Random walk in a stratified independent random environment

Julien Brémont

Random walk in a stratified independent random environment

Julien Brémont

Université Paris-Est Créteil, février 2019

Abstract

We study Markov chains on a lattice in a codimension-one stratified independent random environment, exploiting results established in [2]. The random walk is first shown to be transient in dimension at least three. Focusing on dimension two, we provide sharp sufficient conditions for both recurrence and transience. The critical scale of the local drift in the direction of the stratification corresponding to the frontier between the two regimes is shown to be very small.

1 Introduction

A very first and important question concerning the asymptotic behaviour of a Markov chain on a lattice in inhomogeneous environment is the question of its recurrence/transience. We consider in the present paper the situation where the environment is stratified, continuing the line of research initiated by Matheron and de Marsily [13] in the early 1980'. Such a model of random motion was initially motivated by hydrology and the diffusion of pollutants in the ground. The latter is porous and stratified for geologic reasons, with some heterogeneity among the strata. In 2003, a simplified discrete version was introduced by Campanino and Petritis [3], principally inspired by physical considerations about the study of discrete gauge theories.

Let us detail the model studied in [2], an extension of the one originally introduced in [3]. Fixing $d \geq 1$, we consider a Markov chain $(S_n)_{n\geq 0}$ in $\mathbb{Z}^d \times \mathbb{Z}$, with $S_0 = 0$. Quantities relative to the first (resp. second) coordinate in $\mathbb{Z}^d$ (resp. $\mathbb{Z}$) are declared "horizontal" (resp. "vertical"). In this article, we assume that the transition laws only depend on the vertical coordinate, i.e. are constant on each affine hyperplane $\mathbb{Z}^d \times \{n\}$, $n \in \mathbb{Z}$.

Write $B_d(a, r)$ for the Euclidean ball in $\mathbb{Z}^d$ of center a and radius r. Any $x \in \mathbb{Z}^d$ is written in column, with transpose x^t. For each vertical $n \in \mathbb{Z}$, suppose to be given positive reals p_n, q_n, r_n, with $p_n + q_n + r_n = 1$, and a probability measure μ_n with support in $\mathbb{Z}^d$, satisfying :

Hypothesis 1.1 *Let $d \geq 1$. There exists $\eta > 0$ such that for all vertical $n \in \mathbb{Z}$:*

– $\min\{p_n, q_n, r_n\} \geq \eta$,

– $Supp(\mu_n) \subset B_d(0, 1/\eta)$,

– the spectrum of the real symmetric matrix $\sum_{k\in\mathbb{Z}^d} kk^t\mu_n(k)$ is included in $[\eta, +\infty)$. When $d = 1$, this condition is replaced by $\mu_n(0) \leq 1 - \eta$.

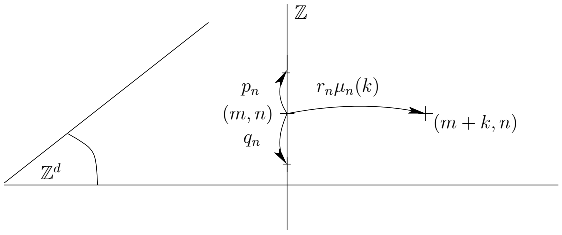
In [2], there was a priori no link between the strata. The second condition was replaced by a weaker one (a uniform in n moment condition), but for the sake of simplicity, we restrict here to the above setting. The transition laws are now defined, for all $(m, n) \in \mathbb{Z}^d \times \mathbb{Z}$ and $k \in \mathbb{Z}^d$, by :

$$(m, n) \xrightarrow{p_n} (m, n + 1),\ (m, n) \xrightarrow{q_n} (m, n - 1),\ (m, n) \xrightarrow{r_n\mu_n(k)} (m + k, n).$$

With a picture :

AMS 2000 *subject classifications : 60G17, 60J10, 60K37.*

Key words and phrases : Markov chain, recurrence, stratification, random environment, independence.



The vertically flat model corresponds to the symmetry assumption $p_n = q_n$, $n \in \mathbb{Z}$. The expectation of μ_n, or "local horizontal drift along the stratum $n \in \mathbb{Z}$", is defined as $\varepsilon_n := \sum_{k\in\mathbb{Z}^d} k\mu_n(k)$. Let us now present former results concerning recurrence/transience for this model.

Considering first the vertically flat case, the Campanino-Petritis model [3] consists in taking $d = 1$, with $p_n = q_n = r_n = 1/3$ and $\mu_n = \delta_{\alpha_n}$, fixing some $(\alpha_n)_{n\in\mathbb{Z}} \in \{\pm 1\}^{\mathbb{Z}}$ and where δ_x is Dirac measure at x. It is shown in [3] that the random walk is recurrent when $\alpha_n = (-1)^n$ and transient when $\alpha_n = 1_{n\geq 1} - 1_{n\leq 0}$ or when (α_n) are typical realizations of independent and identically distributed ($i.i.d$ for the sequel) random variables with $\mathbb{P}(\alpha_n = \pm 1) = 1/2$. Many works followed, close to this setting. Guillotin-Plantard and Le Ny [9] have shown transience results when the (α_n) are independent with marginals described by some dynamical system. Pene [15] proved transience when the (α_n) are stationary, under a decorrelation condition. Castell, Guillotin-Plantard, Pene and Schapira [5] quantified transience when the (α_n) are $i.i.d.$, studying the tail of the annealed return time to 0. Still under the condition $p_n = q_n$, $n \in \mathbb{Z}$, Devulder and Pene [7] consider an extension of the model in [3] and establish transience when the (r_n) are $i.i.d.$ non-constant and $\mu_n = \delta_{\alpha_n}$, with arbitrary (α_n). In [4], Campanino and Petritis, still considering their initial model, study a random perturbation of a periodic (α_n).

In [1] (Theorem 1.2), for the general vertically flat case with $d = 1$, a complete recurrence criterion was given. In this situation, the asymptotics of the random walk is governed by a cocycle related to the local horizontal drift, namely $(r_0\varepsilon_0/p_0 + \cdots + r_{n-1}\varepsilon_{n-1}/p_{n-1})$ when $n \geq 1$, and more precisely by the properties of a two-variables function $\Phi(a, b)$, for vertical $a < b$, built with this cocycle. The latter function measures horizontal dispersion between vertical levels a and b. The form of the criterion in [1], a priori a little abstract, in fact directly comes from the computation of a Poisson kernel in a half-plane. Such a kernel appears in the recurrence criterion for $i.i.d.$ random walks on $\mathbb{Z}^d$ (see Spitzer [19] or Ornstein [14]), an essential ingredient in the proof in [1]. The quantity deciding for the recurrence/transience of (S_n) quantifies some "capacity of dispersion to infinity" of the environment. It is interesting to notice that it involves the level lines of the function $\Phi(a, b)$ and some notion of curvature at infinity of these lines. A priori a geometric point of view of the recurrence criterion has to be developed.

Still for the general vertically flat model when $d = 1$, after the principal result of [1], some (little) extra work is required to treat concrete examples. This was done at the end of [1], giving rather fine recurrent quasi-periodic examples ([1], Prop. 1.5; cf also [2], Prop. 7.1). It is also shown that in this family of models, simple random walk in the plane is in some sense the most recurrent one and, as is well known it is hardly recurrent. For instance a growth condition like $(\log n)^{1+\delta}$ on $(r_0\varepsilon_0/p_0 + \cdots + r_{n-1}\varepsilon_{n-1}/p_{n-1})$ is sufficient for transience. This explains to some extent the prevalence of transience results in the litterature.

