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# CONVEX COMPACT SURFACES WITH NO BOUND ON THEIR SYNTHETIC RICCI CURVATURE

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ABSTRACT. Using refraction in the setting of normed vector spaces allows us to present an example of convex compact surface which admit no lower bound on its Ricci curvature as defined by Lott-Villani and Sturm.

## INTRODUCTION AND STATEMENT OF RESULTS

Many notions of curvature adapted to a metric measure space have been defined to extend the ones existing in Riemannian geometry. Most of them heavily rely on comparison to the Euclidean space and that is why they are quite restrictive. For instance a normed vector space is CAT(0) if and only if it is a Euclidean space, as a consequence the only Finsler spaces which can be CAT(0) are Riemannian (see also [BI13]). The same thing happens with the Alexandroff spaces. It is even more restrictive in that case for an Alexandroff metric space is almost Riemannian manifold (see [BBI01] for a precise statement).

Some older notion, such as the Busemann convexity, are less restrictive. However they might not pass to the Gromov-Hausdorff limit of a sequence of metric measure spaces. In the light of the current interest in understanding the limit spaces arising as limits of Riemannian metric space, with Ricci curvature bounded from below for instance this is a huge flaw.

Following the work of Lott-Villani [LV09] and Sturm [Stu06a, Stu06b], a new family of notion of "curved" spaces arised. They involve the convexity of an operator on the  $L^p$ -Wasserstein space, which is a metrization of the space of probability measures with finite  $p$ -th moment. Among them one finds the space satisfying the curvature dimension condition  $CD(K, N)$  or the measure contraction property  $MCP(K, N)$ . The later may be seen as a measure analog to the Busemann convexity. The former as a generalisation of having Ricci curvature bounded from below by  $(N - 1)K$  and being of dimension less than  $N$ . We will refer to this last notion as synthetic Ricci curvature and describe such spaces as admitting a lower bound on their synthetic Ricci curvature. An example is given by normed vector space of dimension  $n$  which satisfy the curvature dimension condition  $CD(0, n)$  (This follows from proof of the interpolation inequality and concavity of  $\det^{\frac{1}{n}}$ ).

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Another point of view on curvature in metric spaces is based on analytical inequalities. For instance Cordero-Erausquin, McCann and Schmuckenschläger [CEMS01] looked at the Brascamp-Lieb inequality which is a generalisation of the Prekopa-Leindler inequality that can be used to prove the Brunn-Minkowski inequality in the Euclidean space.

The interesting aspect on which this paper is based on is that most notions of curvature deriving from the work of Lott-Villani and Sturm imply a Brunn-Minkowski inequality, hence our focus on this inequality.

Our main result is the following

**Theorem 1.** *There exist a compact  $C^{1,1}$  convex surface in  $R^3$  with the norm  $\|(x, y, z)\| = \sqrt{x^2 + y^2} + |z|$  which admits no lower bounds on its synthetic Ricci curvature.*

The idea of that example came from the study of reflections and refraction in normed (not necessarily reflexive) vector spaces. Section 2 focuses on a specific example which allows us to obtain our convex set in Section 3.

## 1. DEFINITIONS AND NOTATIONS

Let  $(X, d, \mu)$  be a metric measured space. For any pairs of point  $m_0, m_1 \in X$ , we call  $m_s \in X$  an  $s$ -intermediate point from  $m_0$  to  $m_1$  if and only if

$$d(m_0, m_s) = sd(m_0, m_1) \quad \text{and} \quad d(m_s, m_1) = (1 - s)d(m_0, m_1).$$

Let  $K_0$  and  $K_1$  be two compact sets in  $X$ , the set of  $s$ -intermediate points from points of  $K_0$  to points of  $K_1$  will be denoted by

$$(1 - s)K_0 + sK_1.$$

If  $(1 - s)K_0 + sK_1$  is not measurable, we will still denote its outer measure by

$$\mu((1 - s)K_0 + sK_1).$$

Let us first start with the classical Brunn-Minkowski inequality:

