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# EXTREMES FOR MULTIVARIATE EXPECTILES

VÉRONIQUE MAUME-DESCHAMPS, DIDIER RULLIÈRE, AND KHALIL SAID

ABSTRACT. Multivariate expectiles, a new family of vector-valued risk measures, were recently introduced in the literature [23]. Here we investigate the asymptotic behavior of these measures in a multivariate regular variation context. For models with equivalent tails, we propose an estimator of extreme multivariate expectiles in the Fréchet domain of attraction case with asymptotic independence, or for comonotonic marginal distributions.

## INTRODUCTION

In recent years, expectiles have become an important and frequently-used risk measure because they satisfy both coherence and elicibility properties. In the one-dimensional setting, expectiles were introduced by Newey & Powell (1987) [18]. For a random variable  $X$  with a finite second moment, the expectile of level  $\alpha$  is defined by:

$$e_\alpha(X) = \arg \min_{x \in \mathbb{R}} \mathbb{E}[\alpha(X - x)_+^2 + (1 - \alpha)(x - X)_+^2],$$

where  $(x)_+ = \max(x, 0)$ . Expectiles are the only risk measure satisfying both elicibility and coherence properties—see Bellini & Bigozzi (2015) [4].

In higher dimensions, several extensions of expectile measures have been recently proposed. A direct extension using multivariate versions of the elicibility property were introduced in Maume-Deschamps et al. (2017) [23]. Multivariate geometric expectiles were presented in Herrmann et al. (2018) [15], constructed as an extension of the multivariate geometric quantiles introduced in Chaudhuri (1996) [8]. Geometric expectiles are also elicitable by construction, and generalize univariate expectiles.

In this paper we focus on *matrix expectiles*, introduced in [23]. Consider a random vector  $\mathbf{X} = (X_1, \dots, X_d)^T \in \mathbb{R}^d$  whose second order moments exist, and let  $\Sigma = (\pi_{ij})_{1 \leq i, j \leq d}$  be a  $d \times d$  real, symmetric, and positive semi-definite matrix such that  $i \in \{1, \dots, d\}$ ,  $\pi_{ii} = \pi_i > 0$ . A  $\Sigma$ -expectile of  $\mathbf{X}$  is then defined as:

$$\mathbf{e}_\alpha^\Sigma(\mathbf{X}) \in \arg \min_{\mathbf{x} \in \mathbb{R}^d} \mathbb{E}[\alpha(\mathbf{X} - \mathbf{x})_+^T \Sigma (\mathbf{X} - \mathbf{x})_+ + (1 - \alpha)(\mathbf{X} - \mathbf{x})_-^T \Sigma (\mathbf{X} - \mathbf{x})_-],$$

where  $(\mathbf{x})_+ = ((x_1)_+, \dots, (x_d)_+)^T$  and  $(\mathbf{x})_- = (-\mathbf{x})_+$ . We shall concentrate on the case where the above minimization has a unique solution. In [23], conditions on  $\Sigma$  ensuring the uniqueness of the argmin are given; indeed, it is sufficient that  $\pi_{ij} \geq 0$ ,  $\forall i, j \in \{1, \dots, d\}$ . We shall make this assumption throughout the paper. In which case the vector expectile is unique and the solution to the following system of equations:

$$(0.1) \quad \alpha \sum_{i=1}^d \pi_{ki} \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}] = (1 - \alpha) \sum_{i=1}^d \pi_{ki} \mathbb{E}[(x_i - X_i)_+ \mathbb{1}_{\{x_k > X_k\}}], \quad \forall k \in \{1, \dots, d\}.$$

In the case where  $\pi_{ij} = 1$  for all  $(i, j) \in \{1, \dots, d\}^2$ , the corresponding  $\Sigma$ -expectile is called a  $L_1$ -expectile. This coincides with the  $L_1$ -norm expectile defined in [23].

In [23] it is proved that,

$$\lim_{\alpha \rightarrow 1} \mathbf{e}_\alpha^\Sigma(\mathbf{X}) = \mathbf{X}_F, \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \mathbf{e}_\alpha^\Sigma(\mathbf{X}) = \mathbf{X}_I,$$

where  $\mathbf{X}_F \in (\mathbb{R} \cup \{+\infty\})^d$  is the right endpoint vector  $(x_F^1, \dots, x_F^d)^T$ , and  $\mathbf{X}_I \in (\mathbb{R} \cup \{-\infty\})^d$  the left endpoint vector  $(x_I^1, \dots, x_I^d)^T$  of the support of the random vector  $\mathbf{X}$ .

Multivariate expectiles can be estimated in the general setting using stochastic optimization algorithms. The estimation example using the Robbins-Monro algorithm (1951) [28] presented in [23] shows that for extreme levels (i.e.,  $\alpha \rightarrow 1$  or  $\alpha \rightarrow 0$ ), the estimation obtained is not satisfactory in term of speed of convergence. This has

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lead us to a theoretical analysis of the asymptotic behavior of multivariate expectiles. Since—to give a real-world example—solvency thresholds in insurance are generally high (e.g.,  $\alpha = 0.995$  for the Solvency II directive), the study of the asymptotic behavior of risk measures is of natural importance. The goal of the present paper is therefore to establish the asymptotic behaviour of multivariate expectiles.

The study of the extreme behaviour of risk measures in a multivariate regular variation framework is the subject of a number of works. Let us mention as examples Embrechts et al. (2009) [13], Albrecher et al. (2006) [1] in risk aggregation contexts, and Asimit et al. (2011) [2] in risk capital allocation. Similar works has also been done on other multivariate risk measures, as for example in Girard & Stupfler (2015) [14] for geometric quantiles, and recently in Di Bernardino & Prieur [11] for the multivariate conditional-tail-expectation.

Here we work with the *equivalent tails model*. This is often used to model claim amounts in insurance and study dependent extreme events, and is also used in ruin theory models. The equivalent tails model includes in particular portfolios with identically distributed risks, and the case where there is a scale difference in distributions. Here we will study the asymptotic behavior of multivariate expectiles in the multivariate regular variations framework. We shall focus on marginal distributions belonging to the Fréchet domain of attraction, which contains heavy-tailed distributions that represent—to give an example—the most potentially expensive claims in insurance. We remark that the attention given to univariate expectiles is a quite recent phenomenon. In [5], for regularly varying random variables, asymptotic equivalents of expectiles as a function of the quantile of the same level are proved. First and second order asymptotics for the expectile of the sum in the Farlie–Gumbel–Morgenstern (FGM) dependence structure case are given in [22].

The paper is set out as follows. The first section is devoted to a presentation of the multivariate regularly varying distribution framework. The study of the asymptotic behavior of the multivariate expectiles for the Fréchet model with equivalent tails is the subject of Section 2. The case with an asymptotically dominant tail is analyzed in Section 3. Section 4 is devoted to the estimation of extreme multivariate expectiles in the asymptotic independence and comonotonicity settings, and simulation results are also provided.

## 1. THE MRV FRAMEWORK

Regularly varying distributions are well suited to study extreme phenomena. Lots of works have been devoted to the asymptotic behavior of usual risk measures for this class of distributions, and results are given for sums of risks belonging to this family. It is well known that the three domains of attraction of extreme value distributions can be defined using the concept of regular variations (see [12; 26; 9; 6]).

This section is devoted to the classical characterization of multivariate regular variations, which will be used in the study of the asymptotic behavior of multivariate expectiles. We also recall some basic results on the univariate setting that we shall use.

**1.1. Univariate regular variations.** We begin by recalling basic definitions and results on univariate regular variations.

**Definition 1.1** (Regularly varying functions). A measurable positive function  $f$  is *regularly varying of index  $\rho$*  at  $a \in \{0, +\infty\}$ , if for all  $t > 0$ ,

$$\lim_{x \rightarrow a} \frac{f(tx)}{f(x)} = t^\rho,$$

we denote  $f \in \text{RV}_\rho(a)$ .

A slowly varying function is a regularly varying function of index  $\rho = 0$ . Remark that  $f \in \text{RV}_\rho(+\infty)$  if and only if, there exists a slowly varying function at infinity,  $L \in \text{RV}_0(+\infty)$  such that

$$f(x) = x^\rho L(x).$$

**Theorem 1.2** (Karamata’s representation, [27]). *For any slowly varying function  $L$  at  $+\infty$ , there exists a positive measurable function  $c(\cdot)$  that satisfies  $\lim_{x \rightarrow +\infty} c(x) = c \in ]0, +\infty[$ , and a measurable function  $\varepsilon(\cdot)$  with  $\lim_{x \rightarrow +\infty} \varepsilon(x) = 0$ , such that*

$$L(x) = c(x) \exp \left( \int_1^x \frac{\varepsilon(t)}{t} dt \right).$$

**Lemma 1.3** (Integration of RV functions (Karamata's Theorem)), [25]. *For a positive measurable function  $f$ , regularly varying of index  $\rho$  at  $+\infty$ , locally bounded on  $[x_0, +\infty)$  with  $x_0 \geq 0$*

- if  $\rho > -1$ , then

$$\lim_{x \rightarrow +\infty} \frac{\int_{x_0}^x f(t) dt}{xf(x)} = \frac{1}{\rho + 1},$$

- if  $\rho < -1$ , then

$$\lim_{x \rightarrow +\infty} \frac{\int_x^{+\infty} f(t) dt}{xf(x)} = -\frac{1}{\rho + 1}.$$

**Lemma 1.4** (Potter's bounds [6]). *For  $f \in RV_\rho(a)$ , with  $a \in \{0, \infty\}$  and  $\rho \in \mathbb{R}$ . For any  $0 < \epsilon < 1$  and all  $x$  and  $y$  sufficiently close to  $a$ , we have*

$$(1 - \epsilon) \min \left( \left( \frac{x}{y} \right)^{\rho - \epsilon}, \left( \frac{x}{y} \right)^{\rho + \epsilon} \right) \leq \frac{f(x)}{f(y)} \leq (1 + \epsilon) \max \left( \left( \frac{x}{y} \right)^{\rho - \epsilon}, \left( \frac{x}{y} \right)^{\rho + \epsilon} \right).$$

Many other properties of regularly varying functions are presented in [6]. Throughout the paper, we shall consider generalized inverses of non-decreasing functions  $f$ :  $f^{\leftarrow}(y) = \inf\{x \in \mathbb{R}, f(x) \geq y\}$ .

**1.2. Multivariate regular variations.** The multivariate extension of regular variations is introduced in [10]. We denote by  $\mu_n \xrightarrow{v} \mu$  the vague convergence of Radon measures as presented in [19]. The following definitions are given for non negative random variables.

**Definition 1.5** (Multivariate regular variations). The distribution of a random vector  $\mathbf{X}$  on  $[0, \infty]^d$  is said to be regularly varying if there exist a non-null Radon measure  $\mu_{\mathbf{X}}$  on the Borel  $\sigma$ -algebra  $\mathcal{B}_d$  on  $[0, \infty]^d \setminus \mathbf{0}$ , and a normalization function  $b : \mathbb{R} \rightarrow \mathbb{R}$  which satisfies  $\lim_{x \rightarrow +\infty} b(x) = +\infty$ , and such that

$$(1.1) \quad u\mathbb{P} \left( \frac{\mathbf{X}}{b(u)} \in \cdot \right) \xrightarrow{v} \mu_{\mathbf{X}}(\cdot) \text{ as } u \rightarrow +\infty.$$

There exist several equivalent definitions of multivariate regular variations which will be useful in what follows.

**Definition 1.6** (MRV equivalent definitions). Let  $\mathbf{X}$  be a random vector on  $\mathbb{R}^d$ , the following definitions are equivalent:

- The vector  $\mathbf{X}$  has a regularly varying tail of index  $\theta$ .
- There exist a finite measure  $\mu$  on the unit sphere  $\mathbb{S}^{d-1}$ , and a normalization function  $b : (0, \infty) \rightarrow (0, \infty)$  such that

$$(1.2) \quad \lim_{t \rightarrow +\infty} \mathbb{P} \left( \|\mathbf{X}\| > xb(t), \frac{\mathbf{X}}{\|\mathbf{X}\|} \in \cdot \right) = x^{-\theta} \mu(\cdot),$$

for all  $x > 0$ . The measure  $\mu$  depends on the chosen norm, it is called the *spectral measure* of  $\mathbf{X}$ .

- There exist a finite measure  $\mu$  on the unit sphere  $\mathbb{S}^{d-1}$ , a slowly varying function  $L$ , and a positive real  $\theta > 0$  such that

$$(1.3) \quad \lim_{x \rightarrow +\infty} \frac{x^\theta}{L(x)} \mathbb{P} \left( \|\mathbf{X}\| > x, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in B \right) = \mu(B),$$

for all  $B \in \mathcal{B}(\mathbb{S}^{d-1})$  with  $\mu(\partial B) = 0$ .

From now on, MRV denotes the set of multivariate regularly varying distributions, and  $\text{MRV}(\theta, \mu)$  denotes the set of random vectors with regularly varying tail, with index  $\theta$  and spectral measure  $\mu$ .

From (1.3), we may assume that  $\mu$  is normalized i.e.  $\mu(\mathbb{S}^{d-1}) = 1$ , which implies that  $\|\mathbf{X}\|$  has a regularly varying tail of index  $-\theta$ .

On the other hand,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \mathbb{P} \left( \frac{\mathbf{X}}{\|\mathbf{X}\|} \in B \mid \|\mathbf{X}\| > x \right) &= \lim_{x \rightarrow +\infty} \frac{\mathbb{P} \left( \|\mathbf{X}\| > x, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in B \right)}{\mathbb{P}(\|\mathbf{X}\| > x)} \\ &= \lim_{x \rightarrow +\infty} \frac{x^\theta}{L(x)} \mu(B) x^{-\theta} L(x) = \mu(B), \end{aligned}$$

for all  $B \in \mathcal{B}(\mathbb{S}^{d-1})$  with  $\mu(\partial B) = 0$ . That means that conditionally to  $\{\|\mathbf{X}\| > x\}$ ,  $\frac{\mathbf{X}}{\|\mathbf{X}\|}$  converges weakly to  $\mu$ .

The different possible characterizations of the MRV concept are presented in [25].

