha\eta} \\ (1 - \vartheta)(\sigma - s_0) &> m + \vartheta\ell' + \frac{g}{\alpha\eta} \end{aligned}$$



**Lemma 6.** *Take  $\Lambda_1 = K\varepsilon^\eta$  with  $K$  chosen large enough independently of  $0 < \varepsilon \leq 1$ . Then, for all  $z \in B_{\mathcal{N}_n}(0, r_n)$ :*

$$\|\Pi'_n DF_\varepsilon(u_{n-1} + z)(1 - \Pi_n)L_\varepsilon(u_{n-1} + z)\|_{\mathcal{N}_n} \leq \frac{1}{2}$$

*Proof.* We proceed as in the proof of Lemma 3. For  $k \in E'_n$ , we set  $h = (1 - \Pi_n)L_\varepsilon(u_{n-1} + z)k$ . We have:

$$\|h\|_{s_0+m} \lesssim \Lambda_n^{-\sigma+s_0+m} \|L_\varepsilon(u_{n-1} + z)k\|_\sigma$$

By (3.33) and the tame estimates for  $L_\varepsilon$ , we get:

$$\begin{aligned} \|L_\varepsilon(u_{n-1} + z)k\|_\sigma &\lesssim \varepsilon^{-g} (\|u_{n-1} + z\|_{\sigma+\ell} \|k\|'_{s_0+\ell'} + \|k\|_{\sigma+\ell'}) \\ (3.34) \qquad \qquad \qquad &\lesssim \varepsilon^{-g} \left( \Lambda_n^{\beta+\ell} \Lambda_{n-1}^{\ell'} + M_n^{\ell'} \Lambda_{n-1}^{\sigma-s_0} \right) \|k\|'_{\mathcal{N}_n} \end{aligned}$$

where we have used Lemma 5. Substituting in the preceding formula, we get:

$$\|h\|_{s_0+m} \lesssim \varepsilon^{-g} \left( \Lambda_n^{\beta+\ell-\sigma+s_0+m} \Lambda_{n-1}^{\ell'} + M_n^{\ell'} \Lambda_{n-1}^{-(\alpha-1)(\sigma-s_0)+\alpha m} \right) \|k\|'_{\mathcal{N}_n}$$

By (2.4), we have:

$$\|\Pi'_n DF_\varepsilon(u_{n-1} + z)h\|_{s_0} \lesssim \|h\|_{s_0+m}$$

From this it follows that:

$$\|\Pi'_n DF_\varepsilon(u_{n-1} + z)h\|_\sigma \lesssim M_n^{\sigma-s_0} \|h\|_{s_0+m}$$

Hence:

$$\|\Pi'_n DF_\varepsilon(u_{n-1} + z)h\|_{\mathcal{N}_n} \lesssim (1 + \Lambda_{n-1}^{-\sigma+s_0} M_n^{\sigma-s_0}) \|h\|_{s_0+m}$$

We have  $\Lambda_{n-1}^{-\sigma+s_0} M_n^{\sigma-s_0} \lesssim \Lambda_{n-1}^{(\alpha\vartheta-1)(\sigma-s_0)}$ . Since  $\alpha\vartheta > 1$ , the dominant term in the parenthesis is the second one, and:

$$\begin{aligned} \|\Pi'_n DF_\varepsilon(u_{n-1} + z)h\|_{\mathcal{N}_n} &\lesssim \Lambda_{n-1}^{-\sigma+s_0} M_n^{\sigma-s_0} \|h\|_{s_0+m} \\ &\lesssim \varepsilon^{-g} M_n^{\sigma-s_0} \left( \Lambda_{n-1}^{\alpha(\beta+\ell-\sigma+s_0+m)+\ell'-\sigma+s_0} + M_n^{\ell'} \Lambda_{n-1}^{-\alpha(\sigma-s_0)+\alpha m} \right) \|k\|'_{\mathcal{N}_n} \end{aligned}$$