**Definition 2** (Classical Brunn-Minkowski inequality). Let  $N$  be greater than 1. We say that the Brunn-Minkowski inequality  $BM(0, N)$  holds in the metric measured space  $(X, d, \mu)$  if for every pair of compact set of non-zero measure  $K_0$  and  $K_1$ , the following inequality is satisfied

$$(1) \quad \mu^{1/N}((1 - s)K_0 + sK_1) \geq (1 - s)\mu^{1/N}(K_0) + s\mu^{1/N}(K_1).$$

We also say that  $BM(0, +\infty)$  holds if and only if

$$(2) \quad \mu((1 - s)K_0 + sK_1) \geq \mu^{1-s}(K_0)\mu^s(K_1).$$

**Remark 3.** *Notice that if for some  $n \in R^*$ , and  $t, a$  and  $b \in \mathbb{R}$  the inequality  $t \geq (sa^{1/n} + (1 - s)b^{1/n})^n$  holds, then from the concavity of*

the logarithm we have

$$\begin{aligned} \ln t &\geq n \ln sa^{1/n} + (1-s)b^{1/n} \\ &\geq s \ln a + 1 - s \ln b. \end{aligned}$$

Hence any  $B(0, N)$  implies  $B(0, \infty)$ .

Now the general Brunn-Minkowski inequality  $BM(K, N)$  requires the introduction of a family of functions depending on  $K$ ,  $N$  and  $s \in [0, 1]$  denoted by  $\tau_{K,N}^{(s)}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . For a fixed  $s \in [0, 1]$  and  $\theta \in \mathbb{R}^+$ ,  $\tau_{K,N}^{(s)}(\theta)$  is continuous, nonincreasing in  $N$  and nondecreasing in  $K$ . Its exact definition is not important for our applications, refer to [Stu06b].

**Definition 4** (Generalised Brunn-Minkowski inequality). Let  $N$  be greater than 1. We say that the Brunn-Minkowski inequality  $BM(K, N)$  holds in the metric measured space  $(X, d, \mu)$  if for every pair of compact set of non-zero measure  $K_0$  and  $K_1$ , the following inequality is satisfied

$$(3) \quad \mu^{1/N}((1-s)K_0 + sK_1) \geq \tau_{K,N}^{(1-s)}(\vartheta)\mu^{1/N}(K_0) + \tau_{K,N}^{(s)}(\vartheta)\mu^{1/N}(K_1).$$

where  $\vartheta$  is the minimal (resp. maximal) length of a geodesic between a point in  $K_0$  and a point in  $K_1$  if  $K \geq 0$  (resp.  $K < 0$ ).

We can also define the  $BM(K, +\infty)$  as follows:

$$(4) \quad \mu((1-s)K_0 + sK_1) \geq \mu^{1-s}(K_0)\mu^s(K_1)e^{Ks(1-s)\vartheta^2/2}.$$

The curvature dimension property, denoted by  $CD(K, N)$  is generalisation of the following sentence to metric measures spaces:

The space has dimension less than  $N$  and the ricci curvature is bigger than  $K$ .

It is defined in term of a convexity property of the entropy along geodesics in the space of probability of the metric space (see [Stu06b] for more precise statements).

For our purpose we only need to know the following properties of a space satisfying a curvature dimension property (see K.T. Sturm [Stu06b]).

**Property 5.** Let  $(X, d, \mu)$  be a metric measured space. The following implications are valid:

- (1) Suppose  $CD(K, N)$  holds. If  $K' \leq K$ , then  $CD(K', N)$  holds as well. If  $N' > N$ , then  $CD(K, N')$  holds as well.
- (2) Suppose  $CD(K, N)$  holds. Then for any  $\alpha, \beta > 0$ , the metric measured space  $(X, \alpha d, \beta \mu)$  satisfies the  $CD(K/\alpha^2, N)$  condition.
- (3) When  $N \geq 1$ ,  $CD(0, N)$  implies  $BM(0, N)$  and more generally  $CD(K, N)$  implies  $BM(K, N)$ .
- (4) When  $N > 1$ , then  $CD(K, N)$  implies the Bishop-Gromov volume growth inequality with the Riemannian space of constant curvature  $K$  and dimension  $N$ .