In order to try to enlarge the recurrence domain, it is natural to focus on the general model introduced above. A recurrence criterion was shown in [2] (Theorem 2.4) with exactly the same form as in [1], coming from a Poisson kernel. The function $\Phi(a, b)$, defined below, is just a little more complicated. The criterion highlights that the environment (i.e. the set of transition laws) defines a new metrization of $\mathbb{Z}^{d+1}$. The random walk (S_n) has some kind of "product structure" and a key observation (appearing in [3]), following from the invariance of the environment under horizontal translations, is that restricting (S_n) to vertical jumps, the vertical components form a Markov chain. This is not true in general for an inhomogeneous Markov chain, when the environment is not stratified. We call this Markov chain the "vertical random walk". Its transition laws on $\mathbb{Z}$ are :

$$n \overset{p_n/(p_n+q_n)}{\longrightarrow} n+1,\ n \overset{q_n/(p_n+q_n)}{\longrightarrow} n-1.$$

In the vertically flat case, this is simple random walk on $\mathbb{Z}$, but for the general model it can be an arbitrary nearest-neighbour random walk on $\mathbb{Z}$. For instance it can be chosen positive recurrent. In this case, (S_n) is strongly pushed towards $\mathbb{Z}^d \times \{0\}$ and in fact behaves as if it were in $\mathbb{Z}^d$; see section 7.3 on the "half-pipe" in [2]. Several examples of recurrent and transient random walks were given in [2], in dimensions both 2 and 3 ($d = 1$ or 2). In dimension ≥ 4 ($d \geq 3$), the random walk is always transient.

The purpose of the present article is to extend the applications of [2], studying in detail the general model in the important case when the stratifications of the environment are random and independent, with a quenched point of view. On such a model, mention a result by Kochler in his doctoral thesis [12], concerning the case when $\mathbb{P}(\mu_n = \delta_1) = \mathbb{P}(\mu_n = \delta_{-1}) = 1/2$ (hence $\mathbb{P}(\varepsilon_n = \pm 1) = 1/2$), with $r_n/p_n = c$ (a constant independent on n) and $(p_n/q_n, \mu_m)_{(n,m)\in\mathbb{Z}^2}$ independent. Transience is established, after a long and delicate analysis (80 pages) of the Brownian path. Our goal is to recover such a result and to try to touch the frontier between recurrence and transience for the present model. The results probably have some flavour of what should happen for more general models of random walks in independent environments.

2 Independent setting, notations and results

We first recall the situation in the independent setting, including the vertically flat case. For the rest of the article, we assume that $(p_n, q_n, r_n)_{n\in\mathbb{Z}}$ are random variables such that $(p_n/q_n)_{n\in\mathbb{Z}}$ are *i.i.d.*, with arbitrary or random μ_n, verifying Hypothesis 1.1, a.-s.. Recall that $\varepsilon_n = \sum_{k\in\mathbb{Z}^d} k\mu_n(k)$.

In the sequel, randomness is always for the environment. We never enter the mechanism of the random walk itself.

Proposition 2.1
If either $d \geq 2$ or $\mathbb{E}(\log(p_0/q_0)) \neq 0$, then for a.-e. realization, (S_n) is transient.

It is classical that the condition $\mathbb{E}(\log(p_0/q_0)) \neq 0$ implies that the vertical random walk is transient; see Solomon [18]. This hence implies the transience of (S_n). For $d \geq 3$, as said above, this is a general result ([2], Prop. 2.5, 1)*i*)). The remaining case $d = 2$ with $\mathbb{E}(\log(p_0/q_0)) = 0$, which includes the case when $p_0 = q_0$, a.-s., will be proved in section 3.2. Recall here in passing the related conjecture that any random walk in *i.i.d.* random environment in $\mathbb{Z}^3$ is transient, supposing ellipticity conditions on the data; see Kalikow [11] and Sabot [16].

From now on, $d = 1$ and $\mathbb{E}(\log(p_0/q_0)) = 0$. When $p_0/q_0 = 1$, a.-s., (notice then that $r_n = 1 - 2p_n$) i.e. for the vertically flat case, by [1] Prop. 1.4 *i*), then (S_n) is recurrent whenever :

$$\left|\sum_{-n\leq k\leq 0} r_k\varepsilon_k/p_k\right| + \left|\sum_{0\leq k\leq n} r_k\varepsilon_k/p_k\right| = O((\log n)^{1/2}).$$

In the other direction, if the $(r_n\varepsilon_n/p_n)_n$ are independent random variables, with for some $\delta > 0$:

$$\liminf \frac{1}{N}\sum_{n=1}^{N} \mathbb{P}\left(\left|\sum_{0\leq k\leq n} r_k\varepsilon_k/p_k\right| \geq (\log n)^{1+\delta}\right) > 0,$$

then, for a.-e. realization, (S_n) is transient; cf [1] Prop. 1.6. This is an extension in the elliptic setting of former results of [3] and [7]. When the $(r_n\varepsilon_n/p_n)_n$ are independent, the critical growth for $\sum_{0\leq k<n} r_k\varepsilon_k/p_k$ with respect to recurrence/transience is expected to be $\log n$, thus somehow corresponding to ε_n having order $1/n$.

We turn next to the case when $\mathbb{P}(p_0/q_0 = 1) < 1$. The vertical random walk is now Sinaï's random walk [17], very different from simple random walk and with a typical scale of $(\log n)^2$ at time n. We need to introduce some notations, the first ones concerning the exponential of the traditional potential governing the behaviour of Sinaï's random walk.

Definition 2.2
Set $a_n = q_n/p_n$, $n \in \mathbb{Z}$, and :

$$\rho_n = \begin{cases} a_1 \cdots a_n, & n \geq 1, \\ 1, & n = 0, \\ (a_{n+1} \cdots a_{-1} a_0)^{-1} & n \leq -1. \end{cases}$$

For $n \geq 0$, introduce the quantities :

$$v_+(n) = \sum_{0 \leq k \leq n} \rho_k \text{ and } v_-(n) = a_0 \sum_{-n-1 \leq k \leq -1} \rho_k.$$

In the same way, let for $n \geq 0$:

$$w_+(n) = \sum_{0 \leq k \leq n} 1/\rho_k \text{ and } w_-(n) = (1/a_0) \sum_{-n-1 \leq k \leq -1} 1/\rho_k.$$

Notice for the sequel that $\rho_{n+1}/\rho_n = a_{n+1} \in [\eta, 1/\eta]$. If $\mathbb{E}(\log(p_0/q_0)) = 0$, then almost-surely, $v_+(n)$ and $v_-(n)$ increasingly tend to $+\infty$, as $n \to +\infty$. Indeed, $\log \rho_n$ is an *i.i.d.* random walk with integrable and centered step, hence is recurrent, so ρ_n does not go to 0.

We also require notations for functions having comparable orders and for inverse functions. Let us first set $\mathbb{N} = \{0, 1, \cdots\}$.

Definition 2.3
i) Let $f, g : \mathbb{N} \to \mathbb{R}_+$. We write $f \asymp g$ or $f(n) \asymp g(n)$ if there exists $C > 0$ so that for large n, $(1/C)f(n) \leq g(n) \leq Cf(n)$. Set $g \preceq f$ or $g(n) \preceq f(n)$ if $g(n) \leq Cf(n)$ for large n.

ii) Let $f : \mathbb{N} \to \mathbb{R}_+$ be increasing with $\lim f(n) = +\infty$. For large $x \in \mathbb{R}_+$, let :

$$f^{-1}(x) = \max\{n \in \mathbb{N} \mid f(n) \leq x\},$$

convening that $\max \emptyset = 0$. Notice that $f(f^{-1}(x)) \leq x < f(f^{-1}(x) + 1)$, for large $x \in \mathbb{R}_+$. Also $f^{-1}(f(n)) = n$, for large $n \in \mathbb{N}$.

The average horizontal macrodispersion of the environment is described by the following functions, introduced in [2].

