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# PRAMs over integers do not compute maxflow efficiently

Luc Pellissier, University Paris Est Creteil  
luc.pellissier@lACL.fr

Thomas Seiller, CNRS  
seiller@lipn.fr

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## Abstract

This paper presents a new semantic method for proving lower bounds in computational complexity. We use it to prove that `maxflow`, a `PTIME` complete problem, is not computable in polylogarithmic time on parallel random access machines (PRAMs) working with integers, showing that  $\text{NC}_{\mathbf{Z}} \neq \text{PTIME}$ , where  $\text{NC}_{\mathbf{Z}}$  is the complexity class defined by such machines, and `PTIME` is the standard class of polynomial time computable problems (on, say, a Turing machine). On top of showing this new separation result, we show our method captures previous lower bounds results from the literature: Steele and Yao's lower bounds for algebraic decision trees [30], Ben-Or's lower bounds for algebraic computation trees [4], Cucker's proof that  $\text{NC}_{\mathbf{R}}$  is not equal to  $\text{PTIME}_{\mathbf{R}}$  [10], and Mulmuley's lower bounds for "PRAMs without bit operations" [20].

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# 1 Introduction

## 1.1 Computational Complexity

The field of computational complexity was initiated soon after the conception of the first computers. While theoretical results had already established a definition of the notion of "computable function" on the set of natural numbers, it became quickly apparent that computable did not mean practical, as many functions considered computable could not be computed within a reasonable time. (Example?)

The first complexity class defined was that of *feasible functions* [15, 7, 11], which is now known as PTIME: the set of polynomial time computable functions, i.e. functions  $f$  for which there exists a polynomial  $p$  and a machine  $M$  computing  $f$  whose running time on an input  $n$  is bounded by  $p(n)$ . This class, apart from being the first ever complexity class to appear in the literature, is arguably the most important one in computer science. Many fundamental problems concern its relation to other classes, such as NPTIME which is the extension of PTIME if one allows for non-deterministic machines in the above definition. These problems, however, are still open. (lower bounds, barriers)

Beyond the relationship between PTIME and other classes, the general question of *classifying* the complexity classes became one of the main objectives of the field, and a number of important results were obtained within the first years.

## 1.2 Separation, Lower bounds and Barriers

As part of the classification problem, complexity theory has traditionally been concerned with proving *separation results*. Among the numerous open separation problems lies the much advertised PTIME vs. NPTIME problem of showing that some problems considered hard to solve but efficient to verify do not have a polynomial time algorithm solving them.

Proving that two classes  $B \subset A$  are not equal can be reduced to finding lower bounds for problems in  $A$ : by proving that certain problems cannot be solved with less than certain resources on a specific model of computation, one can show that two classes are not equal. Conversely, proving a separation result  $B \subsetneq A$  provides a lower bound for the problems that are *A-complete* [9] – i.e. problems that are in some way *universal* for the class  $A$ .

The proven lower bound results are however very few, and most separation problems remain as generally accepted conjectures. For instance, a proof that the class of non-deterministic exponential problems is not included in what is thought of as a very small class of circuits was not achieved until very recently [32].

The failure of most techniques of proof has been studied in itself, which lead to the proof of the existence of negative results that are commonly called *barriers*. Altogether, these results show that all proof methods we know are ineffective with respect to proving interesting lower bounds. Indeed, there are three barriers: relativisation [3], natural proofs [24] and algebrization [1], and every known proof method hits at least one of them. This shows the need for new methods<sup>1</sup>. However, to this day, only one research program aimed at proving new separation results is commonly believed to have the ability to bypass all barriers: Mulmuley and Sohoni's Geometric Complexity Theory (GCT) program [21].

## 1.3 Algebraic models and Geometric Complexity Theory

Geometric Complexity Theory (GCT) is widely considered to be a promising research program that might lead to interesting results. It is also widely believed to necessitate new and extremely sophisticated pieces of mathematics in order to achieve its goal. The research program aims to prove the PTIME  $\neq$  NPTIME lower bound by showing that certain algebraic surfaces (representing the permanent and the discriminant, which are believed [31] to have different complexity if PTIME  $\neq$  NPTIME) cannot be embedded one into the other. Although this program has led to interesting developments as far as pure mathematics is concerned, it has

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<sup>1</sup>In the words of S. Aaronson and A. Wigderson [1], "We speculate that going beyond this limit [algebrization] will require fundamentally new methods."

not enhanced our understanding of complexity lower bounds for the time being (actually, even for Mulmuley himself, such understanding will not be achieved in our lifetimes [13]).