## 2. BRUNN-MINKOWSKI INEQUALITY IS NOT PRESERVED IN A TWO LAYERS BANACH SPACE

In this section we are going to consider the vector space  $\mathbb{R}^2$ , and the hyperplane  $\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ . We are going to put the classical Euclidean  $\ell^2$  norm  $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$  on the half space  $y > 0$  and the  $\ell^1$  norm  $\|(x, y)\|_1 = |x| + |y|$ , on the half space  $y < 0$ . We will denote by  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$  the metric space obtained this way.

Now let us specify the measures used here. On either half space we want a measure which is invariant by translation. This implies that in each half space it is a multiple of the Lebesgue measure  $\lambda$ . Let us take  $\lambda$  in the upper half space such that  $\pi$  is the measure of the standard Euclidean disk. Let  $\alpha > 0$  and take  $\alpha\lambda$  in the lower half space (The Busemann normalisation would consist in taking  $\alpha = \pi/2$  for instance).

**Properties 6.** *Let  $X_0 = (\rho, \theta)$  be in the upper half plane in polar coordinates and  $X_1 = (0, y)$  be in the lower half plane in cartesian coordinates ( $y < 0$ ), then*

- *the geodesic joining  $X_0$  to  $X_1$  is composed of the line segment from  $X_0$  to the origin and the origin to  $X_1$ . It is unique.*
- *The distance between  $X_0$  to  $X_1$  is equal to  $\rho - y$ .*
- *Let  $X_s$  be the  $s$ -intermediate point between  $X_0$  to  $X_1$ ,*
  - (1) *If  $s(\rho - y) < \rho$ , then  $X_s$  belongs to the upper half plane and lies on the affine segment from  $X_0$  to the origin, and  $X_s = ((1 - s)\rho + sy, \theta)$  in polar coordinates;*
  - (2) *If  $s(\rho - y) > \rho$ , then  $X_s$  belongs to the lower half plane and lies in the line  $x = 0$ , and  $X_s = (0, (1 - s)\rho + sy)$  in cartesian coordinates.*

*Proof.* The fact that this path is a geodesic is an easy computation in that case. The only thing we need to prove is uniqueness. Any geodesic between these points has to pass through the origin. Hence on the upper half plane, as there is only one geodesic between any two points we don't have any choice.

Now on the lower half plane, let  $\gamma: [0, 1] \rightarrow \mathbb{R}^2$  be a piecewise  $C^1$  path between the origin and the point  $(0, y)$ , if  $\gamma(t) = (x(t), y(t))$ , then  $t \rightarrow g(t) = (0, y(t))$  is also a piecewise  $C^1$  path between the origin and the point  $(0, y)$ . Now we have almost everywhere

$$|\dot{\gamma}(t)| \leq |\dot{x}(t)| + |\dot{y}(t)|$$

therefore the length of  $\gamma$  is bigger than the length of  $g$ , except if  $x(t) = 0$  almost everywhere. This implies that the only geodesic between  $(0, 0)$  and  $(0, y)$  is the segment between these two points. The other properties are easy to check.  $\square$

**Proposition 7.** *In the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$  no Brunn-Minkowski inequality holds, i.e., for any  $K \in \mathbb{R}$  and  $N \in \mathbb{N} \cup \{+\infty\}$ ,  $BM(K, N)$  does not hold.*

*Proof.* First one can notice that for  $N < +\infty$  the space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$  is invariant under linear dilations. This implies that if it is  $BM(K, N)$  then it is  $BM(0, N)$ .