Definition 2.4
i) The structure function, depending only on the vertical, is defined for $n \geq 0$ by :

$$\Phi_{str}(n) = \left(n \sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} \frac{1}{\rho_k} \right)^{1/2}.$$

2) For $d = 1$ and $m, n \geq 0$, introduce :

$$\Phi(-m, n) = \left(\sum_{-v_-^{-1}(m) \leq k \leq \ell \leq v_+^{-1}(n)} \rho_k \rho_\ell \left[\frac{1}{\rho_k^2} + \frac{1}{\rho_\ell^2} + \left(\sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right] \right)^{1/2}.$$

For $n \geq 0$, set :

$$\Phi(n) = \Phi(-n, n) \text{ and } \Phi_+(n) = \sqrt{\Phi^2(-n, 0) + \Phi^2(0, n)}.$$

3) For $d = 1$ and $n \geq 0$, set :

$$C(n) = \sum_{-v_-^{-1}(n) \leq k \leq v_+^{-1}(n)} \frac{\varepsilon_k}{\rho_k}.$$

We are now in position for stating the main result of this paper in the remaining situation when $d = 1$, $\mathbb{E}(\log(p_0/q_0)) = 0$ and $\mathbb{P}(p_0/q_0 = 1) < 1$.

Theorem 2.5
Let $d = 1$, with $\mathbb{E}(\log(p_0/q_0)) = 0$ and $\mathbb{P}(p_0/q_0 = 1) < 1$.

i) For a.-e. realization, (S_n) is recurrent, whenever $(\mu_n)_{n\in\mathbb{Z}}$ is a deterministic sequence verifying :

$$\sum_{n\in\mathbb{Z}} \frac{|\varepsilon_n|}{\rho_n} < +\infty. \tag{1}$$

This condition holds if for some $\delta > 0$, $\varepsilon_n = O(\exp(-|n|^{1/2+\delta}))$, as $|n| \to +\infty$.

ii) For a.-e. realization, (S_n) is transient, whenever $(\mu_n)_{n\in\mathbb{Z}}$ is a deterministic sequence verifying $\varepsilon_n \geq 0$ for large $n \in \mathbb{Z}$, with :

$$\sum_{n\in\mathbb{Z}} \frac{\varepsilon_n}{\rho_n} = +\infty \text{ and } \sum_{n\geq N_0} \frac{1}{nC(n)} < +\infty, \tag{2}$$

where $N_0 \geq 0$ is such that $C(n) > 0$, for $n \geq N_0$. These conditions hold if for some $\delta > 0$ and for large enough $n \in \mathbb{Z}$, $\varepsilon_n \geq \exp(-|n|^{1/2-\delta})$.

iii) Let the $(\varepsilon_n)_{n\in\mathbb{Z}}$ be independent random variables, independent from the $(p_n, q_n, r_n)_{n\in\mathbb{Z}}$, such that there exists $1/2 > \delta > 0$, so that for large $n \in \mathbb{N}$, the support of ε_n is a finite set $\{\alpha_{i,n}\}_{1\leq i\leq L_n} \subset [-1/\eta, 1/\eta]$, with $L_n \geq 2$, verifying for all $1 \leq i \neq j \leq L_n$:

$$|\alpha_{i,n} - \alpha_{j,n}| \geq \exp(-n^{1/2-\delta}) \text{ and } \mathbb{P}(\varepsilon_n \neq \alpha_{i,n}) \geq n^{-\delta/5}. \tag{3}$$

Then for a.-e. realization, (S_n) is transient.

Remark. — In *i*), the condition $\sum_{n\in\mathbb{Z}} |\varepsilon_n|/\rho_n < +\infty$ for recurrence can be interpreted as a condition of "finite dispersion to infinity". It is of different nature than $\sum_{n\in\mathbb{Z}} 1/\rho_n < +\infty$, of "finite channel capacity", defining the half-pipe of [2] (section 7.3). This last condition, not true here (as ρ_n does not go to 0, as $|n| \to +\infty$), implied that (S_n) is transient if and only if :

$$\sum_{s\in\mathbb{Z}} \frac{r_s \varepsilon_s}{p_s \rho_s} \neq 0. \tag{4}$$

Cf [2], Prop. 7.4, when $d = 1$. In the present situation, assuming $\sum_{n\in\mathbb{Z}} |\varepsilon_n|/\rho_n < +\infty$, then for any value of $\sum_{s\in\mathbb{Z}} r_s\varepsilon_s/(p_s\rho_s)$, the random walk is recurrent.

Remark. — Concerning *iii*), the conditions easily cover the situation considered in [12], where $\mathbb{P}(\mu_n = \delta_1) = \mathbb{P}(\mu_n = \delta_{-1}) = 1/2$ and $r_n/p_n = c$. When $\mathbb{P}(p_0 = q_0) < 1$, the rough critical scale for ε_n with respect to recurrence/transience is $\exp(-|n|^{1/2})$, hence much smaller than for the vertically flat case. Remembering that the vertical random walk is Sinaï's random walk, the intuition for this is easy. Indeed, the landscape for (S_n) somehow looks like a succession of horizontally invariant canyons, with transversal profile described by the potential of the vertical random walk. Then (S_n) stays confined for a long time at the bottom of the canyons. Just a little bit of horizontal flow is enough to make the random walk transient. The previous theorem quantifies this.

3 Proof of the results

3.1 Preliminary remarks

1) Comparison of Φ_{str}, Φ_+ and Φ. One has $\Phi_{str} \preceq \Phi_+ \preceq \Phi$ and more precisely :

$$\Phi_+(n) \asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq 0 \text{ or } 0\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}, \tag{5}$$

as well as :

$$\Phi(n) \asymp \Phi_{str}(n) + \left(\sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(\sum_{s=k}^{\ell} \frac{r_s \varepsilon_s}{p_s \rho_s} \right)^2 \right)^{1/2}. \tag{6}$$

Indeed, observe first that for $n \geq 0$:

$$\begin{aligned} \sum_{0 \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(1/\rho_k^2 + 1/\rho_\ell^2\right) &= \sum_{0 \le k \le \ell \le v_+^{-1}(n)} (\rho_k/\rho_\ell + \rho_\ell/\rho_k) \\ &\asymp \sum_{0 \le k \le v_+^{-1}(n)} \rho_k \sum_{0 \le \ell \le v_+^{-1}(n)} 1/\rho_\ell. \end{aligned}$$

Observe that $\sum_{0 \le k \le v_+^{-1}(n)} \rho_k = v_+(v_+^{-1}(n)) \le n$ and :

$$\begin{aligned} n \le v_+(v_+^{-1}(n) + 1) &= v_+(v_+^{-1}(n)) + \rho_{v_+^{-1}(n)+1} \\ &\le (1 + 1/\eta) v_+(v_+^{-1}(n)) \le (1 + 1/\eta) n. \end{aligned}$$

Hence $v_+(v_+^{-1}(n)) \asymp n$ and we obtain :

$$\sum_{0 \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(1/\rho_k^2 + 1/\rho_\ell^2\right) \asymp n \sum_{0 \le k \le v_+^{-1}(n)} 1/\rho_k.$$

Proceeding in the same way for $\sum_{-v_-^{-1}(n) \le k \le \ell \le 0}$ and summing, we get :

$$\sum_{-v_-^{-1}(n) \le k \le \ell \le 0 \text{ or } 0 \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(1/\rho_k^2 + 1/\rho_\ell^2\right) \asymp n \sum_{-v_-^{-1}(n) \le k \le v_+^{-1}(n)} 1/\rho_k = \Phi_{str}^2(n).$$

This furnishes (5). In a very similar fashion :

$$\begin{aligned} \sum_{-v_-^{-1}(n) \le k \le \ell \le v_+^{-1}(n)} \rho_k \rho_\ell \left(1/\rho_k^2 + 1/\rho_\ell^2\right) &\asymp \sum_{-v_-^{-1}(n) \le k \le v_+^{-1}(n)} \rho_k \sum_{-v_-^{-1}(n \le \ell \le v_+^{-1}(n)} 1/\rho_\ell \\ &\asymp n \sum_{-v_-^{-1}(n) \le \ell \le v_+^{-1}(n)} 1/\rho_\ell \asymp \Phi_{str}^2(n). \end{aligned}$$

This leads to (6).