The right-hand side is a decreasing function of  $n$ . To check that it is less than  $1/2$  for all  $n \geq 2$ , it is enough to check it for  $n = 2$ . Since  $\Lambda_1 = K\varepsilon^{-\eta}$ , substituting in the right-hand side, we get:

$$\|\Pi'_n DF_\varepsilon(u_{n-1} + z)h\|_{\mathcal{N}_n} \lesssim \left( K^{-\min\{C_1, C_2\}} \right)^{\alpha^{n-2}} \left( \varepsilon^{\min\{C_1, C_2\} - \alpha^{2-n} g/\eta} \right)^{\eta \alpha^{n-2}} \|k\|'_{\mathcal{N}_n}$$

with

$$\begin{aligned} C_1 &= -\alpha(\beta + \ell + m) - \ell' + (1 + \alpha - \alpha\vartheta)(\sigma - s_0) \\ C_2 &= \alpha((1 - \vartheta)(\sigma - s_0) - \vartheta\ell' - m) \end{aligned}$$

By (3.6) and (3.7), both exponents  $C_1$  and  $C_2$  are larger than  $g/\eta$ . As a consequence,  $\|\Pi'_n DF_\varepsilon(u_{n-1} + z)h\|_{\mathcal{N}_n} \leq 1/2$  for  $K$  chosen large enough, independently of  $n$  and  $0 < \varepsilon \leq 1$ .  $\square$

Define  $L_n(z) = \Pi_n L_\varepsilon(u_{n-1} + z)|_{E'_n}$ . Arguing as in subsection 3.2.2, we find that the Neumann series  $\sum_{i \geq 0} (I_{E'_n} - Df_n(u) L_n(u))^i$  converges in operator norm. Its sum is  $S_n(u) = (Df_n(u) L_n(u))^{-1}$  and it has operator norm at most 2. Then  $T_n(u) := L_n(u) S_n(u)$  is a right inverse of  $Df_n(u)$  and  $\|T_n(u)\|_{\mathcal{N}_n} \leq 2 \|L_n(u)\|_{\mathcal{N}_n}$ . We have already derived an estimate (3.34), for  $\|L_n(z)\|_\sigma$ , namely:

$$\|L_n(z)k\|_\sigma \lesssim \varepsilon^{-g} \Lambda_{n-1}^{\sigma-s_0} \left( \Lambda_n^{\beta+\ell} \Lambda_{n-1}^{-\sigma+s_0+\ell'} + M_n^{\ell'} \right) \|k\|'_{\mathcal{N}_n}$$

From the tame estimates and Lemma 5, we also have:

$$\|L_n(z)k\|_{s_0} \lesssim \varepsilon^{-g} \|k\|'_{s_0+\ell'} \lesssim \varepsilon^{-g} \Lambda_{n-1}^{\ell'} \|k\|'_{\mathcal{N}_n}$$

Since  $\alpha\vartheta > 1$ , we have  $\Lambda_{n-1}^{\ell'} \lesssim M_n^{\ell'}$ , and we get the final estimate for the right inverse:

$$(3.35) \quad \|T_n(z)\|_{\mathcal{N}_n} \lesssim \varepsilon^{-g} \left( \Lambda_n^{\beta+\ell} \Lambda_{n-1}^{-\sigma+s_0+\ell'} + M_n^{\ell'} \right)$$

3.3.4. *Finding  $u_n$ .* In this subsection, we use relations (3.5), (3.8), (3.9) and (3.10). We recall them for the reader's convenience:

$$\begin{aligned} \sigma &> \alpha\beta + s_1 \\ \delta &> s_0 + \frac{\alpha}{\vartheta} (\sigma - s_0 - \alpha\beta + \ell''') \\ (\alpha - 1)\beta &> (1 - \vartheta)(\sigma - s_0) + \vartheta m + \ell'' + \frac{g}{\eta} \\ \ell''' &= \max \{ (\alpha - 1)\ell + \ell', \alpha\vartheta\ell' \} \end{aligned}$$

Let us go back to (3.28). By the local surjection theorem, to solve  $\Pi'_n f_n(z) = \Delta_n v + e_n$  with  $z \in B_{\mathcal{N}_n}(0, r_n)$  it is enough that:

$$(3.36) \quad \|T_n(z)\|_{\mathcal{N}_n} (\|\Delta_n v\|_{\mathcal{N}_n} + \|e_n\|_{\mathcal{N}_n}) \leq r_n$$

Here  $\|T_n(z)\|_{\mathcal{N}_n}$  is given by (3.35) and  $r_n$  by (3.31). We need to estimate  $\|\Delta_n v\|_{\mathcal{N}_n}$  and  $\|e_n\|_{\mathcal{N}_n}$ .