Let  $(\rho, \theta)$  be the polar coordinates in  $\mathbb{R}^2$ . Consider the annulus

$$K_0 = \{(\rho, \theta) \mid 6 \leq \rho \leq 8, \pi/3 \leq \theta \leq 2\pi/3\},$$

and the affine segment

$$I = \{(x, y) \in \mathbb{R}^2 \mid -101 \leq y \leq -100, x = 0\}.$$

Now let  $X_0 = (\rho_0, \theta)$  be in  $K_1$ , and  $X_I = (0, -100-t)$  in  $I$ . Following the previous section, there is a unique geodesic from  $X_0$  to  $X_I$ , and is composed of the affine segment joining  $X_0$  to the origin  $O = (0, 0)$  and of the affine segment joining the origin to  $X_I$ . We therefore have  $\|X_0\|_2 = \rho_0$  and  $\|X_I\|_1 = 100 + t$ , from which we deduce that the distance between these two points is  $\rho_0 + 100 + t$ . Now following the Properties 6, as

$$(\rho_0 + 100 + t)/2 > 106/2 = 53 > 8 \geq \rho_0,$$

for  $s \geq 1/2$  the point  $X_s = (0, (1-s)\rho_0 + s(-100-t))$ , is the  $s$ -intermediate point on the geodesic from  $X_0$  to  $X_I$ , From this we easily deduce that the  $1/2$ -intermediate set from  $X_0$  to  $I$  is

$$\frac{1}{2}K_0 + \frac{1}{2}I = \{(x, y) \mid x = 0, -47,5 \leq y \leq -46\}.$$

This suffices to prove that  $BM(0, N)$  is not satisfied as

$$(\alpha\lambda)^{\frac{1}{N}} \left( \frac{1}{2}K_0 + \frac{1}{2}I \right) = 0 < \frac{1}{2}\lambda^{\frac{1}{N}}(K_0)$$

(Actually, we have that the space is not  $MCP(0, N)$ ).

Now let us prove that  $BM(K, +\infty)$  is never satisfied. For  $s > 1/2$  the  $s$ -intermediate set from  $X_0$  to  $I$  is easily seen to be

$$(1-s)K_0 + sI = \{(0, y) \in \mathbb{R}^2 \mid -101s + 6(1-s) \leq y \leq -100s + 8(1-s)\}.$$

We start by considering some  $0 < \varepsilon < 1$ , whose value will be chosen at the end, and replace  $I$  with

$$K_1 = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq \varepsilon, |y + 100, 5| \leq 0, 5\}.$$

The next step is to introduce the slices of  $K_1$ :

$$I_\alpha = \{(x, y) \in \mathbb{R}^2 \mid -101 \leq y \leq -100, x = \alpha\}$$

for  $0 < \alpha \leq \varepsilon < \rho$ , and to identify their intermediate sets  $(1-s)K_0 + sI_\alpha$ . In order to do this we compute the distance between  $X_0$  and  $(\alpha, 0)$ , which gives

$$\rho_\alpha = \sqrt{\rho_0^2 - 2\alpha\rho_0 \cos \theta + \alpha^2},$$

and it is now easy to check that for  $X_0$  in  $K_0$  we have

$$(5) \quad \begin{aligned} (6 - \alpha) \leq (\rho_0 - \alpha) &< \sqrt{\rho_0^2 - 2\alpha\rho_0 + \alpha^2} \leq \\ \rho_\alpha &\leq \sqrt{\rho_0^2 + 2\alpha\rho_0 + \alpha^2} < (\rho_0 + \alpha) \leq (8 + \alpha). \end{aligned}$$

The description we were seeking is therefore (recall that  $s > 1/2$ )

$$(6) \quad \begin{aligned} (1-s)K_0 + sI_\alpha = \\ \left\{ (\alpha, y) \in \mathbb{R}^2 \mid -101s + (1-s)\sqrt{6^2 - 6\alpha + \alpha^2} \leq y \right. \\ \left. \leq -100s + \sqrt{8^2 + 8\alpha + \alpha^2}(1-s) \right\}. \end{aligned}$$

To obtain an upper bound on its area we notice that it can be seen as a subset as follows

$$(7) \quad \begin{aligned} (1-s)K_0 + sI_\alpha \subset \\ \left\{ (\alpha, y) \in \mathbb{R}^2 \mid -101s + (1-s)(6 - \alpha) \leq y \right. \\ \left. \leq -100s + (1-s)(8 + \alpha) \right\}, \end{aligned}$$

Therefore the area of the intermediate set  $K_s = (1-s)K_0 + sK_1$  is less than

$$\varepsilon \cdot (16 - 15s)$$

up to some multiplicative constant  $C$ , depending on the normalisation chosen for the Lebesgue measure.

