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# CONVEX COMPACT SURFACES WITH NO BOUND ON THEIR SYNTHETIC RICCI CURVATURE

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ABSTRACT. The study of the law of reflection and Snell's law of refraction in the setting of (weak) normed vector spaces allows us to present an example of convex compact surface which admit no lower bound on its Ricci curvature as defined by Lott-Villani and Sturm.

## INTRODUCTION AND STATEMENT OF RESULTS

Many notions of curvature adapted to a metric (measured) space have been defined to extends the ones existing in Riemannian geometry. Most of them heavily rely on comparison to the Euclidean space and that is why they are quite restrictive. For instance a normed vector space is CAT(0) if and only if it is a Euclidean space, as a consequence the only Finsler spaces which can be CAT(0) are Riemannian. The same thing happens with the Alexandroff spaces. It is even more restrictive in that case for an Alexandroff metric space is almost Riemannian manifold (see [BBI01] for a precise statement).

Some older notion, such as the Busemann convexity, are less restrictive. However they might not pass to the Gromov-Hausdorff limit of a sequence of metric measured spaces. In the light of the current interest in understanding the limit spaces arising as limits of Riemannian metric space, with Ricci curvature bounded from below for instance this is a huge flow.

Following the work of Villani-Lott and Sturm, a new family of notion of "curved" spaces arised. They involve the convexity of an operator on the Wasserstein space, which is a metrization of the space of probability measures. Among them one finds the space satisfying the curvature dimension condition  $CD(K, N)$  or the measure contraction property  $MCP(K, N)$ . The later may be seen as a measure analog to the Busemann convexity. The former as a generalisation of having Ricci curvature bounded from below by  $(N - 1)K$  and being of dimension less than  $N$ . We will refer to this last notion as synthetic Ricci curvature and describe such spaces as admitting a lower bound on their synthetic Ricci curvature. An exemple is given by normed vector space of dimension  $n$  which satisfy the curvature dimension condition  $CD(0, n)$ . This result is not easily obtained and requires to use the fact that a

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normed vector space can be seen as a limit of  $CD(0, n)$ -metric spaces with respect to the Gromov-Hausdorff convergence (see [Vil09]).

Another point of view on curvature in metric spaces is based on analytical inequalities. For instance Cordero-Erausquin, McCann and Schmuckenschläger looked at the Brascamp-Lieb inequality which is a generalisation of the Prekopa-Leindler that can be used to prove the Brunn-Minkowski inequality in the euclidean space.

The interesting aspect on which this paper is based on is that most notions of curvature deriving from the work of Villani-Lott and Sturm imply a Brunn-Minkowski inequality, hence our focus on this inequality.

Our main result is the following

**Theorem 1.** *There exist a compact  $C^{1,1}$  convex surface in  $R^3$  with the norm  $\|(x, y, z)\| = \sqrt{x^2 + y^2} + |z|$  which admits no lower bounds on its synthetic Ricci curvature.*

The idea of that exemple came from the study of reflections and refraction in normed (not necessarily reflexive) vector spaces. In relation with this we describe and prove what we think is the the correct generalisation of the law of reflection and Snell's law of refractions in any normed vector space. The section 2,3 and 4 are devoted to this generalisation and are interesting on their own. Section 5 studies a specific exemple which allows us to obtain our convex set in Section 6.

#### AKNOWLEDGMENT

The author is strongly indebted to Juan Carlos Álvarez Paiva who introduced him to the reflections in normed spaces with strictly convex unit balls, and shared many hours in Lille on this subject and many other fruitfull mathematical insights.

I therefore have to present my apologies to him to have use this very beautiful geometric construction for my own selfish interest in synthetic ricci curvature, and by using so much analysis to prove all this!

#### 1. DEFINITIONS AND NOTATIONS

Let  $(X, d, \mu)$  be a metric measured space. For any pairs of point  $m_0, m_1 \in X$ , we call  $m_s \in X$  an  $s$ -intermediate point from  $m_0$  to  $m_1$  if and only if

$$d(m_0, m_s) = sd(m_0, m_1) \quad \text{and} \quad d(m_s, m_1) = (1 - s)d(m_0, m_1).$$

Let  $K_0$  and  $K_1$  be two compact sets in  $X$ , the set of  $s$ -intermediate points from points of  $K_0$  to points of  $K_1$  will be denoted by

$$(1 - s)K_0 + sK_1.$$

If  $(1 - s)K_0 + sK_1$  is not measurable, we will still denote its outer measure by

$$\mu((1 - s)K_0 + sK_1).$$

Let us first start with the classical Brunn-Minkowski inequality:

**Definition 2** (Classical Brunn-Minkowski inequality). Let  $N$  be greater than 1. We say that the Brunn-Minkowski inequality  $BM(0, N)$  holds in the metric measured space  $(X, d, \mu)$  if for every pair of compact set of non-zero measure  $K_0$  and  $K_1$ , the following inequality is satisfied

$$(1) \quad \mu^{1/N}((1-s)K_0 + sK_1) \geq (1-s)\mu^{1/N}(K_0) + s\mu^{1/N}(K_1).$$

We also say that  $BM(0, +\infty)$  holds if and only if

$$(2) \quad \mu((1-s)K_0 + sK_1) \geq \mu^{1-s}(K_0)\mu^s(K_1).$$

**Remark 3.** Notice that if for some  $n \in \mathbb{R}^*$ , and  $t, a$  and  $b \in \mathbb{R}$  the inequality  $t \geq (sa^{1/n} + (1-s)b^{1/n})^n$  holds, then from the concavity of the logarithm we have

$$\begin{aligned} \ln t &\geq n \ln sa^{1/n} + (1-s)b^{1/n} \\ &\geq s \ln a + 1 - s \ln b. \end{aligned}$$

Hence any  $B(0, N)$  implies  $B(0, \infty)$ .

Now the generale Brunn-Minkowski inequality  $BM(K, N)$  requires the introduction of a family of functions depending on  $K, N$  and  $s \in [0, 1]$  denoted by  $\tau_{K,N}^{(s)}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . For a fixed  $s \in [0, 1]$  and  $\theta \in \mathbb{R}^+$ ,  $\tau_{K,N}^{(s)}(\theta)$  is continuous, nonincreasing in  $N$  and nondecreasing in  $K$ . Its exact definition is not important for our applications, refer to [Stu06b].

**Definition 4** (Generalised Brunn-Minkowski inequality). Let  $N$  be greater than 1. We say that the Brunn-Minkowski inequality  $BM(K, N)$  holds in the metric measured space  $(X, d, \mu)$  if for every pair of compact set of non-zero measure  $K_0$  and  $K_1$ , the following inequality is satisfied

$$(3) \quad \mu^{1/N}((1-s)K_0 + sK_1) \geq \tau_{K,N}^{(1-s)}(\theta)\mu^{1/N}(K_0) + \tau_{K,N}^{(s)}(\theta)\mu^{1/N}(K_1).$$

where  $\theta$  is the minimal (resp. maximal) length of a geodesic between a point in  $K_0$  and a point in  $K_1$  if  $K \geq 0$  (resp.  $K < 0$ ).

We can also define the  $BM(K, +\infty)$  as follows:

$$(4) \quad \mu((1-s)K_0 + sK_1) \geq \mu^{1-s}(K_0)\mu^s(K_1)e^{Ks(1-s)\theta^2/2}.$$

The curvature dimension property, denoted by  $CD(K, N)$  is generalisation of the following sentence to metric measures spaces:

The space has dimension less than  $N$  and the ricci curvature is bigger than  $K$ .

It is defined in term of a convexity property of the entropy along geodesics in the space of probability of the metric space (see [Stu06b] for more precise statements).

For our purpose we only need to know the following properties of a space satisfying a curvature dimension property (see K.T. Sturm [Stu06b]).

**Property 5.** *Let  $(X, d, \mu)$  be a metric measured space. The following implications are valid:*

- (1) *Suppose  $CD(K, N)$  holds. If  $K' \leq K$ , then  $CD(K', N)$  holds as well. If  $N' > N$ , then  $CD(K, N')$  holds as well.*
- (2) *Suppose  $CD(K, N)$  holds. Then for any  $\alpha, \beta > 0$ , the metric measured space  $(X, \alpha d, \beta \mu)$  satisfies the  $CD(K/\alpha^2, N)$  condition.*
- (3) *When  $N \geq 1$ ,  $CD(0, N)$  implies  $BM(0, N)$  and more generally  $CD(K, N)$  implies  $BM(K, N)$ .*
- (4) *When  $N > 1$ , then  $CD(K, N)$  implies the Bishop-Gromov volume growth inequality with the riemannian space of constant curvature  $K$  and dimension  $N$ .*

## 2. PRELUDES ON CONVEXE SETS

Let  $\mathcal{A}$  be a real affine space of finite dimension. We remind the reader that such a space is naturally endowed with the action of a real vector space  $\vec{D}$ , called its direction, such that:

- (1)  $\forall a, b \in \mathcal{A}$  there exists a unique  $\vec{v} \in \vec{D}$  such that  $a + \vec{v} = b$ . That vector is usually denoted by  $\vec{ab}$  (or  $b - a$  when there is no ambiguity). For  $a$  fixed the vectors  $\vec{ab}$  are onto  $\vec{D}$ .
- (2) The Chasles relation is satisfied:  $\vec{ab} + \vec{bc} = \vec{ac}$ .

