Multi-agent simultaneous formation-tracking and stabilization of nonholonomic vehicles

Mohamed Maghenem, Antonio Loria, Elena Panteley

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Abstract

We solve the open problem of formation control for swarms of nonholonomic vehicles over a very wide class of reference trajectories (vanishing, persistently exciting, and set-points). Our main result applies to the full unicycle model, consituted of velocity kinematics and dynamics equations. Only a handful of works (see below) cover such generality, but in the one-leader-one-follower scenario; none applies to swarms of vehicles. On technical grounds, we first establish uniform global asymptotic stability (UGAS) and strong input-to-state stability for the kinematics equations; moreover, our proofs provide strict Lyapunov functions. Our second statement is on UGAS for the full model under the action of *any* velocity controller guaranteeing that the velocity errors are square integrable. Our third and main result, establishes UGAS for a swarm of vehicles and provides the first solution to the above-mentioned problem. To the best of our knowledge, there does not exist in the literature similar stability and stabilization statements on tracking, stabilization or formation control, under the relaxed hypotheses considered here.

I. Introduction

Leader-follower tracking and set-point stabilization control of nonholonomic systems are well-studied and documented problems —see, *e.g.*, [1]–[4]. In the first case, the control goal is to make the robot follow a (often non-converging) time-varying reference trajectory. In the second case the aim is to stabilize the robot around a set-point. Somewhat in the middle, the *robust stabilization* control problem consists in making the vehicle follow a trajectory whose derivative vanishes. All these problems present different technical difficulties which often lead to the design of controllers that apply to *either* one problem *or* the other. For instance, methods based on an assumption that the reference velocity does not vanish or that it is persistently exciting, have been presented in [5] and [6], but such methods do not solve the set-point stabilization control problem.

Furthermore, some results in the literature apply only to the so-called simplified velocity-controlled model, in which only the kinematics equations are considered —see, *e.g.*, [7] and [8]. Others apply to the more general force-controlled model, which includes the velocity dynamics —see [9], [10], [11].

Designing a universal controller that applies in both scenarios, of tracking and (robust) stabilization, even for the simplified model, is a very challenging problem that has been little addressed in the literature. Indeed, to the best of our knowledge the leader-follower *tracking-stabilization* problem for *one* leader and *one* follower nonholonomic vehicle has only been studied in [7] and [12]–[15].

In [12] a saturated time-varying velocity controller is proposed which applies under fairly relaxed conditions on the leader's velocities. The seminal paper [13] introduced the transverse-functions approach which applies to a wide variety of nonholonomic systems (unicycles, trailers, chained-form systems). A similar result is presented in [14] using an adaptive approach. In the last two references convergence of the tracking errors, albeit in the practical sense, is established. In contrast to this, asymptotic convergence to zero of the tracking errors is proved in [7] and [15] for the system in closed loop with a universal controller that applies both to the *tracking* and *parking* scenarios. In [7], a keen idea is proposed: to combine a tracking controller and a stabilization one via a weighing function that favors the action of one controller over the other, depending on the control objective (converging vs persistently exciting reference velocities).

As already mentioned, the works cited above concern the one-leader-one-follower case, whereas designing a universal controller guaranteeing *asymptotic convergence*, for the multi-agent formation case and in both scenarios, remains open. Indeed, to the best of our knowledge the problem has been addressed only in [16] (via the transverse-function approach), but in this reference only convergence of the tracking errors in a practical sense is established. It is assumed, moreover, that the robots communicate over a directed graph and the reference velocities are available to all agents in the network.

Inspired by the clever idea introduced in [7], of combining two control laws, in this paper we propose a kinematics controller that applies universally to the tracking and robust stabilization scenarios, under fairly relaxed conditions on the reference trajectories. Our main results cover also those in [17] and [18] by showing that, when combined, the stabilization and the tracking controller are robust with respect to the action of one another, as well as with respect to the converging velocity errors. Hence, beyond the controller itself, our primary contribution lies in establishing strong stability and robustness properties, such as *integral input-to-state stability* and *small input-to-state stability* —see [19]–[21] for definitions and statements.

The importance of establishing such input-to-state stability properties (that go well beyond convergence in a practical sense) cannot be overestimated. Only these can lead to establishing uniform global asymptotic stability for swarms of vehicles with full (kinematics-and-dynamics) models, which is another significant contribution of this paper. Moreover, our main results are

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general in the sense that they guarantee that the kinematics controller may be used along with *any* velocity controller that ensures that the velocity errors converge and are square integrable. In addition, if the force-control loop is uniformly globally asymptotically stable, so is the complete closed-loop system. Such statement leads to further interesting unprecedented results such as uniform global asymptotic stability in the case in which the system's lumped parameters are unknown (thereby implying uniform parametric convergence).

Finally, we stress that our stability proofs are constructive as they rely on original constructions of strict Lyapunov functions for classes of nonlinear time-varying systems with persistency of excitation, based on the intricate but powerful methods laid in [22].

The rest of the paper is organized as follows. In next section we describe in detail the simultaneous tracking-and-robust-stabilization problem statement and, to put our contributions in perspective, we give further account of the literature. In Section III we present our main results. Firstly, we present a core stability result for the case of two systems, which is followed by a generalization to the multi-agent case, to end with several interesting corollaries are also stated. For the sake of completeness, we present a brief simulations case-study in Section IV and we conclude with some remarks in Section V. The paper is completed with two appendices: the first one contains some of the proofs of our main results, the second one contains a simple but original technical statement on stability of the origin for integral-input-to-state stable systems with decaying inputs as well as some well-known definitions that are recalled for ease of reference.

II. Model and problem formulation

A. The model

Let us consider a swarm of N nonholonomic vehicles moving on the plane. For each $i \in \{1 \dots N\}$ let $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ correspond to the Cartesian coordinates of a point on the ith vehicle with respect to a fixed reference frame, and $\theta_i \in \mathbb{R}$ denotes the vehicle's orientation. Now, denoting by $\dot{x}_i$ and $\dot{y}_i$ the velocities in the respective Cartesian directions, each vehicle's motion on the plane is subject to the nonholonomic constraint

$$\dot{x}_i \sin(\theta_i) = \dot{y}_i \cos(\theta_i).$$

That is, the vehicle moves about with forward velocity $v_i = [\dot{x}_i + \dot{y}_i]^{1/2}$ and unconstrained angular velocity $\omega_i = \dot{\theta}_i$; this leads to the velocity kinematics equations

$$\left\{ \begin{array}{rcl} \dot{x}_i &=& v_i \cos(\theta_i) \\ \dot{y}_i &=& v_i \sin(\theta_i) \\ \dot{\theta}_i &=& \omega_i. \end{array} \right. \tag{1}$$

Some times in the literature it is assumed that the vehicle's motion is fully described by the so-called simplified model (1) —see, *e.g.*, [7] and [12]. That is, v_i and ω_i are considered to be control inputs. In a more realistic model, however, the control inputs which we denote by $u_i \in \mathbb{R}^2$ are functions of the input torques applied at the steering wheels. In this case, the equations (1) are complemented by velocity-dynamics equations of the generic form

$$\dot{\eta}_i = F_i(t, \eta_i, z_i) + G_i(t, \eta_i, z_i) u_i \tag{2a}$$

$$\eta_i := [v_i \ \omega_i]^\top, \ z_i := [x_i \ y_i \ \theta_i]^\top, \tag{2b}$$

where the functions $F_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \times \mathbb{R}^3$ and $G_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \times \mathbb{R}^3$ may be defined in various ways. Most typically, (2a) are determined by the Euler-Lagrange equations, as for instance in [2], or they are expressed in terms of the system's Hamiltonian —see, *e.g.*, [11]. Our main statements in this paper are not restricted to either form; it is only assumed that F_i and G_i satisfy Caratheodory's conditions of local existence and uniqueness of solutions over compact intervals.

B. The control problems

Let the aforementioned group of N vehicles, modelled by (1)–(2), communicate according to a spanning-tree topology. That is, for each $i \leq N$, the ith robot receives the states of exactly one leader, labeled $(i-1)$. Generally speaking, for such swarm of nonholonomic systems the leader-follower formation control problem consists in: (i) the vehicles acquiring and maintaining a specified physical formation relative to one another and (ii) following reference trajectories generated by a fictitious robot modelled by the equations:

$$\dot{x}_r = v_r \cos(\theta_r) \tag{3a}$$

$$\dot{y}_r = v_r \sin(\theta_r) \tag{3b}$$

$$\dot{\theta}_r = \omega_r. \tag{3c}$$

The vector $z_r := [x_r, \ y_r, \ \theta_r]^\top$ denotes the position and orientation reference coordinates and v_r, ω_r are given piecewise continuous functions mapping $\mathbb{R}_{\geq 0} \to \mathbb{R}$ that represent the forward and angular reference velocities respectively.

The aspect of acquiring and maintaing a formation may be formulated in function of the relative positions, orientations and velocities of all the vehicles. For each $i \leq N$, let d_{xi} and d_{yi} denote given positive numbers and let

$$\begin{aligned} p_{\theta i} &:= \theta_{i-1} - \theta_i, \\ p_{xi} &:= x_{i-1} - x_i - d_{xi}, \\ p_{yi} &:= y_{i-1} - y_i - d_{yi}. \end{aligned}$$

That is, the distances d_{xi} and d_{yi} define the position of any leader vehicle with respect to any follower. Then, as it is customary, we transform the error coordinates (p_θ, p_x, p_y) of the leader vehicle from the global coordinate frame to local coordinates fixed on the vehicle, that is, we define

$$\begin{bmatrix} e_{\theta i} \\ e_{xi} \\ e_{yi} \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_i) & \sin(\theta_i) \\ 0 & -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \begin{bmatrix} p_{\theta i} \\ p_{xi} \\ p_{yi} \end{bmatrix}. \tag{4}$$

In these new coordinates the position errors,

$$e_i := [e_{\theta i}\ e_{xi}\ e_{yi}]^\top, \tag{5}$$

satisfy

$$\begin{aligned} \dot{e}_{\theta i} &= \omega_{i-1} - \omega_i && \text{(6a)} \\ \dot{e}_{xi} &= \omega_i e_{yi} - v_i + v_{i-1}\cos(e_{\theta i}) && \text{(6b)} \\ \dot{e}_{yi} &= -\omega_i e_{xi} + v_{i-1}\sin(e_{\theta i}), && \text{(6c)} \end{aligned}$$

where v_{i-1} and ω_{i-1} are, respectively, the forward and angular velocities of the leader vehicle. By convention, in (6) we set $v_0 := v_r$ and $\omega_0 := \omega_r$ where v_r and ω_r.

Thus, the leader-follower formation control problem reduces to steering the trajectories of (6) to zero, *i.e.*, ensuring that

$$\lim_{t\to\infty} e_i(t) = 0 \qquad \forall i \in \{1 \dots N\}. \tag{7}$$

If the vehicle is considered to be velocity-controlled, this is tantamount to designing a control law $\eta_i^* := [v_i^*\ \omega_i^*]^\top$ such that, setting $[v_i\ \omega_i]^\top = [v_i^*\ \omega_i^*]^\top$ in (6), (7) follow. In the case that the vehicle is force-controlled the leader-follower control problem consists in designing control inputs $u_i := [u_{i1}\ u_{i2}]^\top$, with $i \in \{1 \dots n\}$, such that (7) hold for the system (1)–(2).

The above-formulated problems have a clear practical meaning and have been addressed under different conditions imposed on the reference trajectories (see the section below). From a control-theory perspective, however, it is significant to adress the following, more challenging, open problem.

Definition 1 (UGAS leader-follower formation control): Let $\eta_r := [v_r\ \omega_r]^\top$ be a piece-wise continuous function $\mathbb{R}_{\geq 0} \to \mathbb{R}^2$ that generates, through (3), feasible trajectories $t \mapsto z_r$. For the system (1)–(2), design a controller $(t, z_{i-1}, \eta_{i-1}, z_i, \eta_i) \mapsto u_i$ such that, defining

$$\tilde{v}_i := v_i - v_i^*, \quad \tilde{\omega}_i := \omega_i - \omega_i^*, \quad \text{and} \quad \tilde{\eta}_i := [\tilde{v}_i\ \tilde{\omega}_i]^\top, \tag{8}$$

the origin for the closed-loop system, $\{\,(e_i, \tilde{\eta}_i) = (0,0)\,\}$ is uniformly globally asymptotically stable. □

The UGAS leader-follower formation control problem in *full* generality, is probably impossible to solve even in the case that $N = 1$ (one leader-one folower scenario). Indeed, it is established in [23], which generalizes the seminal results of [24], that for nonholonomic systems arbitrary feasible trajectories are not stabilizable asymptotically. In the following section we identify a wide calss of feasible reference trajectories that are stabilizable and we put our hypotheses in perspective relatively to the literature.

C. On the reference trajectories

Generally speaking, the reference velocity trajectories $\eta_r : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, $\eta_r := [v_r\ \omega_r]^\top$ may be null, in which case, we are confronted to the set-point stabilization problem, or they may be time-vaying, in which case we are confronted to the tracking control problem. Somewhat in middle ground, we find the robust stabilization probelm, in which case $\eta_r \to 0$. In this paper, we address the UGAS leader-follower formation control problem considering reference trajectories of all three types, via one "universal" controller. To put our contributions in better perspective we describe these scenarios below.