From (3.29) we have:

$$\begin{aligned} \|\Delta v\|'_{s_0} &\lesssim M_{n-2}^{s_0-\delta} \|v\|'_\delta \\ \|\Delta v\|'_\sigma &\lesssim M_{n-1}^{\sigma-\delta} \|v\|'_\delta \\ \|\Delta v\|'_{\mathcal{N}_n} &\lesssim \max \left\{ M_{n-2}^{s_0-\delta}, \Lambda_{n-1}^{-\sigma+s_0} M_{n-1}^{\sigma-\delta} \right\} \|v\|'_\delta \end{aligned}$$

An easy calculation yields  $\Lambda_{n-1}^{-\sigma+s_0} M_{n-1}^{\sigma-\delta} \lesssim M_{n-2}^{s_0-\delta}$ . It follows that:

$$(3.37) \quad \|\Delta v\|'_{\mathcal{N}_n} \lesssim M_{n-2}^{s_0-\delta} \|v\|'_\delta$$

From (3.30), we derive:

$$\|e_n\|'_{s_0} \lesssim M_{n-1}^{-\sigma+m+s_0} \|e_n\|'_{\sigma-m}$$

By Lemma 2,  $\|F(u_{n-1})\|_{\sigma-m} \lesssim \|u_{n-1}\|_\sigma$ . So, remembering (3.30) and (??), we get:

$$\begin{aligned} \|e_n\|'_{s_0} &\lesssim M_{n-1}^{-\sigma+m+s_0} \|u_{n-1}\|_\sigma \\ &\lesssim \varepsilon^{-g} M_{n-1}^{-\sigma+m+s_0} \Lambda_{n-1}^\beta \|v\|'_\delta \end{aligned}$$

Similarly,

$$\begin{aligned} \|e_n\|'_\sigma &\lesssim \|u_{n-1}\|_{\sigma+m} \lesssim \Lambda_{n-1}^m \|u_{n-1}\|_\sigma \\ &\lesssim \varepsilon^{-g} \Lambda_{n-1}^{\beta+m} \|v\|'_\delta \end{aligned}$$

Finally, since  $M_{n-1} < \Lambda_{n-1}$  and  $\sigma > m + s_0$ , we get:

$$\|e_n\|'_{\mathcal{N}_n} \lesssim \varepsilon^{-g} \Lambda_{n-1}^\beta M_{n-1}^{-\sigma+m+s_0} \|v\|'_\delta$$

Substituting in (3.36), we get the following sufficient condition:

$$(3.38) \quad \left( \Lambda_n^{\beta+\ell} \Lambda_{n-1}^{-\sigma+s_0+\ell'} + M_n^{\ell'} \right) \left( M_{n-2}^{s_0-\delta} + \varepsilon^{-g} \Lambda_{n-1}^\beta M_{n-1}^{-\sigma+m+s_0} \right) \lesssim \Lambda_{n-1}^{\alpha\beta-\sigma+s_0}$$

Since  $\Lambda_n = \Lambda_{n-1}^\alpha$  and  $M_{n-i} = (\Lambda_{n-1})^{\alpha^{1-i}\vartheta}$  and  $\Lambda_{n-1} = (K\varepsilon^{-\eta})^{\alpha^{n-2}}$ , we have

$$\left( \Lambda_n^{\beta+\ell} \Lambda_{n-1}^{-\sigma+s_0+\ell'} + M_n^{\ell'} \right) \left( M_{n-2}^{s_0-\delta} + \varepsilon^{-g} \Lambda_{n-1}^\beta M_{n-1}^{-\sigma+m+s_0} \right) \lesssim (\varepsilon^{-\eta\alpha^{n-2}})^{\max\{C_3, C_4\} + \max\{C_5, C_6\}}$$

and

$$\Lambda_{n-1}^{\alpha\beta-\sigma+s_0} \gtrsim (\varepsilon^{-\eta\alpha^{n-2}})^{C_7}$$

with

$$\begin{aligned} C_3 &:= \alpha(\beta + \ell) - \sigma + s_0 + \ell' \\ C_4 &:= \alpha\vartheta\ell' \\ C_5 &:= \vartheta\alpha^{-1}(s_0 - \delta) \\ C_6 &:= g/\eta + \beta + \vartheta(-\sigma + m + s_0) \\ C_7 &:= \alpha\beta - \sigma + s_0 \end{aligned}$$

By (3.5), we have  $\sigma - \alpha\beta > s_1 > s_0 + \max\{m, \ell\}$ . It follows that:

$$C_3 < (\alpha - 1)\ell + \ell'.$$

So, defining  $\ell'' = \max\{(\alpha - 1)\ell + \ell', \alpha\vartheta\ell'\}$  as in (3.10), we see that

$$\max\{C_3, C_4\} + \max\{C_5, C_6\} \leq \max\{\ell'' + C_5, \ell'' + C_6\}$$

So condition (3.38) is implied by the inequalities  $\ell'' + C_5 < C_7$  and  $\ell'' + C_6 < C_7$ , which are the same as conditions (3.8) and (3.9).