**1.3. Characterization using tail dependence functions.** Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector. From now on,  $\bar{F}_{X_i}$  denotes the survival function of  $X_i$ . In this paper, we use the definition of the upper tail dependence function, as introduced in [20].

**Definition 1.7** (The tail dependence function). Let  $\mathbf{X}$  be a random vector on  $\mathbb{R}^d$ , with continuous marginal distributions. The tail dependence function is defined by

$$(1.4) \quad \lambda_{\bar{U}}^{\mathbf{X}}(x_1, \dots, x_d) = \lim_{t \rightarrow 0^+} t^{-1} \mathbb{P}(\bar{F}_{X_1}(X_1) \leq tx_1, \dots, \bar{F}_{X_d}(X_d) \leq tx_d),$$

when the limit exists.

For  $k \leq d$ , denote by  $\mathbf{X}^{(k)}$  a  $k$  dimensional sub-vector of  $\mathbf{X}$ ,  $C^{(k)}$  its copula and  $\bar{C}^{(k)}$  its survival copula. The upper tail dependence function is

$$(1.5) \quad \lambda_{\bar{U}}^{\mathbf{X}^{(k)}}(u_1, \dots, u_k) = \lim_{t \rightarrow 0^+} \frac{\bar{C}^{(k)}(tu_1, \dots, tu_k)}{t},$$

if this limit exists. The lower tail dependence function can be defined analogically by

$$\lambda_L^{\mathbf{X}^{(k)}}(u_1, \dots, u_k) = \lim_{t \rightarrow 0^+} \frac{C^{(k)}(tu_1, \dots, tu_k)}{t},$$

when the limit exists. In this paper, our study is limited to the upper version as defined in (1.5).

We now consider the following assumption:

**Assumption 1** (Equivalent tails). *We assume that  $\mathbf{X}$  has equivalent regularly varying marginal tails, which means that both (i) and (ii) below are satisfied*

- (i)  $\bar{F}_{X_1} \in RV_{-\theta}(+\infty)$ , with  $\theta > 0$ .
- (ii) *The tails of  $X_i, i = 1, \dots, d$  are equivalent. That is for all  $i \in \{2, \dots, d\}$ , there is a positive constant  $c_i$  such that*

$$\lim_{x \rightarrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = c_i.$$

Conditions (i) and (ii) in Assumption 1 imply that all marginal tails are regularly varying of index  $-\theta$  at  $+\infty$ .

The following two theorems show that, under Assumption 1, the MRV character of multivariate distributions is equivalent to the existence of the tail dependence functions.

**Theorem 1.8** (Theorem 2.3 in [21]). *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector in  $\mathbb{R}^d$ , with continuous marginal distributions  $F_{X_i}, i = 1, \dots, d$  that satisfy Assumption 1. If  $\mathbf{X}$  has a MRV distribution, the tail dependence function exists, and it is given by*

$$\lambda_{\bar{U}}^{\mathbf{X}^{(k)}}(u_1, \dots, u_k) = \lim_{x \rightarrow +\infty} x \mathbb{P} \left( X_1 > b(x) \left( \frac{u_1}{c_1} \right)^{-1/\theta}, \dots, X_k > b(x) \left( \frac{u_d}{c_d} \right)^{-1/\theta} \right),$$

for any  $k \in \{1, \dots, d\}$ .

**Theorem 1.9** (Theorem 3.2 in [30]). *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector in  $\mathbb{R}^d$ , with continuous marginal distributions  $F_{X_i}, i = 1, \dots, d$  that satisfies Assumption 1. If the tail dependence function  $\lambda_{\bar{U}}^{\mathbf{X}^{(k)}}$  exists for all  $k \in \{1, \dots, d\}$ , then  $\mathbf{X}$  is MRV, its normalization function is given by  $b(u) = \left( \frac{1}{\bar{F}_{X_1}} \right)^{\leftarrow}(u)$  and the spectral measure is*

$$\mu([\mathbf{0}, \mathbf{x}]^c) = \sum_{i=1}^d c_i x_i^{-\theta} - \sum_{1 \leq i < j \leq d} \lambda_{\bar{U}}^{(X_i, X_j)}(c_i x_i^{-\theta}, c_j x_j^{-\theta}) + \dots + (-1)^{d+1} \lambda_{\bar{U}}^{\mathbf{X}}(c_1 x_1^{-\theta}, \dots, c_d x_d^{-\theta}).$$

By construction of the multivariate expectiles, only the bivariate dependence structures are taken into account. We shall use the functions  $\lambda_{\bar{U}}^{(X_i, X_k)}$ , for all  $(i, k) \in \{1, \dots, d\}^2$ . In order to simplify the notation, we denote it by  $\lambda_{\bar{U}}^{ik}$ . If the vector  $\mathbf{X}$  has an MRV distribution, the pairs  $(X_i, X_j)$  have also MRV distributions, for any  $(i, j) \in \{1, \dots, d\}^2$ . So, in the MRV framework, and under Assumption 1, the existence of functions  $\lambda^{ik}$  is insured. In addition, we assume in all the rest of this paper that these functions are continuous.

## 2. FRÉCHET MODEL WITH EQUIVALENT TAILS

In this section, we assume that  $\mathbf{X}$  satisfies Assumption 1 with  $\theta > 1$ . It implies that  $X_1$  belongs to the Fréchet extreme value domain of attraction. This domain contains distributions with infinite endpoint  $x_F = \sup\{x : F(x) < 1\} = +\infty$ , so as  $\alpha \rightarrow 1$  we get  $\mathbf{e}_\alpha^i(\mathbf{X}) \rightarrow +\infty \forall i$ . Also, from Karamata's Theorem (Theorem 1.3), we have for  $i = 1, \dots, d$ ,

$$(2.1) \quad \lim_{x \rightarrow +\infty} \frac{\mathbb{E}[(X_i - x)_+]}{x \bar{F}_{X_i}(x)} = \frac{1}{\theta - 1},$$

for all  $i \in \{1, \dots, d\}$ .

**Proposition 2.1** (Equivalent expectiles components). *Let  $\Sigma = (\pi_{ij})_{i,j=1,\dots,d}$  with  $\pi_{ij} > 0$  for all  $i, j \in \{1, \dots, d\}$ . Under Assumption 1, the components of the multivariate  $\Sigma$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1,\dots,d}$  satisfy*

(i)

$$0 < \lim_{\alpha \rightarrow 1} \frac{\mathbf{e}_\alpha^i(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \leq \overline{\lim}_{\alpha \rightarrow 1} \frac{\mathbf{e}_\alpha^i(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} < +\infty, \forall i \in \{2, \dots, d\}.$$

(ii)

$$0 < \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} \leq \overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} < +\infty, \forall i \in \{2, \dots, d\}.$$

Proposition 2.1 (i) implies that distributions with equivalent tails have asymptotically comparable multivariate expectile components.

Before we prove Proposition 2.1, we shall demonstrate some preliminary results. Firstly, let  $\mathbf{X} = (X_1, \dots, X_d)^T$  satisfy Assumption 1, we denote  $x_i = \mathbf{e}_\alpha^i(\mathbf{X})$  for all  $i \in \{1, \dots, d\}$ . We define the functions  $l_{X_i, X_j}^\alpha$  for all  $(i, j) \in \{1, \dots, d\}^2$  by

$$(2.2) \quad l_{X_i, X_j}^\alpha(x_i, x_j) = \alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_j > x_j\}}] - (1 - \alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_j < x_j\}}],$$

and  $l_{X_i}^\alpha(x_i) = l_{X_i, X_i}^\alpha(x_i, x_i)$ .

The optimality system (0.1) rewrites

$$(2.3) \quad l_{X_k}^\alpha(x_k) = - \sum_{i=1, i \neq k}^d \frac{\pi_{ki}}{\pi_{kk}} l_{X_i, X_k}^\alpha(x_i, x_k) \quad \forall k \in \{1, \dots, d\}.$$

We shall use the following sets:

$$\begin{aligned} J_0^i &= \{j \in \{1, \dots, d\} \setminus \{i\} \mid \lim_{\alpha \rightarrow 1} \frac{x_j}{x_i} = 0\}, \\ J_C^i &= \{j \in \{1, \dots, d\} \setminus \{i\} \mid 0 < \lim_{\alpha \rightarrow 1} \frac{x_j}{x_i} < \overline{\lim}_{\alpha \rightarrow 1} \frac{x_j}{x_i} < +\infty\}, \\ \text{and } J_\infty^i &= \{j \in \{1, \dots, d\} \setminus \{i\} \mid \overline{\lim}_{\alpha \rightarrow 1} \frac{x_j}{x_i} = +\infty\}. \end{aligned}$$

The proof of Proposition 2.1 is written for  $\pi_{ij} = 1$ , for all  $(i, j) \in \{1, \dots, d\}^2$ , i.e. for the  $L_1$ -expectiles. The general case can be treated in the same way, provided that  $\pi_{ij} > 0$  for all  $(i, j) \in \{1, \dots, d\}^2$ . The proof of Proposition 2.1 (i) follows from Lemma 2.2 and Proposition 2.1 (ii).

**Lemma 2.2.** *Assume that Assumption 1 is satisfied.*

(1) *If  $t = o(s)$  then for all  $(i, j) \in \{1, \dots, d\}^2$ ,*

$$\lim_{t \rightarrow +\infty} \frac{s \bar{F}_{X_i}(s)}{t \bar{F}_{X_j}(t)} = 0.$$

(2) *If  $t = \Theta(s)$ ,<sup>1</sup> then for all  $(i, j) \in \{1, \dots, d\}^2$ ,*

$$\frac{\bar{F}_{X_i}(s)}{\bar{F}_{X_j}(t)} \sim \frac{c_i}{c_j} \left(\frac{s}{t}\right)^{-\theta} \quad \text{as } t \rightarrow \infty.$$

The proof is given in Appendix A.1. The proof of Proposition 2.1 (ii) is given in Appendix A.2. We may now prove Proposition 2.1 (i).

<sup>1</sup>Recall that  $t = \Theta(s)$  means that there exist positive constants  $C_1$  and  $C_2$  such that  $C_1 s \leq t \leq C_2 s$

*Proof of Proposition 2.1 (i).* We shall prove that  $J_\infty^1 = \emptyset$ , the fact that  $J_\infty^k = \emptyset$  for all  $k \in \{1, \dots, d\}$  may be proven in the same way. This implies that  $J_0^k = J_\infty^k = \emptyset$  for all  $k \in \{1, \dots, d\}$ , hence the result.

We suppose that  $J_\infty^1 \neq \emptyset$ , let  $i \in J_\infty^1$ , taking if necessary a subsequence, we may assume that  $x_i/x_1 \rightarrow +\infty$  as  $\alpha \rightarrow 1$ .

From Proposition 2.1 (ii), we have

$$0 < \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} \leq \overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} < +\infty, \forall i \in \{2, \dots, d\},$$

so, taking if necessary a subsequence, we may assume that  $\exists \ell \in \mathbb{R}^* \setminus \{+\infty\}$  such that

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} = \ell.$$

In this case,

$$\lim_{\alpha \rightarrow 1} \frac{l_{X_1}^\alpha(x_1)}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \rightarrow 1} \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_1(x_1)} \left(1 - \frac{\mathbb{E}[X_1]}{x_1}\right) \right) = \frac{1}{\theta - 1} - \ell < +\infty.$$

Moreover,

$$\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \leq \frac{\mathbb{E}[(X_i - x_i)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)} \rightarrow 0 \text{ using Lemma 2.2.}$$

We get

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \frac{l_{X_i, X_1}^\alpha(x_i, x_1)}{x_1 \bar{F}_{X_1}(x_1)} &= \lim_{\alpha \rightarrow 1} \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}] - (1 - \alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \\ &= \lim_{\alpha \rightarrow 1} \left( \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \frac{x_i}{x_1} \right) = -\infty, \forall i \in J_\infty^1. \end{aligned}$$

Going through the limit ( $\alpha \rightarrow 1$ ) in the first equation of the optimality System (2.3) divided by  $x_1 \bar{F}_{X_1}(x_1)$ , leads to

$$(2.4) \quad \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} = -\infty.$$

Now, let  $k \in J_0^1$

$$\begin{aligned} \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \frac{\int_{x_k}^{x_1} \mathbb{P}(X_k > t, X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t, X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} \\ &\leq \frac{\int_{x_k}^{x_1} \mathbb{P}(X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t) dt}{x_1 \bar{F}_{X_1}(x_1)}, \end{aligned}$$

Karamata's Theorem (Theorem 1.3) leads to

$$\lim_{\alpha \rightarrow 1} \frac{\int_{x_k}^{x_1} \mathbb{P}(X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t) dt}{x_1 \bar{F}_{X_1}(x_1)} = 1 + \frac{c_k}{\theta - 1}, \forall k \in J_0^1.$$

Consider  $k \in J_C^1$

$$\frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \leq \frac{\mathbb{E}[(X_k - x_k)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} \frac{x_k \bar{F}_{X_k}(x_k)}{x_1 \bar{F}_{X_1}(x_1)},$$

and

$$\frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} \frac{x_k \bar{F}_{X_k}(x_k)}{x_1 \bar{F}_{X_1}(x_1)} \underset{6}{\underset{\alpha \rightarrow 1}{\rightsquigarrow}} \frac{c_k}{\theta - 1} \left( \frac{x_k}{x_1} \right)^{-\theta+1}.$$

Finally, we deduce that

$$\begin{aligned}
- \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \overline{\lim}_{\alpha \rightarrow 1} \frac{x_k}{x_1} &\leq \overline{\lim}_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} \\
&\leq \overline{\lim}_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} \\
&\leq \sum_{k \in J_C^1} \left( \frac{c_k}{\theta - 1} \left( \overline{\lim}_{\alpha \rightarrow 1} \frac{x_k}{x_1} \right)^{-\theta+1} - \ell \overline{\lim}_{\alpha \rightarrow 1} \frac{x_k}{x_1} \right) + \sum_{k \in J_0^1 \setminus J_\infty^1} \left( 1 + \frac{c_k}{\theta - 1} \right).
\end{aligned}$$

This is contradictory with (2.4), and consequently  $J_\infty^1$  is necessarily an empty set. The result follows.  $\square$

**Proposition 2.3** (Extreme multivariate expectile). *Assume that Assumption 1 is satisfied and  $\mathbf{X}$  has a regularly varying multivariate distribution in the sense of Definition 1.5. Consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (e_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . Then any limit vector  $(\eta, \beta_2, \dots, \beta_d)$  of  $\left( \frac{1-\alpha}{\bar{F}_{X_1}(e_\alpha^1(\mathbf{X}))}, \frac{e_\alpha^2(\mathbf{X})}{e_\alpha^1(\mathbf{X})}, \dots, \frac{e_\alpha^d(\mathbf{X})}{e_\alpha^1(\mathbf{X})} \right)$  satisfies the following equation system*

$$(2.5) \quad \frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \left( \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right), \forall k \in \{1, \dots, d\}.$$

By solving the system (2.5), we may obtain an equivalent of the extreme multivariate expectile, using the marginal quantiles: we will see below that this relation allows to get the asymptotic behavior of  $\bar{F}_{X_k}(e_\alpha^k(\mathbf{X}))$ , which gives a link between quantiles and expectiles.