Tracking. The control objective is (7) under the generic assumption that $\eta_r \not\equiv 0$. At least since [25] this problem has been thoroughly addressed in the literature —see *e.g.*, [4]–[7], [26]–[29]. Of special interest here are approaches based on the assumption that η_r is persistently exciting that is, that there exist positive constants μ and T such that

$$\int_t^{t+T} |\eta_r(\tau)|^2 d\tau \geq \mu \quad \forall\, t \geq 0 \tag{9}$$

or, similarly, such that η_r does not vanish, that is,

$$\lim_{t\to\infty} |\eta_r(t)| \neq 0. \tag{10}$$

For instance, in [5] it is established that the tracking errors converge to zero provided that condition (10) holds.

Starting with [6], controllers relying on persistency of excitation of the reference velocities as a mechanism of stabilization have been used —see, *e.g.*, [26]–[28]. Controllers requiring that the reference velocities be persistently exciting obviously fail in the set-point stabilization case nor if the reference trajectories vanish.

Robust stabilization. In this case it is required to guarantee (7) under the generic assumption that

$$\lim_{t\to\infty} |\eta_r(t)| = 0 \tag{11}$$

This case covers the *set-point stabilization* problem (*i.e.*, $\eta_r \equiv 0$), which has been thoroughly studied, motivated by the well-known fact that non-holonomic systems are not stabilizable to a point via smooth autonomous feedback [24]. It also covers the so-called *parking* control problem, in which case the leader vehicle comes to a full stop at a desired point.

The robust stabilization problem is solved for instance, in [12], [18], [2] and [7] under the assumption that the reference velocities converge sufficiently fast, in the sense that there exists $\beta > 0$ such that

$$\int_0^\infty |\eta_r(\tau)| d\tau \leq \beta. \tag{12}$$

It is clear that both scenarios, of tracking control under condition (9) and robust stabilization under condition (11), are mutually exclusive. Therefore, designing a universal controller that applies indistinctly to both cases is a very challenging and little studied problem. Indeed, as we already mentioned, to the best of our knowledge the leader-follower simultaneous *tracking-robust-stabilization* problem, for *one* leader and *one* follower nonholonomic vehicle has only been studied in [7], [12]–[15], and [17].

In [7] it is assumed that η_r is either persistently exciting (for tracking) or integrable (for robust stabilization) and in [12] several scenarios of tracking control (circular paths, straight-line paths, vanishing trajectories) are covered. In both references, however, it is assumed that the vehicles are velocit-controlled (the equations (2) are ignored). The framework laid in [13], however, is very general in the sense that it applies to chain-form systems, a class that includes the unicycle model (1). Full models, including Lagrangian dynamics for the equations (2), are considered in [14] and [15]. In the former the convergence of the error positions e to a steady-state error, albeit under parametric uncertainty, is established. Convergence to zero of the same errors is guaranteed by the controller reported in [15] provided that either v_r is separated from zero (which implies that η_r is persistently exciting) or ω_r is separated from zero and v_r is integrable. In [17] a robust controller that guarantees the stronger properties of uniform global asymptotic stability and integral input to state stability for the kinematics closed-loop equation is proposed. Note that all these are more restricitve conditions than (9) and (11).

In the case of multiple vehicles ($N \geq 2$), to the best of our knowledge the only article in which the simultaneous tracking and robust stabilization control problem has been addressed is [30]. The control design method in the latter reference follows the framework of [16] and it is established that the formation-errors converge to an arbitrarily small compact ball centered at the origin. Moreover, the controller from [30] is centralized hence, it is assumed that the leader's velocities are accessible to *all* the agents in the network.

This is far from the problem described in Definition 1 with η_r satisfying either (9) or (11). Our main result in this paper solves this problem.

III. Cascades-based leader-follower control

Our control approach relies on the separation of two control loops: one involving the kinematics equations (1) and one involving the dynamics equations (2), whence the term "cascades-based". The controllers are decentralized; for each vehicle we design a local controller that uses measurements of its own states z_i and η_i as well as the states of its leader (z_{i-1} and η_{i-1}). For clarity of presentation, we address first the simultaneous tracking and stabilization problems for the case of two vehicles only. In Section III-B we address the general case of formation control for swarms of more than two vehicles.

A. One leader, one follower

Let $i \leq N$ be arbitrary, but fixed. It is required for the ith vehicle to follow its leader, indexed $i-1$ or, equivalently, to guarantee that (7) hold for the system (6). Inspired by the control method proposed in [7] we define

$$v_i^* = v_{i-1}\cos(e_{\theta i}) + k_{xi}e_{xi} \tag{13a}$$

$$\begin{aligned} \omega_i^* = {} & \omega_{i-1} + k_{\theta i}e_{\theta i} + k_{yi}e_{yi}v_{i-1}\phi(e_{\theta i}) \\ & + \rho_i(t)k_{yi}p_i(t)|e_{xyi}|, \end{aligned} \tag{13b}$$

in which we use $e_{xyi} := [e_{xi}\ e_{yi}]^\top$ and the rest of the variables are defined as follows. The function $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is once continuously differentiable, bounded, and with bounded derivative $\dot{p}_i$. The function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ corresponds to the so-called sinc$(\cdot)$ function, which is defined by $\phi(x) = \sin(x)/x$, and k_{xi}, k_{yi}, $k_{\theta i}$ are positive constants. Furthermore, we define

$$\rho_i(t) := \exp\left(-\int_0^t F\big(\eta_{i-1}(\tau)\big)d\tau\right) \tag{14}$$

where $F : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is a piece-wise continuous function that satisfies the following conditions, by construction:

- if (9) holds then there exists $T_1 > 0$ and $\mu_1 > 0$ such that

$$\int_t^{t+T_1} F\big(\eta_r(s)\big)^2 ds \geq \mu_1, \quad \forall t \geq 0; \tag{15}$$

- if (11) holds then there exists $\beta > 0$ such that

$$\int_0^\infty F\big(\eta_r(s)\big)ds \leq \beta. \tag{16}$$

The first three terms on the right-hand side of (13b) guarantee the achievement of the tracking control goal of persistently-exciting trajectories, while the fourth is added to achieve the robust stabilization goal in the case that the leader's velocities converge. That is, the function ρ_i plays the role of a "weighing" function in the sense that if the reference velocities are persistently exciting, $\rho \approx 0$ and the action of the third term in (13b), $k_{yi}e_{yi}v_{i-1}\phi(e_{\theta i})$, is enforced over that of the last. If, on the contrary, the leader velocities converge, the third term is regarded as a vanishing perturbation to be compensated by the term $\rho_i(t)k_{yi}p_i(t)|e_{xyi}|$, in which $\rho_i(t)$ remains separated from zero. In other words, the expression (13b) comprises two appropriately weighted control laws that, as we shall prove, are robust with respect to one another.

The role of the function F in the control design is highlighted by the following statement.

Proposition 1: Let[1] $\bar{\eta}_r \geq |\eta_r|_\infty$ and $\alpha \in \mathcal{K}$. Then, the functional

$$F(\eta_r) := \begin{cases} 0 & \text{if } \eta_r \ \in (0, \frac{\mu}{2T\bar{\eta}_r}] \\ \alpha(|\eta_r|) & \text{otherwise} \end{cases} \tag{17}$$

is persistently exciting (*i.e.* (15) holds) if (9) holds and $F(\eta_r)$ is integrable (*i.e.* (16) holds) if (11) holds. □

Proof. Note that $F(\eta_r(t))$ is integrable if η_r converges since $F(\eta_r(t)) = 0$ for all $\eta_r \leq \frac{\mu}{2T\bar{\eta}_r}$ and (11) holds by assumption.

To prove that $F(\eta_r)$ is persistently exciting under (9) we use [31, Lemma 2], which states that if a function η_r is persistently exciting then, for every $t \geq 0$, there exists a non-null-measure interval

$$I_t := \{\tau \in [t, t+T] : |\eta_r(\tau)| \geq a := \mu/(2T\bar{\eta}_r)\},$$

such that meas$(I_t) \geq b := T\mu/(2T\bar{\eta}_r^2 - \mu)$. Therefore,

$$\int_t^{t+T} F(\eta_r(s))^2 ds \geq \int_{I_t} \alpha(|\eta_r(s)|)^2 ds \geq \alpha(a)^2 b > 0.$$

■

The idea of introducing a weighing function depending on the nature of the reference velocities is borrowed from [7]. The controller (13) is reminiscent of the controller in [17], which is restricted to the case of one leader and one follower in the particular scenarios of tracking and parking. In the robust stabilization scenario, the controller (13) may also be compared, to some extent, to the controller in [18]. However, there are several important differences with respect to these references that must be underlined.

Firstly, the definition of the "weighting" function ρ_i, in terms of F, gives extra degrees of freedom to the control design, relatively to that in [7] and [17], as shown by Proposition 1 above. On the other hand, our conclusions are more general in the sense that we show integral-input-to-state stability —see Proposition 2 below, and uniform global asymptotic stability (UGAS) of the origin —see Corollary 1 and Proposition 3 farther down. The importance of these properties cannot be overestimated; only uniform global asymptotic stability guarantees robustness with respect to small perturbations (total stability [32]); it is therefore a much stronger property than (non-uniform) convergence of the tracking errors to a neighborhood of the origin —*cf.* [30].

Furthermore, while total stability (also known as *local* input-to-state stability) comes for free from UGAS, constructive Lyapunov-based proofs, as we provide, permit to establish *global* properties such as strong-integral-to-state stability which, in turn, lead to establishing general statements for the full-dynamics model (1)–(2) —see Corollary 2 and Proposition 3; this is not possible from weaker statements on non-uniform convergence. To the best of our knowledge results of this nature have not been reported in the literature before.

We are ready to present our first statement whose proof, for clariy of exposition, is included in the Appendix.

[1]For a measurable function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ we use $|\varphi|_\infty := \operatorname{ess\,sup}_{t\geq 0}|\varphi(t)|$

Proposition 2: Let $i \leq N$ be arbitrarily fixed and consider the system (6) with state e_i, exogenous signal $\eta_{i-1} = [v_{i-1}, \omega_{i-1}]$ such that

$$\max\left\{ |\eta_{i-1}|_\infty , |\dot{\eta}_{i-1}|_\infty \right\} \leq \bar{\eta}_{i-1}, \tag{18}$$

and inputs ω_i and v_i. Consider the virtual control laws (v_i^*, ω_i^*) as given by (13), (14)–(16), with the functions p_i and $\dot{p}_i$ being bounded and persistently exciting. Then, if $\tilde{v}_i$ and $\tilde{\omega}_i$ are bounded, the trajectories are forward complete (*i.e.*, they exist on $[t_\circ, \infty)$). Moreover,

1) if (9) holds with η_r replaced by η_{i-1}, the system is integral input-to-state stable with respect to the input $\tilde{\eta}_i$. Moreover, if $\tilde{\eta}_i$ tends to zero and is square integrable, the limit in (7) holds.
2) If, alternatively, (11) holds with η_r replaced by η_{i-1} the system is small input-to-state stable with respect to the input $\tilde{\eta}_i$ and if $\tilde{\eta}_i$ converges to zero the limit in (7) holds.

□

Remark 1: The assumption that $\tilde{v}_i$ and $\tilde{\omega}_i$ are bounded is imposed here for technical reasons and it is not restrictive in the adequate context. For instance, it comes from the design of a stabilizing controller for the dynamics equations (2). The inequality (18) imposes that the leader trajectories be bounded hence, this can also be considered to be met provided the leader vehicle is adequately controlled. We explore a case-study in Section III-C. •

The following statements that cover others from the literature.

Corollary 1 (UGAS of the kinematics model): Under the conditions of Proposition 2, for the system (6) with $\tilde{\eta}_i = 0$, the origin is uniformly globally asymptotically stable. □

Corollary 2 (Robustness of the full model): Under the conditions of Proposition 2, for any bounded reference trajectories, may they satisfy (9) or (11), the control goal (7) is achieved under the action of *any* controller u_i guaranteeing that

$$\lim_{t\to\infty} |\tilde{\eta}_i(t)| = 0 \tag{19}$$

holds and $\tilde{\eta}_i \in \mathcal{L}_2$. □

Finally, our strongest statement for the case of two vehicles in a leader-follower configuration motivates the qualifier "cascades-based" of our control approach. To the best of our knowledge a statement on uniform global asymptotic stability for the full model and under the assumptions considered here has no precedent in the literature.