This also tells us (depending on the sign of  $K$ ) that (see definition 3 for the definition of  $\vartheta$ )

$$(8) \quad 105 \leq 106 - \varepsilon \leq \vartheta(\varepsilon) \leq 108 + \varepsilon \leq 109.$$

The area of  $K_1$  is exactly  $2\varepsilon$ . Hence, for some fixed constant  $C$  we have

$$(9) \quad \frac{\mu((1-s)K_0 + sK_1)}{\mu^s(K_1)} \leq \varepsilon^{1-s} \cdot C^{1-s} \cdot \left(8 - \frac{15}{2}s\right)$$

We need now to compare, as  $s \rightarrow 1$ , the right hand part of (9) with

$$\mu^{1-s}(K_0)e^{Ks(1-s)\vartheta(\varepsilon)^2/2},$$

which is the same as comparing  $\varepsilon \cdot C \cdot \left(8 - \frac{15}{2}s\right)^{1/(1-s)}$  with

$$\mu(K_0)e^{Ks\vartheta(\varepsilon)^2/2}.$$

This last term converges towards  $\mu(K_0)e^{K\vartheta(\varepsilon)^2/2}$ , while the first to  $\varepsilon \cdot C \cdot e^{15/2}$ .

To conclude, as  $\vartheta(\varepsilon)$  stays bounded, we can find and fix an  $\varepsilon$  small enough such that

$$\varepsilon \cdot C \cdot e^{15/2} < \frac{1}{2}\mu(K_0)e^{K\vartheta(\varepsilon)^2/2}.$$

Then, for values of  $s$  close enough to 1, we will obtain

$$(10) \quad \mu^{1-s}(K_0)e^{Ks(1-s)\vartheta(\varepsilon)^2/2} > \frac{\mu((1-s)K_0 + K_1)}{\mu^s(K_1)},$$

which contradicts  $BM(K, +\infty)$ .  $\square$

**Proposition 8.** *There exists a Minkowski norm  $f$  on  $\mathbb{R}^2$  such that that  $BM(-1, +\infty)$  does not hold in the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, f)$ .*

*Proof.* Recall that a Minkowski norm  $f$  is twice differentiable on  $\mathbb{R}^2 \setminus \{0\}$ , with a definite positive Hessian.

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of Minkowski norms, converging towards the  $\ell^1$  norm. Up to a rescaling we can suppose that the intersection of their unit ball with  $\mathcal{H}$  coincide with the intersection of the unit ball of the  $\ell^1$  norm. In any case we will consider the measures  $\mu_n$  such that  $\mu_n = \lambda$  on the upper half plane, and  $\mu_n = \alpha_n \lambda$  on the lower half plane where

$$\alpha_n = \frac{\pi}{\lambda(\{f_n \leq 1\})}.$$

(Observe also that we can chose the norms  $f_n$  such that their tangents at their point of intersection with  $H$  is orthogonal to  $H$ . This will be useful in the last section of this paper.)

Then the sequence of metric spaces  $(\mathbb{R}^2, \ell^2, \mathcal{H}, f_n)$  converges in the Gromov-Hausdorff measured topology towards  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$ .

Consider again the sets  $K_0$  and  $K_1$  and the intermediate set  $K_s$  as in the proof of Proposition 7. Then for any  $n$  we would get another intermediate set  $K_s(n)$ , and another function  $\theta(n)$  which is the maximum (resp. Minimum) between two points from  $K_0$  to  $K_1$  or from  $K_1$  to  $K_0$ . Following our assumption we have that  $\theta(n)$  converges towards the  $\theta$  of the limit,  $\mu(K_0)$  does not change and  $\mu_n(K_1)$  converges towards  $\mu(K_1)$  thanks to the gromov-hausdorff measured convergence.

We suppose that  $s$  is close enough to 1 to be on the lower half plane.