*Proof.* The optimality system (2.3) can be written in the following form

$$\begin{aligned}
(2\alpha - 1) \frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} - \frac{1 - \alpha}{\bar{F}_{X_k}(x_k)} \left( 1 - \frac{\mathbb{E}[X_k]}{x_k} \right) &= \sum_{i=1, i \neq k}^d \left( (1 - \alpha) \frac{\mathbb{E}[(X_i - x_i) - \mathbb{1}_{\{X_k < x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} \right) \\
&- \sum_{i=1, i \neq k}^d \alpha \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)}, \quad \forall k \in \{1, \dots, d\}.
\end{aligned}$$

For all  $k \in \{1, \dots, d\}$ , we have (taking if necessary a subsequence)

$$\lim_{\alpha \rightarrow 1} (2\alpha - 1) \frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} - \frac{1 - \alpha}{\bar{F}_{X_k}(x_k)} \left( 1 - \frac{\mathbb{E}[X_k]}{x_k} \right) = \frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k},$$

and for all  $i \in \{1, \dots, d\} \setminus \{k\}$

$$\begin{aligned}
\lim_{\alpha \rightarrow 1} (1 - \alpha) \frac{\mathbb{E}[(X_i - x_i) - \mathbb{1}_{\{X_k < x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} &= \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_k}(x_k)} \left( \frac{x_i}{x_k} \mathbb{P}(X_i < x_i, X_k < x_k) - \frac{\mathbb{E}[X_i \mathbb{1}_{\{X_i < x_i, X_k < x_k\}}]}{x_k} \right) \\
&= \eta \frac{\beta_k^\theta}{c_k} \frac{\beta_i}{\beta_k} = \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} &= \frac{1}{x_k \bar{F}_{X_k}(x_k)} \int_{x_i}^{+\infty} \mathbb{P}(X_i > t, X_k > x_k) dt \\
&= \int_{\frac{x_i}{x_k}}^{+\infty} \frac{\mathbb{P}(X_i > tx_k, X_k > x_k)}{\bar{F}_{X_k}(x_k)} dt \\
&= \int_{\frac{x_i}{x_k}}^{+\infty} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} dt.
\end{aligned}$$

Firstly, we remark that

$$\left| \int_{\frac{\beta_i}{\beta_k}}^{\frac{x_i}{x_k}} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} dt \right| \leq \left| \frac{x_i}{x_k} - \frac{\beta_i}{\beta_k} \right|.$$



Since the functions  $\lambda_U^{ik}$  are assumed to be continuous,

$$(2.6) \quad \lim_{\alpha \rightarrow 1} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} = \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right).$$

In order to show that

$$\lim_{\alpha \rightarrow 1} \alpha \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} = \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt,$$

we may use the Lebesgue's Dominated Convergence Theorem with Potter's bounds (1942) (Lemma 1.4) for regularly varying functions.

First of all,

$$\frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} \leq \min \left\{ 1, \frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(x_k)} \right\},$$

since  $\frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(x_k)} = \frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(tx_k)} \frac{\bar{F}_{X_k}(tx_k)}{\bar{F}_{X_k}(x_k)}$  and  $\lim_{\alpha \rightarrow 1} \frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(tx_k)} = \frac{c_i}{c_k}$ , using Potter's bounds, for all  $\varepsilon_1 > 0$  and  $0 < \varepsilon_2 < \theta - 1$ , there exists  $x_k^0(\varepsilon_2, \varepsilon_1)$  such that for  $\min\{x_k, tx_k\} \geq x_k^0(\varepsilon_2, \varepsilon_1)$

$$\frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(x_k)} \leq \left( \frac{c_i}{c_k} + 2\varepsilon_1 \right) t^{-\theta} \max(t^{\varepsilon_2}, t^{-\varepsilon_2}).$$

Lebesgue's theorem gives

$$\lim_{\alpha \rightarrow 1} \int_{\frac{x_i}{x_k}}^{+\infty} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} dt = \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt,$$

so, for all  $(i \neq k) \in \{1, \dots, d\}^2$

$$\lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} = \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt.$$

Hence the system announced in this proposition.  $\square$

In the general case of  $\Sigma$ -expectiles, with  $\Sigma = (\pi_{ij})_{i,j=1,\dots,d}$ ,  $\pi_{ij} \geq 0$ ,  $\pi_{ii} = \pi_i > 0$ , System (2.5) becomes

$$\frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \frac{\pi_{ik}}{\pi_k} \left( \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right), \forall k \in \{1, \dots, d\}.$$

Moreover, let us remark that System (2.5) is equivalent to the following system

$$(2.7) \quad \sum_{i=1}^d \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} (c_i t^{-\theta}, c_k \beta_k^{-\theta}) dt = \sum_{i=1}^d \int_{\beta_i}^{+\infty} \lambda_U^{i1} (c_i t^{-\theta}, 1) dt, \forall k \in \{2, \dots, d\}.$$

The limit points  $\beta_i$  are thus completely determined by the asymptotic bivariate dependencies between the marginal components of the vector  $\mathbf{X}$ .

Furthermore, using a substitution in System (2.5) and the positive homogeneity property of the bivariate tail dependence functions  $\lambda_U^{ik}$  (see Proposition 2.2 in [17]), we obtain the following system

$$(2.8) \quad \frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \left( \frac{c_i}{c_k} \left( \frac{\beta_i}{\beta_k} \right)^{-\theta+1} \int_1^{+\infty} \lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right), \forall k \in \{1, \dots, d\}.$$

The main utility of writing the asymptotic optimality system in the form (2.8) is the possibility to give an explicit form to  $(\eta, \beta_2, \dots, \beta_d)$  for some dependence structures. In order to illustrate that point, consider that the dependence structure of  $\mathbf{X}$  is given by an Archimedean copula with generator  $\psi$ . The survival copula is given by

$$\bar{C}(x_1, \dots, x_d) = \psi(\psi^{\leftarrow}(x_1) + \dots + \psi^{\leftarrow}(x_d)),$$

where  $\psi^\leftarrow(x) = \inf\{t \geq 0 \mid \psi(t) \leq x\}$  (see e.g. [24] for more details). Assume that,  $\psi$  is a regularly varying function at  $+\infty$ , with negative index:  $\psi \in \text{RV}_{-\theta_\psi}(+\infty)$ . According to [7] (Theorem 3.1), the right tail dependence functions exist, and one can get their explicit forms

$$\lambda_U^{\mathbf{X}^{(k)}}(x_1, \dots, x_k) = \left( \sum_{i=1}^k x_i^{-\frac{1}{\theta_\psi}} \right)^{-\theta_\psi}.$$

Thus, the bivariate upper tail dependence functions are given by

$$\lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) = \left( t^{\frac{\theta}{\theta_\psi}} + \left( \frac{c_i}{c_k} \right)^{\frac{1}{\theta_\psi}} \left( \frac{\beta_k}{\beta_i} \right)^{\frac{\theta}{\theta_\psi}} \right)^{-\theta_\psi}.$$

In particular, if  $\theta = \theta_\psi$ , we have

$$\int_1^{+\infty} \lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt = \frac{1}{\theta - 1} \left( 1 + \left( \frac{c_i}{c_k} \right)^{\frac{1}{\theta}} \frac{\beta_k}{\beta_i} \right)^{-\theta+1},$$

and System (2.8) becomes

$$\frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \left( \frac{1}{\theta - 1} \frac{c_i}{c_k} \left( \frac{\beta_i}{\beta_k} + \left( \frac{c_i}{c_k} \right)^{\frac{1}{\theta}} \right)^{-\theta+1} - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right).$$

**Lemma 2.4** (The comonotonic Fréchet case). *Under Assumption 1, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . If  $\mathbf{X} = (X_1, \dots, X_d)$  is a comonotonic random vector, then the limit*

$$(\eta, \beta_2, \dots, \beta_d) = \lim_{\alpha \rightarrow 1} \left( \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right),$$

satisfies

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_k}(\mathbf{e}_\alpha^k(\mathbf{X}))} = \frac{1}{\theta - 1} \quad \text{and} \quad \beta_k = c_k^{1/\theta}, \quad \forall k \in \{1, \dots, d\}.$$

The proof is given in Appendix A.3

**Lemma 2.5** (Asymptotic independence case). *Under Assumption 1, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . If  $\mathbf{X} = (X_1, \dots, X_d)$  is such that the pairs  $(X_i, X_j)$  are asymptotically independent, then the limit vector  $(\eta, \beta_2, \dots, \beta_d)$  of  $\left( \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$  satisfies*

$$\eta = \frac{1}{(\theta - 1) \left( 1 + \sum_{j=2}^d c_j^{\frac{1}{\theta-1}} \right)} \quad \text{and} \quad \beta_k = c_k^{\frac{1}{\theta-1}},$$

for all  $k \in \{1, \dots, d\}$ .

*Proof.* The hypothesis of asymptotic bivariate independence means:

$$\lim_{\alpha \rightarrow 1} \frac{\mathbb{P}(X_i > x_i, X_j > x_j)}{\mathbb{P}(X_j > x_j)} = \lim_{\alpha \rightarrow 1} \frac{\mathbb{P}(X_i > tx_j, X_j > x_j)}{\mathbb{P}(X_j > x_j)} = 0,$$

for all  $(i, j) \in \{1, \dots, d\}^2$  and for all  $t > 0$ , then, Lebesgue's Theorem used as in Proposition 2.3 gives

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_j > x_j\}}]}{x_j \bar{F}_{X_j}(x_j)} &= \lim_{\alpha \rightarrow 1} \int_{\frac{x_i}{x_j}}^{+\infty} \frac{\mathbb{P}(X_i > tx_j, X_j > x_j)}{\mathbb{P}(X_j > x_j)} dt \\ &= 0. \end{aligned}$$

The extreme multivariate expectile verifies the following equation system

$$\frac{1}{\theta - 1} - \frac{\eta}{c_k} \beta_k^\theta = + \sum_{i=1, i \neq k}^d \frac{\eta}{c_k} \beta_k^{\theta-1} \beta_i, \quad \forall k \in \{1, \dots, d\},$$

which can be rewritten as

$$(2.9) \quad \frac{c_k}{\eta(\theta-1)\beta_k^{\theta-1}} = \sum_{i=1}^d \beta_i, \quad \forall k \in \{1, \dots, d\},$$

hence  $\beta_k = c_k^{\frac{1}{\theta-1}}$  for all  $k \in \{1, \dots, d\}$ , and

$$\eta = \frac{1}{(\theta-1) \left( 1 + \sum_{j=2}^d c_j^{\frac{1}{\theta-1}} \right)}.$$

□

In the general case of a matrix of positive coefficients  $\pi_{ij}$ ,  $i, j \in \{1, \dots, d\}$ , the limits  $\beta_i, i = 2, \dots, d$  remain the same, but the limit  $\eta$  will change:

$$\lim_{\alpha \rightarrow 1} \frac{\mathbf{e}_\alpha^k(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} = c_k^{\frac{1}{\theta-1}} \quad \text{and} \quad \lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_k}(\mathbf{e}_\alpha^k(\mathbf{X}))} = \frac{c_k^{\frac{1}{\theta-1}}}{(\theta-1) \left( 1 + \sum_{j=2}^d \frac{\pi_{jk}}{\pi_k} c_j^{\frac{1}{\theta-1}} \right)},$$

for all  $k \in \{1, \dots, d\}$ .

We remark that

$$\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_i}(x_i)} \leq \frac{c_i^{\frac{1}{\theta-1}}}{\theta-1},$$

which allows a comparison between the marginal quantile and the corresponding component of the multivariate expectile, and since  $F_{X_k}^{-1}(1-\cdot)$  is a regularly varying function at 0 for all  $k \in \{1, \dots, d\}$  with index  $-\frac{1}{\theta}$  (see Theorem 1.5.12 in [6]), we get

$$e_\alpha^k(\mathbf{X}) \underset{\alpha \rightarrow 1}{\sim} \text{VaR}_\alpha(X_k) (\theta-1)^{-\frac{1}{\theta}} \left( \frac{1 + \sum_{i=2}^d c_i^{\frac{1}{\theta-1}}}{c_k^{\frac{1}{\theta-1}}} \right)^{-\frac{1}{\theta}},$$

where  $\text{VaR}_\alpha(X_k)$  denotes the Value at Risk of  $X_k$  at level  $\alpha$ , i.e. the  $\alpha$ -quantile  $F_{X_k}^{-1}(\alpha)$  of  $X_k$ . These conclusions coincide with the results obtained in dimension 1, for distributions that belong to the Fréchet domain of attraction, in [5]. The values of constants  $c_i$  determine the position of the marginal quantile compared to the corresponding component of the multivariate expectile for each risk.

### 3. FRÉCHET MODEL WITH A DOMINANT TAIL

This section is devoted to the case where  $X_1$  has a dominant tail with respect to the  $X_i$ 's.