Proposition 3 (UGAS of the full model): Consider the system (1), (2) under the action of *any* controller u_i guaranteeing uniform global asymptotic stability of $\{\tilde{\eta}_i = 0\}$ and that $\tilde{\eta}_i \in \mathcal{L}_2$. Then, under the conditions of Proposition 2, the origin $(\tilde{e}_i, \tilde{\eta}_i) = (0, 0)$ is uniformly globally asymptotically stable. □

Proof. We use cascades-systems theory (in particular, [33, Lemma 2]) and Proposition 2. Let u_i be a given controller for the dynamics equations (2), depending on the leader and follower's states, as well as on the virtual control laws (13). Then, by a suitable change of variable the closed-loop equations take the generic form

$$\dot{\tilde{\eta}}_i = F_{\eta i}(t, \tilde{\eta}_i, e_i), \tag{20}$$

while (6) may be written in the compact form (using (8)),

$$\dot{e}_i = F_{ei}(t, e_i) + G_{ei}(t, e_i)\tilde{\eta}_i. \tag{21}$$

Next, we replace e_i in (20) by complete trajectories $e_i(t)$ so the overall closed-loop equations cover a cascaded form

$$\begin{aligned} \dot{e}_i &= F_{ei}(t, e_i) + G_{ei}(t, e_i)\tilde{\eta}_i && (22) \\ \dot{\tilde{\eta}}_i &= \tilde{F}_{\eta i}(t, \tilde{\eta}_i) && (23) \end{aligned}$$

where $\tilde{F}_{\eta i}(t, \tilde{\eta}_i) := F_{\eta i}(t, \tilde{\eta}_i, e_i(t))$ —*cf.* [34], [35, p. 627].

After [33, Lemma 2] the origin $(e_i, \tilde{\eta}_i) = (0, 0)$ is uniformly globally asymptotically stable if so are the respective origins for the systems (23) and $\dot{e}_i = F_{ei}(t, e_i)$ and if the solutions of (22) are uniformly globally bounded. UGAS for (23) holds by assumption. Then, after Proposition 2, if (9) holds the system (22) is integral-input-to-state stable while, if (11) holds it is small input-to-state stable. On the other hand, either of these conditions implies the so-called 0-UGAS property, that is, uniform global asymptotic stability of the origin without input —this corresponds to the statement of Corollary 1.

Finally, uniform global boundedness follows, under condition (9), from the integral-input-to-state-stability property and the assumption that $\tilde{\eta}_i \in \mathcal{L}_2$. Under condition (11) it follows from the property of small input-to-state stability and forward completeness (see Proposition 2).

This completes the proof of Proposition 3. ■

B. Leader-follower formation control

Now let us consider a swarm of autonomous vehicles ($N \geq 2$) which are required to follow a reference vehicle that is modeled by (3) and describes a trajectory that either converges, diverges or has both behaviours sequentially. The standing assumption is that the vehicles communicate with each other over a spanning-tree-topology network hence, each vehicle has only one leader but may have several followers.

Proposition 4: Consider the system (1), (2). Let $\eta_r = [v_r\ \omega_r]^\top$ be a given piece-wise continuous function satisfying either (9) or (11) and assume that there exists $\bar{\eta}_r > 0$ such that

$$\max\left\{|\eta_r|_\infty,\ |\dot{\eta}_r|_\infty\right\} \leq \bar{\eta}_r. \tag{24}$$

For each $i \leq N$ consider the expressions of v_i^* and ω_i^* as in (13) (with $v_0 := v_r$ and $\omega_0 := \omega_r$) where:

(i) k_{xi}, k_{yi}, $k_{\theta i}$ are positive constants;

(ii) the functions p_i and $\dot{p}_i$ are bounded and persistently exciting.

Then, for any given control laws u_{i1} and u_{i2} guaranteeing that $\tilde{\eta}_i$ is square integrable and converges to zero, the control objective (7) holds.

Furthermore, define $\tilde{\boldsymbol{\eta}} := [\tilde{\eta}_1 \cdots \tilde{\eta}_N]^\top$, $\boldsymbol{\eta}^* := [\eta_1^* \cdots \eta_N^*]^\top$, and $\boldsymbol{e} := [e_1 \cdots e_N]^\top$. If $\{\tilde{\boldsymbol{\eta}} = 0\}$ for (23) is uniformly globally asymptotically stable (UGAS) and $\tilde{\boldsymbol{\eta}} \in \mathcal{L}_2$ then, for the closed-loop system (22)-(23), $\{(\boldsymbol{e}, \tilde{\boldsymbol{\eta}}) = (0,0)\}$ is also UGAS. Consequently, if $\tilde{\boldsymbol{\eta}} \equiv 0$ then $\{\boldsymbol{e} = 0\}$ for (1) in closed loop with $\boldsymbol{\eta}^*$ is UGAS. □

Proof. The proof consists in applying recursively the statement of Proposition 2 for each $i \leq N$ that is, for each pair of leader-follower vehicles. Indeed, Proposition 2 guarantees the asymptotic convergence of the formation errors whether the leader velocities are persistently exciting or converging. Therefore, the properties of $(i-1)$th leader's velocities are propagated to the ith follower and, in turn, to the $(i+1)$th vehicle down to the leaf nodes in the graph.

We use $\omega_i = \tilde{\omega}_i + \omega_i^*$ and $v_i = \tilde{v}_i + v_i^*$ in (6), together with (13) to write the error-dynamics equations as

$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i + B_{1i}(t, e_i)\rho_i(t) + B_{2i}(e_i)\tilde{\eta}_i, \tag{25}$$

where

$$A_{v_{i-1}} := \begin{bmatrix} -k_{\theta i} & 0 & -v_{i-1}(t)k_{yi}\phi(e_{\theta i}) \\ 0 & -k_{xi} & \varphi_i(t, e_i) \\ v_{i-1}(t)\phi(e_{\theta i}) & -\varphi_i(t, e_i) & 0 \end{bmatrix},$$

$$B_{1i} := \begin{bmatrix} -k_{yi}p_i(t)|e_{xyi}| \\ k_{yi}p_i(t)|e_{xyi}|e_{yi} \\ -k_{yi}p_i(t)|e_{xyi}|e_{xi} \end{bmatrix}, \quad B_{2i} := \begin{bmatrix} 0 & -1 \\ -1 & e_{yi} \\ 0 & -e_{xi} \end{bmatrix}$$

and $\varphi_i(t, e_i) := \omega_{i-1} + k_{\theta i}e_{\theta i} + k_{yi}e_{yi}v_{i-1}\phi(e_{\theta i})$. We stress that these closed-loop equations have the convenient triangular structure

$$\begin{aligned} \dot{e}_N &= A_{v_{N-1}}(t, e_N)e_N + B_{1N}(t, e_N)\rho_N + B_{2N}(e_N)\tilde{\eta}_N \\ &\vdots && \text{(26a)} \\ \dot{e}_2 &= A_{v_1}(t, e_2)e_2 + B_{12}(t, e_2)\rho_2 + B_{22}(e_2)\tilde{\eta}_2 && \text{(26b)} \\ \dot{e}_1 &= A_{v_r}(t, e_1)e_1 + B_{11}(t, e_1)\rho_1 + B_{21}(e_1)\tilde{\eta}_1 && \text{(26c)} \end{aligned}$$

Note that for the ith vehicle the dynamics equations depend on e_i and, through $\eta_{i-1} = [v_{i-1}\ \omega_{i-1}]^\top$, on the states of the vehicles above in the graph, up to the reference vehicle. However, in view of forward completeness (which can be established as in the proof of Proposition 2), for the purpose of analysis the velocities η_{i-1} may be regarded as exogenous signals. This allows us to consider the system as a multi-*cascaded* time-varying one —see [34]. Then, we may invoke Proposition 2 recursively. However, technically, such reasoning relies on distinct analyses corresponding to each scenario, of tracking and robust stabilization.

Under the tracking scenario. Let (9) hold. Let $i = 1$ and consider the equation (26c) which corresponds to the dynamics of the error trajectories between the virtual vehicle and the swarm leader. By Proposition 2 the system (26c) is integral input-to-state stable with respect to the input $\tilde{\eta}_1 := [\tilde{v}_1\ \tilde{\omega}_1]^\top$. Moreover, since by assumption $\tilde{\eta}_1(t)$ is square-integrable and converges to zero, it follows that $e_1 \to 0$ so, consequently, $v_1^* \to v_r$, $\omega_1^* \to \omega_r$ and, in turn,

$$\lim_{t\to\infty} v_1(t) = v_r(t), \qquad \lim_{t\to\infty} \omega_1(t) = \omega_r(t). \tag{27}$$

Furthermore, there exists $\bar{c}_1 > 0$ such that

$$\max\left\{|v_1|_\infty,\ |\dot{v}_1|_\infty,\ |\omega_1|_\infty,\ |\dot{\omega}_1|_\infty\right\} \leq \bar{c}_1. \tag{28}$$

For $i = 2$ we consider the equation (26b). We see that v_1 and ω_1, regarded as functions of complete solutions, have the same properties as v_r and ω_r. Therefore, A_{v_1} may be considered as a function of time and the state e_2. Consequently, it has similar properties to those of A_{v_r} and, by Proposition 2, we conclude that (26b) is integral input-to-state stable with respect to the input $\tilde{\eta}_2 := [\tilde{v}_2\ \tilde{\omega}_2]^\top$, provided that η_1 is persistently exciting. The latter indeed follows from (9), (27) and (28) —see [36]. Thus, after Proposition 2, the system (26b) is integral input-to-state stable with respect to $\tilde{\eta}_2$. Next, in view of forward completeness, the assumption that $\tilde{\eta}_2(t) = [\tilde{v}_2(t)\ \tilde{\omega}_2(t)]^\top$ is square integrable and converges, we conclude that

$$\lim_{t\to\infty} |e_2(t)| = 0, \quad \lim_{t\to\infty} v_2(t) = v_1(t), \quad \lim_{t\to\infty} \omega_2(t) = \omega_1(t).$$

and, moreover, there exists $\bar{c}_2 > 0$ such that

$$\max\left\{|v_2|_\infty,\ |\dot{v}_2|_\infty,\ |\omega_2|_\infty,\ |\dot{\omega}_2|_\infty\right\} \leq \bar{c}_2. \tag{29}$$

The previous arguments apply for any $i \geq 2$ so the statement of Proposition 4 under condition (9) follows by induction.

Under the robust-stabilization scenario. By assumption, (11) holds. As in the previous scenario, the proof follows using Proposition 2 recursively. Indeed, for $i = 1$, we conclude that the error dynamics corresponding to the swarm leader and the virtual reference vehicle is small-input-to-state stable with respect to $\tilde{\eta}_1 := [\tilde{v}_1\ \tilde{\omega}_1]^\top$. Consequently, after forward completeness of trajectories, we have

$$\tilde{\eta}_1 \to 0 \quad \Longrightarrow \quad e_1 \to 0 \quad \Longrightarrow \quad v_1 \to 0, \quad \omega_1 \to 0.$$

The last inequality follows from (11). In turn, in view of the convergence of v_1 and ω_1, it follows that for $i = 2$ the closed-loop (26b) is small-input-to-state stable with respect to the input $\tilde{\eta}_2 := [\tilde{v}_2\ \tilde{\omega}_2]^\top$. Consequently, after forward completeness of trajectories, we have

$$\tilde{\eta}_2 \to 0 \quad \Longrightarrow \quad e_2 \to 0 \quad \Longrightarrow \quad v_2 \to 0, \quad \omega_2 \to 0.$$

The statement that (7) holds, follows by induction.

The proof of UGAS follows by applying, recursively, the same cascades argument as in the proof of Proposition 3. ■

Remark 2: As the proof of Proposition 3 shows it is the statement of Proposition 2 (integral input-to-state stability and small-input-to-state stability) which may be generalized to the multi-agent case under a spanning tree communication topology, but such extension is not possible for the main results in [7] and [17] which rely on the assumption that the leader velocities are integrable. Indeed, while the convergence of the velocities (in the parking scenario) may be asserted for the swarm leader (first follower in the tree) integrability remains unproved without a Lyapunov function. This poses a fundamental technical obstacle to use recursively the main results of the mentioned references to extend them to the multi-agent setting. •

C. Further results

With the purpose of emphasizing our main statements we wrap up this section by addressing a brief case-study of velocity control using classical passivity-based tracking control, with and without knowledge of the system's physical parameters. Even though they are simply stated the following results are, to the best of our knowledge, other original contributions of this paper. In particular, in contrast to other articles where adaptive controllers are presented, we establish uniform global asymptotic stability of the origin, which implies the uniform convergence of the parameter estimation errors.

Let (2) be in the Lagrangian form —*cf.* [2]; that is, let $M_i = M_i^\top > 0$ denote the inertia matrix and $C_i(\eta_i)$ denote the Coriolis and centrifugal forces matrix, which is skew-symmetric. Then, let the dynamics equations (2) correspond to

$$M_i \dot{\eta}_i + C_i(\eta_i)\eta_i = u_i, \qquad i \leq N, \tag{30}$$

where $u_i := B_i \tau_i$, $B_i \in \mathbb{R}^{2\times 2}$ is a full rank constant matrix of known coefficients, and τ_i is the vector of input torques at the wheels. See [2] for details.