We need to prove that  $\lim \mu_n(K_s(n)) \leq \mu(K_s)$  as  $n$  goes to infinity. First notice that  $K_s$  is a compact closed set, and so are the sets  $K_s(n)$ . Secondly, the geodesics from a point on the upper half space to the lower half space are unique, because both norms are strictly convex. Hence the geodesics are converging to the geodesics, thus  $K_s(n)$  converges to a subset  $K'_s$  of  $K_s$ .

Therefore we get  $\lim \mu_n(K_s(n)) \leq \mu(K'_s) \leq \mu(K_s)$ .

Now let us take  $K_0, K_1$  as in the proof of Proposition 7 and  $s$  close enough to one such that

$$\lambda^{1-s}(K_0)e^{-s(1-s)\vartheta(\varepsilon)^2/2} > \alpha^{1-s} \frac{\lambda((1-s)K_0 + K_1)}{\lambda^s(K_1)},$$



then for any  $n$  large enough we would also get

$$\lambda^{1-s}(K_0)e^{-s(1-s)\vartheta(\varepsilon)^2/2} > \alpha_n^{1-s} \frac{\lambda(K_s(n))}{\lambda^s(K_1)},$$

which concludes our proof, because any  $f_n$  for  $n$  large enough can be chosen.  $\square$

### 3. A COMPACT FINSLER SURFACE WITH NO LOWER RICCI BOUND EMBEDDED IN A MINKOWSKI SPACE

**3.1. First example.** Let us consider in the Euclidean three-dimensional space, the two-dimensional disk

$$\mathcal{S} = \{(x, y, z) \mid z = 0, x^2 + y^2 \leq 1\},$$

and let  $\mathcal{B}$  be the convex hull of

$$\mathcal{S} \cup \{(0, 0, 1), (0, 0, -1)\}.$$

We now endow  $\mathbb{R}^3$  with the norm  $\|\cdot\|_{\mathcal{B}}$  whose unit ball is  $\mathcal{B}$ . In other words for any  $(x, y, z) \in \mathbb{R}^3$ ,

$$\|(x, y, z)\|_{\mathcal{B}} = \sqrt{x^2 + y^2} + |z|.$$

The affine planes normal to the vector  $(0, 0, 1)$  endowed with the norm induced by  $\|\cdot\|_{\mathcal{B}}$  are all isometric to the two dimensional Euclidean plane. In the same way, the affine planes containing the direction  $(0, 0, 1)$  are isometric to the  $\ell^1$ -plane (i.e., that is the manhattan distance).

In this normed vector space, we will consider the cube  $\mathcal{C}_\rho$  obtained as the convex hull of the eight points

$$\{(\pm\rho, \pm\rho, \pm\rho)\}.$$

The cube  $\mathcal{C}_\rho$  admits two faces which are Euclidean, and four faces which are  $\ell^1$ . The measures considered are the Hausdorff measures. In other words,  $\lambda(B) = \frac{4}{3}\pi$  and for any linear subspace  $L$  of dimension 2, the measure is the Lebesgue measure  $\lambda_L$  normalised such that

$$\lambda_L(\mathcal{B} \cap L) = \pi.$$

**Proposition 9.** *Let  $\mathbb{R}^3$  be endowed with the norm  $\|\cdot\|_{\mathcal{B}}$ . Then the cube  $\mathcal{C}_1$  with the metric induced by  $\|\cdot\|_{\mathcal{B}}$  does not satisfy any curvature dimension.*

*Proof.* Let us denote by  $d_\rho$  the distance induced on  $\mathcal{C}_\rho$  by  $\|\cdot\|_{\mathcal{B}}$  and  $\mu_\rho$  the induced Lebesgue measure. Focus on two adjacent faces of  $\mathcal{C}_\rho$ , one Euclidean and the second one  $\ell^1$ . Then we are locally exactly as in section 2, and therefore the same computations as in section 2 show that for any  $\rho \in \mathbb{R}^*$ , the Brunn-Minkowski  $BM(K, N)$  inequality does not hold for any  $N \in \mathbb{N} \cup \{+\infty\}$  and any  $K \in \mathbb{R}$ .