More important yet, for any family of points  $a_1, \dots, a_k$  in  $\mathcal{A}$  and weights  $\lambda_1, \dots, \lambda_k$  in  $\mathbb{R}$  such that  $\sum_{i=1}^k \lambda_i = 1$  there exists a unique point  $g$ , called barycenter, such that

$$\sum_{i=1}^k \lambda_i \vec{ga}_i = 0.$$

The point  $g$  is usually denoted by  $\sum_{i=1}^k \lambda_i a_i$ .

Affine functions from  $\mathcal{A}$  to  $\mathbb{R}$  (resp. any other real affine space) are the maps  $l$  preserving the barycenter, i.e., for any  $a_1, \dots, a_k$  in  $\mathcal{A}$  and weights  $\lambda_1, \dots, \lambda_k$  in  $\mathbb{R}$  such that  $\sum_{i=1}^k \lambda_i = 1$  we have

$$l\left(\sum_{i=1}^k \lambda_i a_i\right) = \sum_{i=1}^k \lambda_i l(a_i).$$

An affine function  $l$  admits a unique linear part  $\vec{l}$  from  $\vec{D}$  to  $\mathbb{R}$  (resp. the direction of the targeted affine space), such that, for any  $a, b \in \mathcal{A}$  we have

$$l(b) = l(a) + \vec{l}(\vec{ab}).$$

For our purpose let us also recall that the inverse image of a real number by such an affine map is an affine hyperplane (resp. affine subspace) whose direction is the kernel of  $\vec{l}$ , denoted by  $\ker \vec{l}$ .

Let us now consider two convex functions  $n_1$  and  $n_2$  from  $\vec{D}$  to  $\mathbb{R}$ . In the sequel these maps will be weak norms, that is positive homogeneous functions satisfying the triangle inequality.

**Definition 6.** For any pair of points  $x, y$  in  $\mathcal{A}$ , we call *generalised ellipsoid* from  $x$  to  $y$  with respect to  $n_1$  and  $n_2$  the sets

$$\mathcal{E}_{x,y}(r, n_1, n_2) = \{z \mid n_1(\vec{xz}) + n_2(\vec{zy}) \leq r\}.$$

In the Euclidean geometry one property of ellipsoids, is that when a ray of light originating from one focus reflects off its inner surface, it always passes through the other focus.

Another way of stating this is that given two points  $A$  and  $B$  and a hyperplane  $\mathcal{H}$ , if a ray starts at the point  $A$  and reflects on the hyperplane such that its reflection passes through the point  $B$ , then the hyperplane is supporting an ellipsoid whose focal points are  $A$  and  $B$ .

To generalise this in an affine setting, we need a completely angle free point of view. That is the scope of the next proposition. The sequel of this paper will show that this is indeed the correct point of view if one requires the ray of lights to take the shortest path.

**Proposition 7** (Generalised law of reflection/refraction). *The generalised ellipsoid  $\mathcal{E}_{x,y}(r, n_1, n_2)$  is a convex subset of  $\mathcal{A}$ . Furthermore,  $\mathcal{H}$  is a supporting hyperplane of this generalised ellipsoid at the point  $z$  if and only if there exists*

- a supporting hyperplane  $h_1$  of  $B_1(x) = \{a \mid n_1(\vec{xa}) \leq n_1(\vec{xz})\}$  at  $z$ , and
- a supporting hyperplane  $h_2$  of  $B_2(x) = \{a \mid n_2(\vec{ay}) \leq n_2(\vec{zy})\}$  at  $z$

whose directions have the same intersection with the direction of  $\mathcal{H}$ .

*Proof.* The convexity follows from the convexity of  $n_1$  and  $n_2$ : Let  $a$  and  $b$  two points in  $\mathcal{E}_{x,y}(r, n_1, n_2)$  and let  $t \in [0, 1]$ , then by definition of the barycentre  $ta + (1-t)b$ :

$$(5) \quad \overrightarrow{x(ta + (1-t)b)} = t\vec{xa} + (1-t)\vec{xb}$$

$$(6) \quad \overrightarrow{(ta + (1-t)b)y} = t\vec{ay} + (1-t)\vec{by}.$$

hence by convexity of  $n_1$  and  $n_2$  we have

$$\begin{aligned} n_1\left(x(ta + (1-t)b)\right) + n_2\left((ta + (1-t)b)y\right) &\leq t\left(n_1(\vec{xa}) + n_2(\vec{ay})\right) \\ &\quad (1-t)\left(n_1(\vec{xb}) + n_2(\vec{by})\right) \leq tr + (1-t)r = r \end{aligned}$$

The rest of the proof is an adaptation of a proof from Rockafellar [Roc97]. let  $\mathcal{H}$  be a supporting hyperplane of  $\mathcal{E}_{x,y}(r, n_1, n_2)$  at  $z$ ,

then by convexity of the function

$$a \mapsto n_1(\vec{x}\vec{a}) + n_2(\vec{a}\vec{y})$$

there exists a supporting hyperplane  $\tilde{\mathcal{H}}$  of

$$\mathcal{E} = \{(a, r) \in \mathcal{A} \times \mathbb{R} \mid n_1(\vec{x}\vec{a}) + n_2(\vec{a}\vec{y}) \leq r\}.$$

at  $(z, r)$  defined by an affine function  $H: \mathcal{A} \rightarrow \mathbb{R}$ , i.e., such that for all  $a \in \mathcal{A}$  we have

$$H(a) \leq n_1(\vec{x}\vec{a}) + n_2(\vec{a}\vec{y}) \text{ and } H(z) = r,$$

$\tilde{\mathcal{H}}$  is the graph of  $H$  in  $\mathcal{A} \times \mathbb{R}$ , and  $\mathcal{H} = H^{-1}(r)$ .

Let us consider the two following subsets of  $\mathcal{A} \times \mathbb{R}$ :

$$\begin{aligned} \mathcal{F} &= \{(a, \mu) \in \mathcal{A} \times \mathbb{R} \mid n_1(\vec{x}\vec{a}) \leq n_1(\vec{z}\vec{x}) + \vec{H}(\vec{z}\vec{a}) + \mu\} \\ \mathcal{G} &= \{(a, \mu) \in \mathcal{A} \times \mathbb{R} \mid n_2(\vec{a}\vec{y}) \leq n_2(\vec{z}\vec{y}) - \mu\} \end{aligned}$$

By convexity of  $n_1$  and  $n_2$  both sets  $\mathcal{F}$  and  $\mathcal{G}$  are convex.

If we consider their relative interior, that is the interior of these sets in the smaller affine space containing them, we can see that they are disjoint. Indeed suppose that we have a common point, that would mean that we have the existence of  $(v, \mu) \in X \times \mathbb{R}$  such that

$$n_1(\vec{x}\vec{a}) < n_1(\vec{z}\vec{x}) + \vec{H}(\vec{z}\vec{a}) + \mu \text{ and } n_2(\vec{a}\vec{y}) \leq n_2(\vec{z}\vec{y}) - \mu$$

which would imply that

$$\begin{aligned} (7) \quad n_1(\vec{x}\vec{a}) + n_2(\vec{a}\vec{y}) &< n_1(\vec{z}\vec{x}) + n_2(\vec{z}\vec{y}) + \vec{H}(\vec{z}\vec{a}) \\ &= H(z) + \vec{H}(\vec{z}\vec{a}) = H(a) \end{aligned}$$

which contradicts the definition of  $H$ .

Hence by the Hahn-Banach separation theorem we have the existence of hyperplane  $\mathcal{L}$  in  $\mathcal{A} \times \mathbb{R}$  which separates these convex sets. That hyperplane can be defined thanks to an affine map  $\Lambda: \mathcal{A} \rightarrow \mathbb{R}$  and two numbers  $\lambda, c$  such that

$$(a, \mu) \in \mathcal{L} \iff \vec{\Lambda}(\vec{z}\vec{a}) + \lambda\mu = c.$$

Notice now the following two facts:

- (1)  $\lambda \neq 0$  because otherwise the hyperplane defined by  $\vec{\Lambda}(\vec{z}\vec{a}) = c$  would separate the projections of  $\mathcal{F}$  and  $\mathcal{G}$  onto  $\mathcal{A}$ , which are both  $\mathcal{A}$ .
- (2)  $c = 0$ , because  $(z, 0)$  belongs to both  $\mathcal{F}$  and  $\mathcal{G}$ .