Note that a variety of tracking controllers for (30), ensuring the *convergence* of $\tilde{\eta}_i$, are available from the literature, *e.g.*, on robot control. For instance, an elementary passivity-based state-feedback control law is

$$u_i = M_i \dot{\eta}_i^* + C_i(\eta_i)\eta_i^* - k_{di}\tilde{\eta}_i, \qquad k_{di} > 0. \tag{31}$$

Proposition 5: Consider the the system (1), (30) in closed-loop with (13) and (31). Let condition (24) as well as items (i) and (ii) of Proposition 4 hold. Then, the origin in the state space of the closed-loop system is uniformly globally asymptotically stable. □

Proof. The closed-loop dynamics (20) is

$$M_i \dot{\tilde{\eta}}_i + C_i(\tilde{\eta}_i + \eta_i^*(t, e_i))\tilde{\eta}_i + k_{di}\tilde{\eta}_i = 0, \qquad i \leq N \tag{32}$$

which may be rewritten along complete solutions $e_i(t)$ in the form (23). Then, a direct computation, using the skew-symmetry of $C_i(\cdot)$, shows that

$$V(\tilde{\eta}_i) := \tilde{\eta}_i^\top M \tilde{\eta}_i \quad \Longrightarrow \quad \dot{V}(\tilde{\eta}_i) = -2k_{di}|\tilde{\eta}_i|^2$$

hence, $\{\tilde{\boldsymbol{\eta}} = 0\}$ is a uniformly (in the initial times $t_\circ$ and in the trajectories $e_i(t)$) globally exponentially stable equilibrium of (32). The result follows from Proposition 4. ■

Let us now assume that the constant lumped parameters in M_i and $C_i(\eta_i)$, denoted $\Theta_i \in \mathbb{R}^m$, are unknown and let $\hat{M}_i$ and $\hat{C}_i$ denote the estimates of the inertia and Coriolis matrices respectively. Let $\hat{\Theta}_i$ correspond to an estimate of Θ_i and consider the controller

$$u_i = \hat{M}_i \dot{\eta}_i^* + \hat{C}_i(\eta_i)\eta_i^* - k_{di}\tilde{\eta}_i, \qquad k_{di} > 0 \tag{33a}$$

$$\dot{\hat{\Theta}}_i = -\gamma \Phi_i(t, \dot{\eta}_i^*, \eta_i^*, \tilde{\eta}_i)^\top \tilde{\eta}_i, \quad \gamma > 0 \tag{33b}$$

where, for any $i \leq N$, Φ_i is a smooth function implicitly defined by the expression

$$\Phi_i(t, \dot{\eta}_i^*, \eta_i^*, \tilde{\eta}_i)\tilde{\Theta}_i := [\hat{C}_i - C_i]\eta_i^* + [\hat{M}_i - M_i]\dot{\eta}_i^*, \tag{34}$$

where $\tilde{\Theta}_i := \hat{\Theta}_i - \Theta_i$ and $[\hat{C}_i - C_i]$ is a function of $\eta_i = \tilde{\eta}_i + \eta_i^*$.

Proposition 6: Consider the system (1), (30) in closed loop with (13) and (33). Then, the origin $\{(e_i, \tilde{\eta}_i, \tilde{\Theta}_i) = (0, 0, 0)\}$, for all $i \leq N$, is a uniformly globally asymptotically stable equilibrium point if $\Phi_1(t, \dot{\eta}_r, \eta_r, 0)$ is persistently exciting. □

Proof. The closed-loop system corresponding to the force equations (2) is

$$M_i \dot{\tilde{\eta}}_i + C_i(\eta_i)\tilde{\eta}_i + k_{di}\tilde{\eta}_i = \Phi_i(t, \dot{\eta}_i^*, \eta_i^*, \tilde{\eta}_i)\tilde{\Theta}_i \tag{35a}$$

$$\dot{\tilde{\Theta}}_i = -\gamma \Phi_i(t, \dot{\eta}_i^*, \eta_i^*, \tilde{\eta}_i)^\top \tilde{\eta}_i. \tag{35b}$$

In view of (the proof of) Proposition 5, uniform global asymptotic stability of the origin $(\tilde{\eta}_i, \tilde{\Theta}_i)$ for (35) follows directly from [37, Theorem 3], provided that $\Phi_i(t, \dot{\eta}_i^*, \eta_i^*, 0)$ is persistently exciting. Now, for $i = 1$, this means that $\Phi_1(t, \dot{\eta}_r, \eta_r, 0)$ must be persistently exciting, which holds by assumption. We conclude that $\eta_1 \to \eta_r$ and $\dot{\eta}_1 \to \dot{\eta}_r$ hence, $\Phi_2(t, \dot{\eta}_1, \eta_1, 0)$ is also persistently exciting. The result follows by induction. ■

Remark 3: We stress that:

- the controllers (31) (resp. (33)) are implemented using the leader velocities η_{i-1}, and the relative errors e_i.
- In Proposition 5 the reference trajectory η_r is only required to be bounded and of bounded derivative. In particular, it may converge at any rate, be a set-point or be persistently exciting. In Proposition 6, however, the reference trajectories are restricted to those rendering the regressor $\Phi_1(t, \dot{\eta}_r, \eta_r, 0)$ persistently exciting —typically, this excludes vanishing trajectories $\eta_r \to 0$.

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IV. Example

For the sake of illustration, we have performed some numerical simulations using Simulink of Matlab. The simulation scenario is as follows. We consider a group of four mobile robots required to follow a virtual leader while assuming a diamond-shape formation, which is designed by imposing desired distances between the robots as follows: $[d_{x_{r,1}}, d_{y_{r,1}}] = [0, 0]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [-1, 0]$ and $[d_{x_{2,3}}, d_{y_{2,3}}] = [1/2, -1/2]$ and $[d_{x_{3,4}}, d_{y_{3,4}}] = [0, 1]$ –see Figure 1 below.

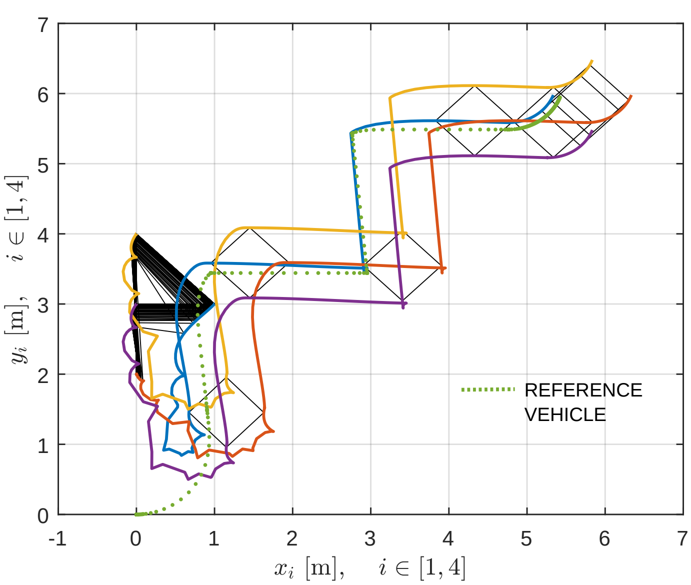

Fig. 1. Path followed by the formation with velocity vanishing slowly to full stop

The reference vehicle trajectories are generated by (3) with $v_r(t)$ and $\omega_r(t)$ such that $|\eta_r(t)|$ is persistently exciting "up to" $t = 62s$. At this instant, the reference velocities abruptly change to asymptotically-converging functions generated by the solutions of $\dot{v}_r = -50v_r^3$ and $\dot{\omega}_r = -100\omega_r^3$ with initial conditions $v_r(62) = \omega_r(62) = 1$ —see Figure 2. That is, in a first stage of the simulation test, the reference trajectories satisfy Inequality (9) (for all $t \lesssim 50$) and in a second stage they satisfy (11), but not (12). On the other hand, (15) and (16) hold with

$$F(\eta_r) := \begin{cases} 0 & \text{if } \eta_r \in (0, 0.1] \\ |\eta_r| & \text{otherwise.} \end{cases}$$

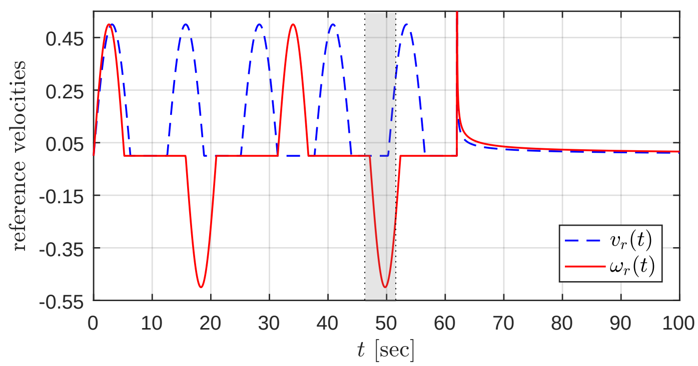

Fig. 2. Reference velocity trajectories $v_r(t)$ and $\omega_r(t)$

Each vehicle is considered to be modeled by (1), (30) with

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}, \quad C(\eta_i) = \begin{bmatrix} 0 & c\omega_i \\ -c\omega_i & 0 \end{bmatrix},$$

$m_1 = 0.6227$, $m_2 = -0.2577$, and $c = 0.2025$ –*cf.* [38] .

Then, we use the adaptive velocity-tracking controller (33) with $\Theta := [m_1\ m_2\ c]^\top$. The initial values for $\hat{\Theta}$ are set to zero. The control gains are set to $k_{x_i} = k_{y_i} = k_{\theta_i} = 1$, $\gamma = 10$, and $k_d = 15$ while $p(t) := 20\sin(0.5t)$, which has a persistently exciting time-derivative.

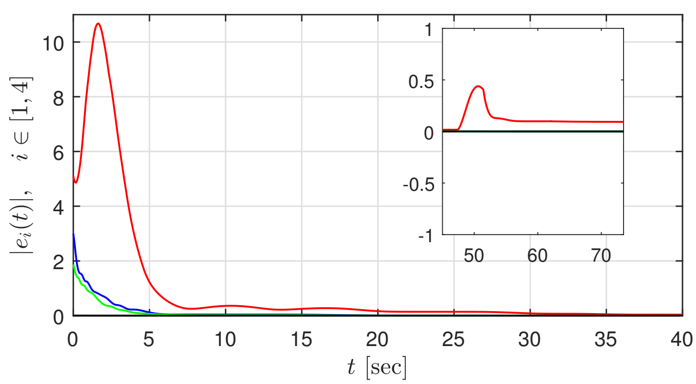

Fig. 3. Normed relative errors for each pair leader-follower

The numerical results are illustrated in Figures 3–5. In Figures 3 and 4 are showed the relative-position errors $e_i(t)$ in norm and the relative velocity errors $\eta_i - \eta_{i-1}$. In Figure 5 are depicted the norms of the parameter-estimation errors for each of the four vehicles. It may be appreciated that in view of the oscillatory behavior of $\eta_r(t)$ during the first 62s, the estimation errors converge to zero; indeed, the regressor evaluated along the reference trajectories,

$$\Phi_1(t, \dot{\eta}_r, \eta_r, 0) := \begin{bmatrix} \dot{v}_r & \dot{\omega}_r & \omega_r^2 \\ \dot{\omega}_r & \dot{v}_r & -v_r\omega_r \end{bmatrix},$$

is persistently exciting. Although the parameters are taken equal for all vehicles, the convergence rates are clearly different and, not surprisingly, the slowest rate corresponds to the parameters of the leaf-node vehicle —see right plot in Figure 5.

In addition, to illustrate the robustness of the closed-loop system, in the simulation setup the communication between the virtual leader and the swarm leader is assumed to be lost in the interval $t \in [47,\ 51.4]$ —see the shadowed region in Figure 2. The effect of the perturbation that this, and the sudden change in the reference trajectories at $t = 62$s, entail in the system's response is appreciated in the zoomed windows in Figures 3 and 4. See also the terminal stage of the test in Figure 1.

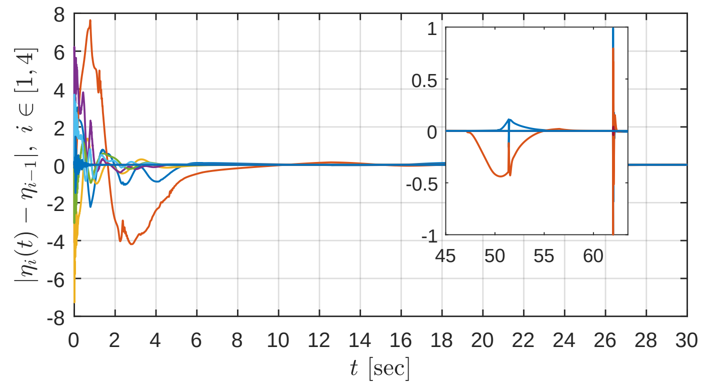

Fig. 4. Normed relative velocities for each pair leader-follower

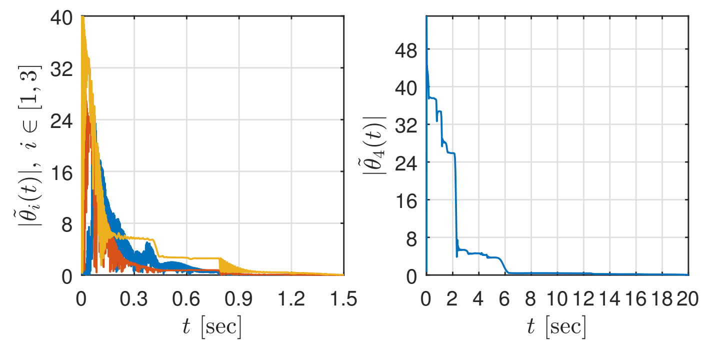

Fig. 5. Normed parameter estimation errors for each of the four vehicles

V. Conclusion

We have established the stability and robustness of a universal controller for tracking and stabilization of swarms of autonomous nonholonomic vehicles interconnected under a spanning-tree configuration topology. Our contributions reside principally in the strength of the properties that are established, such as uniform global asymptotic stability and (integral) input-to-state stability, but also in the methods of proof which, mostly, appeal to Lyapunov's direct method.