2) The functions Φ_{str}^{-1}, Φ_+^{-1} and Φ^{-1} check dominated variation. This property, coming from the structure of the main result in [2] (Theorem 2.4), considerably simplifies the study of the convergence of series. Recall that a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies dominated variation if there exists a constant $C > 0$ so that for large $x > 0$:

$$f(2x) \le C f(x). \tag{7}$$

This directly implies that for any $A > 0$, for $x > 0$ large enough, $f(Ax) \asymp f(x)$. Dominated variation is obvious for Φ_{str}^{-1}, because $\Phi_{str}(n) = \sqrt{n}\psi(n)$, with some non-decreasing ψ. This is a little less clear for Φ_+^{-1} and Φ^{-1} and is shown in Lemma 6.2 in [2] (with the notations of [2], $\Phi_+ = G_{u,+}$ and $\Phi = G_u$, with $u = 1$ when $d = 1$; the functions G_u and G are defined in Definition 6.1 and the involved $T_k^l(u)$ appears in Definition 5.1; also in the present situation $\eta_k^u = r_k \varepsilon_k / p_k$). Notice on the contrary that, although well-defined, the function $v_+^{-1}(n)$ does not always check dominated variation in the non vertically flat case.

3.2 Proof of Proposition 2.1.

As explained after the statement of the proposition, it remains to check transience when $d = 2$ and $\mathbb{E}(\log(p_0/q_0)) = 0$. Using [2], Prop. 2.5 1)ii), it is enough to verify that for some $\delta > 0$:

$$\Phi_{str}(n) \geq \sqrt{n}(\log n)^{1/2+\delta}. \tag{8}$$

When $\mathbb{P}(p_0 = q_0) = 1$, we have $\Phi_{str}(n) \succeq n$, so remains the case when $\mathbb{P}(p_0 = q_0) < 1$. Notice that $\mathrm{Var}(\log(p_0/q_0)) > 0$. We shall repeatedly use that if $(Y_n)_{n\geq 1}$ is an $i.i.d.$ sequence of non-constant random variables with $\mathbb{E}(Y_1^2) < +\infty$ and $\mathbb{E}(Y_1) = 0$, then, a.-s., for any $\varepsilon > 0$, for large n :

$$n^{1/2-\varepsilon} \leq \max_{1\leq k\leq n} (Y_1 + \cdots + Y_k) \leq n^{1/2+\varepsilon}. \tag{9}$$

This may be seen for example as a consequence of classical results relative to the law of the Iterated Logarithm; see Chung [6], chap. 7 for instance.

Consider $\log \rho_n = \sum_{k=1}^n \log(q_k/p_k)$, $n \geq 1$. We obtain that a.-s., for any $\varepsilon > 0$, for n large enough, $v_+(n) \leq n \exp(n^{1/2+\varepsilon}) \leq \exp(n^{1/2+2\varepsilon})$. Hence, using $1/(1/2 + 2\varepsilon) \geq 2(1 - 4\varepsilon)$:

$$v_+^{-1}(n) \geq (\log n)^{2-9\varepsilon}. \tag{10}$$

As a result, using (9) for minima (i.e. with $-\log(q_k/p_k)$) :

$$\sum_{0\leq k\leq v_+^{-1}(n)} 1/\rho_k \geq \exp(\log^{(2-9\varepsilon)(1/2-\varepsilon)} n) \geq \exp(\log^{1-7\varepsilon} n). \tag{11}$$

We obtain that a.-s., $\forall\varepsilon > 0$, for n large enough, $\Phi_{str}(n) \succeq \sqrt{n}\exp(\log^{1-\varepsilon} n)$. Therefore (8) is also verified and this completes the proof of the proposition.

□

Remark. — For the rest of the article, $d = 1$. Also, for all the proofs of Theorem 2.5, we fix an integer $K > 2(1 + 1/\eta)^2$.

3.3 Proof of Theorem 2.5 i)

Consider point i) and $(\varepsilon_n)_{n\in\mathbb{Z}}$ verifying (1). Observe first that this condition holds if $\varepsilon_n = O(\exp(-|n|^{1/2+\delta}))$, $\delta > 0$. Indeed, using (9), $\varepsilon_n/\rho_n = O(\exp(-|n|^{1/2+\delta/2}))$ in this case.

Now, condition (1) implies that $\sum_{u=k}^{l}(r_s\varepsilon_s)/(p_s\rho_s)$ is bounded in (k,l), $k \leq l$, since $\eta \leq r_s/p_s \leq 1/\eta$. Using that (obtained as for getting (6)) :

$$\sum_{-v_-^{-1}(n)\leq k\leq \ell\leq v_+^{-1}(n)} \rho_k\rho_\ell \asymp n^2,$$

it then follows from (6) that $\Phi^2(n) \preceq \Phi^2_{str}(n) + n^2$ and so $\Phi(n) \preceq \Phi_{str}(n) + n$. As a result, because Φ^{-1}, Φ^{-1}_{str} and $n \longmapsto n$ check dominated variation :

$$\Phi^{-1}(n) \succeq \min\{\Phi^{-1}_{str}(n), n\}.$$

We shall also use that $\Phi_+^{-1}(n) \preceq \Phi^{-1}_{str}(n)$, coming from the general relation $\Phi_+(n) \succeq \Phi_{str}(n)$ and dominated variation for Φ_+^{-1} and Φ^{-1}_{str}.

In view of [2], Theorem 2.4, the recurrence of (S_n) is equivalent to :

$$\sum_{n\geq 1} \frac{1}{n^2} \frac{(\Phi^{-1}(n))^2}{\Phi_+^{-1}(n)} = +\infty.$$

Given the previous remarks, it is sufficient to show the divergence of :

$$\sum_{n\geq 1}\frac{1}{n^2}\frac{(\min\{\Phi_{str}^{-1}(n),n\})^2}{\Phi_{str}^{-1}(n)}=\sum_{n\geq 1}\frac{1}{n}\min\left\{\frac{\Phi_{str}^{-1}(n)}{n},\frac{n}{\Phi_{str}^{-1}(n)}\right\}.$$

Splitting the sum according to the intervals $[K^n,K^{n+1})$, $n\geq 1$, and using that Φ_{str}^{-1} verifies dominated variation, the previous condition is equivalent to checking that :

$$\sum_{n\geq 1}\min\left\{\frac{\Phi_{str}^{-1}(K^n)}{K^n},\frac{K^n}{\Phi_{str}^{-1}(K^n)}\right\}=+\infty. \tag{12}$$

We prove below that the general term in the above series does not go to 0.

Lemma 3.1
Almost-surely, $\limsup_{n\to+\infty}\dfrac{w_+(n)}{v_+(n)}=+\infty$ *and* $\limsup_{n\to+\infty}\min\left\{\dfrac{v_+(n)}{w_+(n)},\dfrac{v_-(n)}{v_+(n)},\dfrac{w_+(n)}{w_-(n)}\right\}=+\infty.$

Proof of the lemma :
Let us consider the second point. Fix an increasing sequence (k_n) with $k_n/k_{n-1}^2\to+\infty$. For $n\in\mathbb{Z}$, let $R_n=\log\rho_n$. Let :

$$U_n=\max_{k\in[-k_n,-k_{n-1})}(R_k-R_{-k_{n-1}}),\ V_n=\max_{k\in(k_{n-1},k_n]}(R_k-R_{k_{n-1}}).$$

In the same way, introduce :

$$W_n=\min_{k\in(k_{n-1},k_n]}(R_k-R_{k_{n-1}}),\ X_n=\min_{k\in[-k_n,-k_{n-1})}(R_k-R_{-k_{n-1}}).$$