Therefore, if  $\vec{l} = -\vec{\Lambda}/\lambda$ , we can suppose that

$$\begin{cases} \vec{l}(\vec{z}\vec{a}) \leq \mu & \forall (v, \mu) \in \mathcal{F} \\ \vec{l}(\vec{a}\vec{z}) \leq -\mu & \forall (v, \mu) \in \mathcal{G} \end{cases}$$

Which implies that

$$(8) \quad \vec{l}(\vec{z}\vec{a}) + \vec{H}(\vec{z}\vec{a}) \leq n_1(\vec{x}\vec{a}) - n_1(\vec{x}\vec{z}), \forall a \in \mathcal{A}$$

$$(9) \quad \vec{l}(\vec{a}\vec{z}) \leq n_2(\vec{a}\vec{y}) - n_2(\vec{z}\vec{y}), \forall a \in \mathcal{A}.$$

Therefore, if we consider

$$(10) \quad F(a) = \vec{l}(\vec{z}\vec{a}) + \vec{H}(\vec{z}\vec{a}) + n_1(\vec{x}\vec{z})$$

$$(11) \quad G(a) = \vec{l}(\vec{a}\vec{z}) + n_2(\vec{z}\vec{y})$$

we have on one hand that for any  $a$  in  $\mathcal{A}$

$$(12) \quad F(a) \leq n_1(\vec{x}\vec{a}) \text{ and } G(a) \leq n_2(\vec{a}\vec{y})$$

and

$$(13) \quad F(z) = n_1(\vec{x}\vec{z}) \text{ and } G(z) = n_2(\vec{z}\vec{y}).$$

which proves that the two hyperplanes  $F = n_1(\vec{x}\vec{z})$  and  $G = n_2(\vec{z}\vec{y})$  are respectively tangent to  $B_1$  and  $B_2$  at  $z$ .

Furthermore the linear parts of  $F$  is  $\vec{l} + \vec{h}$  and the linear part of  $G$  is  $-\vec{l}$ . Therefore, the intersections of the directions of these two hyperplanes with the kernel of  $\vec{h}$  coincides .

Reciprocally: Let  $F$  and  $G$  ne affine maps from  $\mathcal{A}$  to  $\mathbb{R}$  such that  $F^{-1}(n_1(\vec{x}\vec{z}))$  is a supporting hyperplane of  $B_1$  at  $z$ , and  $G^{-1}(n_2(\vec{z}\vec{y}))$  is a supporting hyperplane of  $B_2$  at  $z$ . If we define  $H(a) = \vec{F}(\vec{x}\vec{a}) + \vec{G}(\vec{a}\vec{y})$ , then we have

$$H(z) = n_1(\vec{x}\vec{z}) + n_2(\vec{z}\vec{y}) = r$$

and by assumption for any  $a \in \mathcal{E}_{x,y}(r, n_1, n_2)$

$$H(a) = \vec{F}(\vec{x}\vec{a}) + \vec{G}(\vec{a}\vec{y}) \leq n_1(\vec{x}\vec{a}) + n_2(\vec{a}\vec{y}) \leq r.$$

Therefore  $H = r$  is a supporting hyperplane of  $\mathcal{E}_{x,y}(r, n_1, n_2)$  at  $z$ .

Finally notice that  $\vec{H}(\vec{v}) = \vec{F}(\vec{v}) - \vec{G}(\vec{v})$  and therefore the intersection of  $\ker \vec{F}$  with  $\ker \vec{H}$  concides with the intersection of  $\ker \vec{G}$  with  $\ker \vec{H}$ .

□

### 3. REFLECTION IN A MINKOWSKI SPACE

Consider a Minkowski space  $(X, \|\cdot\|)$  and let  $H: X \rightarrow \mathbb{R}$  be a continuous affine map and denote by  $\vec{H}$  its linear part. We are going to study reflection on the hyperplane  $\mathcal{H} = \{x \in X \mid H(x) = 1\}$ . The unit ball associated to the norm  $\|\cdot\|$  will be denoted by  $B_X$ , in a more formal way

$$B_X = \{x \in X \mid \|x\| \leq 1\}.$$

In the case of a Minkowski space we can define ellipsoids, as a particular case of the generalised ellipsoid seen in the definition 6.



**Definition 8** (Minkowski ellipsoids). Let  $x$  and  $y$  be two points in the Minkowski space  $(X, \|\cdot\|)$ . We define the *ellipsoid with focal points from  $x$  to  $y$  and detour  $\rho \geq \|y - x\|$* , the set

$$E_{(x,y)}(\rho) = \{z \mid \|y - z\| + \|z - x\| \leq \rho\}.$$

We will call *intermediate points* from  $x$  to  $y$  the set of points  $z$  such that  $\|y - z\| + \|z - x\| = \|y - x\|$ , that is the ellipsoid  $E_{(x,y)}(\|y - x\|)$ .

The generalised law of reflection of Proposition 7 can therefore be stated in this particular case as follows.

**Property 9** (Fundamental property of Minkowski ellipsoids). *For any two points  $x$  and  $y$  in the Minkowski space  $(X, \|\cdot\|)$  and  $\rho > \|y - x\|$ , the Minkowski ellipsoid  $E_{(x,y)}(\rho)$  is a convex set of non empty interior, and it contains  $x$  and  $y$ .*

*Furthermore for any point  $z$  on the boundary of the Minkowski ellipsoid,  $\mathcal{H}$  is a supporting hyperplane to the ellipsoid  $E_{(x,y)}(\rho)$  at the point  $z$  if and only if there exists*

- *a supporting hyperplane  $\mathcal{H}_v$  to the unit ball  $B_X$  at the point  $\vec{v} = \vec{xz}/\|\vec{xz}\|$ , and*
- *a supporting hyperplane  $\mathcal{H}_w$  to the unit ball  $B_X$  at the point  $\vec{w} = \vec{zy}/\|\vec{zy}\|$*

*whose directions have the same intersection with  $\mathcal{H}$ .*

*Proof.* Let us show that it is of non-empty interior. First of all notice that the whole segment  $[x, y]$  is contained in the interior of the ellipsoid  $E_{(x,y)}(\rho)$ . Take any point  $z$  such that

$$\|y - z\| + \|z - x\| < \rho$$

then for some small enough  $\epsilon$  we have

$$\|y - z\| + \|z - x\| < (1 - 2\epsilon)\rho$$

hence for any  $v$  such that  $\|t - z\| < \epsilon\rho$  and  $\|z - t\| < \epsilon\rho$ , we obtain

$$\|y - t\| + \|t - x\| < \rho.$$

The rest is a straightforward application of the generalised law of reflection (Proposition 7) with  $n_1 = n_2 = \|\cdot\|$  and  $\mathcal{A} = X$ .  $\square$

In a Euclidean vector space, as mentioned in the previous section, the property 9 is known to characterise reflections. Hence in analogy with the Euclidean case we are lead to introduce the following definition in a Minkowski space.

**Definition 10.** Let  $\vec{v}$  be a unit vector in  $X$  and  $\mathcal{H}$  a hyperplane. A unit vector  $\vec{w}$  such that  $\vec{H}(\vec{v}) \cdot \vec{H}(\vec{w}) < 0$  will be called a *reflection* of the vector  $\vec{v}$  with respects to the hyperplane  $\mathcal{H}$  if and only if the following occurs: the unit ball  $B_X$  admits two supporting hyperplanes,  $\mathcal{H}_v$  at  $\vec{v}$  and  $\mathcal{H}_w$  at  $\vec{w}$  whose intersections with the direction of  $\mathcal{H}$  coincide.

**Remark 11.** In this definition, when the hyperplane  $\mathcal{H}_v$  is parallel to  $\mathcal{H}$  then so is the hyperplane  $\mathcal{H}_w$ . The vector  $\vec{w}$  may not be unique if the unit ball is not strictly convex, neither the pair of hyperplanes  $\mathcal{H}_v$  and  $\mathcal{H}_w$  if the unit sphere is not  $C^1$ .

In the sequel we fix an hyperplane  $\mathcal{H}$ , a unit vector  $\vec{v}$  and  $\vec{w}$  one of its reflections with respect to  $\mathcal{H}$ . Therefore  $\mathcal{H}_v$  and  $\mathcal{H}_w$  will be two hyperplanes satisfying the conditons of the definition 10.

Our next step consists in proving that if we think of the hyperplanes parallel to  $\mathcal{H}_v$  as wave fronts, then the hyperplanes parallel to  $\mathcal{H}_w$  can be seen as their reflection on the hyperplane  $\mathcal{H}$ .

To do so, let us point out that given given the vector  $\vec{v}$  as above, the hyperplane  $\mathcal{H}$  determines two half spaces. One of those half spaces is composed of those points  $x$  where the ‘‘positive’’ half line originating at  $x$  and directed by  $\vec{v}$  meets the hyperplane  $\mathcal{H}$ . We shal denote by  $\mathcal{H}^+$  this half space, that is,

$$\mathcal{H}^+ = \{x | \exists t > 0, x + t\vec{v} \in \mathcal{H}\}.$$

Given the unit vector  $\vec{v}$  as above, let us consider four points  $x, x', y, y'$  in this half space  $\mathcal{H}^+$  and two points  $z, z'$  on the hyperplane  $\mathcal{H}$  such that

- $\overrightarrow{xz}$  and  $\overrightarrow{x'z'}$  are parallel to the vector  $\vec{v}$ ,
- $\overrightarrow{zy}$  and  $\overrightarrow{z'y'}$  are parallel to the same vector  $\vec{w}$ , which is a reflection of the vector  $\vec{v}$ ,
- $x$  and  $x'$  belong to the same hyperplane parallel to  $\mathcal{H}_v$ ,
- $y$  and  $y'$  belong to the same hyperplane parallel to  $\mathcal{H}_w$ .