Our results are fairly general since they are not bound even to a particular choice of dynamic model. In that regard, we believe that they may contribute to pave the way to generalizations and relaxations of certain hypotheses, in orther to incorporate realistic scenarios, such as output-feedback control, more general interconnection topologies, and time-varying graphs. Indeed, such problems may be addressed on the solid basis provided by the construction of Lyapunov functions. Research in such directions is being carried out.

References

[1] C. Samson, "Control of chained system: Application to path following and time-varying point stabilization of mobile robots," *IEEE Trans. on Automatic Control*, vol. 40, no. 1, pp. 64–77, 1995.

[2] K. D. Do, Z.-P. Jiang, and J. Pan, "A global output-feedback controller for simultaneous tracking and stabilization of unicycle-type mobile robots," *IEEE Trans. on Robotics Automat.*, vol. 20, no. 3, pp. 589–594, 2004.

[3] A. Dong and J. A. Farrell, "Cooperative control of multiple nonholonomic mobile agents," *IEEE Trans. on Automatic Control*, vol. 53, no. 6, pp. 1434–1447, 2008.

[4] J. Huang, C. Wen, W. Wang, and Z.-P. Jiang, "Adaptive output feedback tracking control of a nonholonomic mobile robot," *Automatica*, vol. 50, no. 3, pp. 821–831, 2014.

[5] C. C. de Wit, H. Khennouf, C. Samson, and O. J. Sørdalen, "Nonlinear control design for mobile robots", vol. 11 of *Robotics and Automated Systems*, ch. Recent Trends in Mobile Robots. Y. F. Zheng, ed., London: World Scientific, 1993.

[6] E. Panteley, E. Lefeber, A. Loria, and H. Nijmeijer, "Exponential tracking of a mobile car using a cascaded approach," in *IFAC Workshop on Motion Control*, (Grenoble, France), pp. 221–226, 1998.

[7] Y. Wang, Z. Miao, H. Zhong, and Q. Pan, "Simultaneous stabilization and tracking of nonholonomic mobile robots: A lyapunov-based approach," *IEEE Trans. on Control Systems Technology*, vol. 23, pp. 1440–1450, July 2015.

[8] A. Sadowska, T. van den Broek, H. Huijberts, N. van de Wouw, D. Kostić, and H. Nijmeijer, "A virtual structure approach to formation control of unicycle mobile robots using mutual coupling," *Int. J. of Contr.*, vol. 84, no. 11, pp. 1886–1902, 2011.

[9] K. D. Do, "Formation tracking control of unicycle-type mobile robots with limited sensing ranges," *IEEE Trans. Contr. Syst. Technol.*, vol. 16, no. 3, pp. 527–538, 2008.

[10] M. Defoort, T. Floquet, and V. Perruquetti, "Sliding-mode formation control for cooperative autonomous mobile robots," *IEEE Trans. Industrial Electronics*, vol. 55, pp. 3944–3953, 2008.

[11] E. Vos, A. J. van der Schaft, and J. M. A. Scherpen, "Formation control and velocity tracking for a group of nonholonomic wheeled robots," *IEEE Trans. on Automatic Control*, vol. 61, pp. 2702–2707, Sept 2016.

[12] T. C. Lee, K. T. Song, C. H. Lee, and C. C. Teng, “Tracking control of unicycle-modeled mobile robots using a saturation feedback controller,” *IEEE Trans. Contr. Syst. Technol.*, vol. 9, pp. 305–318, Mar 2001.

[13] P. Morin and C. Samson, “Control of nonholonomic mobile robots based on the transverse function approach,” *IEEE Trans. on robotics*, vol. 25, no. 5, pp. 1058–1073, 2009.

[14] W. E. Dixon, M. S. de Queiroz, D. M. Dawson, and T. J. Flynn, “Adaptive tracking and regulation of a wheeled mobile robot with controller/update law modularity,” *IEEE Trans. on control systems technology*, vol. 12, no. 1, pp. 138–147, 2004.

[15] K. D. Do, Z.-P. Jiang, and J. Pan, “Simultaneous tracking and stabilization of mobile robots: an adaptive approach,” *IEEE Trans. on Automatic Control*, vol. 49, no. 7, pp. 1147–1152, 2004.

[16] P. Morin and C. Samson, “Practical stabilization of driftless systems on lie groups: the transverse function approach,” *IEEE Trans. on Automatic control*, vol. 48, no. 9, pp. 1496–1508, 2003.

[17] M. Maghenem, A. Loria, and E. Panteley, “A universal adaptive controller for tracking and stabilization control of nonholonomic vehicles,” *Int. Journal of Robust and nonlinear Control (under review)*, 2017. Based on the technical report: https://hal.archives-ouvertes.fr/view/index/docid/1609231 hal-01609231.

[18] M. Maghenem, A. Loria and E. Panteley, “A robust δ-persistently exciting controller for leader-follower tracking-agreement of multiple vehicles,” *European Journal of Control*, vol. 40, pp. 1–12, 2018. Appeared online: sept. 2017. DOI: 10.1016/j.ejcon.2017.09.001.

[19] D. Angeli, E. D. Sontag, and Y. Wang, “A characterization of integral input-to-state stability,” *IEEE Trans. on Automatic Control*, vol. 45, no. 6, pp. 1082–1097, 2000.

[20] A. Chaillet, D. Angeli, and H. Ito, “Combining iISS and ISS with respect to small inputs: the strong iISS property,” *IEEE Trans. on Automatic Control*, vol. 59, no. 9, pp. 2518–2524, 2014.

[21] A. Chaillet, D. Angeli, and H. Ito, “Strong iISS is preserved under cascade interconnection,” *Automatica*, vol. 50, no. 9, pp. 2424–2427, 2014.

[22] M. Malisoff and F. Mazenc, *Constructions of Strict Lyapunov functions*. London: Springer Verlag, 2009.

[23] D. A. Lizárraga., “Obstructions to the existence of universal stabilizers for smooth control systems,” *Mathematics of Control, Signals and Systems*, vol. 16, p. 255277, 2004.

[24] R. Brockett, “Asymptotic stability and feedback stabilization,” in *Differential geometric control theory* (R. S. M. R. W. Brocket and H. J. Sussmann, eds.), pp. 181–191, Birkhäuser, 1983.

[25] Y. Kanayama, Y. Kimura, F. Miyazaki, and T. Naguchi, “A stable tracking control scheme for an autonomous vehicle,” in *Proc. IEEE Conf. Robotics Automat.*, pp. 384–389, 1990.

[26] H. Yamaguchi, T. Arai, and G. Beni, “A distributed control scheme for multiple robotic vehicles to make group formations,” *Robotics and Autonomous Systems*, vol. 36, no. 4, pp. 125–147, 2001.

[27] A. A. J. Lefeber, *Tracking control of nonlinear mechanical systems*. PhD thesis, University of Twente, Enschede, The Netherlands, 2000.

[28] W. E. Dixon, D. M. Dawson, F. Zhang, and E. Zergeroglu, “Global exponential tracking control of a mobile robot system via a pe condition,” *IEEE Trans. Systems, Man, and Cybernetics B*, vol. 30, no. 1, pp. 129–142, 2000.

[29] A. Loria, J. Dasdemir, and N. Alvarez-Jarquin, “Leader-follower formation control of mobile robots on straight paths,” *IEEE Trans. on Contr. Syst. Techn.*, vol. 24, no. 2, pp. 727–732, 2016.

[30] W. Wang, J. Huang, C. Wen, and H. Fan, “Distributed adaptive control for consensus tracking with application to formation control of nonholonomic mobile robots,” *Automatica*, vol. 50, no. 4, pp. 1254–1263, 2014.

[31] A. Loria and E. Panteley, “Uniform exponential stability of linear time-varying systems: revisited,” *Syst. & Contr. Letters*, vol. 47, no. 1, pp. 13–24, 2002.

[32] I. Malkin, “ Об устойчивости при постоянно действуюсчих возмысчениях,” *Prikl. Mat. i Mekh.*, vol. Tom. VIII, pp. 241–245, 1944. In Russian. Engl. translation: *On stability under constantly acting disturbances*.

[33] E. Panteley and A. Loria, “Growth rate conditions for stability of cascaded time-varying systems,” *Automatica*, vol. 37, no. 3, pp. 453–460, 2001.

[34] A. Loria, “From feedback to cascade-interconnected systems: Breaking the loop,” in *Proc. of the 47th. IEEE Conf. on Decision Control*, (Cancun, Mex.), pp. 4109–4114, 2008.

[35] H. Khalil, *Nonlinear systems*. New York: Macmillan Publishing Co., 2nd ed., 1996.

[36] K. S. Narendra and A. M. Annaswamy, “Persistent excitation in adaptive systems,” *Int. J. of Contr.*, vol. 45, no. 1, pp. 127–160, 1987.

[37] A. Loria E. Panteley, D. Popovic, and A. Teel, “δ-persistency of excitation: a necessary and sufficient condition for uniform attractivity,” in *Proc. of the 41st. IEEE Conf. on Decision Control*, (Las Vegas, CA, USA), pp. 3506–3511, 2002. Paper no. REG0623 .

[38] T. Fukao, H. Nakagawa, and N. Adachi, “Adaptive tracking control of a nonholonomic mobile robot,” *IEEE Trans. on Robotics Automat.*, vol. 16, no. 5, pp. 609–615, 2000.

[39] M. Maghenem, A. Loria, and E. Panteley, “iISS formation tracking control of autonomous vehicles,” tech. rep., CentraleSupelec, 2016. Available online: https://hal.archives-ouvertes.fr/hal-01364791.

[40] H. Ito, “A Lyapunov approach to cascade interconnection of integral input-to-state stable systems,” *IEEE Trans. on Automatic Control*, vol. 55, no. 3, pp. 702–708, 2010.

[41] S. Srikant and M. R. Akella, “Persistence filter-based control for systems with time-varying control gains,” *Syst. & Contr. Letters*, vol. 58, no. 6, pp. 413–420, 2009.

[42] P. Ioannou and J. Sun, *Robust adaptive control*. New Jersey, USA: Prentice Hall, 1996.

[43] E. Sontag, “Input to state stability: Basic concepts and results,” in *Nonlinear and optimal control theory*, pp. 163–220, Springer Berlin Heidelberg, 2008.

Appendix

A. Proof of Proposition 2

Forward completenss may be established by evaluating the total derivative of the positive-definite function

$$V_{1i}(t, e_i) := \frac{1}{2}\left[e_{xi}^2 + e_{yi}^2 + \frac{1}{k_{yi}} e_{\theta i}^2\right], \quad i \in \{1, \ldots, N\}$$

to obtain

$$\dot{V}_{1i} = -\, k_{xi} e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}} e_{\theta i}^2 - p_i \rho_i e_{\theta i} |e_{xyi}| - \frac{1}{k_{yi}} e_{\theta i} \tilde{\omega}_i - e_{xi} \tilde{v}_i.$$

After the uniform boundedness of p_i, ρ_i, $\tilde{\omega}_i$, and $\tilde{v}_i$ and the triangle inequality, it follows that there exist positive constants a_i and b_i such that

$$\dot{V}_{1i}(t, e_i(t)) \leq a_i V_{1i}(t, e_i(t)) + b_i$$

which, upon integration from any $t_\circ$ to ∞ leads to the conclusion that the solutions have no finite escape-time.

1) In the tracking scenario: Let i be arbitrarily fixed. By assumption, (9), and consequently (15), hold with η_r replaced by[2] η_{i-1}. The analysis of the closed-loop equation (25) follows the following steps:

1) to design a strict Lyapunov function $V_i(t, e_i)$ for $\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i$;
2) based on the latter, to construct a strict Lyapunov function $W_i(t, e_i)$ for

$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i + B_{1i}(t, e_i)\rho_i; \tag{36}$$

3) in turn, to use $W_i(t, e)$ to construct a Lyapunov function W_{1i} to establish integral ISS of (25) with respect to $\tilde{\eta}_i$, as well as the boundedness of the trajectories of (25) under the assumption that $\tilde{\eta}_i \in \mathcal{L}_2$.