**Assumption 2** (Dominant tail). *We assume that both (i) and (ii) below are satisfied*

- (i)  $\bar{F}_{X_1} \in RV_{-\theta}(+\infty)$ , with  $\theta > 0$ .
- (ii) *The tail of  $X_1$  is dominant, i.e.*

$$\lim_{x \uparrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = 0, \quad \forall i \in \{2, \dots, d\}.$$

**Proposition 3.1** (Dominant expectile component). *Under Assumption 2, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . Then*

(i)

$$\beta_i = \lim_{\alpha \uparrow 1} \frac{\mathbf{e}_\alpha^i(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} = 0, \quad \forall i \in \{2, \dots, d\}$$

(ii)

$$\lim_{\alpha \uparrow 1} \frac{1-\alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = \frac{1}{\theta-1} \quad \text{and} \quad \lim_{\alpha \uparrow 1} \frac{1-\alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} = 0, \quad \forall i \in \{2, \dots, d\}$$

The proof of Proposition 3.1 follows from the following lemma.

**Lemma 3.2.** *Under Assumption 2, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1,\dots,d}$ . Then*

(1)

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall i \in \{2, \dots, d\},$$

(2) and

$$0 < \lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \leq \overline{\lim}_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} < +\infty.$$

The proofs of (1) and (2) of Lemma 3.2 are given in Appendix A.4 and A.5 respectively. Now, we have all necessary tools to prove Proposition 3.1.

*Proof of Proposition 3.1.* From Lemma 3.2, and taking if necessary a subsequence  $(\alpha_n)_{n \in \mathbb{N}} \alpha_n \rightarrow 1$ , we suppose that  $\frac{1-\alpha}{\bar{F}_{X_1}(x_1)}$  is converging to a limit denoted  $0 < \eta < +\infty$  and that the limits  $\lim_{\alpha \rightarrow 1} \frac{x_i}{x_1} = \beta_i$  exist for all  $i = 1, \dots, d$ . Going through the limit ( $\alpha \rightarrow 1$ ) in the 1<sup>st</sup> equation of System (0.1) divided by  $x_1 \bar{F}_{X_1}(x_1)$ , leads using Lemma 3.2 to

$$(3.1) \quad \lim_{\alpha \uparrow 1} \left( \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \sum_{i \in J_C \cup J_\infty} \frac{x_i}{x_1} \right) = \lim_{\alpha \uparrow 1} \left( \eta \sum_{i \in J_C \cup J_\infty} \frac{x_i}{x_1} \right) = \frac{1}{\theta - 1} - \eta,$$

we deduce that  $J_\infty = \emptyset$ .

We suppose that  $J_C \neq \emptyset$ , so there exists at least one  $i \in \{2, \dots, d\}$  such that  $i \in J_C$ , and for all  $j \in \{1, \dots, d\} \setminus \{i\}$ , we have

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0.$$

Indeed, if  $j \in J_C \setminus \{i\}$

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \uparrow 1} \int_{\beta_j}^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt,$$

because

$$\frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} = \mathbb{P}(X_j > tx_1 | X_i > x_i) \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)},$$

$$\text{and } \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)} = \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_i)} \frac{\mathbb{P}(X_1 > x_i)}{\mathbb{P}(X_1 > x_1)} = \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_i)} \left( \frac{x_i}{x_1} \right)^{-\theta} = 0.$$

Furthermore, since

$$\frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} \leq \min \left( \frac{\mathbb{P}(X_j > tx_1)}{\mathbb{P}(X_1 > x_1)}, \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)} \right),$$

and using Potter's Bounds associated to  $\bar{F}_{X_1}$  as regularly varying function in order to apply the dominated convergence Theorem, we get

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \int_{\beta_j}^{+\infty} \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt = 0, \quad \forall j \in J_C \setminus \{i\}.$$

Now, if  $j \in J_0$  then

$$\begin{aligned} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \int_{x_j}^{x_1} \frac{\mathbb{P}(X_j > t, X_i > x_i)}{x_1 \mathbb{P}(X_1 > x_1)} dt + \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt \\ &\leq \left( 1 - \frac{x_j}{x_1} \right) \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)} + \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt, \end{aligned}$$

thus

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall j \in J_0,$$

because  $\lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)} = 0$  and  $\lim_{\alpha \uparrow 1} \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt = 0$ .

Going through the limit ( $\alpha \rightarrow 1$ ) in the  $i^{\text{th}}$  equation of System (0.1) divided by  $x_1 \bar{F}_{X_1}(x_1)$ , leads to

$$-\eta \beta_i = \eta \left( 1 + \sum_{j \in J_C \setminus \{i\}} \beta_j \right),$$

which is absurd, and consequently  $J_C = \emptyset$ .

We have thus proved that  $J_0 = \{2, \dots, d\}$  which means

$$\lim_{\alpha \uparrow 1} \frac{\mathbf{e}_\alpha^i(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} = \beta_i = 0, \quad \forall i \in \{2, \dots, d\}.$$

And from Equation (3.1) we deduce also that

$$\eta = \lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = \frac{1}{\theta - 1},$$

and by Lemma 2.2 that

$$\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_k}(\mathbf{e}_\alpha^i(\mathbf{X}))} = 0, \quad \forall i \in \{2, \dots, d\}.$$

□

Proposition 3.1 shows that the dominant risk behaves asymptotically as in the univariate case, and its component in the extreme multivariate expectile satisfies

$$\mathbf{e}_\alpha^1(\mathbf{X}) \stackrel{\alpha \uparrow 1}{\sim} (\theta - 1)^{-\frac{1}{\theta}} \text{VaR}_\alpha(X_1) \stackrel{\alpha \uparrow 1}{\sim} \mathbf{e}_\alpha(X_1),$$

the right equivalence is proved, in the univariate case, in [5], Proposition 2.3.

As an illustration, consider Pareto distributions,  $X_i \sim Pa(a_i, b)$ ,  $i = 1, \dots, d$ , such that  $a_i > a_1 > 0$  for all  $i \in \{1, \dots, d\}$ , and  $b > 0$ . The survival function of  $X_i$  is  $\bar{F}_{X_i}(x) = (b/x)^{a_i}$ ,  $x \geq b$ , where  $b$  is the scale parameter, which is the minimum possible value of  $X_i$ , and  $a_i$  is the shape parameter of  $X_i$ , known as the tail index. For all  $i \in \{2, \dots, d\}$ , the tail of  $X_1$  dominates the tail of each  $X_i$  and Proposition 3.1 applies.

#### 4. ESTIMATION OF THE EXTREME EXPECTILES

In this section, we propose some estimators of the extreme multivariate expectile. We focus on the Assumption 1 of equivalent tails, and more specifically on the cases of asymptotic independence and comonotonicity, for which the equation system is more tractable. We begin with the main ideas of our approach, then, we construct the estimators using the extreme values statistical tools and prove its consistency. We terminate this section with a simulation study.

**Proposition 4.1** (Estimation's idea). *Using notations of previous sections, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . Under Assumption 1 and the assumption that the vector  $\left( \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$  has a unique limit point  $(\eta, \beta_2, \dots, \beta_d)$ ,*

$$\mathbf{e}_\alpha(\mathbf{X}) \underset{\alpha \rightarrow 1}{\sim} \text{VaR}_\alpha(X_1) \eta^{\frac{1}{\theta}} (1, \beta_2, \dots, \beta_d)^T.$$

*Proof.* Let  $(\eta, \beta_2, \dots, \beta_d) = \lim_{\alpha \rightarrow 1} \left( \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$ , we have

$$\mathbf{e}_\alpha(\mathbf{X}) \underset{\alpha \rightarrow 1}{\sim} \mathbf{e}_\alpha^1(\mathbf{X}) (1, \beta_2, \dots, \beta_d)^T.$$

Moreover,  $\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = \eta$ , and Theorem 1.5.12 in [6] states that  $F_{X_1}^{\leftarrow}(1 - \cdot)$  is regularly varying at 0, with index  $-\frac{1}{\theta}$ . This leads to

$$\mathbf{e}_\alpha^1(\mathbf{X}) \underset{\alpha \rightarrow 1}{\sim} F_{X_1}^{\leftarrow}(\alpha) \left( \frac{1}{\eta} \right)^{-\frac{1}{\theta}},$$

and the result follows. □

Proposition 4.1 gives a way to estimate the extreme multivariate expectile. Let  $\mathbf{X} = (X_1, \dots, X_d)^T$  be an independent sample of size  $n$ , with  $\mathbf{X}_i = (X_{1,i}, \dots, X_{d,i})^T$  for all  $i \in \{1, \dots, n\}$ . We denote by  $X_{i,1,n} \leq X_{i,2,n} \leq \dots \leq X_{i,n,n}$  the ordered sample corresponding to  $X_i$ .

**4.1. Estimator's construction.** We begin with the case of asymptotic independence. Proposition 4.1 and Lemma 2.5 are the key tools in the construction of the estimator. We have for all  $i \in \{1, \dots, d\}$

$$\beta_i = c_i^{\frac{1}{\theta-1}}, \text{ and } \lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} = \frac{c_i^{\frac{1}{\theta-1}}}{(\theta-1) \left(1 + \sum_{j=2}^d c_j^{\frac{1}{\theta-1}}\right)}.$$

Proposition 4.1 gives

$$\mathbf{e}_\alpha(\mathbf{X})^T \stackrel{\alpha \uparrow 1}{\sim} \text{VaR}_\alpha(X_1) (\theta-1)^{-\frac{1}{\theta}} \left(1 + \sum_{i=2}^d c_i^{\frac{1}{\theta-1}}\right)^{-\frac{1}{\theta}} \left(1, c_2^{\frac{1}{\theta-1}}, \dots, c_d^{\frac{1}{\theta-1}}\right)^T.$$

So, in order to estimate the extreme multivariate expectile, we need an estimator of the univariate quantile of  $X_1$ , of the tail equivalence parameters. and of  $\theta$ .

In the same way, and for the case of comonotonic risks, we may use Lemma 2.4

$$\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} = \frac{1}{\theta-1} \text{ and } \beta_i = c_i^{1/\theta}, \quad \forall i \in \{1, \dots, d\},$$

and by Proposition 4.1 we obtain

$$\mathbf{e}_\alpha(\mathbf{X})^T \stackrel{\alpha \uparrow 1}{\sim} \text{VaR}_\alpha(X_1) (\theta-1)^{-\frac{1}{\theta}} (1, c_2^{\frac{1}{\theta}}, \dots, c_d^{\frac{1}{\theta}})^T.$$

The  $X_i$ 's have all the same index  $\theta$  of regular variation, which is also the same as the index of regular variation of  $\|\mathbf{X}\|$ . We propose to estimate  $\theta$  by using the Hill estimator  $\hat{\gamma}$ . We shall denote  $\hat{\theta} = \frac{1}{\hat{\gamma}}$ . See [16] for details on the Hill estimator. In order to estimate the  $c_i$ 's, we shall use the GPD approximation: for  $u$  a large threshold, and  $x \geq u$ ,

$$\bar{F}(x) \simeq \bar{F}(u) \left(\frac{x}{u}\right)^{-\theta}.$$

Let  $k \in \mathbb{N}$  be fixed and consider the thresholds  $u_i$ :

$$\bar{F}_{X_i}(u_i) = \bar{F}_{X_1}(u_1) = \frac{k}{n}, \quad \forall i \in \{1, \dots, d\}.$$

The  $u_i$  are estimated by  $X_{i,n-k+1,n}$  with  $k \rightarrow \infty$  and  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . Using Lemma 2.2, we get

$$c_i = \lim_{n \rightarrow \infty} \left(\frac{u_i}{u_1}\right)^\theta.$$

We shall consider

$$(4.1) \quad \hat{c}_i = \left(\frac{X_{i,n-k+1,n}}{X_{1,n-k+1,n}}\right)^{\frac{1}{\hat{\gamma}(k)}},$$

where  $\hat{\gamma}(k)$  is the Hill estimator of the extreme values index constructed using the  $k$  largest observations of  $\|\mathbf{X}\|$ . Let  $\hat{\theta} = \frac{1}{\hat{\gamma}(k)}$ .

**Proposition 4.2.** *Let  $k = k(n)$  be such that  $k \rightarrow \infty$  and  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . Under Assumption 1, for any  $i = 2, \dots, d$ ,*

$$\hat{c}_i \xrightarrow{\mathbb{P}} c_i.$$

*Proof.* The results in [3] page 86 imply that for any  $i = 1, \dots, d$

$$\frac{X_{i,n-k+1,n}}{u_i} \xrightarrow{\mathbb{P}} 1.$$

Moreover, it is well known (see [16]) that the Hill estimator is consistent. Using (4.1), and the fact that

$$\frac{X_{i,n-k+1,n}}{X_{1,n-k+1,n}} \sim \frac{u_i}{u_1} \text{ in probability and thus is bounded in probability,}$$

we get the result. □

To estimate the extreme quantile, we will use Weissman's estimator (1978) [29]:

$$\widehat{\text{VaR}}_\alpha(X_1) = X_{1, n-k(n)+1, n} \left( \frac{k(n)}{(1-\alpha)n} \right)^{\hat{\gamma}}.$$

The properties of Weissman's estimator are presented in Embrechts et al. (1997) [12] and also in [3] page 119. In order to prove the consistency of our estimators of extreme multivariate expectiles, we shall need the following second order condition (see [3] Section 4.4).

**Definition 4.3.** A random variable  $X$  satisfying  $\bar{F}_X \in \text{RV}_{-\theta}(+\infty)$  with  $\theta = \frac{1}{\gamma} > 0$  will be said to verify the second order condition  $\text{SOC}_{-\beta}(b)$  with  $\beta > 0$  and  $b \in \mathcal{RV}_{-\beta}(+\infty)$  if the function  $U : y \rightarrow F_X^{\leftarrow}(1 - \frac{1}{y})$  satisfies for  $u > 0$ :

$$\frac{U(ux)}{U(x)} = u^\gamma (1 + h_{-\beta}(u)b(x) + o(b(x))), \text{ as } x \text{ goes to infinity.}$$

where  $h_{-\beta}(u) = \frac{1-u^{-\beta}}{\beta}$ .

Now, we can deduce some estimators of the extreme multivariate expectile, using the previous ones, in the cases of asymptotic independence and perfect dependence.