Intuitively, the program develops a proof method for proving lower bounds in *algebraic complexity* based on algebraic geometry: separation of the Valiant complexity classes VP and VNP could be deduced from the impossibility of embedding an algebraic variety  $\mathcal{P}$  defined from the permanent into an algebraic variety defined from the determinant  $\mathcal{D}$  (with constraints on the dimensions). Two main approaches were proposed, based on the decomposition of varieties w.r.t. irreducible varieties: *occurrence obstructions* aims to show such an embedding does not exist by exhibiting an irreducible variety occurring in  $\mathcal{P}$  but not in  $\mathcal{D}$ , while *multiplicity obstruction* aims to prove the same result by exhibiting an irreducible variety whose multiplicity of occurrence in  $\mathcal{P}$  is strictly greater than its multiplicity of occurrence in  $\mathcal{D}$ . Obviously, the first approach is easier, as a particular case of the second. Recently, some negative results [17] have shown this easiest path proposed by GCT fails. Some positive results on a toy model were however obtained regarding multiplicity obstructions [18]: although the obtained results are not new, they use the multiplicity obstruction method.

The GCT program was inspired, according to its creators, by a lower bound result obtained by Mulmuley [20].

#### 1.4 Lower bounds for PRAMS without bit operations

Indeed, Mulmuley showed in 1999 [20] that a notion of machine introduced under the name “PRAMS without bit operations” does not compute `maxflow` in polylogarithmic time. This notion of machine, quite exotic at first sight, corresponds to an algebraic variant of PRAMS, where registers contain integers and individual processors are allowed to perform sums, subtractions and products of integers. It is argued by Mulmuley that this notion of machine provides an expressive model of computation, able to compute some non trivial problems in NC such as Neff’s algorithm for computing approximate roots of polynomials [23]. Although Mulmuley’s result represents a big step forward in the quest for a proof that PTIME and NC are not equal, the result was not strengthened or reused in the last 20 years, and remained the strongest known lower bound result.

The `maxflow` problem is quite interesting as it is known to be in PTIME (by reduction to linear programming, or the Ford-Fulkerson algorithm [12]), but there are no known efficiently parallel algorithm solving it. This lower bound proof, despite being the main inspiration of the well-known GCT research program, remains seldom cited and has not led to variations applied to other problems.

#### 1.5 Contributions.

The main contribution of this work is a strengthening of Mulmuley’s lower bounds result, making an additional step in the direction of a potential proof that NC is different from PTIME. While the latter proves that `maxflow` is not computable in polylogarithmic time in the model of “PRAMS without bit operations”, we show here that `maxflow` is not computable in polylogarithmic time in the more expressive model of PRAMS over integers, in which processors are allowed to perform arbitrary divisions and square roots in addition to the basic operations allowed in Mulmuley’s model (addition, subtraction, multiplication). Indeed, our result can be stated as

**Theorem 1.**

$$\text{NC}_{\mathbf{Z}} \neq \text{PTIME},$$

where  $\text{NC}_{\mathbf{Z}}$  is the set of problems decidable in polylogarithmic time by a (not necessarily uniform) family of PRAMS over  $\mathbf{Z}$ .

The second contribution of the paper is the proof method itself, which is based on *dynamic semantics* for programs by means of *graphings*, a notion introduced in ergodic theory and recently used to define models of linear logic by the second author [26, 27, 28, 29]. The dual nature of graphings, both continuous and discrete, is essential in the present work, as it enables invariants from continuous mathematics, in particular

the notion of *topological entropy* for dynamical systems, while the finite representability of graphings is used in the key lemma (as the number of *edges* appears in the upper bounds of Lemma 57 and Theorem 59).

In particular, we show how this proof method captures known lower bounds and separation results in algebraic models of computation, namely Steele and Yao’s lower bounds for algebraic decision trees [30], Ben-Or’s lower bounds on algebraic computation trees [4], Cucker’s proof that  $\text{NC}_{\mathbf{R}}$  is not equal to  $\text{PTIME}_{\mathbf{R}}$  (i.e. answering the NC vs PTIME problem for computation over the real numbers).