Step 1. Uniform global asymptotic stability for

$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i \tag{37}$$

is established in [39] via Lyapunov's direct method. Indeed, after [39, Proposition 1], there exists a positive definite radially unbounded function $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$,

$$\begin{aligned} V_i(t, e_i) \;\; &:= \;\; P_{[2]}(t, V_{1i})V_{1i}(e_i) - \omega_{i-1}(t)e_{xi}e_{yi} \\ &\quad\; + v_{i-1}(t)P_{[1]}(t, V_{1i})e_{\theta i}e_{yi}, \end{aligned} \tag{38}$$

satisfying

$$F_{[3]}(V_{1i}(e_i)) \leq V_i(t, e_i) \leq S_{[3]}(V_{1i}(e_i)), \tag{39}$$

where

$$V_{1i}(e_i) := \frac{1}{2}\left[e_{xi}^2 + e_{yi}^2 + \frac{1}{k_{yi}}e_{\theta i}^2\right], \tag{40}$$

$F_{[k]}$, $S_{[k]}$, and $P_{[k]}(t, \cdot)$ are smooth polynomials in V_{1i} with strictly positive coefficients of degree k, and $P_{[k]}(\cdot, V_{1i})$ is uniformly bounded. Furthermore, a direct computation shows that the total derivative of V_{1i} along the trajectories of (37) satisfies

$$\dot{V}_{1i}(e_i) = -k_{xi}e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}}e_{\theta i}^2. \tag{41}$$

Hence, mimicking [39] one finds that the total derivative of $V_i(t, e_i)$ along the trajectories of (37) satisfies

$$\dot{V}_i(t, e_i) \leq -\sigma V_{1i}(e_i) - k_{xi}e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}}e_{\theta i}^2 \tag{42}$$

where $\sigma > 0$ is a design parameter that depends on μ and T introduced in (9). Uniform global asymptotic stability follows.

Remark 4: This establishes Corollary 1 •

Step 2. Let $Q_{[3]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be another third order polynomial in V_{1i} with strictly positive coefficients and define

$$Z_i(t, e_i) := Q_{[3]}(V_{1i})V_{1i}(e_i) + V_i(t, e_i). \tag{43}$$

In view of the fact that $\partial Q_{[3]}/\partial V_{1i} \geq 0$, and after (41), the total derivative of Z_i along the trajectories of (37) yields

$$\dot{Z}_i(t, e_i) \;\leq\; -Y_i(e_i) \tag{44}$$

$$Y_i(e_i) \;:=\; \sigma V_{1i}(e_i) + Q_{[3]}(V_{1i}(e_i))\big[k_{xi}e_{xi}^2 + k_{\theta i}e_{\theta i}^2\big]. \tag{45}$$

Note that Y_i is positive definite and radially unbounded.

On the other hand, from (14) we see that $\dot{\rho}_i = -F(\eta_{i-1}(t))\rho_i$. From this and (15) (in which we replace η_r with η_{i-1}) it follows that $\rho_i \to 0$ exponentially fast (and is uniformly integrable). Therefore, for any $\gamma > 0$, the function

$$G_i(t) := \exp\left(-\gamma\int_0^t \rho_i(s)ds\right) \quad \forall t \geq 0 \tag{46}$$

is bounded from above and below. Indeed, defining $G_m := \lim_{t\to\infty} G_i(t) > 0$ and we have $G(t) \in [G_m, 1]$ for all $t \geq 0$. In addition, since $Z_i(t, e_i)$ and $V_i(t, e_i)$ are positive definite radially unbounded —see (39) and (43), so is the function

$$W_i(t, e_i) \;\; := \;\; G_i(t)Z_i(t, e_i), \tag{47}$$

whose total derivative along the trajectories of (36) verifies

$$\dot{W}_i(t, e_i) \leq -\, G_i(t)Y_i(e_i) + \dot{G}_i(t)Z_i(t, e_i) \tag{48}$$

$$+\, G_i(t)\frac{\partial Z_i(t, e_i)}{\partial e_i}B_{1i}(t, e_i)\rho_i(t). \tag{49}$$

[2]Recall that in this proof η_{i-1} is an exogenous signal.

Next, we develop

$$\frac{\partial Z_i(t,e_i)}{\partial e_i}B_{1i}(t,e_i) = \frac{\partial\left[Q_{[3]}(V_{1i})V_{1i}+V_i\right]}{\partial V_{1i}}\frac{\partial V_{1i}}{\partial e_i}B_{1i}(t,e_i)$$
$$+\,\omega_{i-1}k_{yi}p_i(t)|e_{xyi}|\left[e_{xi}^2-e_{yi}^2\right]$$
$$-\,v_{i-1}P_{[1]}(t,V_{1i})k_{yi}p_i(t)|e_{xyi}|\left[e_{\theta i}e_{xi}+e_{yi}\right] \quad (50)$$

and we decompose $B_{1i}(t,e_i)$ into

$$B_{1i}(t,e_i) = k_{yi}p(t)|e_{xyi}|\left[\begin{bmatrix}-1\\0\\0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}e_i\right].$$

Then, since

$$\frac{\partial V_{1i}}{\partial e_i}\begin{bmatrix}0&0&0\\0&0&k_{yi}p(t)|e_{xyi}|\\0&-k_{yi}p(t)|e_{xyi}|&0\end{bmatrix}e_i = 0,$$

it follows that

$$\frac{\partial V_{1i}}{\partial e_i}B_{1i}(t,e_i) = -\frac{\partial V_{1i}}{\partial e_{\theta i}}k_{yi}p_i(t)|e_{xyi}| = -e_{\theta i}p_i(t)|e_{xyi}|.$$

Therefore, in view of the boundedness of v_{i-1}, ω_{i-1}, p_i, and $P_{[1]}(\cdot,V_{1i})$ there exists a polynomial $R_{[3]}(V_{1i})$ with non-negative coefficients, such that

$$\frac{\partial Z_i(t,e_i)}{\partial e_i}B_{1i}(t,e_i) \le R_{[3]}(V_{1i})V_{1i} \quad (51)$$

and, since the coefficients of $F_{[3]}(V_{1i})$ are strictly positive there exists $\gamma>0$ such that

$$\gamma Q_{[3]}(V_{1i}) \ge R_{[3]}(V_{1i}).$$

Consider now (46) with such γ; we have $\dot{G}_i(t) = -\gamma G_i(t)\rho_i(t) \le 0$. Therefore, since $Z_i(t,e_i) \ge Q_{[3]}(V_{1i})V_{1i}$ —see (43), it follows that the last two terms on the right-hand side of (48) are bounded from above by

$$-\gamma G_i(t)\rho_i(t)Q_{[3]}(V_{1i})V_{1i} + G(t)\rho_i(t)R_{[3]}(V_{1i})V_{1i} \le 0.$$

Consequently, $\dot{W}(t,e_i) \le -G_mY_i(e_i)$ for all $t\ge 0$ and all $e_i\in\mathbb{R}^3$. Uniform global asymptotic stability of the null solution of (36) follows.

Remark 5: In words, we have established UGAS for the kinematics closed-loop system, even in the presence of vanishing persturbations ρ_i. •

Step 3. In order to establish iISS with respect to $\tilde{\eta}_i$ we introduce the positive definite radially unbounded function $W_{1i}:\mathbb{R}_{\ge 0}\times\mathbb{R}^3\to\mathbb{R}_{\ge 0}$, defined by

$$W_{1i}(t,e_i) := \ln\left(1+W_i(t,e_i)\right). \quad (52)$$

The derivative of W_{1i} along trajectories of (25) satisfies

$$\dot{W}_{1i} \le -\frac{1}{1+W_i(t,e_i)}\left[G_mY_i(e_i) - \left|\frac{\partial W_i}{\partial e_i}B_{2i}\tilde{\eta}_i\right|\right]. \quad (53)$$

We proceed to evaluate and bound the last term on the right-hand side of (53). To that end, let us introduce

$$\zeta_i := \begin{bmatrix}\frac{e_{\theta i}}{k_{yi}}\\e_{xi}\end{bmatrix},$$

$$H_i(t,e_i) := Q_{[3]}+P_{[3]}+\frac{\partial Q_{[3]}}{\partial V_{1i}}V_{1i}+\frac{\partial P_{[3]}}{\partial V_{1i}}V_{1i} + v_{i-1}e_{\theta i}e_{yi}\frac{\partial P_{[1]}}{\partial V_{1i}},$$

and let us decompose $B_{2i}(e_i)\tilde{\eta}_i$ from (25) into

$$B_{2i}(e_i)\tilde{\eta}_i := B_{21i}(\tilde{\eta}_i) + B_{22i}(\tilde{\eta}_i)e_i$$

where

$$B_{21i}(\tilde{\eta}_i) := \begin{bmatrix}-\tilde{\omega}_i\\-\tilde{v}_i\\0\end{bmatrix},\quad B_{22i}(\tilde{\eta}_i) := \begin{bmatrix}0&0&0\\0&0&\tilde{\omega}_i\\0&-\tilde{\omega}_i&0\end{bmatrix}.$$

Then, we see that
$$\frac{\partial V_{1i}}{\partial e_i}B_{21i}=\zeta_i^\top\tilde{\eta}_i,\qquad \frac{\partial V_{1i}}{\partial e_i}B_{22i}e_i=0$$
and, therefore,
$$\frac{1}{G_i}\frac{\partial W_i}{\partial e_i}B_{21i}=-H_i(t,e_i)\zeta_i^\top\tilde{\eta}_i+e_{yi}\big[v_{i-1}P_{[1]}\tilde{\omega}_i-\omega_{i-1}\tilde{v}_i\big]$$
$$\frac{1}{G_i}\frac{\partial W_i}{\partial e_i}\ =\ \omega_{i-1}[0\ e_{yi}\ e_{xi}]+v_{i-1}P_{[1]}[e_{yi}\ 0\ e_{\theta i}].$$
Hence, using $|G_i(t)|\leq 1$ and $|\eta_{i-1}|\leq\bar{\eta}_{i-1}$ it follows that
$$\begin{aligned}\Big|\frac{\partial W_i}{\partial e_i}B_{2i}\tilde{\eta}_i\Big|\ &\leq\ |\tilde{\eta}_i|\Big[|H_i||\zeta_i|+\bar{\eta}_{i-1}|e_{yi}|+\bar{\eta}_{i-1}P_{[1]}|e_{yi}|\\
&\qquad+\bar{\eta}_{i-1}V_{1i}+\bar{\eta}_{i-1}P_{[1]}|e_{\theta i}||e_{yi}|\Big]\\
&\leq\ |H_i|\Big[\frac{1}{2\epsilon}|\zeta_i|^2+\frac{\epsilon}{2}|\tilde{\eta}_i|^2\Big]+\bar{\eta}_{i-1}\Big[\frac{1}{2\epsilon}V_{1i}+\frac{\epsilon}{2}|\tilde{\eta}_i|^2\Big]\\
&\quad+\frac{\bar{\eta}_{i-1}}{2}\bigg[\Big[\frac{1}{\epsilon}V_{1i}+\epsilon P_{[1]}^2|\tilde{\eta}_i|^2\Big]+\Big[\frac{1}{\epsilon}V_{1i}+\epsilon V_{1i}|\tilde{\eta}_i|^2\Big]\bigg]\\
&\quad+\bar{\eta}_{i-1}P_{[1]}\Big[\frac{1}{2\epsilon}V_{1i}|e_\theta|^2+\frac{\epsilon}{2}|\tilde{\eta}_i|^2\Big]\\
&\leq\ \big[|H_i|+\bar{\eta}_{i-1}P_{[1]}k_y^2V_{1i}\big]\ \frac{1}{2\epsilon}|\zeta_i|^2+\frac{3\bar{\eta}_{i-1}}{2\epsilon}V_{1i}\\
&\quad+\frac{\epsilon}{2}|\tilde{\eta}_i|^2\Big[|H_i|+\bar{\eta}_{i-1}\big[V_{1i}+1+P_{[1]}^2+P_{[1]}\big]\Big].\end{aligned}$$
Next, we introduce a third-order polynomial $D_{[3]}(V_{1i})$ satisfying
$$|H_i|+\bar{\eta}_{i-1}\big[V_{1i}+1+P_{[1]}^2+P_{[1]}\big]\leq D_{[3]}$$
and we choose $\epsilon>0$ such that $3\bar{\eta}_{i-1}\leq\epsilon\sigma G_m$ and
$$\big[|H_i|+\bar{\eta}_{i-1}P_{[1]}k_y^2V_{1i}\big]\ \frac{|\zeta_i|^2}{\epsilon}\leq G_mQ_{[3]}\big[k_{xi}e_{xi}^2+k_{\theta i}e_{\theta i}^2\big].$$
Such $\epsilon>0$ exists because $Q_{[3]}$ is and $|H_i|$ is bounded by third-order polynomials of V_{1i} with strictly positive coefficients. Thus, (53) becomes
$$\dot{W}_1\leq-\frac{1}{2}\frac{1}{1+W_i(t,e_i)}\big[G_mY_i(e_i)-\epsilon D_{[3]}(V_{1i})|\tilde{\eta}_i|^2\,\big]\tag{54}$$
On the other hand, from (43) and (47) it follows that
$$G_mQ_{[3]}(V_{1i})V_{1i}\leq W_i(t,e_i)\leq Q_{[3]}(V_{1i})V_{1i}\tag{55}$$
hence,
$$\begin{aligned}\dot{W}_{1i}\ &\leq\ -\frac{G_m}{2}\frac{Y_i(e_i)}{1+Q_{[3]}(V_{1i}(e_i))V_{1i}(e_i)}\\
&\qquad+\frac{\epsilon}{2}\frac{D_{[3]}(V_{1i})}{1+G_mQ_{[3]}(V_{1i})V_{1i}}\,|\tilde{\eta}_i|^2\end{aligned}$$
and we conclude that there exist a constant $c>0$ and a positive definite function $\alpha:\mathbb{R}^3\to\mathbb{R}_{\geq 0}$ such that
$$\dot{W}_{1i}\leq-\alpha(e_i)+c\,|\tilde{\eta}_i|^2\tag{56}$$
The statement follows from [40].