**3.4. Conclusion.** For every  $v \in E'$  with  $\|v\|'_\delta \lesssim \varepsilon^g \left( N_1^\ell + M_1^{\ell'} \right)^{-1}$  we have constructed a sequence  $u_n, n \geq 1$ , with  $\|u_n\|_{s_0+\max\{m, \ell\}} \leq 1$ , such that  $\Pi'_n F(u_n) = \Pi'_{n-1} v$ . We note that, by condition (3.5),  $s_1 < \sigma - \alpha\beta$ . It follows from Lemma 4 that the sequence  $u_n$  is Cauchy for the  $s_1$ -norm, so that it converges to some  $u$  with  $\|u\|_{s_1} \lesssim \varepsilon^{-g'} \|v\|'_\delta$ . On the right-hand side,  $\Pi'_n F(u_n)$  converges to  $v$  for all norms. Since  $s_1 \geq s_0 + m$ , the map  $F$  is continuous from the  $s_1$ -norm to the  $(s_1 - m)$ -norm, so  $F(u_n)$  converges to  $F(u)$  and  $F(u) = v$ , as desired.

It remains to check the estimate on  $v$ . We have  $\|v\|'_\delta \lesssim \varepsilon^g \left( \Lambda_1^\ell + M_1^{\ell'} \right)^{-1}$  and  $\varepsilon^{-\vartheta\eta} \lesssim M_1 \leq \Lambda_1 \lesssim \varepsilon^{-\eta}$ . It follows that  $\|v\|'_\delta \lesssim \varepsilon^{g+\eta \max\{\vartheta\ell', \ell\}}$ . By (3.2), this yields  $\|v\|'_\delta \lesssim \varepsilon^{g'}$  as desired.

In the alternative case, when one directly assumes that  $\Pi'_n DF_\varepsilon(u)|_{E_n}$  has a right inverse for every  $n$ , and the inverse satisfies tame estimates, one can dispense with subsections 3.2.2 and 3.3.3. Condition (3.4), (3.6) and (3.7) become unnecessary,

together with the parameter  $\vartheta$ , and the proof goes through with these simplifications.

#### 4. AN APPLICATION OF THE SINGULAR PERTURBATION THEOREM

**4.1. The result.** In this section, we consider a Cauchy problem for nonlinear Schrödinger systems arising in nonlinear optics, a question recently studied by Métivier-Rauch [21] and Texier-Zumbrun [26]. Métivier-Rauch proved the existence of local in time solutions, with an existence time  $T$  converging to 0 when the  $H^s$  norm of the initial datum goes to infinity. Texier-Zumbrun, thanks to their version of the Nash-Moser theorem adapted to singular perturbation problems, were able to find a uniform lower bound on  $T$  for certain highly oscillatory initial data. The  $H^s$  norm of these initial data could go to infinity. By applying our "semiglobal" version of the Nash-Moser theorem, we are able to extend Texier-Zumbrun's result to even larger initial data. In the sequel we follow closely their exposition, but some parameters are named differently to avoid confusions with our other notations.

The problem takes the following form:

$$(4.1) \quad \begin{cases} \partial_t u + iA(\partial_x)u = B(u, \partial_x)u, \\ u(0, x) = \varepsilon^\kappa (u_\varepsilon(x), \bar{u}_\varepsilon(x)) \end{cases}$$

with  $u(t, x) = (\psi(t, x), \bar{\psi}(t, x)) \in \mathbf{C}^{2n}$ ,  $(t, x) \in [0, T] \times \mathbf{R}^d$ ,

$$A(\partial_x) = \text{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)\Delta_x$$

and

$$B = \begin{pmatrix} \mathcal{B} & \mathcal{C} \\ \bar{\mathcal{C}} & \bar{\mathcal{B}} \end{pmatrix}$$