## 1.6 A more detailed view of the proof method

One of the key ingredients in the proof is the representation of programs as graphings, and *quantitative soundness* results. We refer to the next section for a formal statement, and we only provide an intuitive explanation for the moment. Since a program  $P$  is represented as a graphing  $\llbracket P \rrbracket$ , which is in some way a dynamical system, the computation  $P(a)$  on a given input  $a$  is represented as a sequence of values  $\llbracket a \rrbracket, \llbracket P \rrbracket(\llbracket a \rrbracket), \llbracket P \rrbracket^2(\llbracket a \rrbracket), \dots$ . Quantitative soundness states that not only  $\llbracket P \rrbracket$  computes exactly as  $P$ , but it does so with a constant overhead, i.e. if  $P(a)$  terminates on a value  $b$  in time  $k$ , then  $\llbracket P \rrbracket^{Ck}(\llbracket a \rrbracket) = \llbracket b \rrbracket$ , where  $C$  is a constant fixed once and for all for the model of computation.

The second ingredient is the dual nature of graphings, both continuous and discrete objects. Indeed, a graphing *representative* is a graph-like structure whose edges are represented as continuous maps, i.e. a finite representation of a (partial) continuous dynamical system. Given a graphing, we define its *kth cell decomposition*, which separates the input space into cells such that two inputs in the same cell are indistinguishable in  $k$  steps, i.e. the graphing’s computational traces on both inputs are equal. We can then use both the finiteness of the graphing representatives and the *topological entropy* of the associated dynamical system to provide upper bounds on the size of a further refinement of this geometric object, namely the *kth entropic co-tree* of a graphing – a kind of final approximation of the graphing by a computational tree<sup>2</sup>

As we deal with algebraic models of computation, this implies a bound on the representation of the  $k$ th cell decomposition as a semi-algebraic variety. In other words, the  $k$ th cell decomposition is defined by polynomial in-equalities and we provide bounds on the number and degree of the involved polynomials. The corresponding statement is the main technical result of this paper (Theorem 59).

This lemma can then be used to obtain lower bounds results. Using the Milnor-Thom theorem to bound the number of connected components of the  $k$ th cell decomposition, we then recover the lower bounds of Steele and Yao on algebraic decision trees, and the refined result of Ben-Or providing lower bounds for algebraic computation trees. A different argument based on invariant polynomials provides a proof of Cucker’s result that  $\text{NC}_{\mathbf{R}} \neq \text{PTIME}_{\mathbf{R}}$  by showing that a given polynomial that belongs to  $\text{PTIME}_{\mathbf{R}}$  cannot be computed within  $\text{NC}_{\mathbf{R}}$ . Lastly, following Mulmuley’s geometric representation of the `maxflow` problem, we are able to strengthen his celebrated result to obtain lower bounds on the size (depth) of a PRAM over the integers computing this problem. This proves the following theorem, which has Theorem 1 as a corollary.

**Theorem 2.** *Let  $c$  be a positive integer,  $M$  a PRAM over  $\mathbf{Z}$  with  $2^{O((\log N)^c)}$  processors, with  $N$  the length of the inputs. Then  $M$  does not decide `maxflow` in  $O((\log N)^c)$  steps.*

## 2 Contents of the paper

### 2.1 Computation models as graphings.

The present work reports on the first investigations into how the interpretation of programs as graphings could lead to separation techniques, by rephrasing two well-known lower bound proofs. The interpretation of programs rely on two ingredients:

- the interpretation of models of computation as monoid actions. In our setting, we view the computational principles of a computational model as elements that act on a configuration space. As these

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<sup>2</sup>Intuitively, the  $k$ -th entropic co-tree mimicks the behaviour of the graphing for  $k$  steps of computation.

actions can be composed, but are not necessarily reversible, it is natural to interpret them as composing a monoid acting on a configuration space. As, moreover, we are interested in having control in our computations (knowing whether it is finished, failed, succeeded, . . .), we consider actions that can be decomposed as a part that computes using the principles of computation and a part that just modifies a control state;

- the realization of programs as graphings. We abstract programs as graphs whose vertices are subspaces of the product of the configuration space and the control states and edges are labelled by elements of the acting monoid, acting on subspaces of vertices.