**Definition 4.4** (Multivariate expectile estimator, Asymptotic independence). Under Assumption 1, in the case of bivariate asymptotic independence of the random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ , we define the estimator of the  $L_1$ -expectile as follows

$$\begin{aligned} \hat{\mathbf{e}}_\alpha^\perp(\mathbf{X}) &= X_{1, n-k(n)+1, n} \left( \frac{k(n)}{(1-\alpha)n} \right)^{\hat{\gamma}} \left( \frac{\hat{\gamma}}{1-\hat{\gamma}} \right)^{\hat{\gamma}} \left( \frac{1}{1 + \sum_{\ell=2}^d \hat{c}_\ell^{\frac{\hat{\gamma}}{1-\hat{\gamma}}}} \right)^{\hat{\gamma}} \left( 1, \hat{c}_2^{\frac{\hat{\gamma}}{1-\hat{\gamma}}}, \dots, \hat{c}_d^{\frac{\hat{\gamma}}{1-\hat{\gamma}}} \right)^T \\ &= \widehat{\text{VaR}}_\alpha(X_1) \left( \frac{\hat{\gamma}}{1-\hat{\gamma}} \right)^{\hat{\gamma}} \left( \frac{1}{1 + \sum_{\ell=2}^d \hat{c}_\ell^{\frac{\hat{\gamma}}{1-\hat{\gamma}}}} \right)^{\hat{\gamma}} \left( 1, \hat{c}_2^{\frac{\hat{\gamma}}{1-\hat{\gamma}}}, \dots, \hat{c}_d^{\frac{\hat{\gamma}}{1-\hat{\gamma}}} \right)^T. \end{aligned}$$

**Definition 4.5** (Multivariate expectile estimator, comonotonic risks). Under the assumptions of the Fréchet model with equivalent tails, for a comonotonic random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ , we define the estimator of  $L_1$ -expectile as follows

$$\begin{aligned} \hat{\mathbf{e}}_\alpha^+(\mathbf{X}) &= X_{1, n-k(n)+1, n} \left( \frac{k(n)}{(1-\alpha)n} \right)^{\hat{\gamma}} \left( \frac{\hat{\gamma}}{1-\hat{\gamma}} \right)^{\hat{\gamma}} \left( 1, \hat{c}_2^{\hat{\gamma}}, \dots, \hat{c}_d^{\hat{\gamma}} \right)^T \\ &= \widehat{\text{VaR}}_\alpha(X_1) \left( \frac{\hat{\gamma}}{1-\hat{\gamma}} \right)^{\hat{\gamma}} \left( 1, \hat{c}_2^{\hat{\gamma}}, \dots, \hat{c}_d^{\hat{\gamma}} \right)^T. \end{aligned}$$

We prove below that if the second order condition  $\text{SOC}_{-\beta}(b)$  is satisfied, then the term by term ratio  $\hat{\mathbf{e}}_\alpha^\perp(\mathbf{X})/\mathbf{e}_\alpha(\mathbf{X})$  goes to 1 in probability in the asymptotically independent case and  $\hat{\mathbf{e}}_\alpha^+(\mathbf{X})/\mathbf{e}_\alpha(\mathbf{X})$  goes to 1 in probability in the comonotonic case. More work is required to get the asymptotic normality.

**Theorem 4.6.** Under Assumption 1, assume that furthermore  $X_1$  satisfies  $\text{SOC}_{-\beta}(b)$  condition. Choose  $k = k(n)$  and  $\alpha = \alpha(n)$  such that

- (i)  $k(n) \rightarrow \infty$ ,  $k(n)/n \rightarrow 0$  and  $\alpha(n) \rightarrow 1$ ,  $n(1-\alpha(n)) \rightarrow c > 0$  as  $n \rightarrow \infty$ ,
- (ii)  $\sqrt{k(n)} \left( 1 + \log^2 \frac{k(n)}{n(1-\alpha(n))} \right)^{-\frac{1}{2}} \rightarrow \infty$  as  $n \rightarrow \infty$ .

Then, if each pair of the random vector  $\mathbf{X}$  is asymptotically independent,

$$\hat{\mathbf{e}}_\alpha^\perp(\mathbf{X})/\mathbf{e}_\alpha(\mathbf{X}) \rightarrow 1 \text{ in probability, as } n \rightarrow \infty.$$

If the random vector  $\mathbf{X}$  is comonotonic, then

$$\hat{\mathbf{e}}_\alpha^+(\mathbf{X})/\mathbf{e}_\alpha(\mathbf{X}) \rightarrow 1 \text{ in probability, as } n \rightarrow \infty.$$

*Proof.* Following [3] p. 120, we use the  $\text{SOC}_{-\beta}(b)$  hypothesis and Assumption (i). We write  $k$  for  $k(n)$  and  $\alpha$  for  $\alpha(n)$  and we get:

$$\sqrt{k} \left( 1 + \log^2 \frac{k}{n(1-\alpha)} \right)^{-\frac{1}{2}} \left( \frac{\widehat{\text{VaR}}_\alpha(X_1)}{\text{VaR}_\alpha(X_1)} - 1 \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \gamma^2) \text{ as } n \rightarrow \infty.$$

Now, the choice (ii) of  $k(n)$ , leads to

$$\frac{\widehat{\text{VaR}}_\alpha(X_1)}{\text{VaR}_\alpha(X_1)} \rightarrow 1 \text{ in probability as } n \rightarrow \infty.$$

Then the announced results follow from Proposition 4.2 and Lemma 2.5.  $\square$

**4.2. Numerical illustration.** The Fréchet domain of attraction contains the usual distributions of Pareto, Student, Burr and Cauchy. In order to illustrate the convergence of the proposed estimators, we study numerically, the cases of Pareto, Burr and Student distributions.

In the independence case which is a special case of asymptotic independence, the functions  $l_{X_i, X_j}^\alpha$  defined in (2.2) have the following expression

$$l_{X_i, X_j}^\alpha(x_i, x_j) = \alpha \left( \bar{F}_{X_j}(x_j) \mathbb{E}[(X_i - x_i)_+] \right) - (1 - \alpha) \left( F_{X_j}(x_j) \mathbb{E}[(X_i - x_i)_-] \right).$$

In the comonotonic case we have

$$l_{X_i, X_j}^\alpha(x_i, x_j) = \alpha \left( \bar{F}_{X_j}(x_j) (\mu_{i,j} - x_i)_+ + \mathbb{E}[(X_i - \max(x_i, \mu_{i,j}))_+] \right) - (1 - \alpha) \left( F_{X_j}(x_j) (x_i - \mu_{i,j})_+ + \mathbb{E}[(X_i - \min(x_i, \mu_{i,j}))_-] \right),$$

where  $\mu_{i,j} = F_{X_i}^{\leftarrow} (F_{X_j}(x_j))$ .

From these expressions, the exact value of the extreme multivariate expectile is obtained using numerical optimization, and we can confront it to the estimated values. The choice of  $k(n)$  is function of the distributions parameters, and it is done in our simulations using graphical illustrations. We present the estimators for different values of  $k(n)$  that belong to the common convergence range of the estimators of tail equivalence coefficients, in order to verify the stability of the expectile estimator's convergence.

**4.2.1. Pareto distributions.** We consider a bivariate Pareto model  $X_i \sim Pa(a, b_i)$ ,  $i \in \{1, 2\}$ . Both distributions have the same shape parameter  $a$ , so they have equivalent tails with equivalence parameter

$$c_2 = \lim_{x \rightarrow +\infty} \frac{\bar{F}_{X_2}(x)}{\bar{F}_{X_1}(x)} = \lim_{x \rightarrow +\infty} \frac{\left(\frac{b_2}{x}\right)^a}{\left(\frac{b_1}{x}\right)^a} = \left(\frac{b_2}{b_1}\right)^a.$$

In what follows, we consider two models for which the exact values of the  $L_1$ -expectiles are computable. In the first model, the  $X_i$ 's are independent. In the second one, the  $X_i$ 's are comonotonic and Pareto distributed. In that case,  $\mu_{i,j} = \frac{b_i}{b_j} x_j$ . In the simulations below, we have taken the same  $k = k(n)$  to get  $\hat{\gamma}$  and  $\hat{e}_\alpha(\mathbf{X})$ . For  $X_i \sim Pa(2, 5 \cdot (i + 1))_{i=1,2}$ , and  $n = 100,000$ , Figure 1 illustrates the convergence of estimator  $\hat{c}_2$ . On the left, the shaded area indicates suitable values of  $k(n)$  for  $n = 100,000$ . The boxplots are obtained for different values of  $n$  and a fixed  $k \in k(n)$ .

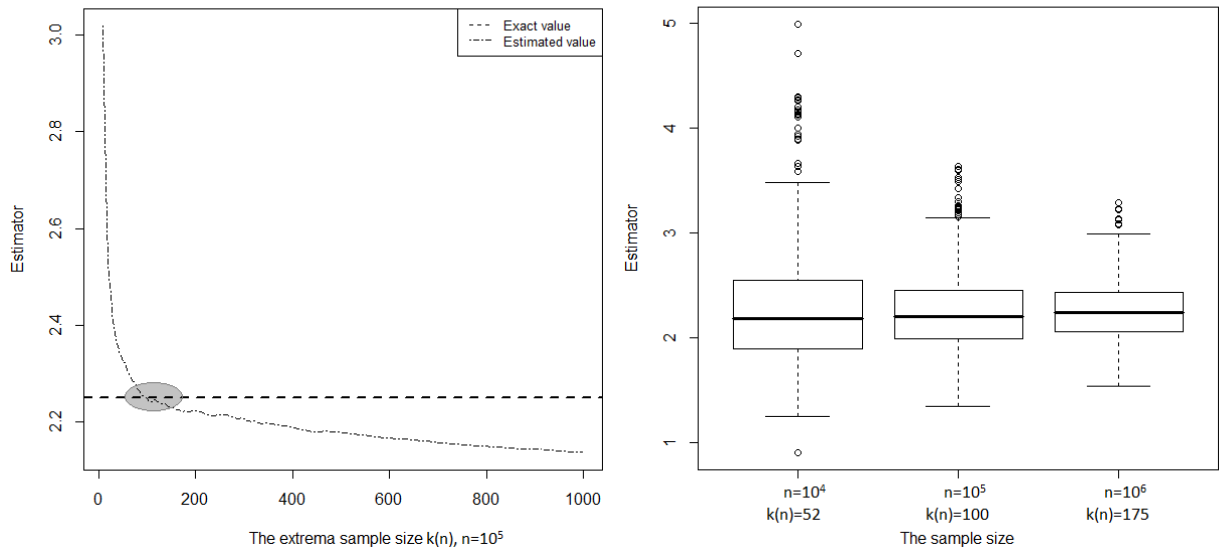


FIGURE 1. Convergence of  $\hat{c}_2$ .  $X_i \sim Pa(2, 5 \cdot (i + 1))_{i=1,2}$ .



Figure 2 presents the obtained results for different  $k(n)$  values in the independence case where  $n = 100,000$ . A multivariate illustration in dimension 4 is given in Figure 3. The comonotonic case is illustrated in Figure 4. The simulations parameters are  $a = 2$ ,  $b_1 = 10$  and  $b_2 = 15$ .

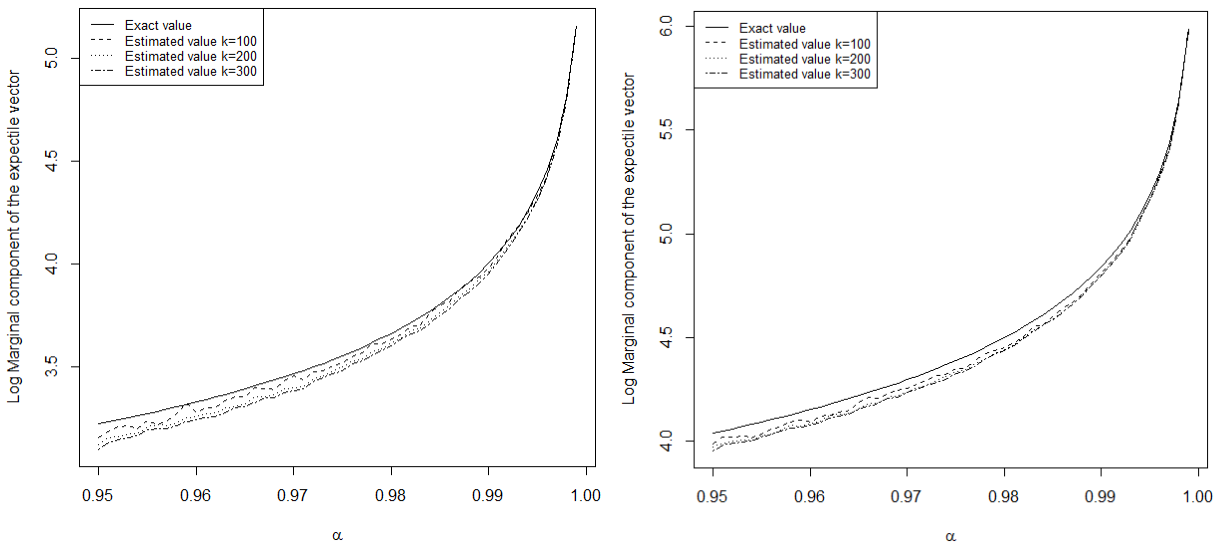


FIGURE 2. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  (asymptotic independence case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