Let $\sigma>0$ be such that $\sigma^2=\mathrm{Var}(\log(q_0/p_0))$. Let $c=2+2|\log(1/\eta)|$ and notice that $|R_k|\leq c|k|$, $k\in\mathbb{Z}$. Using functional convergence to standard Brownian motion $(B_t)_{t\in[-1,1]}$:

$$\begin{aligned}&\mathbb{P}\left(\frac{U_n}{\sigma\sqrt{k_n-k_{n-1}}}\geq 1+\frac{V_n}{\sigma\sqrt{k_n-k_{n-1}}}\geq 2-\frac{W_n}{\sigma\sqrt{k_n-k_{n-1}}}\geq 3-\frac{X_n}{\sigma\sqrt{k_n-k_{n-1}}}\right)\\ &\longrightarrow_{n\to+\infty}\mathbb{P}\left(\max_{t\in[-1,0]}B_t\geq 1+\max_{t\in[0,1]}B_t\geq 2-\min_{t\in[0,1]}B_t\geq 3-\min_{t\in[-1,0]}B_t\right)=:\alpha>0.\end{aligned}$$

Using independence and the second Borel-Cantelli lemma, almost-surely the event appearing in the first probability is realized for infinitely many n. For such a n, we have :

$$\begin{aligned}v_+(k_n)&\geq\exp(-ck_{n-1}+\max_{k\in(k_{n-1},k_n]}(R_k-R_{k_{n-1}}))\\&\geq\exp(-ck_{n-1}+\sigma\sqrt{k_n-k_{n-1}}-\min_{k\in(k_{n-1},k_n]}(R_k-R_{k_{n-1}}))\\&\geq\exp(-3ck_{n-1}+\sigma\sqrt{k_n-k_{n-1}}-\min_{k\in[1,k_n]}R_k)\\&\geq\frac{w_+(k_n)}{k_n}\exp(-3ck_{n-1}+\sigma\sqrt{k_n-k_{n-1}}).\end{aligned}$$

If n is large, $v_+(k_n)/w_+(k_n)\geq\exp(\sigma\sqrt{k_n}/2)$, going to $+\infty$. Similar lower bounds are proved, with the same n, for $v_-(k_n)/v_+(k_n)$ and $w_+(k_n)/w_-(k_n)$. The first property stated in the lemma, namely $\limsup_{n\to+\infty}w_+(n)/v_+(n)=+\infty$, is proved in a similar (simpler) way.

□

We complete the proof of $i)$. Using the first point of the previous lemma, a.-s., for any $A>0$, infinitely often, $w_+(n)\geq Av_+(n)$, i.e. $w_+(v_+^{-1}(v_+(n)))\geq Av_+(n)$. Taking $\ell\in\mathbb{N}$ so that $\ell\leq v_+(n)<\ell+1$, we obtain :

$$\Phi_{str}(\ell+1)\geq\sqrt{\ell+1}\sqrt{A\ell}\geq\ell\sqrt{A}.$$

Hence $\Phi_{str}^{-1}(\ell\sqrt{A}) \leq \ell + 1$. This implies that, a.-s., $\liminf \Phi_{str}^{-1}(n)/n = 0$.

Using now the second point of the previous lemma, a.-s., for any $A > 1$, for infinitely many $n \in \mathbb{N}$, we have $v_-(n) \geq v_+(n) \geq Aw_+(n)$ and $w_+(n) \geq w_-(n)$. For such a n, $v_-^{-1}(v_+(n)) \leq n$ and :

$$\begin{aligned}
\frac{v_+(n)}{\sum_{-v_-^{-1}(v_+(n)) \leq k \leq v_+^{-1}(v_+(n))} 1/\rho_k} &\geq \frac{v_+(n)}{\sum_{-n \leq k \leq n} 1/\rho_k} \geq \frac{v_+(n)}{a_0 w_-(n) + w_+(n)} \\
&\geq \frac{v_+(n)}{w_+(n)}(1+a_0)^{-1} \geq \frac{A}{1+a_0}.
\end{aligned}$$

Choosing $\ell \in \mathbb{N}$ such that $\ell \leq v_+(n) < \ell + 1$, we obtain :

$$\frac{\ell+1}{\Phi_{str}^2(\ell)/\ell} \geq \frac{A}{1+a_0}, \text{ or } \Phi_{str}(\ell) \leq \frac{\sqrt{1+a_0}}{\sqrt{A}}(\ell+1).$$

Hence $\Phi_{str}^{-1}(\sqrt{1+a_0}(\ell+1)/\sqrt{A}) \geq \ell$. Hence, a.-s., $\limsup \Phi_{str}^{-1}(n)/n = +\infty$.

Let finally $b_n = \Phi_{str}^{-1}(K^n)/K^n$. Because Φ_{str}^{-1} checks dominated variation (7), a.-s., the previous results give $\liminf b_n = 0$ and $\limsup b_n = +\infty$. Dominated variation also implies that, a.-s. for a constant $H > 1$, for all n, we have $b_n/b_{n+1} \in [1/H, H]$. Thus $b_n \in [1, H]$ for infinitely many n. For such a n, $\min\{b_n, 1/b_n\} = 1/b_n \geq 1/H$. This shows (12) and completes the proof of i).

□

3.4 Proof of Theorem 2.5 *ii*)

Assume here that the ε_n are non-negative for large $n \in \mathbb{Z}$ and verify (2). The monotonicity of $C(n)$ for large n, hence of $nC(n)$, implies that the condition $\sum_{n \geq N_0} 1/(nC(n)) < +\infty$ is equivalent to $\sum_{n \geq N_1} 1/C(K^n) < +\infty$, for another $N_1 > 0$.

We first observe that $\sum_{n \geq N_1} 1/C(K^n) < +\infty$, for some $N_1 > 0$, in the case when there exists $\delta > 0$ so that $\varepsilon_n \geq \exp(-|n|^{1/2-\delta})$, for large $n \in \mathbb{Z}$.

Indeed, using (9), a.-s., for any $\varepsilon > 0$ and large n, $\exp(n^{1/2-\varepsilon}) \leq v_+(n) \leq \exp(n^{1/2+\varepsilon})$. Therefore a.-s., for any $\varepsilon > 0$ and large n, $(\log n)^{2+\varepsilon} \geq v_+^{-1}(n) \geq (\log n)^{2-\varepsilon}$. Via again (9), a.-s., for any $\varepsilon > 0$, for n large enough :

$$\min_{0 \leq k \leq v_+^{-1}(n)} \rho_k \leq \exp(-\log^{1-\varepsilon} n).$$

Let u_n be the first point in $[0, v_+^{-1}(n)]$ realizing the minimum of ρ_k on this interval. Necessarily, $u_n \to +\infty$. Hence, a.-s., $\forall \varepsilon > 0$, for large n, using that $u_n \leq v_+^{-1}(n) \leq (\log n)^{2+\varepsilon}$, we obtain :

$$\varepsilon_{u_n}/\rho_{u_n} \geq \exp(-u_n^{1/2-\delta}) \exp(\log^{1-\varepsilon} n) \geq \exp(\log^{1-2\varepsilon} n).$$

Hence, a.-s., $\forall \varepsilon > 0$, for large n, $C(n) \geq (1/2)\varepsilon_{u_n}/\rho_{u_n} \geq \exp(\log^{1-\varepsilon} n)$. Hence, a.-s., $\forall \varepsilon > 0$, for large n, $C(K^n) \geq \exp(n^{1-\varepsilon})$, giving $\sum_{n \geq N_1} 1/C(K^n) < +\infty$.

We now prove transience for (S_n), assuming the conditions of Theorem 2.5 *ii*). Reformulating [2], Prop. 2.5 1), a sufficient condition for transience is $\sum_{n \geq 1} 1/\Phi(n) < +\infty$ or $\sum_{n \geq 1} K^n/\Phi(K^n) < +\infty$, as Φ is increasing.