Then we have the following lemma, which is a generalisation of the Huygens-Fresnel principle.

**Lemma 12** (Wave front lemma). *Let  $\vec{w}$  be a reflection of the vector  $\vec{v}$  on the hyperplane  $\mathcal{H}$ , if the six points  $x, y, z$  and  $x', y'$  and  $z'$  are in the situation described above, we have*

$$\|z' - x'\| + \|y' - z'\| = \|z - x\| + \|y - z\|.$$

*Proof.* Let us denote by  $H_w(y)$  the hyperplane parallel to  $\mathcal{H}_w$  containing both  $y$  and  $y'$ . We will then denote by  $H_v(y)$  the hyperplane parallel to  $\mathcal{H}_v$  whose interection with  $H_w(y)$  lies on  $\mathcal{H}$ , and by  $Y, Y'$  the interesections of  $H_v(y)$  with, respectively, the lines  $(xz)$  and  $(x'z')$ . Then on the one hand as  $xx'Y'Y$  is a parallelogram we have

$$\|Y' - x'\| = \|Y - x\|.$$

Furthermore, as  $x', Y', z'$  (resp,  $x, z, y$ ) are on the same line we have

$$\|Y' - z'\| + \|z' - x'\| = \|Y - z\| + \|z - x\|.$$

On the other hand, by definition of the vectors  $\vec{w}$ , both  $Y'$  and  $y'$  (resp.  $Y$  and  $y$ ) lie on the same sphere centred at  $z'$  (resp.  $z$ ) which

implies

$$\|Y' - z'\| = \|y' - z'\|, \quad \text{and} \quad \|Y - z\| = \|y - z\|.$$

□

The following proposition states that the length minimization property of a reflection in a Euclidean ellipsoid is also true in a Minkowski one. Recall that  $\mathcal{H} = \{x \in X \mid H(x) = 1\}$ .

**Proposition 13.** *Let  $x$  and  $y$  be two points in the half space  $H < 1$ . For any  $z \in \mathcal{H}$  define the path from  $x$  to  $y$  composed of the straight segments  $[x, z]$  and  $[z, y]$ . This path has minimal length as a function of  $z$  if and only if one of the following occurs*

- $z$  is an intermediate point from  $x$  to  $y$ , i.e.,

$$(14) \quad \|z - x\| + \|y - z\| = \|y - x\|,$$

- or

$$(15) \quad \frac{y - z}{\|y - z\|} \text{ is a reflection of } \frac{z - x}{\|z - x\|} \text{ with respect to } \mathcal{H}.$$

Such a minimizing path exists.

*Proof.* Let  $l_z$  be the length of the path defined as above. We are going to show that this path has minimal length. If  $z$  is an intermediate point from  $x$  to  $y$ , this follows from the triangle inequality.

Hence we can suppose that  $z$  satisfies the property (15). Therefore, let us consider (see Figure 3)

- $z'$  any other point in  $\mathcal{H}$ , and  $l_{z'}$  the length of the path  $[xz'] \cup [z'y]$ .
- $\xi'$  be the hyperplane passing by  $x$  and parallel to  $\xi$  the supporting hyperplane of the unit ball in the direction of  $z - x$ .
- $\eta'$  be the hyperplane passing by  $y$  and parallel to  $\eta$  the supporting hyperplane of the unit ball in the direction of  $y - z$ .
- $\zeta$  be the hyperplane parallel to  $\xi$  passing by  $z'$ .

Then we can define

- $x''$  the intersection of the segment  $[x, z]$  with  $\zeta$ , and
- $x'$  the intersection of  $\xi'$  with the line passing by  $z'$  and parallel to the line  $(xz)$ .

Then on the one hand, as the hyperplane  $\zeta$  is tangent at  $z'$  to the sphere centred at  $x'$  and passing by  $z'$  we have

$$(16) \quad \|z' - x''\| \leq \|z' - x\|.$$

On the other hand as  $\eta'$  is tangent at  $y'$  to the sphere centred at  $z'$  and passing by  $y'$  we have

$$(17) \quad \|y' - z'\| \leq \|y - z'\|.$$

From these two equations it follows that

$$(18) \quad \|y' - z'\| + \|z' - x''\| \leq l_{z'}.$$

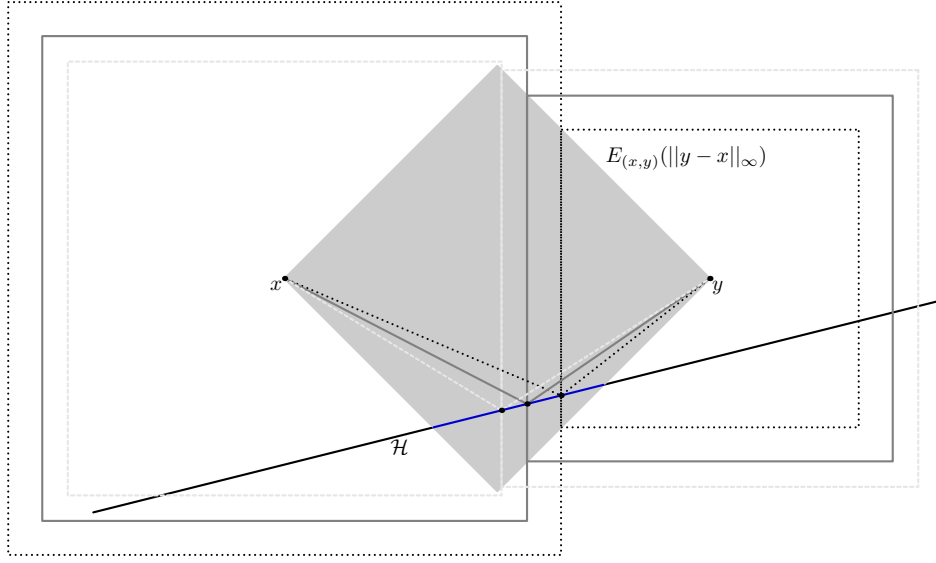


FIGURE 1. An example where  $E_{(x,y)}(\|y-x\|)$  is not a segment

Now it suffices to see that by construction and following lemma 12 we have  $p_z = \|y' - z'\| + \|z' - x'\| \leq l_{z'}$ .

Reciprocal and existence: Let the direction of  $\mathcal{H}$  be  $\vec{H}$  (which is the kernel of  $H$ ), and let us suppose, without loss of generality, that  $x$  and  $y$  lie on the half space  $H < 1$ .

Let us define for any  $z \in X$

$$(19) \quad N(z) = \|y - z\| + \|z - x\| - \|y - x\|.$$

Let  $r \geq 0$ . By convexity there exists at least two points on the boundary of the Minkowski ellipsoid  $E_{(x,y)}(r + \|y - x\|)$  which admits a supporting hyperplane directed by  $\vec{H}$ , i.e., parallel to  $\mathcal{H}$ .

Therefore these hyperplanes can be identified with a level set of  $H$ , Let us call  $h(r)$  the biggest real number corresponding to one of these supporting hyperplanes.

First case: Suppose that  $h(0) \geq 1$ , in other words  $E_{(x,y)}(\|y - x\|)$  intersects the hyperplane  $\mathcal{H}$ . Then for any point of this intersection (which can be reduced to a point), the assumption is satisfied. These corresponds to intermediate points from  $x$  to  $y$  on the hyperplane.

Notice that this case does not occur if the norm is strictly convex.

Second case: We suppose that  $h(0) < 1$ . then given a fixed  $z'$  in the half space  $H > 1$  we have  $h(N(z')) > 1$ .