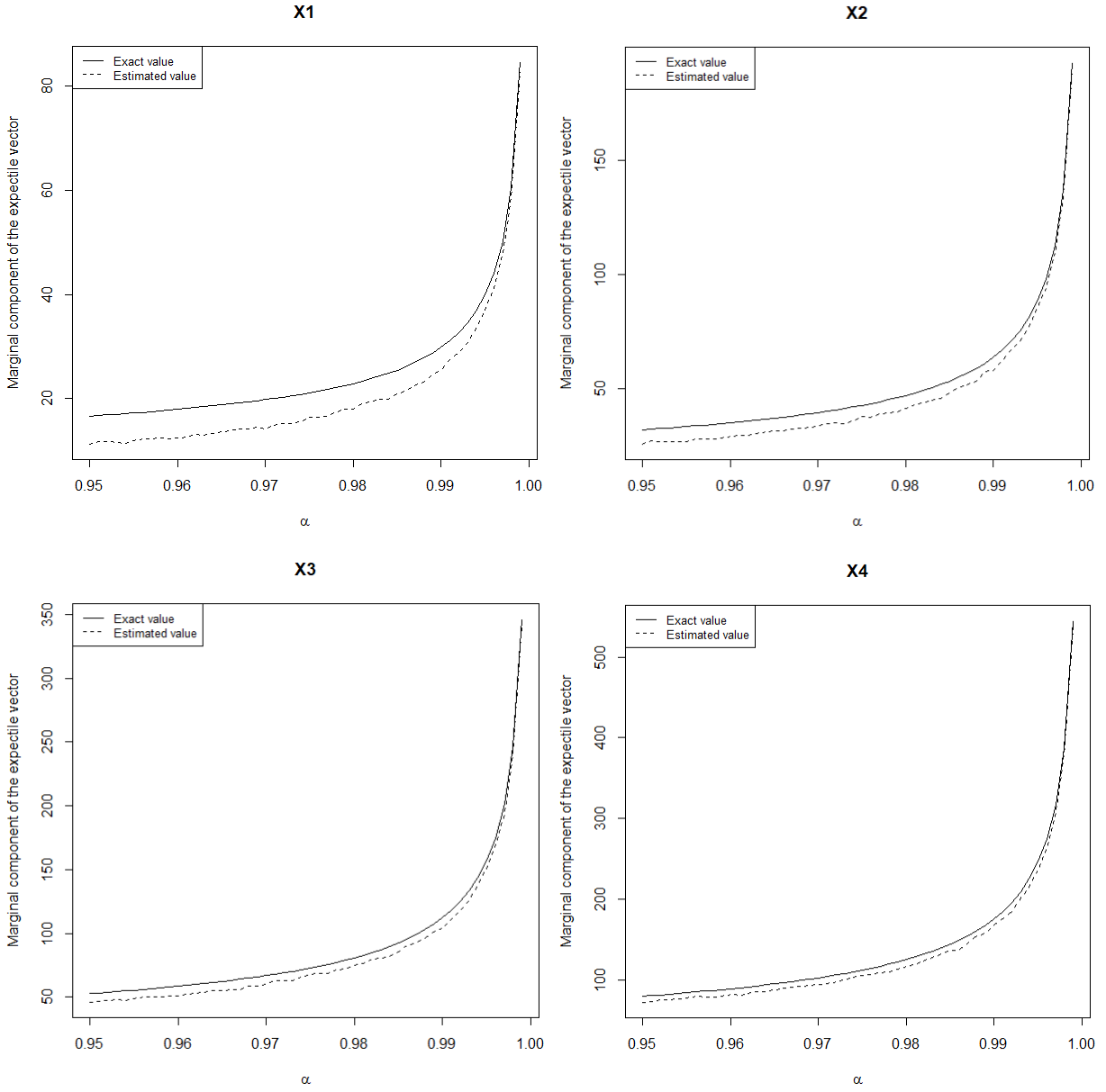


FIGURE 3. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  (asymptotic independence case). The coordinates of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  in dimension  $d = 4$ ,  $n = 100,000$  and  $k(n) = 100$ .

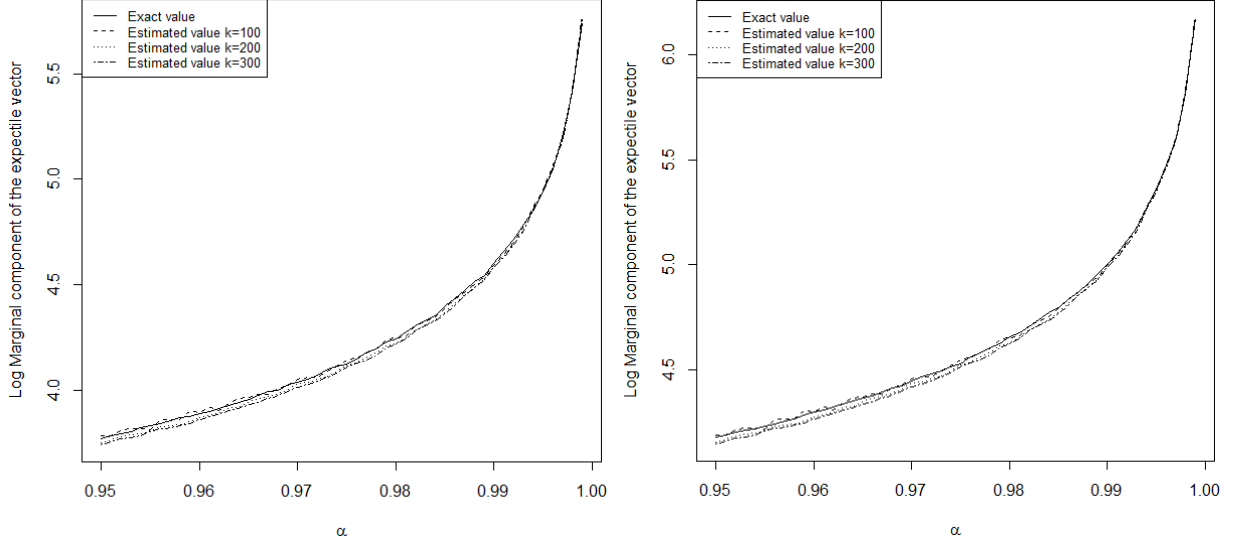


FIGURE 4. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  (comonotonic case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

4.2.2. *Burr distributions.* We consider a multivariate Burr model  $X_i \sim \text{Burr}(a, b_i, \tau)$ ,  $i \in \{1, \dots, d\}$ . In this case, the tails are equivalent with equivalence parameter

$$c_i = \lim_{x \rightarrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = \lim_{x \rightarrow +\infty} \frac{\left(\frac{b_i}{b_i + x^{\tau}}\right)^a}{\left(\frac{b_1}{b_1 + x^{\tau}}\right)^a} = \left(\frac{b_i}{b_1}\right)^a, \forall i \in \{2, \dots, d\},$$

and  $\bar{F}_{X_i} \in RV_{a\tau}(+\infty)$  for all  $i$  in  $\{1, \dots, d\}$ . In the Burr comonotonic case  $\mu_{i,j} = \left(\frac{b_i}{b_j}\right)^{\frac{1}{\tau}} x_j$ . The model is asymptotically equivalent to the Pareto one, but the margins are different, which helps to test the pertinence of the estimation processes compared to the theoretical results. Figures 5 and 6 present the obtained results for different  $k(n)$  values in the independence and the comonotonic cases respectively. The simulations parameters are  $a = 4$ ,  $b_1 = 10$ ,  $b_2 = 15$ ,  $\tau = 0.75$  and  $n = 10,000$ .

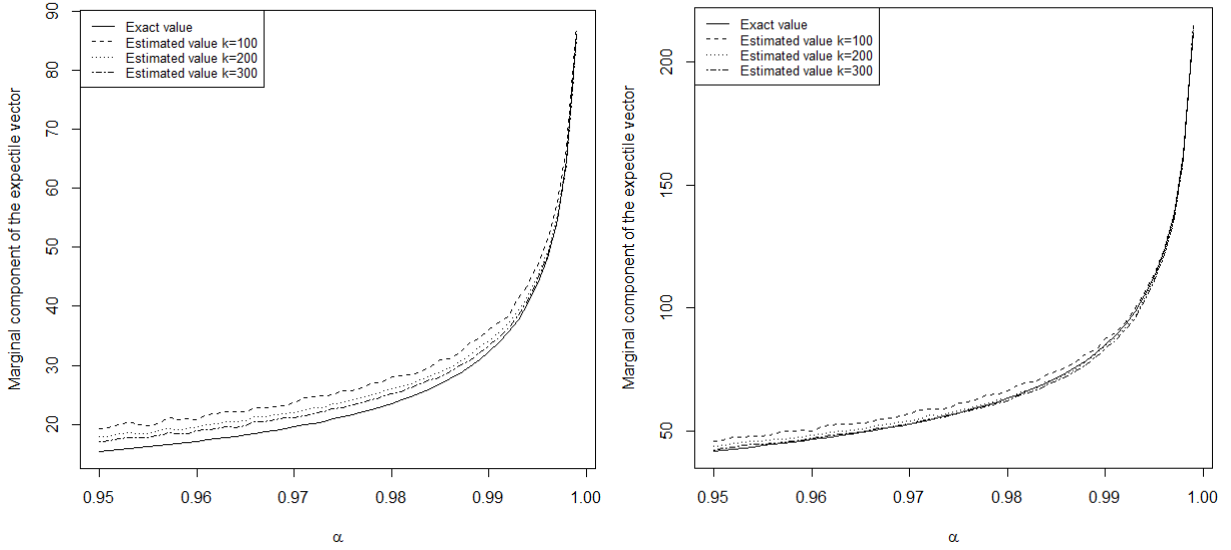


FIGURE 5. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  (asymptotic independence case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

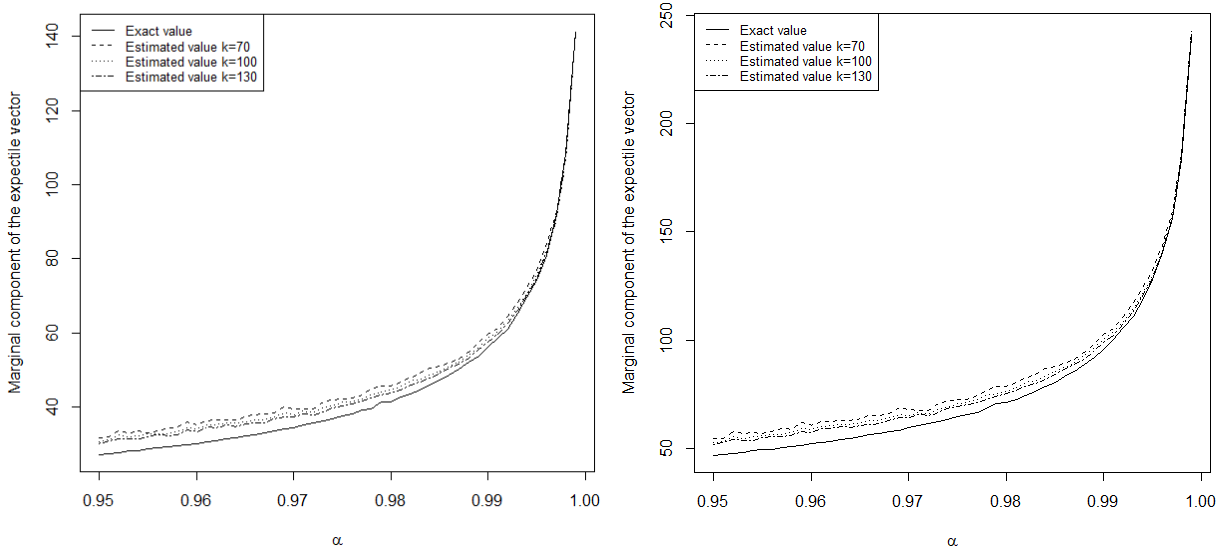


FIGURE 6. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  (comonotonic case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

4.2.3. *Student distributions.* In order to illustrate the convergence of the two estimators for other distributions nature, we close this subsection by a Student model. We consider a risk vector  $(X_1, \dots, X_d)$  such that  $X_i = a_i T_i$  for all  $i \in \{1, \dots, d\}$  and  $(T_i)_i$  are identically distributed following a t-distribution with  $z$  degrees of freedom. Each  $T_i$  has density  $f_{T_i}(t) = \frac{1}{\sqrt{z\pi}} \frac{\Gamma((z+1)/2)}{\Gamma(z/2)} (1 + t^2/z)^{-(z+1)/2}$ , where  $\Gamma$  is the Euler Gamma function.  $(a_i)_{i=1, \dots, d}$  are positive constants. Using L'Hôpital's rule, the tails are equivalent since

$$\lim_{x \rightarrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = \lim_{x \rightarrow +\infty} \frac{\bar{F}_{T_1}(x/a_i)}{\bar{F}_{T_1}(x)} = \lim_{x \rightarrow +\infty} \frac{a_1 f_{T_1}(x/a_i)}{a_1 f_{T_1}(x/a_1)} = \left(\frac{a_i}{a_1}\right)^z = c_i, \forall i \in \{2, \dots, d\}.$$

The marginal tails are all  $RV_{-(z+1)}(+\infty)$ . For the Student comonotonic model  $\mu_{i,j} = \frac{a_i}{a_j} x_j$ .

For the numerical illustration the parameters are  $a_i = 2^{i-1}$  for  $i = 1, \dots, d$  and  $z = 2$ . In the case of the independence  $(T_i)_i$  are supposed independent, and they are comonotonic in the comonotonic case.

Figures 7 and 8 present an illustration of the obtained results in the two cases.