Therefore in  $(\mathcal{C}_\rho, d_\rho, \lambda_\rho)$  the curvature dimension  $CD(K, N)$  does not hold for any  $K$  and any  $N$ .  $\square$

**Corollary 10.** *There exists a  $C^{1,1}$  compact and convex surface in  $(\mathbb{R}^3, \|\cdot\|_B)$  such that for any  $N \in \mathbb{N} \cup \{+\infty\}$  and any  $K \in \mathbb{R}$ ,  $CD(K, N)$  does not hold.*

Remark here that in our example both the  $C^{1,1}$  assumption and the fact that the norm is not smooth restricted to the surface are important. If the objects are too smooth, there is always some  $K$  and  $N$  for which it is  $CD(K, N)$ .

*Proof.* let  $B(\varepsilon)$  be the Euclidean ball of radius  $\varepsilon$ . Consider the Minkowski sum of the cube and this ball, that is,

$$C(\varepsilon) = B(\varepsilon) + \mathcal{C}_1 = \{x + y \mid x \in B(\varepsilon), y \in \mathcal{C}_1\},$$

with the unduced metric by  $\|\cdot\|_B$ . Then  $C(\varepsilon)$  is  $C^{1,1}$ , and as  $\varepsilon$  goes to zero, it converges in the Gromov-Hausdorff measured topology towards  $\mathcal{C}_1$ .

Actually  $C(\varepsilon)$  is obtained by translating the faces of the cube  $\mathcal{C}_1$  outward at a Euclidean distance  $\varepsilon$  and then closing by rolling the Euclidean ball of radius  $\varepsilon$  along the edges, from the inside.

Hence the difference is on the surface obtained along these curved edges. On the flat section we have the same distance than in  $\mathcal{C}_1$ .

Fix some  $K = -1$  and  $N = +\infty$ . We can use the annulus  $K_0$  and the rectangle  $K_1$  from the proof of Proposition 7, the only thing that will change is the  $s$ -intermediate set from  $K_0$  to  $K_1$ , denoted by  $K_s(\varepsilon)$ .

Fix an  $s$  such that we get the inequality (10) as in proof of Proposition 7 for  $K_0$ ,  $K_1$  and  $K_s$ .

Then as  $\varepsilon$  goes to zero the corresponding sequence of  $s$ -intermediate sets  $K_s(\varepsilon)$  converges towards a subset of  $K_s$  and thus

$$\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon(K_s(\varepsilon)) \leq \lambda(K_s),$$

where  $K_s$  is the same as in the proof of Proposition 7. Hence for some  $\varepsilon$  small enough, we would get the same contradiction.

Let us now fix such an  $\varepsilon$  for  $K = -1$ .

Let  $h_\rho$  be the dilation of ratio  $\rho$  of center the origin. Consider the images of  $K_0$ ,  $K_1$  and  $K_s(\varepsilon)$  by  $h_\rho$ , they all lie on  $C_\rho + B(\rho \cdot \varepsilon)$ . Furthermore the image of  $K_s(\varepsilon)$  by  $h_\rho$  is the  $s$ -intermediate set from  $h_\rho(K_0)$  to  $h_\rho(K_1)$  on  $C_\rho + B(\rho \cdot \varepsilon)$ . Therefore we still get the inequality 10 which is invariant by dilations, which proves that  $C_\rho + B(\rho \cdot \varepsilon)$  is not  $BM(-1, +\infty)$  as well.

Hence for any  $\rho > 0$ ,  $C_\rho + B(\rho \cdot \varepsilon)$  is not  $BM(-1, +\infty)$  (and not  $CD(-1, +\infty)$ ).