The continuity of  $h$  and the intermediate value theorem gives us therefore the existence of a real number  $0 < r < N(z')$  such that  $h(r) = 1$ , in other words, there exists some finite real number  $\rho > \|y - x\|$  such that  $\mathcal{H}$  is a supporting hyperplane of  $E_{(x,y)}(\rho)$  at some point  $z$  (which need not be unique). We can now conclude thanks to the

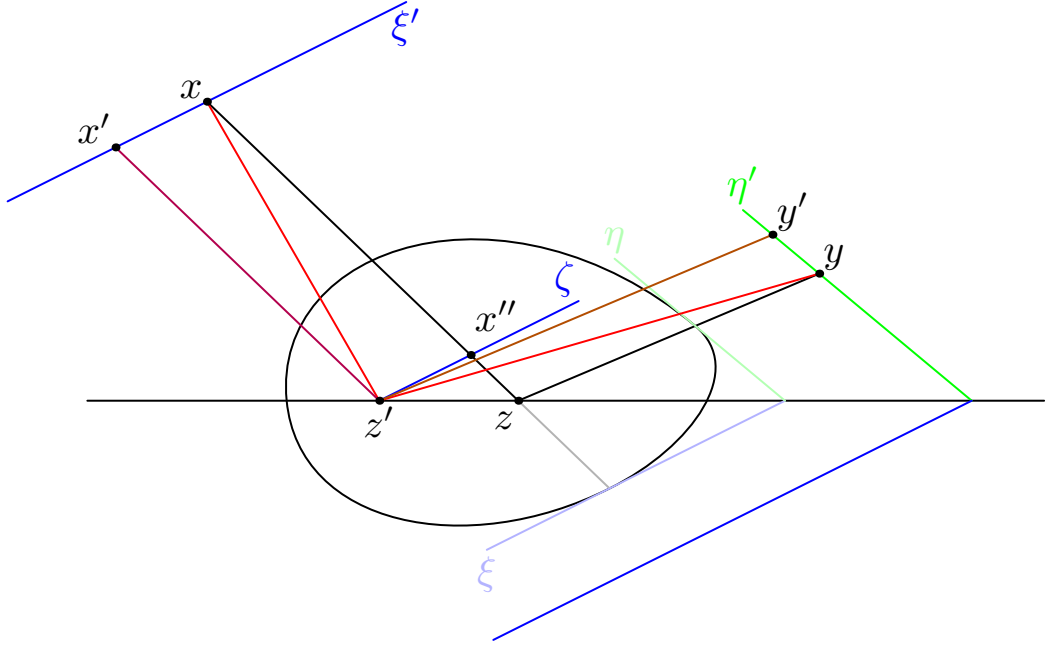


FIGURE 2. Proof of Proposition 13, a.k.a Huygens-Fresnel principle

fundamental property of Minkowski ellipsoids 9 that the property (15) is satisfied.  $\square$

#### 4. CHANGE OF LAYERS IN BANACH SPACES, A GENERALISATION OF SNELL-DESCARTES LAW

Consider a real vector space  $X$  endowed with two Minkowski norms  $\|\cdot\|_1, \|\cdot\|_2$ .

Let us denote by  $B_1 = \{x \in X \mid \|x\|_1 \leq 1\}$  and  $B_2 = \{x \in X \mid \|x\|_2 \leq 1\}$  the unit balls associated to  $\|\cdot\|_1$  and  $\|\cdot\|_2$ .

We can study the generalised ellipsoids in that setting as well, which can be defined as follows:

**Definition 14** (Bi-Minkowski ellipsoids). Let  $x$  and  $y$  be two points in the "Bi-Minkowski" vector space  $(X, \|\cdot\|_1, \|\cdot\|_2)$ . We define the *ellipsoid with focal points from  $x$  to  $y$  and detour*

$$\rho \geq \rho_0(x, y) = \min\{\|y - z\|_2 + \|z - x\|_1 \mid z \in X\}$$

as the set

$$E_{(x,y)}(\rho) = \{z \in X \mid \|y - z\|_2 + \|z - x\|_1 \leq \rho\}.$$

The set  $E_{(x,y)}(\rho_0(x, y))$  will be called the set of *intermediate points* from  $x$  to  $y$ .

Even if the convexity of the norms guaranties the convexity of these ellipsoids, their shape is rather unpredictable. The easy case is when one norm dominates the other. For instance if  $\|v\|_1 < \|v\|_2$  for all  $v \in X$ , then  $E_{(x,y)}(\rho_0)$  restricts to the point  $y$ . Otherwise anything can happen, i.e.,  $E_{(x,y)}(\rho_0)$  may contain both  $x$  and  $y$  or neither of them.

We now translate in that setting the generalized law of reflection.

**Property 15** (Fundamental property of Bi-Minkowski ellipsoids). *For any two points  $x$  and  $y$  in the Bi-Minkowski space  $(X, \|\cdot\|_1, \|\cdot\|_2)$  and  $\rho > \rho_0(x, y)$ ,  $E_{(x,y)}(\rho)$  is convex of non empty interior.*

*Furthermore for any point  $z$  on the boundary of the Bi-Minkowski ellipsoid,  $\mathcal{H}$  is a supporting hyperplane to the ellipsoid  $E_{(x,y)}(\rho)$  at the point  $z$  if and only if there exists*

- *a supporting hyperplane  $\mathcal{H}_v$  to the unit ball  $B_1$  at the point  $\vec{v} = \vec{xz}/\|\vec{xz}\|$ , and*
- *a supporting hyperplane  $\mathcal{H}_w$  to the unit ball  $B_2$  at the point  $\vec{w} = \vec{zy}/\|\vec{zy}\|$*

*whose directions have the same intersection with the direction of  $\mathcal{H}$ .*

*Proof.* The fact that  $E_{(x,y)}(\rho)$  is not empty comes from the definition of  $\rho_0(x, y)$ . Let us show that it is of non-empty interior.

Take any point  $z$  such that

$$\|y - z\|_2 + \|z - x\|_1 < \rho$$

then for some small enough  $\epsilon$  we have

$$\|y - z\|_2 + \|z - x\|_1 < (1 - 2\epsilon)\rho$$

hence for any  $v$  such that  $\|t - z\|_1 < \epsilon\rho$  and  $\|z - t\|_2 < \epsilon\rho$ , we obtain

$$\|y - t\|_2 + \|t - x\|_1 < \rho.$$

The rest is a straightforward application of the generalised law of reflection 7 with  $n_1 = \|\cdot\|_1$ ,  $n_2 = \|\cdot\|_2$  and  $\mathcal{A} = X$ .  $\square$

Let  $H: X \rightarrow \mathbb{R}$  be a continuous affine map and  $\vec{H}$  its linear part. We are going to study change of layer through the hyperplane  $\mathcal{H} = \{x \in X \mid H(x) = 1\}$ . That is to say that we will endow the half space  $H < 1$  with a first minkowski norm  $\|\cdot\|_1$  and the half space  $H > 1$  with a second minkowski norm  $\|\cdot\|_2$ . Let us denote this metric space by

$$(X, \|\cdot\|_1, \mathcal{H}, \|\cdot\|_2)$$

An exemple of such a phenomenon is when the two norms are proportional to the same euclidean norm. In that case our problem is related to the refraction of light and the ratio of these norms is the ratio of the indices of refraction of the two half spaces.

That is why we introduce the following definition in the space  $(X, \|\cdot\|_1, \mathcal{H}, \|\cdot\|_2)$ .

**Definition 16.** Let  $\vec{v}$  be a unit vector for the first norm in the space  $X$  pointing toward the half space  $H > 1$  (in other words  $\vec{H}(\vec{v}) > 1$ ).

A unit vector  $\vec{w}$  with respect to the second norm, also pointing toward the half space  $H > 1$  will be called a *refraction* of the vector  $\vec{v}$  with respects to the hyperplane  $\mathcal{H}$  if and only if the following occurs:

There exists a supporting hyperplane  $\mathcal{H}_v$  to the first unit ball  $B_1$  at  $\vec{v}$ , and a supporting hyperplane  $\mathcal{H}_w$  to the second unit ball  $B_2$  at  $\vec{w}$ . whose intersections with the direction of  $\mathcal{H}$  coincide.

Let us keep the notations of the definition 16. Given the unit vector  $\vec{v}$  as above, let us consider two points  $x, x'$  in the half space  $H < 1$ , two points  $y, y'$  in the half space  $H > 1$  and two points  $z, z'$  on the hyperplane  $\mathcal{H}$  such that

- $\vec{xz}$  and  $\vec{x'z'}$  are parallel to the vector  $\vec{v}$ ,
- $\vec{zy}$  and  $\vec{z'y'}$  are parallel to the same vector  $\vec{w}$ , which is a refraction of the vector  $\vec{v}$ ,
- $x$  and  $x'$  belong to the same hyperplane parallel to  $\mathcal{H}_v$ ,
- $y$  and  $y'$  belong to the same hyperplane parallel to  $\mathcal{H}_w$ .

Once again we have a Huygens-Fresnel principle in the following lemma:

**Lemma 17** (Wave front lemma). *Let  $\vec{w}$  be a refraction of the vector  $\vec{v}$  on the hyperplane  $\mathcal{H}$ , if the six points  $x, y, z$  and  $x', y'$  and  $z'$  are in the situation described above, we have*

$$\|z' - x'\|_1 + \|y' - z'\|_2 = \|z - x\|_1 + \|y - z\|_2.$$

*Proof.* Almost same proof as the previous wave front lemma 12.

Let  $H_w(y)$  be the hyperplane parallel to  $H_w$  containing both  $y$  and  $y'$ . Let us denote by  $H_v(y)$  the hyperplane parallel to  $H_v$  whose intersection with  $H_w(y)$  lies of  $\mathcal{H}$ . Let  $Y$  and  $Y'$  the intersections of  $H_v(y)$  with, respectively, the lines  $(xz)$  and  $(x'z')$ .