2) In the stabilization scenario: For any *fixed* i we rewrite the closed-loop equation (25) in the form
$$\dot{e}_i=A_i(t,e_i)e_i+B_i(e_i)\xi_i\tag{57}$$
where $\xi_i^\top=[\eta_{i-1}^\top\quad\tilde{\eta}_i^\top]$,
$$A_i:=\begin{bmatrix}-k_{\theta i} & -k_{yi}q_i(t)\frac{|e_{xyi}|}{e_{xi}} & -k_{yi}q_i(t)\frac{|e_{xyi}|}{e_{yi}}\\ 0 & -k_{xi} & \pi_i(t,e_i)\\ 0 & -\pi_i(t,e_i) & 0\end{bmatrix},$$

$$B_i := \begin{bmatrix} -k_{yi}e_{yi}\phi(e_{\theta i}) & 0 & 0 & -1 \\ k_{yi}e_{yi}^2\phi(e_{\theta i}) & e_{yi} & -1 & e_{yi} \\ \sin(e_{\theta i}) - k_{yi}e_{xi}e_{yi}\phi(e_{\theta i}) & -e_{xi} & 0 & -e_{xi} \end{bmatrix},$$

$q_i(t) := \rho_i(t)p_i(t)$, ρ_i is defined in (14), $\pi_i := k_{\theta i}e_{\theta i} + k_{yi}q_i(t)|e_{xyi}|$, and $e_{xyi} = [e_{xi}\ e_{yi}]^\top$. Then, we shall establish the following:

Claim 1: The system (57) is small-input-to-state stable respect to ξ_i.

Claim 2: The system (57) is integral-input-to-state stable with respect to ξ_i.

If these claims hold the system (57) is strong integral-input-to-state stable with respect to the input ξ_i, hence the property also holds with respect to the input $\xi_i^\circ := [\eta_{i-1}^\top\ 0]^\top$. By virtue of Lemma 3 (see farther below) and the condition that $\eta_{i-1} \to 0$, which holds by assumption, it follows that the system subject to $\tilde{\eta}_i = 0$ is uniformly globally asymptotically stable. Then, to establish small-input-to-state stability of the system (57) with respect to $\tilde{\eta}_i$, it is left to show that it possesses the so-called *small-input-bounded-state* property with respect to $\tilde{\eta}_i$, for any converging $t \mapsto \eta_{i-1}$. To that end, pick any small $\epsilon > 0$ and let $|\tilde{\eta}_i| \leq \epsilon/2$. Since the system is forward complete and $\eta_{i-1}(t) \to 0$ it follows that there exists a sufficiently large $T_\epsilon > 0$ such that $|\eta_{i-1}(t)| \leq \epsilon/2$ for all $t \geq t_\circ + T_\epsilon$ and $|\xi_i(t)| \leq \epsilon$. On the other hand, the system (57) is small-input-to-state stable with respect to ξ_i hence, the solutions are bounded. This concludes the proof of small-input-to-state stability with respect to $\tilde{\eta}_i$.

We proceed now to prove Claims 1 and 2 above. To that end, we first construct a strict Lyapunov function for the nominal closed-loop system $\dot{e}_i = A_i(t, e_i)$ —*cf.* Eq. (57).

Let $\psi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a twice-continuously-differentiable function, satisfying the differential equation

$$\dot{\psi}_i = -k_{\theta i}\psi_i + k_{yi}q_i(t) \tag{58}$$

and let $e_{zi} := e_{\theta i} + \psi_i(t)|e_{xyi}|$. Then, the nominal system $\dot{e}_i = A_i(t, e_i)e_i$ becomes

$$\begin{bmatrix}\dot{e}_{xi} \\ \dot{e}_{yi}\end{bmatrix} = \begin{bmatrix} -k_{xi} & \dot{\psi}_i|e_{xyi}| \\ -\dot{\psi}_i|e_{xyi}| & 0 \end{bmatrix}\begin{bmatrix}e_{xi} \\ e_{yi}\end{bmatrix} + e_{zi}\begin{bmatrix} 0 & k_{\theta i} \\ -k_{\theta i} & 0 \end{bmatrix}\begin{bmatrix}e_{xi} \\ e_{yi}\end{bmatrix} \tag{59a}$$

$$\dot{e}_{zi} = -k_{\theta i}e_{zi} - \psi_i k_{xi}\frac{e_{xi}^2}{|e_{xyi}|} \tag{59b}$$

We stress that, by construction, ρ_i and $\dot{\rho}_i$ are bounded and, by assumption, so are p_i and $\dot{p}_i$. It follows that q_i and $\dot{q}_i$, and in turn ψ_i and $\dot{\psi}_i$, are also bounded. Moreover, since p_i and $\dot{p}_i$ are persistently exciting, so is $\dot{q}_i$ —*cf.* [36] and, consequently, there exist $\psi_M > \psi_m > 0$ such that $\psi_i(t) \in [\psi_m, \psi_M]$ for all $t \geq 0$ —see [41]. Furthermore, since $\dot{q}_i$ is persistently exciting and $\dot{\psi}_i$ satisfies

$$\ddot{\psi}_i = -k_{\theta i}\dot{\psi}_i + \dot{q}_i, \tag{60}$$

it follows that $\dot{\psi}_i$ is also persistently exciting —see [42, Lemma 4.8.3]. Thus, one can show that the following is a strict Lyapunov function for (59):

$$V_{2i} := P_{[1]}(V_{1i})V_{1i} + \Upsilon(t)V_{1i}^2 - \dot{\psi}_i\sqrt{V_{1i}}e_{xi}e_{yi} + Q_{[1]}(V_{1i})e_{zi}^2 \tag{61}$$

where $V_{1i} := e_{xi}^2 + e_{yi}^2$,

$$\Upsilon(t) := 1 + \bar{\psi}_i^2 T - \frac{1}{T}\int_t^{t+T}\int_t^m \dot{\psi}_i(s)^2 ds dm, \tag{62}$$

$\bar{\psi}_i \geq \max\{|\psi_i|_\infty,\ |\dot{\psi}_i|_\infty,\ |\ddot{\psi}_i|_\infty\}$, and $P_{[1]}$ and $Q_{[1]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are first-order polynomials of V_{1i}. Indeed, let

$$P_{[1]}(V_{1i}) := \bar{\psi}_i[V_{1i} + 1] \tag{63}$$

$$Q_{[1]}(V_{1i}) := \frac{P_{[1]}(V_{1i})}{2} + V_{1i} \tag{64}$$

then, since $-\dot{\psi}_i\sqrt{V_{1i}}e_{xi}e_{yi} \geq -(\bar{\psi}_i/2)V_{1i}[V_{1i} + 1]$ and $\Upsilon(t) \geq 1$, we obtain

$$V_{2i}(t, e_i) \geq Q_{[1]}(V_{1i})[V_{1i} + e_{zi}^2] \tag{65}$$

so V_{2i} is positive definite and radially unbounded. Furthermore, mimicking the proof of [18, Proposition 2], one finds that there exists $\sigma > 0$ such that the derivative of V_{2i} satisfies

$$\dot{V}_{2i}(t, e_i) \leq -\frac{1}{2}k_{\theta i}Q_{[1]}(V_{1i})e_{zi}^2 - \sigma V_{1i}^2. \tag{66}$$

Proof of Claim 1. The proof of small ISS for the system (57) with respect to ξ_i relies on the function V_{2i} above; specifically on its (second) order of growth in V_{1i}. We proceed to evaluate the total derivative of V_{2i} along trajectories of (57) to obtain, from (66),

$$\dot{V}_{2i}(t,e_i) \leq -\frac{1}{2}k_{\theta i}Q_{[1]}(V_{1i})e_{zi}^2 - \sigma V_{1i}^2 + \frac{\partial V_{2i}}{\partial e_i}B_i(e_i)\xi_i.$$

Then, we decompose $B_i(e)\xi_i$ into

$$B_i(e_i)\xi_i := B_{1i}(\xi_i,e_i)e_i + B_{2i}(\xi_i,e_i), \tag{67}$$

where

$$B_{1i}(\xi_i,e_i) := \left[\tilde{\omega}_i + \omega_{i-1} + k_{yi}v_{i-1}e_{yi}\phi(e_{\theta i})\right]\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}$$

and

$$B_{2i}(\xi_i,e_i) = \begin{bmatrix}-\tilde{\omega}_i - k_{yi}v_{i-1}e_{yi}\phi(e_{\theta i})\\ \tilde{v}_i\\ v_{i-1}\sin(e_{\theta i})\end{bmatrix}.$$

so, using

$$\frac{\partial V_{1i}}{\partial e_i}B_{1i}e_i = 0, \tag{68}$$

we obtain

$$\begin{aligned}\dot{V}_{2i} \leq & -\sigma V_{1i}^2 - \frac{1}{2}k_{\theta i}Q_{[1]}(V_{1i})e_{zi}^2\\ & - 2\dot{\psi}_i[\omega_{i-1}+\tilde{\omega}_i]\sqrt{V_{1i}}\left[e_{yi}^2 - e_{xi}^2\right]\\ & - \dot{\psi}_i\sqrt{V_{1i}}e_{yi}v_{i-1}\left[e_{yi}^2 - e_{xi}^2\right] + \frac{\partial V_{2i}}{\partial e_i}B_{2i}\\ \leq & -\sigma V_{1i}^2 - \frac{1}{2}k_{\theta i}Q_{[1]}(V_{1i})e_{zi}^2 + 2\bar{\psi}\left|\omega_{i-1}+\tilde{\omega}\right|\sqrt{V_{1i}}V_{1i}\\ & + \bar{\psi}_i\sqrt{V_{1i}}\left|e_{yi}\right|\left|v_{i-1}\right|V_{1i} + \frac{\partial V_{2i}}{\partial e_i}B_{2i}.\end{aligned} \tag{69}$$

Moreover, the last term satisfies

$$\begin{aligned}&\left|\frac{\partial V_{2i}}{\partial e_i}B_{2i}\right| \leq\\ &\quad \left[P_{[1]}(V_{1i}) + \frac{\partial P_{[1]}}{\partial V_{1i}}V_{1i} + 2\bar{\psi}_i\sqrt{V_{1i}}\right]\left|e_{xyi}\right|\left|[\tilde{v}\ v_{i-1}]\right|\\ &\quad + \left[\frac{\partial Q_{[1]}}{\partial V_{1i}}e_{zi}^2\left|e_{xyi}\right| + Q_{[1]}(V_{1i})\left|\psi_i e_{zi}\right|\right]\left|[\tilde{v}\ v_{i-1}]\right|\\ &\quad + Q_{[1]}(V_{1i})\left|e_{zi}\right|\left|e_{yi}\right|\left|v_{i-1}\right| + Q_{[1]}(V_{1i})\left|e_{zi}\right|\left|\tilde{\omega}_i\right|\end{aligned} \tag{70}$$

so, using the latter in (69), we obtain

$$\begin{aligned}\dot{V}_{2i} \leq & -\sigma V_{1i}^2 - \frac{1}{2}k_{\theta i}Q_{[1]}(V_{1i})e_{zi}^2 + 4\bar{\psi}_i\left|\xi_i\right|V_{1i}^{3/2} + \bar{\psi}_i\left|\xi_i\right|V_{1i}^2\\ & + \left[P_{[1]}(V_{1i}) + \frac{\partial P_{[1]}}{\partial V_{1i}}V_{1i} + 2\bar{\psi}_i\sqrt{V_{1i}}\right]\sqrt{V_{1i}}\left|\xi_i\right|\\ & + \left[\frac{\partial Q_{[1]}}{\partial V_{1i}}\sqrt{V_{1i}}e_{zi}^2 + Q_{[1]}(V_{1i})\bar{\psi}^2 + Q_{[1]}(V_{1i})e_{zi}^2\right]\left|\xi_i\right|\\ & + Q_{[1]}(V_{1i})[2e_{zi}^2 + V_{1i} + 1]\left|\xi_i\right|\end{aligned} \tag{71}$$