Now let us suppose that  $C_1 + B(\varepsilon)$  is  $CD(K, N)$  for some  $K < -1$ . Then  $(C_1 + B(\varepsilon), \rho d, \rho^2 \lambda)$  is  $CD(K/\rho^2, N)$ . Observe now that  $h_\rho$  is an isometry between  $(C_1 + B(\varepsilon), \rho d, \rho^2 \lambda)$  and  $C_\rho + B(\rho \cdot \varepsilon)$ , because,

$$d(h_\rho(x), h_\rho(y)) = \rho \cdot d(x, y)$$

but then for  $\rho^2 > -K$ , we get that  $C_\rho + B(\rho \cdot \varepsilon)$  is  $C(-1, N)$ , which contradicts the choice of  $\varepsilon$ .  $\square$

The question I am often asked with this example is why  $C(\varepsilon)$  does not satisfies some  $CD(K, N)$  with  $K \rightarrow -\infty$  as  $\varepsilon \rightarrow 0$ ? In the proof one can see that this is due to the very nature of all the objects defined here which behave nicely with respect to dilation on one side, and the translation on the other side. That is to say that the very specificity of the Lebesgue measure, that its homogeneity by dilation and invariance by translation are important here.

Another point of view should be from the point of view of optics, as was explained to me a long time ago. The laws of refraction are an approximation, that is to say that in reality there is no discontinuity of the differential of a ray of light, but to our eyes it looks like that. In other words the intersection between two media behaves as  $C(\varepsilon)$  for  $\varepsilon$  small, but our  $CD(K, N)$ -eyes see  $C(0) = \mathcal{C}_1$ .

**3.2. Second example.** This second example is to justify that one can get an example with a smoother norm.

Let  $\mathcal{H}$  in  $\mathbb{R}^3$  be the  $x$ -axis (that is the line  $z = 0$  and  $y = 0$ ). Consider  $f$  a norm in the plane  $y = 0$  such that  $(y = 0, \ell^2, \mathcal{H}, f)$  does not satisfy  $CD(-1, +\infty)$  as in proposition 8. Then consider  $\mathcal{B}_f$  the convex obtained by rotating the norm  $f$  around the  $z$ -axis.

Then let us denote by  $\|\cdot\|_f$  the norm whose unit ball coincide with  $\mathcal{B}_f$ .

**Proposition 11.** *There exists a  $C^{1,1}$  compact and convex surface in  $(\mathbb{R}^3, \|\cdot\|_f)$  such that for any  $N \in \mathbb{N} \cup \{+\infty\}$  and any  $K \in \mathbb{R}$ ,  $CD(K, N)$  does not hold.*

*Proof.* Again, let us consider the family of cubes  $C_\rho$  with our two sets  $K_0$  and  $K_1$ . Then for any  $\rho$ ,  $C_\rho$  with the induced metric in not  $BM(-1, +\infty)$ .

Then let us also consider  $C_1 + B(\varepsilon)$ , then for some  $\varepsilon$  small enough it will not be  $BM(-1, +\infty)$  as in the previous example. And again, by homotating the sets contradicting  $BM(-1, +\infty)$ , we obtain that for any  $\rho > 0$ ,  $C_\rho + B(\varepsilon \cdot \rho)$  is not  $BM(-1, +\infty)$ .

Again the same reasoning by contradiction as in the proof of Corollary 10 shows that  $C_1 + B(\varepsilon)$  can not satisfy any  $CD(K, N)$ , for any  $K$  and any  $N$ .  $\square$

#### 4. CONCLUDING REMARKS

The current work has been the subject of various talk and discussions with many colleagues having their own idea of what is a good notion of curvature in metric measured spaces.

The first main problem which forbids the notion of synthetic Ricci curvature to apply in our first example is the branching occuring when

one passes from one media to another. It is also related to the Finslerian nature of our spaces.

Both these problems excludes all the notions of curvatures that have been presented to us by our various colleagues. For instance one could decide to work with spaces admitting a Gromov-Bishop comparison theorem, as some nice theorem and results in Riemannian geometry are actually based on the fact that manifolds with Ricci curvature bounded from below admits such a comparison. An easy computation shows that the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$  does not satisfy such a comparison with the standard hyperbolic plane.

Notice that by smoothing our norm, we still got a surface without synthetic Ricci curvature bounded from below, but without branching. This illustrate the fact that by being close to a branching space is also problematic.

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