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# Harmonic morphisms and Hermitian complex structures on $\mathbb{S}^2 \times \mathbb{S}^2$

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## Abstract

In this paper, we investigate the structure of a harmonic morphism  $F$  from  $\mathbb{S}^2 \times \mathbb{S}^2$  to a 2-surface  $\mathbb{S}^2$ . Baird-Ou construct a family of harmonic morphism from an open set of  $\mathbb{S}^2 \times \mathbb{S}^2$  into  $\mathbb{S}^2$ . We check that they are holomorphic with respect to one of the canonical complex structure.

## 1 Introduction

A harmonic morphism  $F : M \rightarrow N$  between two Riemannian manifolds  $(M, g)$  and  $(N, h)$  is a map which pulls back local harmonic functions on  $N$  to local harmonic functions on  $M$ . Although harmonic morphisms can be traced back to Jacobi, their study in modern times was initiated by Fuglede and Ishihara who characterized them using the notion of horizontal weak conformality, or semiconformality:

**Definition 1.** (see [B-W] p.46)

Let  $F : (M, g) \rightarrow (N, h)$  be a smooth map between Riemannian manifolds and let  $x \in M$ . Then  $F$  is called horizontally weakly conformal at  $x$  if either

1)  $dF_x = 0$

2)  $dF_x$  maps the space  $\text{Ker}(dF_x)^\perp$  conformally onto  $T_{F(x)}N$ , i.e. there exists a number  $\lambda(x)$  called the dilation of  $F$  at  $x$  such that

$$\forall X, Y \in \text{Ker}(dF_x)^\perp, h(dF_x(X), dF_x(Y)) = \lambda^2(x)g(X, Y).$$

The space  $\text{Ker}(dF_x)$  (resp.  $\text{Ker}(dF_x)^\perp$ ) is called the vertical (resp. horizontal) space at  $x$ .

Fuglede and Ishihara proved independently

**Theorem 1.** ([Fu],[Is])

Let  $F : (M, g) \rightarrow (N, h)$  be a smooth map between Riemannian manifolds. The following two statements are equivalent:

- 1) For every harmonic function  $f : V \rightarrow \mathbb{R}$  defined on an open set  $V$  of  $N$ , the function  $f \circ F$  defined on the open set  $F^{-1}(V)$  of  $M$  is harmonic.
- 2) The map  $F$  is harmonic and horizontally weakly conformal.  
Such a map is called a harmonic morphism

When the target is 2-dimensional, Baird and Eells proved.

**Theorem 2.** ([B-E])

Let  $F : (M^m, g) \rightarrow (N^2, h)$  be a smooth non constant horizontally weakly conformal map between a Riemannian manifold  $(M^m, g)$  and a Riemannian 2-surface  $(N^2, h)$ . Then  $F$  is harmonic (hence a harmonic morphism) if and only if the fibres of  $F$  at regular points are minimal submanifolds of  $M$ .

**Remark 1.** In Makki-Ville ([Ma-Vi]) we extend Th.2 to the singular fibres if  $M$  is compact.

## 1.1 Background

Let  $M$  be a  $2n$ -dimensional real manifold. An almost complex structure  $J$  on  $M$  is a tensor field  $J : TM \rightarrow TM$  such that  $J^2 = -I$ .

**Definition 2.** Let  $M$  be a  $2n$ -dimensional manifold. An almost complex structure  $J$  on  $M$  is called integrable if there exists an atlas  $\{U_\alpha, \alpha : U_\alpha \rightarrow \mathbb{R}^{2n}\}$  such that

$$d\alpha \circ J = J_0 \circ d\alpha,$$

and the transition functions verify  $d(\beta \circ \alpha^{-1})(z) \in GL(n; \mathbb{C})$ . Here  $J_0$  is the standard complex structure on  $\mathbb{R}^{2n}$ .

For any two vector fields  $X, Y$  on  $M$ , the Nijenhuis tensor  $N_J$  is defined as

$$NJ(X; Y) = [JX; JY] - J[JX; Y] - J[X; JY] - [X; Y].$$

One can prove that  $N_J$  is actually a tensor. We have the following theorem of Newlander-Nirenberg:

**Theorem 3.** (Integrability theorem). An almost complex structure  $J$  is integrable if and only if the Nijenhuis tensor  $N_J$  vanishes.

Now the vanishing of the Nijenhuis tensor can be viewed as a Frobenius integrability condition:

**Lemma 1.** The set of the type  $(1, 0)$  vector fields is closed under the operation of Lie bracket if and only if  $N_J \equiv 0$ .