The coefficients  $b_{jj'}$ ,  $c_{jj'}$  of the  $n \times n$  matrices  $\mathcal{B}$ ,  $\mathcal{C}$  are first-order operators with smooth coefficients:  $b_{jj'} = \sum_{k=1}^d b_{kjj'}(u)\partial_{x_k}$ ,  $c_{jj'} = \sum_{k=1}^d c_{kjj'}(u)\partial_{x_k}$ , with  $b_{kjj'}$  and  $c_{kjj'}$  smooth complex-valued functions of  $u$  satisfying, for some integer  $p \geq 2$ , some  $C > 0$ , all  $0 \leq |\alpha| \leq p$  and all  $u = (\psi, \bar{\psi}) \in \mathbf{C}^{2n}$ :

$$|\partial^\alpha b_{kjj'}(u)| + |\partial^\alpha c_{kjj'}(u)| \leq C|u|^{p-|\alpha|}.$$

Moreover, we assume that the following "transparency" conditions hold: the functions  $b_{kjj'}$  are real-valued, the coefficients  $\lambda_j$  are real and pairwise distinct, and for any  $j, j'$  such that  $\lambda_j + \lambda_{j'} = 0$ ,  $c_{jj'} = c_{j'j}$ .

We consider initial data of the form  $\varepsilon^\kappa(a_\varepsilon(x), \bar{a}_\varepsilon(x))$  with either  $a_\varepsilon(x) = a_1(x/\varepsilon)$  (*concentrating case*) or  $a_\varepsilon(x) = a_1(x)e^{i\xi_1 x/\varepsilon}$  (*oscillating case*) with  $0 < \varepsilon \leq 1$ ,  $a_1 \in H^{\bar{s}}(\mathbf{R}^d)$  for some  $\bar{s}$  large enough and  $\|a_1\|_{H^{\bar{s}}}$  small enough.

Our goal is to prove that the Cauchy problem has a solution on  $[0, T] \times \mathbf{R}^d$  for all  $0 < \varepsilon \leq 1$ , with  $T > 0$  independent of  $\varepsilon$ . Texier-Zumbrun obtain existence and uniqueness of the solution, under some conditions on  $\kappa$ , which should be large enough. This corresponds to a smallness condition on the initial datum when  $\varepsilon$  approaches zero. Our local surjection theorem only provides existence, but our condition on  $\kappa$  is less restrictive, so our initial datum is allowed to be larger. Note that, once existence is proved, uniqueness is easily obtained for this Cauchy problem, indeed local-in-time uniqueness implies global-in-time uniqueness. Our result is the following:

**Theorem 6.** *Under the above assumptions and notations, let us impose the additional condition*

$$(4.2) \quad \kappa > \frac{1 - \kappa_a}{p + 1} + \frac{d}{2} \frac{p}{(p + 1)(p - 1)}$$

where  $\kappa_a = d/2$  in the concentrating case, and  $\kappa_a = 0$  in the oscillating case.

Let  $s_1 > \frac{d}{2} + 4$ . If  $0 < \varepsilon \leq 1$ ,  $a_1 \in H^{\bar{s}}(\mathbf{R}^d)$  for  $\bar{s}$  large enough, and  $\|a_1\|_{H^{\bar{s}}}$  is small enough, then the Cauchy problem (4.1) has a unique solution in the functional space  $C^1([0, T], H^{s_1-2}(\mathbf{R}^d)) \cap C^0([0, T], H^{s_1}(\mathbf{R}^d))$ .

In order to compare our results with those of Texier-Zumbrun [26], we consider the same particular values as in their Remark 4.7 and Examples 4.8, 4.9 pages 517-518. Metivier-Rauch already provide existence for a fixed positive  $T$  when  $\kappa \geq 1 + \frac{d}{2} - \kappa_a$ .

So we obtain something new in comparison with Metivier-Rauch when the following double inequality holds:

$$\frac{1 - \kappa_a}{p + 1} + \frac{d}{2} \frac{p}{(p + 1)(p - 1)} < \kappa < 1 + \frac{d}{2} - \kappa_a.$$

In particular:

**In two space dimensions,  $d = 2$**  (Example 4.8 in [26]):

*In the concentrating case*, our conditions become  $\frac{p}{(p+1)(p-1)} < \kappa < 1$ . They are compatible for any  $p \geq 2$ . In their paper, Texier-Zumbrun need  $p \geq 4$  and  $\frac{9}{2(p+1)} < \kappa < 1$ .