The basic intuitions here can be summarised by the following slogan: "Computation, as a dynamical process, can be modelled as a dynamical system". Of course, the above affirmation cannot be true of all computational processes; for instance the traditional notion of dynamical system is deterministic. In practice, one works with a generalisation of dynamical systems named *graphings*; introduced as part of a family of models of linear logic, graphings have been shown to model non-deterministic and probabilistic computation.

To do so, we consider that a computation model is given by a set of generators (that correspond to computation principles) and its actions on a space (representing the configuration space). So, in other words, we define a computation model as an action of a monoid (presented by its generators and relations) on a space  $\alpha : M \curvearrowright \mathbf{X}$ . This action can then be specified to be continuous, measurable, . . . depending on the properties we are interested in.

A program in such a model of computation is then viewed as a graph, whose vertices are subspace of the configuration space and edges are generators of the monoid: in this way, both the partiality of certain operations and branching is allowed. This point of view is very general, as it can allow to study, as special model of computations, models that can be discrete or continuous, algebraic, rewriting-based, . . .

## 2.2 Entropy

We fix an action  $\alpha : M \curvearrowright \mathbf{X}$  for the following discussion. One important aspect of the representation of abstract programs as graphings is that restrictions of graphings correspond to known notions from mathematics. In a very natural way, a deterministic  $\alpha$ -graphing defines a partial dynamical system. Conversely, a partial dynamical system whose graph is contained in the *measured preorder*  $\{(x, y) \in \mathbf{X}^2 \mid \exists m \in M, \alpha(m)(x) = y\}$  [25] can be associated to an  $\alpha$ -graphing.

The study of deterministic models of computations can thus profit from the methods of the theory of dynamical systems. In particular, the methods employed in this paper relate to the classical notion of *topological entropy*. The topological entropy of a dynamical system is a value representing the average exponential growth rate of the number of orbit segments distinguishable with a finite (but arbitrarily fine) precision. The definition is based on the notion of open covers: for each finite open cover  $\mathcal{C}$ , one can compute the entropy of a map w.r.t.  $\mathcal{C}$ , and the entropy of the map is then the supremum of these values when  $\mathcal{C}$  ranges over the set of all finite covers. As we are considering graphings and those correspond to partial maps, we explain how the techniques adapt to this more general setting and define the entropy  $h(G, \mathcal{C})$  of a graphing  $G$  w.r.t. a cover  $\mathcal{C}$ , as well as the topological entropy  $h(G)$  defined as the supremum of the values  $h(G, \mathcal{C})$  where  $\mathcal{C}$  ranges over all finite open covers.

While the precise results described in this paper use the entropy  $h_0(G)$  w.r.t. a specific cover (similar bounds could be obtained from the topological entropy, but would lack precision), the authors believe entropy could play a much more prominent role in future proofs of lower bound. Indeed, while  $h_0(G)$  somehow quantifies over one aspect of the computation, namely the branchings, the topological entropy computed by considering all possible covers provides a much more precise picture of the dynamics involved. In particular, it provides information about the computational principles described by the AMC; this information may lead to more precise bounds based on how some principles are much more complex than some others, providing some lower bounds on possible simulations of the former with the latter.

All the while only the entropy w.r.t. a given cover will be essential in this work, the overall techniques related to entropy provide a much clearer picture of the techniques. In particular, the definition of *entropic*

*co-trees* (Theorem 52) are quite natural from this point of view and clarifies the methods employed by e.g. Ben-Or and Mulmuley.

### 2.3 Ben-Or’s proof

One lower bounds result related to Mulmuley’s techniques is the bounds obtained by Steele and Yao [30] on *Algebraic Decision Trees*. Algebraic decision trees are defined as finite ternary trees describing a program deciding a subset of  $\mathbf{R}^n$ : each node verifies whether a chosen polynomial, say  $P$ , takes a positive, negative, or null value at the point considered. A  $d$ -th order algebraic decision tree is an algebraic decision tree in which all polynomials are of degree bounded by  $d$ .