On the one hand as  $xx'Y'Y$  is a parallelogram we have

$$\|Y' - x'\|_1 = \|Y - x\|_1.$$

Furthermore, as  $x', z', Y'$  (resp.  $x, z, Y$ ) are on the same line we have

$$\|Y' - z'\|_1 + \|z' - x'\|_1 = \|Y - z\|_1 + \|z - x\|_1.$$

On the other hand, by definition of the vector  $\vec{w}$ , and the hyperplanes  $H_w(y)$  and  $H_v(y)$  we have

$$\|Y' - z'\|_1 = \|y' - z'\|_2, \quad \text{and} \quad \|Y - z\|_1 = \|y - z\|_2.$$

To see this one can apply a dilation centred at  $z'$  (resp.  $z$ ).

□

**Proposition 18** (Generalized minkowskian law of refraction). *Let  $x, y$  be two points in  $X$  which are not in the same half space defined*

by  $\mathcal{H}$ , say  $H(x) < 1$  and  $H(y) > 1$ . For any  $z \in \mathcal{H}$  define the path from  $x$  to  $y$  composed of the straight segments  $[x, z]$  and  $[z, y]$  and let us denote by  $p_z = \|y - z\|_2 + \|z - x\|_1$  the length of that path in  $(X, \|\cdot\|_1, \mathcal{H}, \|\cdot\|_2)$ .

This path has minimal length as a function of  $z$  if and only if one of the following occurs:

- $z$  is an intermediate point from  $x$  to  $y$ , i.e.,

$$(20) \quad \|z - x\|_1 + \|y - z\|_2 = \rho_0(x, y),$$

- or

$$(21) \quad \frac{y - z}{\|y - z\|_2} \text{ is a refraction of } \frac{z - x}{\|z - x\|_1} \text{ with respect to } \mathcal{H}.$$

Such a minimizing path exists.

*Proof.* If  $z$  is an intermediate point from  $x$  to  $y$  it is a global minimum over all  $X$ , hence it is a minimum path.

Suppose now that  $z$  satisfies the property (21) and let  $z'$  any other point in  $\mathcal{H}$ .

Let  $\xi'$  be the hyperplane passing by  $x$  and parallel to  $\xi$  the supporting hyperplane of the unit ball in the direction of  $z - x$ .

Let  $\eta'$  be the hyperplane passing by  $y$  and parallel to  $\eta$  the supporting hyperplane of the unit ball in the direction of  $y - z$ .

let  $\zeta$  be the hyperplane parallel to  $\xi$  passing by  $z'$ .

Then let  $x''$  be the intersection of the segment  $[x, z]$  with  $\zeta$  and let  $x'$  be the intersection of  $\xi'$  with the line passing by  $z'$  and parallel to the line  $(xz)$ .

Then on one hand we have  $\|z' - x'\|_1 \leq \|z' - x\|_1$  because the hyperplane  $\zeta$  is tangent at  $z'$  to the sphere centred at  $x'$  and passing by  $z'$ .

On the second hand we have

$$\|y' - z'\|_2 \leq \|y - z'\|_2$$

because  $\eta'$  is tangent at  $y'$  to the sphere centred at  $z'$  and passing by  $y'$ .

From this it follows that

$$\|y' - z'\|_2 + \|z' - x'\|_1 \leq p_{z'}.$$

Now it suffices to see that by construction and following lemma 17 we have  $p_z = \|y' - z'\|_2 + \|z' - x'\|_1$ .

Reciprocal and existence: Let  $\mathcal{H}$  be the level set of the affine map  $H: X \rightarrow \mathbb{R}$  associated to 1, whose direction is  $\vec{H}$  and let us define for any  $z \in X$  and  $\rho_0(x, y) = \min\{\|t - x\|_1 + \|y - t\|_2 \mid t \in X\}$ ,

$$N(z) = \|y - z\|_2 + \|z - x\|_1 - \rho_0(x, y).$$



First case: suppose that  $E_{(x,y)}(\rho_0(x,y))$  intersects  $\mathcal{H}$ , then any point of this intersection satisfies our assumption. These are intermediate points from  $x$  to  $y$ .

Second case: We suppose now that  $E_{(x,y)}(\rho_0(x,y))$  does not intersect  $\mathcal{H}$ . Without loss of generality we may assume that it lies on the half space  $H < 1$ . Let  $r \geq 0$ , then  $E_{(x,y)}(r + \rho_0(x,y))$  admits two supporting hyperplanes directed by  $\vec{H}$ .

Therefore these hyperplanes can be identified with a level set of  $H$ . As in the minkowski case, let us denote by  $h(r)$  the biggest real number corresponding to one of this supporting hyperplanes. Then by assumption  $h(0) < 1$  and for some  $z'$  in the half space  $H > 1$  we have  $h(N(z')) > 1$ .

The continuity of  $h$  and the intermediate value theorem gives us the existence of  $0 < r < N(z')$  such that  $h(r) = 1$ , in other words, there exists some finite real number  $\rho > \rho_0(x,y)$  such that  $\mathcal{H}$  is a supporting hyperplane of  $E_{(x,y)}(\rho)$  at some point  $z$  (which need not be unique). We can now conclude thanks to the fundamental property of Bi-Minkowski Ellipsoids 15 that the property (21) is satisfied.  $\square$

## 5. BRUNN-MINKOWSKI INEQUALITY IS NOT PRESERVED IN A TWO LAYERS BANACH SPACE

In this section we are going to consider the vector space  $\mathbb{R}^2$ , and the hyperplane  $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ . We are going to put the classical euclidean  $\ell^2$  norm  $\|(x,y)\|_2 = \sqrt{x^2 + y^2}$  on the half space  $y > 0$  and the  $\ell^1$  norm  $\|(x,y)\|_1 = |x| + |y|$ , on the half space  $y < 0$ . Hence we are going to work, using the notation of the previous section, on the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$ .

**Properties 19.** *Let  $X_0 = (\rho, \theta)$  be in the upper half plane and  $X_1 = (0, y)$  be in the lower half plane ( $y < 0$ ), then*

- *the geodesic joining  $X_0$  to  $X_1$  is composed of the line segment from  $X_0$  to the origin and the origin to  $X_1$ . It is unique.*
- *The distance between  $X_0$  to  $X_1$  is equal to  $\rho - y$ .*
- *Let  $X_s$  be the  $s$ -intermediate point between  $X_0$  to  $X_1$ ,*
  - (1) *If  $s(\rho - y) < \rho$ , then  $X_s$  belongs to the upper half plane and lies on the affine segment from  $X_0$  to the origin, and  $X_s = ((1-s)\rho + sy, \theta)$  in polar coordinates;*
  - (2) *If  $s(\rho - y) > \rho$ , then  $X_s$  belongs to the lower half plane and lies in the line  $x = 0$ , and  $X_s = (0, (1-s)\rho + sy)$  in cartesian coordinates.*

*Proof.* The fact that this path is a geodesic is a direct consequence of Proposition 18. The only thing we need to prove is uniqueness. Any geodesic between these points has to pass through the origin. Hence

on the upper half plane, as there is only one geodesic between any two points we don't have any choice.

Now on the lower half plane, let  $\gamma: [0, 1] \rightarrow \mathbb{R}^2$  be a piecewise  $C^1$  path between the origin and the point  $(0, y)$ , if  $\gamma(t) = (x(t), y(t))$ , then  $t \rightarrow g(t) = (0, y(t))$  is also a piecewise  $C^1$  path between the origin and the point  $(0, y)$ . Now we have almost everywhere

$$|\dot{y}(t)| \leq |\dot{x}(t)| + |\dot{y}(t)|$$

therefore the length of  $\gamma$  is bigger than the length of  $g$ , except if  $x(t) = 0$  almost everywhere. This implies that the only geodesic between  $(0, 0)$  and  $(0, y)$  is the segment between these two points. The other properties are easy to check.  $\square$

**Proposition 20.** *In the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$  no Brunn-Minkowski inequality holds, i.e., for any  $K \in \mathbb{R}$  and  $N \in \mathbb{N} \cup \{+\infty\}$ ,  $BM(K, N)$  does not hold.*

*Proof.* Let  $(\rho, \theta)$  be the polar coordinates in  $\mathbb{R}^2$ . Consider the annulus

$$K_0 = \{(\rho, \theta) \mid 6 \leq \rho \leq 8, \pi/3 \leq \theta \leq 2\pi/3\},$$

and the affine segment

$$I = \{(x, y) \in \mathbb{R}^2 \mid -101 \leq y \leq -100, x = 0\}.$$

Now let  $X_0 = (\rho_0, \theta)$  be in  $K_1$ , and  $X_I = (0, -100 - t)$  in  $I$ . Following the previous section, there is a unique geodesic from  $X_0$  to  $X_I$ , and is composed of the affine segment joining  $X_0$  to the origin  $O = (0, 0)$  and of the affine segment joining the origin to  $X_I$ . We therefore have  $\|X_0\|_2 = \rho_0$  and  $\|X_I\|_1 = 100 + t$ , from which we deduce that the distance between these two points is  $\rho_0 + 100 + t$ . Now following the Properties 19, as

$$(\rho_0 + 100 + t)/2 > 106/2 = 53 > 8 \geq \rho_0,$$

for  $s \geq 1/2$  the point  $X_s = (0, (1-s)\rho_0 + s(-100-t))$ , is the  $s$ -intermediate point on the geodesic from  $X_0$  to  $X_I$ . From this we easily deduce that the  $1/2$ -intermediate set from  $X_0$  to  $I$  is

$$\frac{1}{2}K_0 + \frac{1}{2}I = \{(x, y) \mid x = 0, -47, 5 \leq y \leq -46\}.$$

This suffices to prove that no  $BM(K, N)$  is satisfied.