*In the oscillating case*, our conditions become  $\frac{2p-1}{(p+1)(p-1)} < \kappa < 2$ . They are compatible for any  $p \geq 2$ , while Texier-Zumbrun need  $p \geq 3$  and  $\frac{5}{p+1} < \kappa < 2$ .

**In three space dimensions,  $d = 3$**  (Example 4.9 in [26]):

*In the concentrating case*, our conditions become  $\frac{2p+1}{2(p+1)(p-1)} < \kappa < 1$ . They are compatible for any  $p \geq 2$ . In their paper, Texier-Zumbrun need  $p \geq 4$  and  $\frac{4}{p+1} < \kappa < 1$ .

*In the oscillating case* our conditions become  $\frac{5p-2}{2(p+1)(p-1)} < \kappa < \frac{5}{2}$ . They are compatible for any  $p \geq 2$ . In their paper, Texier-Zumbrun need either  $p = 2$  and  $2 < \kappa < 5/2$ , or  $p \geq 3$  and  $\frac{11}{2(p+1)} < \kappa < \frac{5}{2}$ .

We see that in the above cases, we can deal with smaller values of  $p$ , which means larger nonlinear terms, and smaller exponents  $\kappa$ , which means larger initial data.

**Remark.** As we mentioned in the Introduction, after reading our paper, Baldi and Haus [6] found an alternative proof of Theorem 6 based on their version [5] of the classical Newton scheme in the spirit of Hörmander. A key point in their proof is a clever modification of the norms considered by Texier-Zumbrun, allowing better  $C^2$  estimates on the functional. They also explain that their approach can be extended to other  $C^2$  functionals consisting of a linear term perturbed by a nonlinear term of homogeneity at least  $p + 1$ . Our abstract theorem, however, seems more general since we do not need such a structure.

**4.2. Proof of Theorem 6.** We have to show that Corollary 4 applies. Our functional setting is the same as in [26], with similar notations.

We introduce the norm  $\|f\|_{H_\varepsilon^s(\mathbf{R}^d)} = \|(-\varepsilon^2\Delta + 1)^{s/2}f\|_{L^2(\mathbf{R}^d)}$ , and we take

$$X_s = \mathcal{C}^1([0, T], H^{s-2}(\mathbf{R}^d)) \cap \mathcal{C}^0([0, T], H^s(\mathbf{R}^d)),$$

$$\|u\|_s = \sup_{0 \leq t \leq T} \left\{ \|\varepsilon^2 \partial_t u(t, \cdot)\|_{H_\varepsilon^{s-2}(\mathbf{R}^d)} + \|u(t, \cdot)\|_{H_\varepsilon^s(\mathbf{R}^d)} \right\}$$

and

$$Y_s = \mathcal{C}^0([0, T], H^s(\mathbf{R}^d)) \times H^{s+2}(\mathbf{R}^d),$$

$$\|(v_1, v_2)\|'_s = \sup_{0 \leq t \leq T} \left\{ \|v_1(t, \cdot)\|_{H_\varepsilon^s(\mathbf{R}^d)} \right\} + \|v_2\|_{H_\varepsilon^{s+2}(\mathbf{R}^d)}$$

Our projectors are

$$\Pi_\Lambda u = \mathcal{F}_x^{-1}(1_{|\varepsilon\xi| \leq \Lambda} \mathcal{F}_x u(t, \xi)),$$

$$\Pi'_\Lambda(v_1, v_2) = (\mathcal{F}_x^{-1}(1_{|\varepsilon\xi| \leq \Lambda} \mathcal{F}_x v_1(t, \xi)), \mathcal{F}^{-1}(1_{|\varepsilon\xi| \leq \Lambda} \mathcal{F} v_2(\xi)))$$

We take

$$\Phi_\varepsilon(u) = (\varepsilon^2 \partial_t u + iA(\varepsilon \partial_x)u - \varepsilon B(a_f + u, \varepsilon \partial_x)(a_f + u), u(0, \cdot))$$

with

$$a_f(t, x) = \varepsilon^\kappa (\exp(-itA(\partial_x))a_\varepsilon, \exp(itA(\partial_x))\bar{a}_\varepsilon).$$