*Proof.* Let  $X$  and  $Y$  be two real vector fields and we define the projections

$$P^{1,0}X = \frac{1}{2}(I - iJ)X; P^{0,1}Y = \frac{1}{2}(I + iJ)Y.$$

It is easy to show that

$$[P^{1,0}X; P^{1,0}Y] + iJ[P^{1,0}X; P^{1,0}Y] = -N_J(X, Y) - iJN_J(X, Y),$$

which is equivalent to

$$P^{0,1}([P^{1,0}X, P^{1,0}Y]) = P^{0,1}(-N_J(X, Y)).$$

Hence  $N_J(X, Y)$  vanishes if and only if  $[P^{1,0}X, P^{1,0}Y]$  is a  $(1, 0)$  vector. □

In  $\mathbb{C}^n$ , we have the expression

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

or written in terms of  $J_0$ :

$$\frac{\partial}{\partial z_j} = \frac{1}{2} (1 - iJ_0) \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} (1 + iJ_0) \frac{\partial}{\partial x_j}.$$

Hence in the local coordinates  $z_j$ , the type (1,0) vector can be written as the combination  $\sum_k a_k \frac{\partial}{\partial z_k}$  where  $a_k$  are complex valued functions on  $M$ . The Lie bracket of two (1,0) type vectors is still of the (1,0) type. If  $\Sigma$  is a 2-surface, then the Lie bracket of two (1,0) vectors is of type (1,0), hence we have

**Theorem 4.** *Every almost complex structure on a Riemann surface is integrable.*

**Definition 3.** *An integrable complex structure  $J$  on  $M$  is said to be hermitian if and only if for every  $p$*

$$g(JX, JY) = g(X, Y) \text{ for all } X, Y \in T_p M.$$

## 1.2 The results

In this paper we investigate harmonic morphisms on open sets of  $\mathbb{S}^2 \times \mathbb{S}^2$ . J. Wood shows that the harmonic morphisms from an open subset of  $\mathbb{C}P^1 \times \mathbb{C}P^1$  to a Riemann surfaces are holomorphic with respect to a Hermitian structure.

On  $\mathbb{C}P^1$  we have two canonical complex structures  $\pm J_0$  then on  $\mathbb{C}P^1 \times \mathbb{C}P^1$  we obtain four canonical complex structures. Burns-Bartolomeis show that the only Hermitian structures on open subsets of  $\mathbb{C}P^1 \times \mathbb{C}P^1$  are one of the four products of canonical complex structure  $\pm J_0$ .

On other hand Baird-Ou define harmonic morphism from open dense sets of  $\mathbb{S}^2 \times \mathbb{S}^2$  into  $\mathbb{S}^2$ . We check that they are holomorphic w.r.t. one of these four structures.

**Theorem 5.** *([B-O]) There are two tori  $T_1^2$  and  $T_2^2$  in  $\mathbb{S}^3$  such that there exists a family of harmonic morphisms  $F : \mathbb{S}^3 \times \mathbb{S}^3 - (T_1^2 \cup T_2^2) \rightarrow (\mathbb{S}^2, \text{can})$  parametrized by quadruples of non-zero integers  $(k, l, m, n)$ , given explicitly by the formula:*

$$F \left( \left( \cos se^{ia}, \sin se^{ib} \right), \left( \cos te^{ic}, \sin te^{id} \right) \right) = \left( \cos \alpha(s, t), \sin \alpha(s, t) e^{i(ka+lb+mc+nd)} \right),$$

where  $\alpha$  is given by the solution of Equation

$$\frac{1}{\sin^2 \alpha} \left( \left( \frac{\partial \alpha}{\partial s} \right)^2 + \left( \frac{\partial \alpha}{\partial t} \right)^2 \right) = \frac{k^2}{\cos^2 s} + \frac{l^2}{\sin^2 s} + \frac{m^2}{\cos^2 t} + \frac{n^2}{\sin^2 t}. \quad (1.1)$$

For  $|k|=|l|$  and  $|m|=|n|$  the equation (1.1) is solved by

$$\alpha(s, t) = 2 \tan^{-1} (A \tan^k s \tan^m t),$$

for a positive real constant  $A$ .

**Corollary 1.** *In the case  $|k|=|l|$  and  $|m|=|n|$ , the harmonic morphism of the theorem factors to a harmonic morphism*

$$F : (\mathbb{S}^2 \times \mathbb{S}^2) \setminus \{\text{two points}\} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$$

**Proposition 1.** *Let  $F : \mathbb{S}^2 \times \mathbb{S}^2 \setminus \{\text{two points}\} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$  a harmonic morphism defined by*

$$F \left( \left( \cos s, \sin s e^{iA} \right) \times \left( \cos t, \sin t e^{iC} \right) \right) = \left( \cos \alpha \left( \frac{s}{2}, \frac{t}{2} \right), \sin \alpha \left( \frac{s}{2}, \frac{t}{2} \right) e^{i(kA+mC)} \right),$$

with

$$\alpha \left( \frac{s}{2}, \frac{t}{2} \right) = 2 \tan^{-1} \left( A \tan^k \frac{s}{2} \tan^m \frac{t}{2} \right).$$

Then  $F$  is holomorphic with respect to one of the four canonical complex structures on  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ .

*Proof.* To prove the theorem, we need to prove that for two orthogonal vertical vectors  $V_1$  and  $V_2$ , we have  $J_0(V_1) = \lambda V_2$  where  $J_0$  is one of the four canonical complex structures.

The tangent space  $T(\mathbb{S}^2 \times \mathbb{S}^2)$  for  $s \neq 0, \pi$  and  $t \neq 0, \pi$  is generated by the vector fields  $\frac{\partial}{\partial s}, \frac{\partial}{\partial A}, \frac{\partial}{\partial t}, \frac{\partial}{\partial C}$ . We have

$$\begin{aligned} \frac{\partial F}{\partial s} &= \left( -\sin \alpha \frac{\partial \alpha}{\partial s}, \cos \alpha \frac{\partial \alpha}{\partial s} e^{i(kA+mC)} \right), \\ \frac{\partial F}{\partial A} &= \left( 0, \sin \alpha i k e^{i(kA+mC)} \right), \\ \frac{\partial F}{\partial t} &= \left( -\sin \alpha \frac{\partial \alpha}{\partial t}, \cos \alpha \frac{\partial \alpha}{\partial t} e^{i(kA+mC)} \right), \\ \frac{\partial F}{\partial C} &= \left( 0, i m \sin \alpha e^{i(kA+mC)} \right). \end{aligned}$$

So we need to calculate  $\frac{\partial \alpha}{\partial s}$  and  $\frac{\partial \alpha}{\partial t}$ . By a small calculation we obtain the following

$$\begin{aligned} \frac{\partial \alpha}{\partial s} &= \frac{A \tan^m \frac{t}{2} k (1 + \tan^2 \frac{s}{2}) \tan^{(k-1)} \frac{t}{2}}{1 + (A \tan^k \frac{s}{2} \tan^m \frac{t}{2})^2} \\ \frac{\partial \alpha}{\partial t} &= \frac{A \tan^k \frac{s}{2} m (1 + \tan^2 \frac{t}{2}) \tan^{(m-1)} \frac{t}{2}}{1 + (A \tan^k \frac{s}{2} \tan^m \frac{t}{2})^2} \\ \frac{\partial \alpha}{\partial t} &= \frac{\sin s m}{\sin t k} \frac{\partial \alpha}{\partial s}. \end{aligned} \tag{1.2}$$

We can choose and easily check that the following vector fields are mutually orthogonal and vertical ( $dF(V_1) = dF(V_2) = 0$ ) :

$$\begin{aligned} V_1 &= m \frac{\partial}{\partial A} - k \frac{\partial}{\partial C} \\ V_2 &= -\frac{\partial \alpha}{\partial t} \frac{\partial}{\partial s} + \frac{\partial \alpha}{\partial s} \frac{\partial}{\partial t}, \end{aligned}$$

Using (1.2) we find that

$$\begin{aligned} V_2 &= \frac{\partial \alpha}{\partial s} \left[ -\frac{\sin s m}{\sin t k} \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right] \\ &= k^{-1} \sin^{-1} t \frac{\partial \alpha}{\partial s} \left[ -\sin s m \frac{\partial}{\partial s} + \sin t k \frac{\partial}{\partial t} \right]. \end{aligned}$$

We let  $J_0$  be one the four canonical structures defined by

$$\begin{aligned}
J_0\left(\frac{\partial}{\partial s}\right) &= \frac{1}{\sin s} \frac{\partial}{\partial A} \\
J_0\left(\frac{\partial}{\partial t}\right) &= \frac{1}{\sin t} \frac{\partial}{\partial C} \\
J_0\left(\frac{\partial}{\partial A}\right) &= -\sin s \frac{\partial}{\partial s} \\
J_0\left(\frac{\partial}{\partial C}\right) &= -\sin t \frac{\partial}{\partial t}.
\end{aligned}$$

Back to  $V_1$  and  $V_2$ , applying  $J_0$  on  $V_1$  then for some constant  $\lambda(k, m)$

$$\begin{aligned}
J_0(V_1) &= mJ_0\left(\frac{\partial}{\partial A}\right) - kJ_0\left(\frac{\partial}{\partial C}\right) \\
&= -m \sin s \frac{\partial}{\partial s} + k \sin t \frac{\partial}{\partial t} \\
&= \lambda(k, m)V_2.
\end{aligned}$$

□

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