Let $n_0 \geq 0$ be such that $\varepsilon_s \geq 0$ whenever $s \geq v_+^{-1}(n_0)$ or $s \leq -v_-^{-1}(n_0)$. For large n and $k \leq -v_-^{-1}(K^{n-1})$ and $\ell \geq V_+^{-1}(K^{n-1})$, using also that $\eta \leq r_s/p_s \leq 1/\eta$, we have :

$$
\begin{aligned}
\sum_{k\le s\le \ell}\frac{r_s\varepsilon_s}{p_s\rho_s} &\geq \sum_{-v_-^{-1}(K^{n-1})\le s\le v_+^{-1}(K^{n-1})}\frac{r_s\varepsilon_s}{p_s\rho_s}\\
&\geq \sum_{s\in[-v_-^{-1}(K^{n-1}),v_+^{-1}(K^{n-1})]\setminus[-v_-^{-1}(n_0),v_+^{-1}(n_0)]}\frac{r_s\varepsilon_s}{p_s\rho_s}+\sum_{s\in[-v_-^{-1}(n_0),v_+^{-1}(n_0)]}\frac{r_s\varepsilon_s}{p_s\rho_s}\\
&\geq \eta\sum_{s\in[-v_-^{-1}(K^{n-1}),v_+^{-1}(K^{n-1})]\setminus[-v_-^{-1}(n_0),v_+^{-1}(n_0)]}\frac{\varepsilon_s}{\rho_s}+\sum_{s\in[-v_-^{-1}(n_0),v_+^{-1}(n_0)]}\frac{r_s\varepsilon_s}{p_s\rho_s}\\
&\geq (\eta/2)\sum_{s\in[-v_-^{-1}(K^{n-1}),v_+^{-1}(K^{n-1})]}\frac{\varepsilon_s}{\rho_s}=(\eta/2)C(K^{n-1})>0.
\end{aligned}
$$

We obtain that for large $n\geq 0$:

$$
\begin{aligned}
\Phi^2(K^n) &\geq \sum_{\substack{-v_-^{-1}(K^n)\le k<-v_-^{-1}(K^{n-1})\\ v_+^{-1}(K^{n-1})<\ell\le v_+^{-1}(K^n)}}\rho_k\rho_\ell\left(\sum_{k\le s\le\ell}\frac{r_s\varepsilon_s}{p_s\rho_s}\right)^2\\
&\succeq (C(K^{n-1}))^2\sum_{\substack{-v_-^{-1}(K^n)\le k<-v_-^{-1}(K^{n-1})\\ v_+^{-1}(K^{n-1})<\ell\le v_+^{-1}(K^n)}}\rho_k\rho_\ell.
\end{aligned}
$$

Next $\sum_{k=v_+^{-1}(K^{n-1})+1}^{v_+^{-1}(K^n)}\rho_k=v_+(v_+^{-1}(K^n))-v_+(v_+^{-1}(K^{n-1}))\le v_+(v_+^{-1}(K^n))\le K^n$. Also :

$$
v_+(v_+^{-1}(K^n))-v_+(v_+^{-1}(K^{n-1}))\geq\frac{K^n}{1+1/\eta}-K^{n-1}\geq\frac{K^n}{2(1+1/\eta)},
$$

using that $K>2(1+1/\eta)$. In the same way :

$$
\begin{aligned}
\sum_{-v_-^{-1}(K^n)\le k<-v_-^{-1}(K^{n-1})}\rho_k &= (1/a_0)[v_-(v_-^{-1}(K^n)-1)-v_-(v_-^{-1}(K^{n-1})-1)]\\
&\le \frac{1}{\eta}v_-(v_-^{-1}(K^n))\le\frac{K^n}{\eta}.
\end{aligned}
\tag{13}
$$

On the other hand, because $v_-(p)\geq v_-(p+1)/(1+1/\eta)$, the left-hand side is :

$$
\geq\eta\left(\frac{v_-(v_-^{-1}(K^n)+1)}{(1+1/\eta)^2}-K^{n-1}\right)\geq\eta\left(\frac{K^n}{(1+1/\eta)^2}-K^{n-1}\right)\geq\eta\frac{K^n}{2(1+1/\eta)^2}. \tag{14}
$$

Consequently, $\Phi(K^n)\succeq K^nC(K^{n-1})$. The hypothesis $\sum_{n\geq N_1}1/C(K^n)<+\infty$ therefore implies that $\sum_{n\geq1}K^n/\Phi(K^n)<+\infty$. This completes the proof of point *ii*).

□

3.5 Proof of Theorem 2.5 *iii*)

Here the $(\varepsilon_n)_{\in\mathbb{Z}}$ are independent random variables, independent from the $(p_n,q_n,r_n)_{n\in\mathbb{Z}}$, under the conditions of point *iii*) of the theorem.

First of all, as for getting (13) and (14) for instance :

$$
\sum_{v_+^{-1}(K^{n-1})<\ell\le v_+^{-1}(K^n)}\rho_\ell\asymp K^n\ \text{and}\ \sum_{-v_-^{-1}(K^n)\le k\le-1}\rho_k\asymp K^n.
$$

In order to control the proportion of (k,l) in the definition of $\Phi(K^n)$ such that $\sum_{k\le s\le l}(r_s\varepsilon_s/(p_s\rho_s))$ is large, we introduce the following probability measure on $\mathbb{Z}^2$:

$$\nu_n = \frac{1}{Z_n}\sum_{-v_-^{-1}(K^n)\le k\le -1,\, v_+^{-1}(K^{n-1})<\ell\le v_+^{-1}(K^n)} \rho_k\rho_\ell\delta_{(k,\ell)},$$

where the normalizing constant Z_n thus verifies $Z_n \asymp K^{2n}$.

It follows from (9) that, a.-s., for all $\varepsilon>0$, for large n, $(\log n)^{2-\varepsilon}\le v_+^{-1}(n)\le(\log n)^{2+\varepsilon}$. Hence, a.-s., for all $\varepsilon>0$, for large n, $n^{2-\varepsilon}\le v_+^{-1}(K^n)\le n^{2+\varepsilon}$. Let now $u_n\in[0,v_+^{-1}(K^{n-1})]$ be the first minimum of ρ_k on this interval. Then, from (9), a.-s., $\forall\varepsilon>0$, $\forall\varepsilon'>0$:

$$\exp(u_n^{1/2+\varepsilon'}) \ge \max_{0\le k\le u_n}\frac{1}{\rho_k}\ge\frac{1}{\rho_{u_n}}\ge \exp(n^{1-\varepsilon}).$$

Hence, a.-s., $\forall\varepsilon>0$, for large n, $n^{2+\varepsilon}\ge v_+^{-1}(K^{n-1})\ge u_n\ge n^{2-\varepsilon}$.

Let $\delta/2>\gamma>2\delta/5$. We next assume that $\varepsilon>0$ is small enough so that $\gamma+(2+\varepsilon)(1/2-\delta)\le 1-3\varepsilon$ and $(2+\varepsilon)\delta/5<\gamma<\delta/2$.