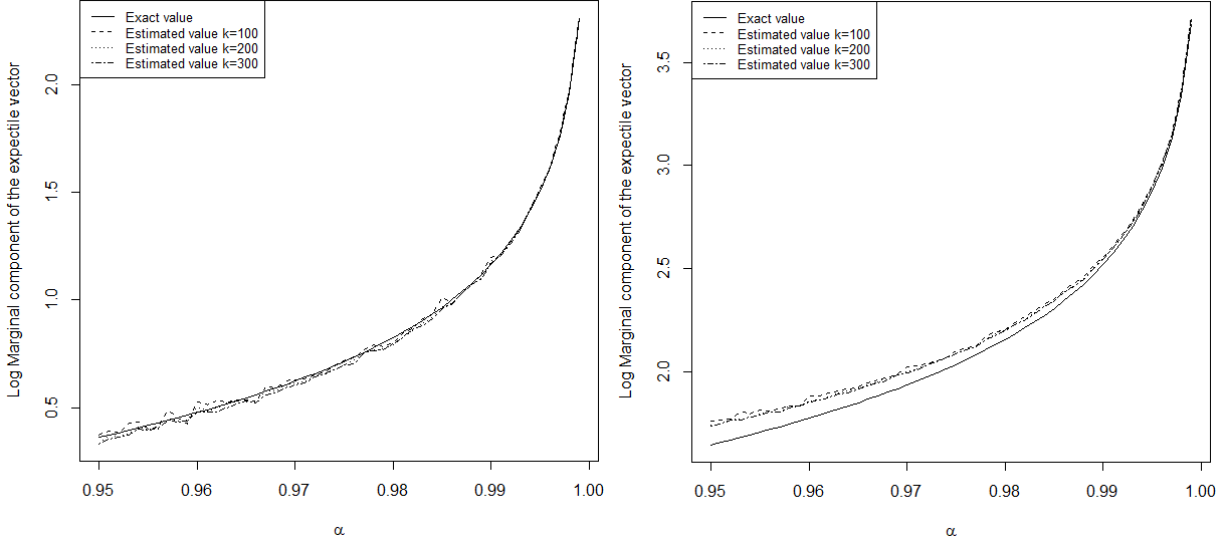


FIGURE 7. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  (asymptotic independence case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{\perp}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

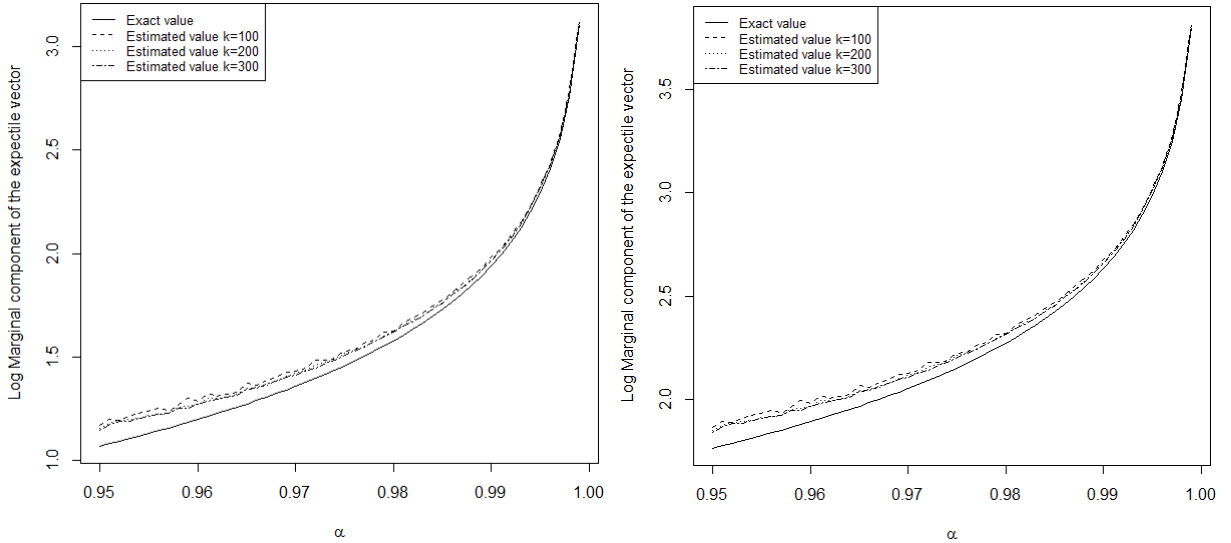


FIGURE 8. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  (comonotonic case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

For the three Fréchet models, Pareto, Burr, and Student, the different illustrations show that the convergence is better for values of  $\alpha$  close to 1. This is natural since we are approaching the extreme level and therefore the estimate value converges towards the theoretical value. The convergence seems to be stable for values of  $k(n)$  in the convergence zone. When  $\alpha$  moves away from 1, the difference with the theoretical value is apparently a function of the marginal risk level represented by the coefficients of tails' equivalence  $c_i$ .

## CONCLUSION

We have studied properties of extreme multivariate expectiles in a regular variations framework. We have seen that the asymptotic behavior of expectile vectors strongly depends on the marginal tail behavior and on the nature of the asymptotic dependence. The main conclusion of our analysis is that the equivalence of marginal tails leads to the equivalence of extreme expectile components.

A statistical estimation of the integrals of the tail dependence functions would allow estimators of the extreme expectile vectors to be constructed. This paper's estimations are limited to the asymptotic independence and comonotonicity cases, which do not require estimation of the tail dependence functions.

The asymptotic normality of the estimators proposed in the last section of this paper would require a careful technical analysis which is not considered in this paper.

A natural follow-up to this work would be to study the asymptotic behavior of  $\Sigma$ -expectiles in the case of marginal distributions with equivalent tails in the Weibull and Gumbel domains of attraction. The Fréchet domain we have considered here contains heavy-tailed distributions, which is interesting for modeling large risks. The Gumbel's domain contains many other distributions (Exponential, Weibull, Gamma, Normal, Lognormal, etc.), so that extensions to this family would be of practical importance.

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APPENDIX A. PROOFS

A.1. **Lemma 2.2.**

*Proof.* We give some details on the proof for the first item, the second one may be obtained in the same way. Under Assumption 1, for all  $i \in \{1, \dots, d\}$

$$\bar{F}_{X_i} \in \text{RV}_{-\theta}(+\infty),$$

there exists for all  $i$  a positive measurable function  $L_i \in \text{RV}_0(+\infty)$  such that

$$\bar{F}_{X_i}(x) = x^{-\theta} L_i(x), \forall x > 0,$$

then for all  $(i, j) \in \{1, \dots, d\}^2$  and all  $t, s > 0$

$$(A.1) \quad \frac{s \bar{F}_{X_i}(s)}{t \bar{F}_{X_j}(t)} = \left(\frac{s}{t}\right)^{-\theta+1} \frac{L_i(s)}{L_j(t)} = \left(\frac{s}{t}\right)^{-\theta+1} \frac{L_i(s)}{L_i(t)} \frac{L_i(t)}{L_j(t)},$$

and under H2

$$(A.2) \quad \lim_{x \uparrow +\infty} \frac{L_i(x)}{L_j(x)} = \frac{c_i}{c_j}.$$

Using Karamata's representation for slowly varying functions (Theorem 1.2), there exist a constant  $c > 0$ , a positive measurable function  $c(\cdot)$  with  $\lim_{x \uparrow +\infty} c(x) = c > 0$ , such that  $\forall \epsilon > 0, \exists t_0$  such that  $\forall t > t_0$

$$\frac{L_i(s)}{L_i(t)} \leq \left(\frac{s}{t}\right)^\epsilon \frac{c(s)}{c(t)}.$$

Taking  $0 < \epsilon < \theta - 1$ , we conclude

$$\lim_{t \uparrow +\infty} \frac{s \bar{F}_{X_i}(s)}{t \bar{F}_{X_j}(t)} = 0, \forall (i, j) \in \{1, \dots, d\}^2.$$

□

A.2. **Proposition 2.1 (ii).**

*Proof.* We start by proving that

$$\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} < +\infty, \forall i \in \{2, \dots, d\}.$$

Using H2, it is sufficient to show that

$$\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} < +\infty.$$

Assume that  $\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = +\infty$ , we shall prove that, in that case, (2.3) cannot be satisfied. Taking if necessary a subsequence  $(\alpha_n \rightarrow 1)$ , we may assume that  $\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = +\infty$ .

We have

$$\begin{aligned} \frac{l_{X_1}^\alpha(x_1)}{(1 - \alpha)x_1} &= \left( \frac{\alpha \mathbb{E}[(X_1 - x_1)_+] - (1 - \alpha) \mathbb{E}[(X_1 - x_1)_-]}{(1 - \alpha)x_1} \right) \\ &= \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{(1 - \alpha)x_1} - \frac{(1 - \alpha)(x_1 - \mathbb{E}[X_1])}{(1 - \alpha)x_1} \right) \\ &= \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} \frac{\bar{F}_{X_1}(x_1)}{1 - \alpha} - 1 + \frac{\mathbb{E}[X_1]}{x_1} \right) \xrightarrow{\alpha \uparrow 1} -1 \text{ recall (2.1)}. \end{aligned}$$

Furthermore, for all  $i \in \{2, \dots, d\}$

$$\begin{aligned} \frac{l_{X_i, X_1}^\alpha(x_i, x_1)}{(1 - \alpha)x_1} &= \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}] - (1 - \alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{(1 - \alpha)x_1} \right) \\ &= \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} - \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} - \frac{x_i \mathbb{P}(X_1 < x_1)}{x_1} + \frac{\mathbb{E}[X_i \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} \right). \end{aligned}$$

On one side,

$$\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} \leq \frac{\mathbb{E}[(X_i - x_i)_+]}{(1 - \alpha)x_1} = \frac{\bar{F}_{X_1}(x_1)}{1 - \alpha} \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)}, \forall i \in \{2, \dots, d\}.$$



So that, Lemma 2.2 implies

$$(A.3) \quad \lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = 0, \forall k \in J_C^1 \cup J_\infty^1.$$

Let  $i \in J_0^1$ , taking if necessary a subsequence, we may assume that  $\frac{x_i}{x_1} \rightarrow 0$ .

$$(A.4) \quad \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = \frac{\int_{x_i}^{x_1} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1}.$$

Now,

$$\frac{\int_{x_i}^{x_1} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} \leq \frac{\int_{x_i}^{x_1} \mathbb{P}(X_1 > x_1) dt}{(1 - \alpha)x_1} = \frac{\bar{F}_{X_1}(x_1)}{1 - \alpha} \left(1 - \frac{x_i}{x_1}\right).$$

Thus,

$$(A.5) \quad \lim_{\alpha \rightarrow 1} \frac{\int_{x_i}^{x_1} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} = 0.$$

Consider the second term of (A.4)

$$\frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} \leq \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t) dt}{(1 - \alpha)x_1},$$

Karamata's Theorem (Theorem 1.3) gives

$$\frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t) dt}{(1 - \alpha)x_1} \stackrel{\alpha \uparrow 1}{\sim} \frac{1}{\theta - 1} \frac{\bar{F}_{X_k}(x_1)}{1 - \alpha},$$

which leads to

$$(A.6) \quad \lim_{\alpha \rightarrow 1} \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} = 0.$$

Finally, we get

$$(A.7) \quad \lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = 0, \forall i \in J_0^1.$$

We have shown that

$$\lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = 0, \forall k \in \{2, \dots, d\},$$

so, the first equation of optimality system (2.3) implies that

$$-\lim_{\alpha \rightarrow 1} \left( \sum_{k \in J_0^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{(1 - \alpha)x_1} + \sum_{k \in J_C^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{(1 - \alpha)x_1} + \sum_{k \in J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{(1 - \alpha)x_1} \right) = \lim_{\alpha \rightarrow 1} \sum_{k=2}^d \frac{x_k}{x_1} = -1,$$

this is absurd since the  $x_k$ 's are non negative, and consequently

$$\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} < +\infty.$$

Now, we prove that the components of the extreme multivariate expectile satisfy also

$$0 < \underline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))}, \forall i \in \{2, \dots, d\}.$$

Using H2, it is sufficient to show that

$$0 < \underline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}.$$

Let us assume that  $\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = 0$ , we shall see that in that case, (2.3) cannot be satisfied. Taking if necessary a convergent subsequence, we may assume that  $\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = 0$ . In this case,

$$\frac{l_{X_1}^\alpha(x_1)}{x_1 \bar{F}_{X_1}(x_1)} = \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1-\alpha}{\bar{F}_1(x_1)} \left(1 - \frac{\mathbb{E}[X_1]}{x_1}\right) \right) \xrightarrow{\alpha \uparrow 1} \frac{1}{\theta - 1} > 0.$$

On another side, let  $i \in J_\infty^1$ , taking if necessary a subsequence, we may assume that  $x_1 = o(x_i)$ . Lemma 2.2 and Proposition 2.1 (ii) give:

$$\frac{1-\alpha}{\bar{F}_1(x_1)} \frac{x_i}{x_1} = \frac{1-\alpha}{\bar{F}_{X_i}(x_i)} \cdot \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)} \rightarrow 0 \text{ as } \alpha \rightarrow 1.$$

Moreover,

$$\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_i > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \leq \frac{\mathbb{E}[(X_i - x_i)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)},$$

We deduce

$$\begin{aligned} \frac{l_{X_i, X_1}^\alpha(x_i, x_1)}{x_1 \bar{F}_{X_1}(x_1)} &= \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_i > x_1\}}] - (1-\alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_i < x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \\ &\rightarrow 0, \forall i \in J_\infty^1. \end{aligned}$$

Going through the limit ( $\alpha \rightarrow 1$ ) in the first equation of the optimality system (2.3) divided by  $x_1 \bar{F}_{X_1}(x_1)$ , leads to

$$\lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} = -\frac{1}{\theta - 1},$$

which is absurd because

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} &= \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \left( \frac{\alpha \mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_k > x_1\}}] - (1-\alpha) \mathbb{E}[(X_k - x_k)_- \mathbb{1}_{\{X_k < x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \\ &= \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \left( \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_k > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1-\alpha}{\bar{F}_{X_1}(x_1)} \frac{x_k}{x_1} \right) \\ &= \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \left( \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_k > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \geq 0. \end{aligned}$$

We can finally conclude that

$$\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_1}(x_1)} > 0.$$

□

### A.3. Lemma 2.4.

*Proof.* Since the random vector  $\mathbf{X}$  is comonotonic, its survival copula is

$$\bar{C}_{\mathbf{X}}(u_1, \dots, u_d) = \min(u_1, \dots, u_d), \quad \forall (u_1, \dots, u_d) \in [0, 1]^d.$$

We deduce the expression of the functions  $\lambda_U^{ij}$

$$\lambda_U^{ij}(x_i, x_j) = \min(x_i, x_j), \quad \forall (x_i, x_j) \in \mathbb{R}_+^2, \forall i, j \in \{1, \dots, d\}.$$

So,

$$\begin{aligned} \int_1^{+\infty} \lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt &= \int_1^{+\infty} \min \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt \\ &= \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \\ &\quad + \frac{1}{\theta - 1} \left( 1 + \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \right)^{-\theta + 1}. \end{aligned}$$

Under Assumption 1, and by Proposition 2.3, let  $(\eta, \beta_2, \dots, \beta_d)$  be a solution of the following equation system.

$$\begin{aligned} \eta \sum_{i=1}^d \beta_i - \frac{1}{\theta-1} \sum_{i=1}^d c_i \beta_i^{-\theta+1} &= \sum_{i=1, i \neq k}^d c_k \beta_k^{-\theta} \beta_i \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \\ &+ \frac{1}{\theta-1} \sum_{i=1, i \neq k}^d c_i \beta_i^{-\theta+1} \left[ \left( 1 + \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \right)^{-\theta+1} - 1 \right], \end{aligned}$$

$\forall k \in \{1, \dots, d\}$ .  $\eta = \frac{1}{\theta-1}$  and  $\beta_k = c_k^{\frac{1}{\theta}}$  is the only solution to this system.  $\square$

#### A.4. Lemma 3.2 (1).

*Proof.* Taking if necessary a convergent subsequence  $(\alpha_n)_{n \in \mathbb{N}}$   $\alpha_n \rightarrow 1$ , we consider that the limits  $\lim_{\alpha \uparrow 1} \frac{x_i}{x_1} = \beta_i$  exist.