Now let us prove that  $BM(K, +\infty)$  is never satisfied. For  $s > 1/2$  the  $s$ -intermediate set from  $X_0$  to  $I$  is easily seen to be

$$(1-s)K_0 + sI = \{(0, y) \in \mathbb{R}^2 \mid -101s + 6(1-s) \leq y \leq -100s + 8(1-s)\}.$$

We start by considering some  $0 < \varepsilon < 1$ , whose value will be chosen at the end, and replace  $I$  with

$$K_1 = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq \varepsilon, |y + 100, 5| \leq 0, 5\}.$$

The next step is to introduce the slices of  $K_1$ :

$$I_\alpha = \{(x, y) \in \mathbb{R}^2 \mid -101 \leq y \leq -100, x = \alpha\}$$

for  $0 < \alpha \leq \varepsilon < \rho$ , and to identify their intermediate sets  $(1-s)K_0 + sI_\alpha$ . In order to do this we compute the distance between  $X_0$  and  $(\alpha, 0)$ , which gives

$$\rho_\alpha = \sqrt{\rho_0^2 - 2\alpha\rho_0 \cos \theta + \alpha^2},$$

and it is now easy to check that for  $X_0$  in  $K_0$  we have

$$(22) \quad \begin{aligned} (6 - \alpha) &\leq (\rho_0 - \alpha) < \sqrt{\rho_0^2 - \alpha\rho_0 + \alpha^2} \leq \\ &\rho_\alpha \leq \sqrt{\rho_0^2 + \alpha\rho_0 + \alpha^2} < (\rho_0 + \alpha) \leq (8 + \alpha). \end{aligned}$$

The description we were seeking is therefore

$$(23) \quad \begin{aligned} (1-s)K_0 + sI_\alpha = \\ \left\{ (\alpha, y) \in \mathbb{R}^2 \mid -101s + (1-s)\sqrt{6^2 - 6\alpha + \alpha^2} \leq y \right. \\ \left. \leq -100s + \sqrt{8^2 + 8\alpha + \alpha^2}(1-s) \right\}. \end{aligned}$$

To obtain an upper bound on its area we notice that it can be seen as a subset as follows

$$(24) \quad \begin{aligned} (1-s)K_0 + sI_\alpha \subset \\ \left\{ (\alpha, y) \in \mathbb{R}^2 \mid -101s + (1-s)(6 - \alpha) \leq y \right. \\ \left. \leq -100s + (1-s)(8 + \alpha) \right\}, \end{aligned}$$

Therefore the area of the intermediate set  $K_s = (1-s)K_0 + sK_1$  is less than

$$\varepsilon \cdot (16 - 15s)$$

up to some multiplicative constant  $C$ , depending on the normalisation chosen for the Lebesgue measure.

This also tells us (depending on the sign of  $K$ ) that

$$(25) \quad 105 \leq 106 - \varepsilon \leq \theta(\varepsilon) \leq 108 + \varepsilon \leq 109.$$

The area of  $K_1$  is exactly  $2\varepsilon$ . Hence, for some fixed constant  $C$  we have

$$(26) \quad \frac{\mu((1-s)K_0 + K_1)}{\mu^s(K_1)} \leq \varepsilon^{1-s} \cdot C^{1-s} \cdot \left(8 - \frac{15}{2}s\right)$$

We need now to compare, as  $s \rightarrow 1$ , the right hand part of (26) with

$$\mu^{1-s}(K_0)e^{Ks(1-s)\theta(\varepsilon)^2/2},$$

which is the same as comparing  $\varepsilon \cdot C \cdot \left(8 - \frac{15}{2}s\right)^{1/(1-s)}$  with

$$\mu(K_0)e^{Ks\theta(\varepsilon)^2/2}.$$

This last term converges towards  $\mu(K_0)e^{K\theta(\varepsilon)^2/2}$ , while the first to  $\varepsilon \cdot C \cdot e^{15/2}$ .

To conclude, as  $\theta(\varepsilon)$  stays bounded, we can find and fix an  $\varepsilon$  small enough such that

$$\varepsilon \cdot C \cdot e^{15/2} < \frac{1}{2}\mu(K_0)e^{K\theta(\varepsilon)^2/2}.$$

Then, for values of  $s$  close enough to 1, we will obtain

$$\mu^{1-s}(K_0)e^{Ks(1-s)\theta(\varepsilon)^2/2} > \frac{\mu((1-s)K_0 + K_1)}{\mu^s(K_1)},$$

which contradicts  $BM(K, +\infty)$ .  $\square$

**Proposition 21.** *There exists a Minkowski norm  $f$  on  $\mathbb{R}^2$  such that that  $BM(-1, +\infty)$  does not hold in the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, f)$ .*

*Proof.* Recall that a Minkowski norm  $f$  is twice differentiable on  $\mathbb{R}^2 \setminus \{0\}$ , with a definite positive Hessian. Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of Minkowski norm, converging towards the  $\ell^1$  norm. Up to a rescaling we can suppose that the intersection of their unit ball with  $\mathcal{H}$  coincide with the intersection of the unit ball of the  $\ell^1$  norm.

Observe also that we can chose the norms  $f_n$  such that their tangents at their point of intersection with  $H$  is orthogonal to  $H$ . This will be useful in the last section of this paper.

Then the sequence of metric spaces  $(\mathbb{R}^2, \ell^2, \mathcal{H}, f_n)$  converges in the Gromov-Hausdorff measured topology towards  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$ .

Let us denote by  $\mu_n$  the Hausdorff measure associated to  $f_n$ . On the upper half space it does not change, but it varies on the lower half space.

Consider again the sets  $K_0$  and  $K_1$  and the intermediate set  $K_s$  as in the proof of Proposition 20. Then for any  $n$  we would get another intermediate set  $K_s(n)$ , and another function  $\theta(n)$  which is the maximum (resp. Minimum) between two points from  $K_0$  to  $K_1$  or from  $K_1$  to  $K_0$ . Following our assumption we have that  $\theta(n)$  converges towards the  $\theta$  of the limit,  $\mu(K_0)$  does not change and  $\mu_n(K_1)$  converges towards  $\mu(K_1)$  thanks to the gromov-hausdorff measured convergence.

We suppose that  $s$  is close enough to 1 to be on the lower half plane.

We need to prove that  $\mu_n(K_s(n))$  converges towards  $\mu(K_s)$  as  $n$  goes to infinity. First notice that  $K_s$  is a compact closed set, and so are the sets  $K_s(n)$ . Secondly, the geodesics from a point on the upper half space to the lower half space are unique, because both norms are strictly convex. Hence the geodesics are converging to the geodesics.

Now consider a finite covering of  $K_0$  by balls of radius  $\varepsilon$  centered at points  $x_1, \dots, x_k$  in and a finite covering of  $K_1$  by balls of radius  $\varepsilon$  centered at points  $y_1, \dots, y_m$  in  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$ . Then consider the  $m \times k$  geodesics between the  $x'_i$ s and the  $y'_j$ s, and the  $m \times k$  corresponding intermediate points  $z_{ij}$ . Then there exists a constant  $c \geq 1$  such that

the balls of radius  $c\epsilon$  centered at the points  $z_{ij}$  form a covering of the  $K_s$ . To see this, look at the  $s$ -intermediate set from a ball of radius  $\epsilon$  on the upper half space to a ball of radius  $\epsilon$  on the lower half space.

Now for any  $n$  we can consider the corresponding family  $z_{ij}(n)$  of  $s$ -intermediate points. The uniqueness of the geodesic and the fact that the norms are strictly convex and smooth implies the uniqueness of these points and the gromov-hausdorff convergences implies that the  $z_{ij}(n)$  converge uniformly towards the  $z_{ij}$ . But for any  $n$  we will also have the existence of some constant  $c(n) \geq 1$  such that the ball of radius  $c(n)\epsilon$  center at  $z_{ij}(n)$  cover  $K_s(n)$ , and as  $n \rightarrow \infty$ ,  $c(n)$  will converge towards  $c$  because these constants are related to the ratio between the  $\ell^2$  norm and the  $f_n$  norm respectively the  $\ell^1$  norm for  $c$ .

This tells us that the sequence of sets  $K_s(n)$  converge in the gromov-Hausdorff sense towards  $K_s$ . Hence in our case, the measures converging as well we can say that they converge in the measured Gromov-Hausdorff sense.