Recall that, a.-s., $\varepsilon_n\in[-1/\eta,1/\eta]$. Let $C_0=2/\eta+1$ and $l(n)=n^\gamma$. Starting from $u_{n,0}=u_n$ and going left, we shall choose recursively in a decreasing order in the interval $[0,u_n]$, points $u_{n,0}>u_{n,1}>\cdots>u_{n,l(n)}>0$, with increasing $\rho_{u_{n,0}}<\cdots<\rho_{u_{n,l(n)}}$. For $0\le l<l(n)$, let $u_{n,l+1}$ be the closest point on the left side of $u_{n,l}$ with :

$$\left(\frac{1}{4}\frac{C_0/\eta^2}{\rho_{u_{n,l}}}>\right)\frac{1}{4}\frac{\exp(-u_{n,l}^{1/2-\delta})}{\rho_{u_{n,l}}}\ge\frac{C_0/\eta^2}{\rho_{u_{n,l+1}}}\ge\frac{1}{4K}\frac{\exp(-u_{n,l}^{1/2-\delta})}{\rho_{u_{n,l}}}. \tag{15}$$

Recall first for this that $\rho_k/\rho_{k+1}\in[1/K,K]$, for $k\in\mathbb{Z}$. Recursively notice that for $0\le l<l(n)$, $u_{n,l+1}$ is a well-defined point in $(0,u_{n,l})$, since we require :

$$\begin{aligned}\rho_{u_{n,l+1}}\le(4C_0K/\eta^2)\rho_{u_{n,l}}\exp(u_{n,l}^{1/2-\delta}) &\le (4C_0K/\eta^2)^{l(n)}\rho_{u_{n,0}}\exp(l(n)u_{n,0}^{1/2-\delta})\\ &\le \exp(n^\gamma\log(4C_0K/\eta^2)-n^{1-\varepsilon}+n^\gamma n^{(2+\varepsilon)(1/2-\delta)})\\ &\le \exp(-n^{1-2\varepsilon}),\end{aligned}$$

for large n. The last upper-bound is uniform in $0\le l<l(n)$. In particular, for large n :

$$\rho_{u_{n,l(n)}}\le\exp(-n^{1-2\varepsilon}). \tag{16}$$

Since $u_{n,l(n)}>0$, this implies that $u_{n,l(n)}\to+\infty$. Remark also that conditions (15) imply that $\rho_{u_{n,l+1}}\ge 4\rho_{u_{n,l}}$, for $0\le l<l(n)$.

We now reason conditionally to the $(p_n,q_n,r_n)_{n\in\mathbb{Z}}$ and make a measurable construction. The probability conditional to the $(p_n,q_n,r_n)_{n\in\mathbb{Z}}$ and relative only to the $(\varepsilon_n)_{n\in\mathbb{Z}}$, is written as $\mathbb{P}'$, with corresponding expectation $\mathbb{E}'$. Consider the event A_n, where $(\varepsilon_i)_{i\in\mathbb{Z}}\in A_n$ if :

$$\nu_n\left\{(k,l)\in\mathbb{Z}^2,\ \left|\sum_{k\le u\le l}\frac{r_u\varepsilon_u}{p_u\rho_u}\right|\le\frac{C_0}{\eta\rho_{u_{n,l(n)}}}\right\}\ge 3/4. \tag{17}$$

Let us write :

$$\mathbb{P}'(A_n)=\mathbb{E}'\left(\mathbb{E}'(1_{(\varepsilon_k)_{k\in\mathbb{Z}}\in A_n}\mid(\varepsilon_l)_{l\ne u_{n,s},0\le s<l(n)})\right). \tag{18}$$

Fixing $(\varepsilon_l)_{l\ne u_{n,s},0\le s<l(n)}$, suppose that there exists $(\varepsilon_{u_{n,s}})_{0\le s<l(n)}$ with $\varepsilon=(\varepsilon_i)_{i\in\mathbb{Z}}\in A_n$. Take next any $(\varepsilon'_{u_{n,s}})_{0\le s<l(n)}\ne(\varepsilon_{u_{n,s}})_{0\le s<l(n)}$ and call ε' the point obtained from ε by replacing $(\varepsilon_{u_{n,s}})_{0\le s<l(n)}$ by $(\varepsilon'_{u_{n,s}})_{0\le s<l(n)}$, without changing the other coordinates.

We claim that $\varepsilon' \notin A_n$. Indeed, let $0 \le p < l(n)$ be the smallest index so that :

$$\varepsilon'_{u_{n,p}} \neq \varepsilon_{u_{n,p}}.$$

Take any (k,l) in the set involved in (17) and notice that $k < 0$ and $v_+^{-1}(K^{n-1}) < l$. Using the hypothesis on the laws of the ε_n, that $\eta \le r_u/p_u \le 1/\eta$, $u \in \mathbb{Z}$, and the conditions in (15) :

$$\begin{aligned}
\left|\sum_{k\le u\le l} \frac{r_u \varepsilon'_u}{p_u \rho_u}\right| &= \left|\frac{r_{u_{n,p}}}{p_{u_{n,p}}} \frac{\varepsilon'_{u_{n,p}} - \varepsilon_{u_{n,p}}}{\rho_{u_{n,p}}} + \sum_{p<q<l(n)} \frac{r_{u_{n,q}}}{p_{u_{n,q}}} \frac{\varepsilon'_{u_{n,q}} - \varepsilon_{u_{n,q}}}{\rho_{u_{n,q}}} + \sum_{k\le u\le l} \frac{r_u}{p_u}\frac{\varepsilon_u}{\rho_u}\right| \\
&\ge \eta \frac{\exp(-u_{n,p}^{1/2-\delta})}{\rho_{u_{n,p}}} - \sum_{p<q<l(n)} \frac{C_0}{\eta \rho_{u_{n,q}}} - \frac{C_0}{\eta \rho_{u_{n,l(n)}}} \\
&\ge 4\frac{C_0}{\eta \rho_{u_{n,p+1}}} - \frac{C_0}{\eta \rho_{u_{n,p+1}}} \sum_{r\ge 0} 4^{-r} \ge 2\frac{C_0}{\eta \rho_{u_{n,p+1}}} \ge 2\frac{C_0}{\eta \rho_{u_{n,l(n)}}}. \qquad (19)
\end{aligned}$$

This being true for any (k,l) appearing in the set in (17), this furnishes :

$$\nu_n \left\{ (k,l) \in \mathbb{Z}^2,\ \left|\sum_{k\le u\le l} \frac{r_u \varepsilon'_u}{p_u \rho_u}\right| \le \frac{C_0}{\eta \rho_{u_{n,l(n)}}} \right\} \le 1/4. \qquad (20)$$

Hence $\varepsilon' \notin A_n$. As a result, for large n, using (3), independence, $u_n \le n^{2+\varepsilon}$ and at the end $1 - x \le e^{-x}$, $x \in \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}'(1_{(\varepsilon_k)_{k\in\mathbb{Z}} \in A_n} \mid (\varepsilon_l)_{l \neq u_{n,s}, 0 \le s < l(n)}) &\le (1 - u_{n,0}^{-\delta/5}) \cdots (1 - u_{n,l(n)-1}^{-\delta/5}) \\
&\le (1 - u_n^{-\delta/5})^{l(n)} \le \exp(-n^{\gamma - (2+\varepsilon)\delta/5}).
\end{aligned}$$

By (18), we deduce that $\mathbb{P}'(A_n) \le \exp(-n^{\gamma-(2+\varepsilon)\delta/5})$. Now $\gamma - (2+\varepsilon)\delta/5 > 0$, leading to $\sum \mathbb{P}'(A_n) < \infty$. By (17) and the first Borel-Cantelli lemma, for $\mathbb{P}'$-almost-all $(\varepsilon_i)_{i\in\mathbb{Z}}$, for large n :

$$\nu_n \left\{ (k,l) \in \mathbb{Z}^2,\ \left|\sum_{k\le u\le l} \frac{r_u \varepsilon_u}{p_u \rho_u}\right| \ge \frac{C_0}{\eta \rho_{u_{n,l(n)}}} \right\} \ge 1/4. \qquad (21)$$

We now conclude the argument. First, we shall write $\mathcal{E}_n(f(X,Y))$ for $\sum_{(k,l)\in\mathbb{Z}^2} f(k,l) d\nu_n(k,l)$. Using [2], Prop. 2.5 1), the random walk (S_n) is transient whenever $\sum_{n\ge1} 1/\Phi(n) < +\infty$, or equivalently $\sum_{n\ge1} K^n/\Phi(K^n) < +\infty$, as $\Phi(n)$ is increasing.