Using the notation  $J_C = \{i \in \{2, \dots, d\} | 0 < \beta_i < +\infty\}$ , for all  $i \in J_C$

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \uparrow 1} \int_{\beta_i}^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt,$$

because

$$\frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} = \mathbb{P}(X_i > tx_1 | X_1 > x_1) \leq 1.$$

On another hand,

$$\frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} \leq \min\{1, \frac{\mathbb{P}(X_i > tx_1)}{\mathbb{P}(X_1 > x_1)}\},$$

and

$$\frac{\mathbb{P}(X_i > tx_1)}{\mathbb{P}(X_1 > x_1)} = \frac{\bar{F}_{X_i}(tx_1) \bar{F}_{X_1}(tx_1)}{\bar{F}_{X_1}(tx_1) \bar{F}_{X_1}(x_1)},$$

then, using  $\lim_{\alpha \uparrow 1} \frac{\bar{F}_{X_i}(tx_1)}{\bar{F}_{X_1}(tx_1)} = 0$  and the Potter's bounds (1.4) associated to  $\bar{F}_{X_1}$ , we deduce that for all  $\epsilon_1 > 0$  and  $0 < \epsilon_2 < 1$ , there exists  $x_1^0(\epsilon_1, \epsilon_2)$  such that for  $x_1 \geq \frac{x_1^0(\epsilon_1, \epsilon_2)}{\min\{1, \beta_i\}}$

$$\frac{\mathbb{P}(X_i > tx_1)}{\mathbb{P}(X_1 > x_1)} \leq \epsilon_1 (1 + \epsilon_2) \max(t^{-\theta+\epsilon_2}, t^{-\theta-\epsilon_2}).$$

And the application of the Dominated Convergence Theorem leads to

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \int_{\beta_i}^{+\infty} \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt = 0, \quad \forall i \in J_C.$$

We denote by  $J_\infty$  the set  $J_\infty = \{i \in \{2, \dots, d\} | \beta_i = +\infty\}$ . So, for all  $i \in J_\infty$ ,  $x_1 = o(x_i)$  and

$$\begin{aligned} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \int_{x_i}^{+\infty} \frac{\mathbb{P}(X_i > t, X_1 > x_1)}{x_1 \mathbb{P}(X_1 > x_1)} dt \\ &\leq \int_{x_1}^{+\infty} \frac{\mathbb{P}(X_i > t, X_1 > x_1)}{x_1 \mathbb{P}(X_1 > x_1)} dt = \int_1^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt. \end{aligned}$$

In the same way as in the previous case, and using the Potter's bounds, we show that

$$\lim_{\alpha \uparrow 1} \int_1^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt = \int_1^{+\infty} \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt = 0,$$

from which we deduce that

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall i \in J_\infty.$$

Let  $J_0$  be the set  $J_0 = \{i \in \{2, \dots, d\} | \beta_i = 0\}$ . For all  $i \in J_0$  we have  $x_i = o(x_1)$ , then

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \uparrow 1} \int_{x_i}^{+\infty} \frac{\mathbb{P}(X_i > t, X_1 > x_1)}{x_1 \mathbb{P}(X_1 > x_1)} dt = \lim_{\alpha \uparrow 1} \int_0^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt,$$

because  $\lim_{\alpha \uparrow 1} \frac{x_i}{x_1} = 0$  and  $\frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} \leq 1$ .

In addition, for all  $\epsilon > 0$ , we have

$$\lim_{\alpha \uparrow 1} \int_{\epsilon}^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt = 0,$$

because the Dominated Convergence Theorem is applicable using the Potter's bounds, and  $\lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} = 0$  for all  $t > 0$  since  $c_i = 0$ .

Let  $\kappa > 0$ ,  $\forall \epsilon > 0 \exists \alpha_0$  such that  $\forall \alpha > \alpha_0$

$$\int_{\epsilon}^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt < \kappa,$$

then

$$\int_0^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt = \int_0^{\epsilon} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt + \int_{\epsilon}^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt < \epsilon + \kappa,$$

we deduce that

$$\lim_{\alpha \uparrow 1} \int_0^{+\infty} \frac{\mathbb{P}(X_i > tx_1, X_1 > x_1)}{\mathbb{P}(X_1 > x_1)} dt = 0,$$

so,

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall i \in J_0.$$

We have therefore shown that

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall i \in \{2, \dots, d\}.$$

□

#### A.5. Lemma 3.2 (2).

*Proof.* We suppose that  $\lim_{\alpha \uparrow 1} \frac{1-\alpha}{\bar{F}_{X_1}(\mathbf{e}_1^{\alpha}(\mathbf{X}))} = +\infty$ . Taking if necessary a convergent subsequence  $(\alpha_n)_{n \in \mathbb{N}}$  with  $\alpha_n \rightarrow 1$ , we consider that the limits  $\lim_{\alpha \rightarrow 1} \frac{x_i}{x_1} = \beta_i$  exist and that  $\lim_{\alpha \uparrow 1} \frac{1-\alpha}{\bar{F}_{X_1}(x_1)} = +\infty$ .

We use the notations  $J_C = \{i \in \{2, \dots, d\} \mid 0 < \beta_i < +\infty\}$ ,  $J_0 = \{i \in \{2, \dots, d\} \mid \beta_i = 0\}$ , and  $J_{\infty} = \{i \in \{2, \dots, d\} \mid \beta_i = +\infty\}$ .

The first equation of the optimality system (0.1) divided by  $x_1 \bar{F}_{X_1}(x_1)$  can be written

$$\begin{aligned} \frac{(2\alpha - 1)\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} + \sum_{i=2}^d \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \left( \frac{x_1 - \mathbb{E}[X_1]}{x_1} \right) \\ &+ \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \left( \sum_{i=2}^d \frac{\mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} \right). \end{aligned}$$

By (2.1)

$$\lim_{\alpha \uparrow 1} \frac{(2\alpha - 1)\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{1}{\theta - 1},$$

and by Lemma 3.2

$$\lim_{\alpha \uparrow 1} \sum_{i=2}^d \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0,$$

so, going through the limit ( $\alpha \rightarrow 1$ ) in the previous equation leads to

$$\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \left( \frac{x_1 - \mathbb{E}[X_1]}{x_1} + \sum_{i=2}^d \frac{\mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} \right) = \frac{1}{\theta - 1},$$

nevertheless,

$$\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \left( \frac{x_1 - \mathbb{E}[X_1]}{x_1} + \sum_{i=2}^d \frac{\mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} \right) = \lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \left( 1 + \sum_{i=2}^d \frac{x_i}{x_1} \right) = +\infty.$$

From this contradiction, we deduce that the case  $\lim_{\alpha \uparrow 1} \frac{1-\alpha}{F_{X_1}(x_1)} = +\infty$  is absurd.

Now, we suppose that  $\lim_{\alpha \uparrow 1} \frac{1-\alpha}{F_{X_1}(e_1^1(\mathbf{X}))} = 0$ . Taking if necessary a subsequence  $(\alpha_n)_{n \in \mathbb{N}}$  with  $\alpha_n \rightarrow 1$ , we consider that the limits  $\lim_{\alpha \rightarrow 1} \frac{x_i}{x_1} = \beta_i$  exist and that  $\lim_{\alpha \uparrow 1} \frac{1-\alpha}{F_{X_1}(x_1)} = 0$ .

We denote  $J_C = \{i \in \{2, \dots, d\} \mid 0 < \beta_i < +\infty\}$ ,  $J_0 = \{i \in \{2, \dots, d\} \mid \beta_i = 0\}$ , and  $J_\infty = \{i \in \{2, \dots, d\} \mid \beta_i = +\infty\}$ . Going through the limit ( $\alpha \rightarrow 1$ ) in the first equation of System 0.1 divided by  $x_1 \bar{F}_{X_1}(x_1)$ , and using Lemma 3.2 and Equation 2.1, leads to

$$(A.8) \quad \lim_{\alpha \uparrow 1} \left( \frac{1-\alpha}{\bar{F}_{X_1}(x_1)} \sum_{i \in J_\infty} \frac{x_i}{x_1} \right) = \frac{1}{\theta - 1}.$$

If  $J_\infty \neq \emptyset$ , so, there exists  $i \in \{2, \dots, d\}$  such that  $i \in J_\infty$ . In this case,

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \uparrow 1} \frac{\bar{F}_{X_i}(x_i)}{\bar{F}_{X_1}(x_1)} \frac{x_i \bar{F}_{X_1}(x_1)}{x_1 \bar{F}_{X_1}(x_1)} \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} = 0,$$

because  $\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} = \frac{1}{\theta - 1}$ ,  $\lim_{\alpha \uparrow 1} \frac{\bar{F}_{X_i}(x_i)}{\bar{F}_{X_1}(x_1)} = 0$ , and by Lemma 2.2 ( $X_i = X_j = X_1$ )  $\lim_{\alpha \uparrow 1} \frac{x_i \bar{F}_{X_1}(x_1)}{x_1 \bar{F}_{X_1}(x_1)} = 0$ .

On another hand, for all  $j \in \{1, \dots, d\} \setminus \{i\}$ ,

$$\frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \int_{x_j}^{+\infty} \frac{\mathbb{P}(X_j > t, X_i > x_i)}{x_1 \mathbb{P}(X_1 > x_1)} dt,$$

so if  $j \in J_C$ , then

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \uparrow 1} \int_{x_j}^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt = \lim_{\alpha \uparrow 1} \int_{\beta_j}^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt,$$

because  $\frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} \leq \frac{\mathbb{P}(X_j > tx_1)}{\mathbb{P}(X_1 > x_1)}$  and  $\lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_j > tx_1)}{\mathbb{P}(X_1 > x_1)} = 0$  for all  $t > 0$ . We apply the dominated convergence Theorem, using Potter's bounds associated to  $\bar{F}_{X_1}$ , to get

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = \int_{\beta_j}^{+\infty} \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt,$$

and since

$$\frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} \leq \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)} = \frac{x_1 \bar{F}_{X_1}(x_i)}{x_i \bar{F}_{X_1}(x_1)} \frac{\bar{F}_{X_i}(x_i)}{\bar{F}_{X_1}(x_i)},$$

so, by Lemma 2.2

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} = \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_i > x_i)}{\mathbb{P}(X_1 > x_1)} = 0,$$

we deduce finally that

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall j \in J_C.$$

If  $j \in J_\infty \setminus \{i\}$ , then

$$\begin{aligned} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \int_{x_j}^{+\infty} \frac{\mathbb{P}(X_j > t, X_i > x_i)}{x_1 \mathbb{P}(X_1 > x_1)} dt \\ &\leq \int_{x_1}^{+\infty} \frac{\mathbb{P}(X_j > t, X_i > x_i)}{x_1 \mathbb{P}(X_1 > x_1)} dt = \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt, \end{aligned}$$

we show in the same way as in the previous case that

$$\lim_{\alpha \uparrow 1} \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt = \int_1^{+\infty} \lim_{\alpha \uparrow 1} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt = 0,$$

then

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall j \in J_\infty \setminus \{i\}.$$

If  $j \in J_0$ , then

$$\begin{aligned} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \int_{x_j}^{x_1} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{x_1 \mathbb{P}(X_1 > x_1)} dt + \int_{x_1}^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{x_1 \mathbb{P}(X_1 > x_1)} dt \\ &\leq \frac{x_1 - x_j}{x_1} \frac{\bar{F}_{X_i}(x_i)}{\bar{F}_{X_1}(x_1)} + \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt, \end{aligned}$$

since  $\lim_{\alpha \uparrow 1} \int_1^{+\infty} \frac{\mathbb{P}(X_j > tx_1, X_i > x_i)}{\mathbb{P}(X_1 > x_1)} dt = 0$ , so, by Lemma 2.2, we get

$$\lim_{\alpha \uparrow 1} \frac{x_1 - x_j}{x_1} \frac{\bar{F}_{X_i}(x_i)}{\bar{F}_{X_1}(x_1)} = \lim_{\alpha \uparrow 1} \frac{x_1 - x_j}{x_1} \frac{\bar{F}_{X_i}(x_i)}{\bar{F}_{X_1}(x_i)} \frac{x_i \bar{F}_{X_1}(x_i)}{x_1 \bar{F}_{X_1}(x_1)} \frac{x_1}{x_i} = 0,$$

we obtain from that

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall j \in J_0,$$

and consequently

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall j \in \{1, \dots, d\} \setminus \{i\}.$$

The  $i^{\text{th}}$  equation of System 0.1 divided by  $x_1 \bar{F}_{X_1}(x_1)$  can be written in the form

$$\begin{aligned} \frac{(2\alpha - 1)\mathbb{E}[(X_i - x_i)_+]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \frac{x_i - \mathbb{E}[X_i]}{x_1} &= \sum_{\substack{j=1 \\ j \neq i}}^d \frac{(1 - \alpha)\mathbb{E}[(X_j - x_j)_- \mathbb{1}_{\{X_i < x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)} \\ &\quad - \sum_{\substack{j=1 \\ j \neq i}}^d \frac{\alpha \mathbb{E}[(X_j - x_j)_+ \mathbb{1}_{\{X_i > x_i\}}]}{x_1 \bar{F}_{X_1}(x_1)}, \end{aligned}$$

going through the limit ( $\alpha \rightarrow 1$ ) in this equation leads to

$$-\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \frac{x_i}{x_1} = \lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \sum_{\substack{j=1 \\ j \neq i}}^d \frac{x_j}{x_1},$$

which is possible only if  $\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \frac{x_i}{x_1} = 0$ , and that is contradictory with Equation A.8. □

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