Now, since $Q_{[1]}$ and $P_{[1]}$ are polynomials of first order we have

$$Q_{[1]}(V_{1i}) := Q_{11}V_{1i} + Q_{12}, \qquad P_{[1]}(V_{1i}) := P_{11}V_{1i} + P_{12}$$

where Q_{11}, Q_{12}, P_{11}, and P_{12} are positive constants. Therefore,

$$\begin{aligned}\dot{V}_{2i} \leq & -V_{1i}^2\left[\frac{\sigma}{2} - \left(\bar{\psi}_i + Q_{11}\right)\left|\xi_i\right|\right] + 2P_{11}V_{1i}\sqrt{V_{1i}}\left|\xi_i\right|\\ & + 4\bar{\psi}_i\left|\xi_i\right|\sqrt{V_{1i}}V_{1i} + 2\bar{\psi}_i\left|\xi_i\right|V_{1i} + P_{12}\sqrt{V_{1i}}\left|\xi_i\right|\end{aligned}$$

$$
\begin{aligned}
&+ Q_{[1]}(V_{1i}) \left[\bar{\psi}_i^2 + 1\right] |\xi_i| + Q_{12} V_{1i} |\xi_i| \\
&- Q_{[1]}(V_{1i}) e_{zi}^2 \left[\frac{k_{\theta i}}{2} - 5 |\xi_i|\right].
\end{aligned} \tag{72}
$$

Let c_i with $i \le 5$ be positive constants satisfying the following:

$$
\begin{aligned}
c_1 &:= \frac{1}{5} \min\left\{\frac{k_{\theta i}}{2}, \frac{\sigma}{\bar{\psi}_i + Q_{11}}\right\} \\
\frac{\sigma}{8} &\ge 2[P_{11} + 2\bar{\psi}_i] c_2 + [2\bar{\psi}_i + Q_{12}] c_2 + c_4 P_{12} \\
&\quad + c_5 Q_{12} [1 + \bar{\psi}_i]
\end{aligned}
$$

and let

$$
\chi(|e_i|) := \min\left\{c_2 V_{1i}(e_i)^{1/2},\, c_3 V_{1i}(e_i),\, c_4 V_{1i}(e_i)^{3/2}\right\}.
$$

Then, we conclude that

$$
|\xi_i| \le \min\{c_4,\, c_5,\, \chi(|e_i|)\} \Longrightarrow \dot{V}_{2i} \le -\frac{\sigma}{4} V_{1i}(e_i)^2
$$

so small-input-to-state stability with respect to ξ follows.

Proof of Claim 2. Consider the positive-definite radially unbounded function $W_{2i} : \mathbb{R}_{\ge 0} \times \mathbb{R}^3 \to \mathbb{R}_{\ge 0}$, defined as

$$
W_{2i}(t, e_i) = \ln\left(1 + V_{2i}(t, e_i)\right). \tag{73}
$$

Let $\Pi(|\xi_i|, V_{1i})$ correspond to the positive terms on the right-hand side of (71) hence, the total derivative along trajectories of (57) satisfies

$$
\dot{W}_{2i} \le -\frac{1}{2} \frac{\left[\sigma V_{1i}^2 + k_{\theta i} Q_{[1]}(V_{1i}) e_{zi}^2 - \Pi(V_{1i}, e_{zi}^2)|\xi_i|\right]}{1 + V_{2i}} \tag{74}
$$

and further straight-forward computations, for which we use $V_{1i}^{1/2} \le V_{1i} + 1$, (63), and (64), show that

$$
\Pi(|\xi_i|, V_{1i}) \le P'_{[2]}(V_{1i}) + Q'_{[1]}(V_{1i}) e_{zi}^2
$$

where $P'_{[2]}$ and $Q'_{[1]}$ are second and first-order polynomials defined as

$$
\begin{aligned}
P'_{[2]}(V_{1i}) &:= \bar{\psi}_i [6 V_{1i}^2 + 8 V_{1i} + 1] \\
Q'_{[1]}(V_{1i}) &:= 3 Q_{[1]}(V_{1i}) + \left[\frac{\bar{\psi}_i}{2} + 1\right] \left[V_{1i} + 1\right]
\end{aligned}
$$

Thus, in view of (65), on one hand there exists $\alpha \in \mathcal{K}$ such that

$$
\alpha(|e_i|) \le \frac{1}{2} \frac{\sigma V_{1i}^2 + k_{\theta i} Q_{[1]}(V_{1i}) e_{zi}^2}{Q_{[1]}(V_{1i}(e_i)) \left[V_{1i}(e_i) + e_{zi}^2\right]} \tag{75}
$$

and, on the other hand, there exists a constant $c > 0$ such that

$$
\frac{P'_{[2]}(V_{1i}) + Q'_{[1]}(V_{1i}) e_{zi}^2}{Q_{[1]}(V_{1i}(e_i)) \left[V_{1i}(e_i) + e_{zi}^2\right]} \le c \qquad \forall e_i \in \mathbb{R}^3.
$$

Thus,

$$
\dot{W}_2 \le -\alpha(|e_i|) + c\, |\xi_i| \tag{76}
$$

so the result follows invoking Lemma 2. ■

B. Technical statements

For the sake of completeness and clarity of presentation we recall some well-known concepts and results related to input-to-state stability. To the best of our knowledge, however, Lemma 3 below is original.

Definition 2: (ISS [43]) The dynamical system

$$
\dot{x} = f(x, u) \tag{77}
$$

is input-to-state stable (ISS) with respect to the input u if there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$, and a class $\mathcal{K}_\infty$ function $\gamma(\cdot)$, such that

$$
|x(t)| \le \beta(|x(t_0)|, t - t_0) + \gamma\left(\sup_{t_0 \le s \le \infty} |u(s)|\right) \tag{78}
$$

Furthermore, (77) is small-input-to-state stable (sISS), with respect to the input u if there exists $r > 0$, such that (78) holds for all $|u| \leq r$. □

Definition 3: (integral ISS [43]) The dynamical system (77) is integral-input-to-state stable (iISS) with respect to the input u if there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, and a class $\mathcal{K}_\infty$ function $\gamma(\cdot)$, such that

$$|x(t)| \leq \beta(|x_0|\,, t - t_0) + \int_{t_0}^{t} \gamma\left(|u(s)|\right) ds. \tag{79}$$

□

Definition 4: (strong iISS [20]) The system (77) is said to be strongly iISS with respect to u if it is iISS and sISS. □

Lemma 1: (characterization of ISS and sISS) The system

$$\dot{x} = f(t, x, u) \tag{80}$$

is input-to-state stable if and only if there exist: a continuously differentiable Lyapunov function $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$, a class $\mathcal{K}$ function ρ, and a continous positive definite function W such that

$$\underline{\alpha}\left(|x|\right) \leq V(t, x) \leq \overline{\alpha}\left(|x|\right) \qquad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \tag{81}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W(x), \quad \forall\, |x| \geq \rho\left(|u|\right) > 0. \tag{82}$$

Moreover, if there exists $r > 0$ such that (82) holds for all $|u| \leq r$ then the system (80) is small ISS with respect to the input u. □

Lemma 2: (characterization of iISS —[40]) The system (80) is integral-input-to-state stable with respect to the input u if there exist: a continuously differentiable Lyapunov function $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and ρ, and a positive definite $\mathcal{K}$ function α such that (81) and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha_1(|x|) + \rho(|u|) \tag{83}$$

hold. □

Lemma 3: Consider the dynamical system

$$\dot{x} = f(x, u(t)), \quad x \in \mathbb{R}^n \tag{84}$$

$$0 = f(0, v), \quad \forall\ v \in \mathcal{D} \subset \mathbb{R}^m, \tag{85}$$

where $u : \mathbb{R}_{\geq 0} \to \mathcal{D} \subset \mathbb{R}^m$ is locally integrable and the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x uniformly in v for all $v \in \mathcal{D}$.

Assume that $\dot{x} = f(x, u)$ is strong iISS with respect to u. Then, if in addition $u(t) \to 0$ as $t \to \infty$ then the origin of (84) is uniformly globally asymptotically stable. □

Proof. The system is small input-to-state stable and integral-input-to-state stable hence, in view of Lemma 2, (83) holds. Let $t \mapsto u$ be arbitrary but fixed and satisfy $u(t) \to 0$ as $t \to \infty$. Integrating on both sides of the latter along trajectories, from t_0 to (any) t, invoking (81), and the uniform local integrability property of u which is due to its global boundedness, uniform forward completeness follows.

In addition, the system is small-input to state stable and $u(t) \to 0$ so for any $r > 0$ there exists $T > 0$ such that $|u(t)| \leq r$ for all $t \geq T$. Next, assume that $|x(t)| \to \infty$ then, there exists $T' \geq T$ such that $|x(t_0 + T')| \geq \rho(r) \geq \rho(|u(t_0 + T')|)$. It follows from (82) that $\dot{V}(x(t)) \leq -W(x(t)) \leq 0$ for all $t \geq t_0 + T'$ and all $t_0 \geq 0$, so the solutions are, under uniform forward completeness, uniformly globally bounded and the origin is uniformly globally attractive under the convergence of $u(t)$.

It is left to show that the origin is uniformly stable. This follows from the following properties:

P1) The continuity of the flow uniformly in t_0 under a locally-Lipschitz in x uniformly in u, for all $u \in \mathcal{D}$, nonlinear vector field. Indeed, since the origin is an equilibrium —see Eq. (85), it follows that for any $\gamma > 0$ and $T > 0$, there exists $\delta(T, \gamma) > 0$ such that, for all $t_0 \geq 0$,

$$|x_0| \leq \delta(T, \gamma) \Rightarrow |x(t, x_0, t_0)| \leq \gamma, \quad \forall t \in [t_0, t_0 + T]. \tag{86}$$

P2) The small ISS property with respect to the input $u(t)$. Indeed, there exist $r > 0$, a $\mathcal{C}^1$ ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\bar{\alpha}$, α, and χ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \tag{87}$$

$$\frac{\partial V}{\partial x}(x) f(x, u) \leq -\alpha(|x|), \quad \forall\ |u| \leq \min\left\{r, \chi(|x|)\right\}. \tag{88}$$

Now, let the strong iISS property generate $r > 0$ —see Defs. 2–4. Since $u(t) \to 0$, for any given $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$|u(t)| \leq \min\left\{r, \chi \circ \bar{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon/2)\right\} \qquad \forall\, t \geq t_0 + T_\varepsilon,\ \forall\, t_0 \geq 0. \tag{89}$$

Then, by virtue of P1 above, let $\gamma(\varepsilon) := \bar{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon)$ and T_ε generate $\delta(T_\varepsilon, \gamma) > 0$ such that

$$\forall\, |x_0| \leq \delta \Rightarrow x(t, t_0, x_0) \in \mathcal{D}_\varepsilon, \quad \forall t \in [t_0, t_0 + T_\varepsilon] \tag{90}$$

where

$$\mathcal{D}_\varepsilon := \{x \in \mathbb{R}^n \ :\ V(x) < \underline{\alpha}(\varepsilon)\}\,.$$

Suppose that $x(t_0 + T_\varepsilon) \notin \mathcal{D}_{\varepsilon/2}$. Then, $|x(t_0 + T_\varepsilon)| \geq \bar{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon/2)$ and $\chi(|x(t_0 + T_\varepsilon)|) \geq \chi \circ \bar{\alpha}^{-1} \circ \underline{\alpha}(\varepsilon/2) \geq |u(t)|$ for all $t \geq t_0 + T_\varepsilon$, $t_0 \geq 0$ (from Eq. (89)). Furthermore, in view of (88), $\dot{V}(x(t_0 + T_\varepsilon)) < 0$ therefore $x(t) \in \mathcal{D}_\varepsilon$ for all $t \geq t_0 + T_\varepsilon$. The latter and (90) lead to concluding that

$$\forall\, |x_0| \leq \delta(T_\varepsilon, \gamma(\varepsilon)) \Rightarrow |x(t, t_0, x_0)| \leq \varepsilon, \quad \forall t \geq t_0 \tag{91}$$

so the result follows. ■

Remark 6: It is worth to point out the importance of the robustness properties used in the lemma. In general, global asymptotic stability for a given bounded and converging function $t \mapsto u$ does not imply uniform global asymptotic stability. Indeed, for the system

$$\dot{x} = -u(t)x, \quad u(t) = \frac{1}{(1+t)}$$

the origin is globally asymptotically stable, uniformly globally stable, not uniformly globally attractive. On the other hand, a system of the form (84) with a fixed bounded and converging function $t \mapsto u$ and such that the origin is globally stable, but not uniformly, is given by

$$\dot{x} = -u(t)x$$

with

$$u(t) = \begin{cases} \frac{1}{1+k} & \text{if } t \in [2^{2k} - 1, 2^{2(k+1)} - 1], \\ -\frac{1}{1+k} & \text{if } t \in [2^{2(k+1)} - 1, 2^{2(k+2)} - 1]. \end{cases}$$

Indeed, defining $t_0 = (2k)$, at $t_1 = 2^{2(k+1)} - 1$ we have

$$x(t_1) = e^{\frac{2^{(2k)}}{(k+1)}} x(t_0) = e^{\frac{2^{t_0}}{(t_0/2+1)}} x(t_0),$$

so the origin is not uniformly globally stable •