As a result, from (21), $\mathbb{P}'$-a.-s., there exists $N_0 \ge 0$, so that for $n \ge N_0$:

$$\mathcal{E}_n \left[\left(\sum_{X \le u \le Y} \frac{r_u \varepsilon_u}{p_u \rho_u} \right)^2 \right] \ge \frac{1}{4} \frac{C_0^2}{\eta^2 \rho_{u_{n,l(n)}}^2}.$$

This then furnishes :

$$
\begin{aligned}
\sum_{n\geq N_0}\frac{K^n}{\Phi(K^n)} &\leq \sum_{n\geq N_0}K^n\left(\sum_{-v_-^{-1}(K^n)\leq k\leq \ell\leq v_+^{-1}(K^n)}\rho_k\rho_\ell\left(\sum_{k\leq u\leq \ell}\frac{r_u\varepsilon_u}{p_u\rho_u}\right)^2\right)^{-1/2}\\
&\leq \sum_{n\geq N_0}K^n\left(\sum_{-v_-^{-1}(K^n)\leq k\leq -1,\, v_+^{-1}(K^{n-1})<\ell\leq v_+^{-1}(K^n)}\rho_k\rho_\ell\left(\sum_{k\leq u\leq \ell}\frac{r_u\varepsilon_u}{p_u\rho_u}\right)^2\right)^{-1/2}\\
&\leq \sum_{n\geq N_0}\frac{K^n}{(Z_n)^{1/2}}\left(\mathcal{E}_n\left[\left(\sum_{X\leq u\leq Y}\frac{r_u\varepsilon_u}{p_u\rho_u}\right)^2\right]\right)^{-1/2}\\
&\leq \sum_{n\geq N_0}\frac{K^n}{(Z_n)^{1/2}}\frac{2\eta\rho_{u_{n,l(n)}}}{C_0}\preceq \sum_{n\geq N_0}\rho_{u_{n,l(n)}}<+\infty,
\end{aligned}
$$

as from (16), we have $\rho_{u_{n,l(n)}}\leq \exp(-n^{1-2\varepsilon})$. Hence $\sum_{n\geq 1}K^n/\Phi(K^n)<+\infty$ and the proof of point *iii*) of the theorem is complete.

$\square$

We conclude this article with a remark on the expected normalization for (S_n), taking $d=1$ and the context of Theorem 2.5. Introduce as in [1, 2] the random times $0=\sigma_0<\tau_0<\sigma_1<\tau_1<\cdots$, where $\tau_k=\min\{n>\sigma_k \mid S_n\notin\mathbb{Z}\times\{0\}\}$ and $\sigma_{k+1}=\{n>\tau_k\mid S_n\in\mathbb{Z}\times\{0\}\}$. Setting $D_n=S_{\sigma_n}-S_{\sigma_{n-1}}$, the stratification of the environment implies that the $(D_n)_{n\geq 1}$ are *i.i.d.*.

Standardly (cf Gnedenko-Kolmogorov [8]), the correct normalization for $D_1+\cdots+D_n$ can be read on the behaviour at the origin of the characteristic function $\chi_{D_1}(t)=\mathbb{E}(e^{itD_1})$, $t\in\mathbb{R}$. More precisely, if m_N is the empirical mean and d_N^2 the empirical variance of $(D_1+\cdots+D_n)_{1\leq n\leq N}$, then d_N and $|m_N|+d_N$ are respectively related to $1-\mathrm{Re}(\chi_D(t))$ and $|1-\chi_D(t)|$, as $t\to 0$. By [2], Prop. 6.5 and Prop. 6.8, one may informally guess that d_N is like $\Phi_+(N)$ and $|m_N|+d_N$ like $\Phi(N)$. Taking the special case $\varepsilon_n=1$, the correct normalization for σ_n should be $\tilde\Phi(n)$, with :

$$
\tilde\Phi(n)\asymp\left(\sum_{-v_-^{-1}(n)\leq k\leq\ell\leq v_+^{-1}(n)}\rho_k\rho_\ell\left(\sum_{s=k}^{\ell}\frac{1}{\rho_s}\right)^2\right)^{1/2}.
$$

Noticing that $\tilde\Phi^{-1}(n)\asymp\Psi^{-1}(n)$, where $\Psi(n)=n\sum_{-v_-^{-1}(n)\leq k\leq v_+^{-1}(n)}1/\rho_k$, a natural conjecture for the normalization for S_n is then :

$$
(\Phi\circ\Psi^{-1}(n),\Psi^{-1}(n)).
$$

Since $\Psi^{-1}(n)$ has rough order $(\log n)^2$, one recovers in the second coordinate the scaling of Sinaï's random walk [17]. Recall that in the vertically flat case and when the $(r_n\varepsilon_n/p_n)_{n\in\mathbb{Z}}$ are *i.i.d.* centered and non-constant, then $\Phi(n)\asymp n^{3/2}$ and $\Psi(n)\asymp n^2$. From these informal considerations, one recovers that the correct normalization for S_n in this case is $(n^{3/4},\sqrt{n})$; see [7] and [10].

Acknowledgments. We thank Jacques Printems and Alexis Devulder for useful discussions. We are also grateful to Françoise Pène for bringing reference [12] to our attention. We also wish to thank the anonymous referee for very carefully reading the first version of this article.

References

[1] J. Brémont, *On planar random walks in environments invariant by horizontal translations*, Markov Processes and Related Fields 2016, issue 2, vol. 22, 267-310.

[2] J. Brémont, *Markov chains in a stratified environment*, ALEA, Lat. Am. J. Probab. Math. Stat. 14, 751-798 (2017).

[3] M. Campanino, D. Petritis, *Random walks on randomly oriented lattices*, Markov Processes and Related Fields 9 (2003), no. 3, 391-412.

[4] M. Campanino, D. Petritis, *Type transitions of simple random walks on randomly directed regular lattices*, Journal of Applied Probability 51, 1065-1080 (2014).

[5] F. Castell, N. Guillotin, F. Pene, B. Schapira, *A local limit theorem for random walks in random scenery and on randomly oriented lattices*, Annals of Proba. 39 (2011), no. 6, 2079-2118.

[6] K. Chung, *A course in probability theory.* Second edition. Academic Press.1974.

[7] A. Devulder, F. Pene, *Random walk in random environment in a two-dimensional stratified medium with orientations*, Electron. Journal of Probab. 18 (2013), no. 88, 1-23.

[8] B. V. Gnedenko, A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Addison-Wesley Publishing Company, revised edition, 1968.

[9] N. Guillotin-Plantard, A. Le Ny, *Transient random walks on* $2D$*-oriented lattices*, Theory Probab. Appl. 52 (2008), no. 4, 699-711.

[10] N. Guillotin-Plantard, A. Le Ny, *A functional limit theorem for a 2d- random walk with dependent marginals.* Electronic Communications in Probability (2008), Vol. 13, 337–351.

[11] S. Kalikow, *Generalized random walk in a random environment.* Ann. Probab. 9, 753-768.

[12] M. Kochler, *Random walks in random environment : random orientations and branching*, Ph-D thesis, Technische Universität München, 2012.

[13] M. Matheron, G. de Marsily, *Is transport in porous media always diffusive ? A counterexample.* Water resources Res. 16:901-907, 1980.

[14] D. Ornstein, *Random walks. I, II.* Trans. Amer. Math. Soc. 138 (1969), 1-43; ibid. 138 (1969), 45-60.

[15] F. Pene, *Transient random walk in* $\mathbb{Z}^2$ *with stationary orientations*, ESAIM Probability and Statistics 13 (2009), 417-436.

[16] C. Sabot, *Random Walks in Random Dirichlet Environment are transient in dimension* $d \geq 3$, Probab. Theory Related Fields 151 (2011), no. 1-2, 297-317.

[17] Y. Sinaï, *The limit behavior of a one-dimensional random walk in a random environment*, Teor. Veroyatnost. i Primenen. **27** (1982), no. 2, 247–258.

[18] F. Solomon, *Random walks in a random environment.* Ann. Probability 3 (1975), 1-31.

[19] F. Spitzer, *Principles of random walks.* Second edition. Graduate Texts in Mathematics, vol. 34. Springer-Verlag, New-York Heidelberg, 1976.

Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est, Faculté des Sciences et Technologies, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, FRANCE

E-mail address : julien.bremont@u-pec.fr