We have  $\Phi_\varepsilon(0) = (-\varepsilon B(a_f, \varepsilon \partial_x)a_f, 0)$ . A solution  $u_\varepsilon$  of the functional equation  $\Phi_\varepsilon(u) = 0$  provides a solution  $\tilde{u}_\varepsilon = a_f + u_\varepsilon$  on  $[0, T] \times \mathbf{R}^d$  of the Cauchy problem  $\partial_t \tilde{u} + iA(\partial_x)\tilde{u} = B(\tilde{u}, \partial_x)\tilde{u}$ ,  $\tilde{u}(0, x) = \varepsilon^\kappa (a_\varepsilon(x), \bar{a}_\varepsilon(x))$ .

Corollary 4 requires a direct estimate (2.9) and an estimate (2.11) on the inverse  $L_\varepsilon$ .

Taking  $s_0 > d/2 + 2$ ,  $m = 2$ ,  $\gamma = \frac{p}{p-1} \frac{d}{2}$ , the direct estimate (2.9) follows from an estimate numbered (2.10) in Assumption 2.4, Subsection 2.2 of [26] (“Tame direct bounds”). Its proof can be found in Subsection 4.2 page 514 of [26]. In that subsection, another estimate numbered (2.11) involving the second derivative of  $\Phi_\varepsilon(\cdot)$  is also proved, but we do not need such an estimate.

Choosing, in addition,  $\ell = 2$ ,  $\ell' = 0$ ,  $g = 2$ , our inverse estimate (2.11) follows from an estimate numbered (2.16) in Assumption 2.10, Subsection 2.3 of [26] (“Tame inverse bounds”).

To summarize, the assumptions (2.9, 2.10, 2.11) of our Corollary 4 are satisfied for  $s_0 > d/2$ ,  $m = 2$ ,  $\gamma = \frac{p}{p-1} \frac{d}{2}$ ,  $g = 2$ ,  $\ell = 2$ ,  $\ell' = 0$ . The regularity parameter  $S$  can be made as large as needed by taking  $\bar{s}$  large enough.

Moreover, in [26], *Proof of Theorem 4.6*, one finds an estimate which can be written in the form

$$\|\Phi_\varepsilon(0)\|'_{\bar{s}-1} \leq r \varepsilon^{1+\kappa(p+1)+\kappa_a}$$

where  $r$  is small when  $\|a_1\|_{H^{\bar{s}}}$  is small.

So, using our Corollary 4, provided  $\bar{s}$  is large enough, we can solve the equation  $\Phi_\varepsilon(u) = 0$  in  $X_{s_1}$  for any  $\kappa$  such that  $1 + \kappa(p+1) + \kappa_a > \gamma + g$ , which means that

$$\kappa > \frac{1 - \kappa_a}{p+1} + \frac{d}{2} \frac{p}{(p+1)(p-1)}.$$

The uniqueness of this solution comes from the local-in-time uniqueness of solutions to the Cauchy problem. This proves Theorem 6 as a consequence of Corollary 4.

## 5. CONCLUSION

The purpose of this paper has been to introduce a new algorithm into the "hard" inverse function theorem, where both  $DF(u)$  and its right inverse  $L(u)$  lose derivatives, in order to improve its range of validity. To highlight this improvement, we have considered singular perturbation problems with loss of derivatives. We have shown that, on the specific example of a Schrödinger-type system of PDEs arising from nonlinear optics, our method leads to substantial improvements of known results. We believe that our method has the potential of improving the known estimates in many other "hard" inversion problems.

In the statement and proof of our abstract theorem, our main focus has been the existence of  $u$  solving  $F(u) = v$  in the case when  $S$  is large and the regularity of  $v$  is as small as possible. We haven't tried to give an explicit bound on  $S$ , but with some additional work, it can be done. In an earlier version [13] of this paper, the reader will find a study of the intermediate case of a tame Galerkin right-invertible differential  $DF$ , with precise estimates on the parameter  $S$  depending on the loss of regularity of the right-inverse, in the special case  $s_0 = m = 0$  and  $\ell = \ell'$ .

Several natural questions are left open in this paper. One of them is the regularity of the inverse map  $v \rightarrow u$ . Another one is the additional regularity on  $u$  that one can get if  $v$  satisfies the assumptions of our main Theorem, and in addition is in  $W_s$  for some  $s > \delta$ . Another question is the domain of uniqueness of  $u$ .

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