Hence for any  $n$  large enough, we would get a contradiction to  $BM(-1, +\infty)$  if we choose  $K_0$ ,  $K_1$  and  $s$  as in the proof of Proposition 20 to contradict  $BM(-1, +\infty)$  in  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$ . □

## 6. A COMPACT FINSLER SURFACE WITH NO LOWER RICCI BOUND EMBBEDED IN A MINKOWSKI SPACE

**6.1. First example.** Let us consider in the Euclidean three-dimensional space, the two-dimensional disk

$$\mathcal{S} = \{(x, y, z) \mid z = 0, x^2 + y^2 \leq 1\},$$

and let  $\mathcal{B}$  be the conve hull of

$$\mathcal{S} \cup \{(0, 0, 1), (0, 0, -1)\}.$$

We now endow  $\mathbb{R}^3$  with the norm  $\|\cdot\|_{\mathcal{B}}$  whose unit ball is  $\mathcal{B}$ . In other words for any  $(x, y, z) \in \mathbb{R}^3$ ,

$$\|(x, y, z)\|_{\mathcal{B}} = \sqrt{x^2 + y^2} + |z|.$$

The affine planes normal to the vector  $(0, 0, 1)$  endowed with the norm induced by  $\|\cdot\|_{\mathcal{B}}$  are all isometric to the two dimensional euclidean plane. In the same way, the affines planes containing the direction  $(0, 0, 1)$  are isometric to the  $\ell^1$ -plane (i.e., that is the manhattan distance).

In this normed vector space, we will consider the cube  $\mathcal{C}_\rho$  obtained as the convex hull of the eight points

$$\{(\pm\rho, \pm\rho, \pm\rho)\}.$$

The cube  $\mathcal{C}_\rho$  admits two faces which are euclidean, and four faces which are  $\ell^1$ .

**Proposition 22.** *Let  $\mathbb{R}^3$  be endowed with the norm  $\|\cdot\|_{\mathcal{B}}$ . Then the cube  $\mathcal{C}_1$  with the metric induced by  $\|\cdot\|_{\mathcal{B}}$  does not satisfy any curvature dimension.*

*Proof.* Let us denote by  $d_\rho$  the distance induced on  $\mathcal{C}_\rho$  by  $\|\cdot\|_{\mathcal{B}}$  and  $\mu_\rho$  the induced Lebesgue measure. Focus on two adjacent faces of  $\mathcal{C}_\rho$ , one euclidean and the second one  $\ell^1$ . Then we are locally exactly as in section 5, and therefore the same computations as in section 5 show that for any  $\rho \in \mathbb{R}^*$ , the Brunn-Minkowski  $BM(K, N)$  inequality does not hold for any  $N \in \mathbb{N} \cup \{+\infty\}$  and any  $K \in \mathbb{R}$ .

Therefore in  $(\mathcal{C}_\rho, d_\rho, \mu_\rho)$  the curvature dimension  $CD(K, N)$  does not hold for any  $K$  and any  $N$ .  $\square$

**Corollary 23.** *There exists a  $C^{1,1}$  compact and convex surface in  $(\mathbb{R}^3, \|\cdot\|_{\mathcal{B}})$  such that for any  $N \in \mathbb{N} \cup \{+\infty\}$  and any  $K \in \mathbb{R}$ ,  $CD(K, N)$  does not hold.*

*Proof.* let  $B(\varepsilon)$  be the euclidean ball of radius  $\varepsilon$ . Consider the Minkowski sum of the cube and this ball, that is,

$$C(\varepsilon) = B(\varepsilon) + \mathcal{C}_1 = \{x + y \mid x \in B(\varepsilon), y \in \mathcal{C}_1\},$$

with the unduced metric by  $\|\cdot\|_{\mathcal{B}}$ . Then  $C(\varepsilon)$  is smooth, and as  $\varepsilon$  goes to zero, it converges in the Gromov-Hausdorff measured topology towards  $\mathcal{C}_1$ .

Actually  $C(\varepsilon)$  is obtained by translating the faces of the cube  $\mathcal{C}_1$  outward at a euclidean distance  $\varepsilon$  and then close by rolling the euclidean ball of radius  $\varepsilon$  along the edges, from the inside. Hence the difference is on the surface obtained along these curved edges. On the flat section we have the same distance than in  $\mathcal{C}_1$ .

Fix some  $K$  and  $N = +\infty$ . We can use the annulus  $K_0$  and the rectangle  $K_1$  from the proof of Proposition 20, the only thing which will change is the  $s$ -intermediate set from  $K_0$  to  $K_1$ , denoted by  $K_s(\varepsilon)$ . Fix an  $s$  such that we get the contradiction as in proof of Proposition 20.

Then as  $\varepsilon$  goes to zero  $K_s(\varepsilon)$  the corresponding  $s$ -intermediate set and in particular its measure converges towards the measure of the  $s$ -intermediate  $K_s$  of the proof of Proposition 20. Hence for some  $\varepsilon$  small enough, we would get the same contradiction.

Let us fix such a  $\varepsilon$  for  $K = -1$ . The images by the dilation  $h_\rho$  of ratio  $\rho$  of center the origin of  $K_0$ ,  $K_1$  and  $K_s(\varepsilon)$  lie on  $C_\rho + B(\rho \cdot \varepsilon)$  where they give a counter example to  $BM(-1, +\infty)$  as well, because the image of  $K_s(\varepsilon)$  by  $h_\rho$  is the  $s$ -intermediate set from  $h_\rho(K_0)$  to  $h_\rho(K_1)$  on  $C_\rho + B(\rho \cdot \varepsilon)$ . Hence for any  $\rho > 0$ ,  $C_\rho + B(\rho \cdot \varepsilon)$  is not  $BM(-1, +\infty)$ , hence not  $CD(-1, +\infty)$ .

Now let us suppose that  $\mathcal{C}_1 + B(\varepsilon)$  is  $CD(K, N)$  for some  $K < -1$ . Then  $(\mathcal{C}_1 + B(\varepsilon), \rho d, \rho^{n-1} \mu)$  is  $CD(K/\rho^2, N)$ . Observe now that  $h_\rho$  is an isometry between  $(\mathcal{C}_1 + B(\varepsilon), \rho d, \rho^{n-1} \mu)$  and  $C_\rho + B(\rho \cdot \varepsilon)$ , because,

$$d(h_\rho(x), h_\rho(y)) = \rho \cdot d(x, y)$$

but then for  $\rho^2 > -K$ , we get that  $C_\rho + B(\rho \cdot \varepsilon)$  is  $C(-1, N)$ , which contradicts the choice of  $\varepsilon$ .  $\square$

**6.2. Second example.** This second example is to justify that one can get an example with a smoother norm.

Let  $\mathcal{H}$  in  $\mathbb{R}^3$  be the  $x$ -axis (that is the line  $z = 0$  and  $y = 0$ ). Consider  $f$  a norm in the plane  $y = 0$  such that  $(y = 0, \ell^2, \mathcal{H}, f)$  does not satisfy  $CD(-1, +\infty)$  as in proposition 21. Then consider  $\mathcal{B}_f$  the convex obtained by rotating the norm  $f$  around the  $z$ -axis.

Then let us denote by  $\|\cdot\|_f$  the norm whose unit ball coincide with  $\mathcal{B}_f$ .

**Proposition 24.** *There exists a  $C^{1,1}$  compact and convex surface in  $(\mathbb{R}^3, \|\cdot\|_f)$  such that for any  $N \in \mathbb{N} \cup \{+\infty\}$  and any  $K \in \mathbb{R}$ ,  $CD(K, N)$  does not hold.*

*Proof.* Again, let us consider the family of cubes  $C_\rho$  with our two sets  $K_0$  and  $K_1$ . Then for any  $\rho$ ,  $C_\rho$  with the induced metric in not  $BM(-1, +\infty)$ .

Then let us also consider  $C_1 + B(\varepsilon)$ , then for some  $\varepsilon$  small enough it will not be  $BM(-1, +\infty)$  as in the previous example. And again, by homotating the sets contradicting  $BM(-1, +\infty)$ , we obtain that for any  $\rho > 0$ ,  $C_\rho + B(\varepsilon \cdot \rho)$  is not  $BM(-1, +\infty)$ .

Again the same reasoning by contradiction as in the proof of Corollary 23 shows that  $C_1 + B(\varepsilon)$  can not satisfy any  $CD(K, N)$ , for any  $K$  and any  $N$ .  $\square$

## 7. CONCLUDING REMARKS

The current work has been the subject of various talk and discussions with many colleagues having their own idea of what is a good notion of curvature in metric measured spaces.

The first main problem which forbids the notion of synthetic Ricci curvature to apply in our first example is the branching occuring when one passes from one media to another. It is also related to the Finslerian nature of our spaces.

Both these problems excludes all the notions of curvatures that have been presented to us by our various colleagues. For instance one could decide to work with spaces admitting a Gromov-Bishop comparison theorem, as some nice theorem and results in Riemannian geometry are actually based on the fact that manifolds with Ricci curvature bounded from below admits such a comparison. An easy computation shows that the metric space  $(\mathbb{R}^2, \ell^2, \mathcal{H}, \ell^1)$  does not satisfy such a comparison with the standard hyperbolic plane.

Notice that by smoothing our norm, we still got a surface without synthetic Ricci curvature bounded from below, but without branching.

This illustrates the fact that by being close to a branching space is also problematic.

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