In a very natural manner, an algebraic decision tree can be represented as an  $\iota$ -graphings, when  $\iota$  is the trivial action on the space  $\mathbf{R}^n$ . We use entropy to provide a bound on the number of connected components of subsets decided by  $\iota$ -graphings. These bounds are obtained by combining a bound in terms of entropy and a variant of the Milnor-Thom theorem due to Ben-Or. The latter, which we recall below (Theorem 47) bounds the number of connected components of a semi-algebraic set in terms of the number of polynomial inequalities, their maximal degree, and the dimension of the space considered.

**Theorem 29.** *Let  $T$  be a  $d$ -th order algebraic decision tree deciding a subset  $W \subseteq \mathbf{R}^n$ . Then the number of connected components of  $W$  is bounded by  $2^h d(2d - 1)^{n+h-1}$ , where  $h$  is the height of  $T$ .*

This result of Steele and Yao adapts in a straightforward manner to a notion of algebraic computation trees describing the construction of the polynomials to be tested by mean of multiplications and additions of the coordinates. The authors remarked this result uses techniques quite similar to that of Mulmuley’s lower bounds for the model of PRAMS *without bit operations*. It is also strongly similar to the techniques used by Cucker in proving that  $\text{NC}_{\mathbf{R}} \neq \text{PTIME}_{\mathbf{R}}$  [10].

However, a refinement of Steele and Yao’s method was quickly obtained by Ben-Or so as to obtain a similar result for an extended notion of algebraic computation trees allowing for computing divisions and taking square roots. We here adapt Ben-Or techniques within the framework of graphings, in order to apply this refined approach to Mulmuley’s framework, leading to a strengthened lower bounds result.

Adapting Ben-Or’s method, we obtain a proof of the following result on *computational graphings* in the AMC of algebraic computational trees. The class of computational graphings contains the interpretation of algebraic computational trees and the result generalises that of Ben-Or by giving a bound on the number of connected components of the subset decided by a computational graphing. This bound depends on the number of edges of the computational graphing, as well as its algebraic degree (Theorem 58).

**Theorem 43.** *Let  $G$  be a computational graphing representative,  $\text{Card}(E)$  its number of edges, and  $D$  its algebraic degree. Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps, i.e. for each element of  $\mathbf{R}^n$ ,  $\pi_{\mathbf{S}}(G^k(x)) = \top$  if and only if  $x \in W$ . Then  $W$  has at most  $2^{h_0(|G|)+1} 3^{2kD+n+1}$  connected components.*

This reformulation of Ben-Or techniques is then applied to strengthen a lower bound obtained by Mulmuley [20]. While Mulmuley’s model of “PRAMS without bit operations” is a restriction of the usual notion of algebraic PRAMS over the integers, we obtain here similar lower bounds for the non-restricted model. For this purpose, we first need to show how parallelism can be accommodated within the framework of AMCs and graphings.

### 2.4 PRAMS and the crew

We are able to introduce PRAMS acting over integers in this setting. They can be described as having a finite number of processors, each having access to a private memory on top of the shared memory, and able to perform the operations  $+$ ,  $-$ ,  $\times$ ,  $/$  as well as branching and indirect addressing. Interestingly, we can represent these machines in the graphings framework in two steps: first, by defining the RAM model, with just one processor; and then by performing an algebraic operation at the level of the algebraic models of computation.



So, in a way, parallel computation is modelled per se, at the level of models. As usual, one is bound to choose a mode of interaction between the different processes when dealing with shared memory. We will consider here only the case of *Concurrent Read Exclusive Write* (CREW), i.e. all processes can read the shared memory concurrently, but if several processes try to write in the shared memory only the process with the smallest index is allowed to do so.

The heart of our approach of parallelism is based on commutation. Among all the instructions, the ones affecting only the private memory of distinct processors can commute, while it is not the case of two instructions affecting the central memory. We do so by considering a notion of product for monoids that generalizes both the direct product and the free product: we specify, through a conflict relation, which of the generators can and can not commute, allowing us to build a monoid representing the simultaneous action.

## 2.5 Mulmuley’s geometrization

Contrarily to Ben-Or’s model, the PRAM machines do not decide sets of reals but of integers, making the use of algebraico-geometric results to uncover their geometry much less obvious. The mechanisms of Mulmuley’s proof rely on twin geometrizations: one of a special optimization problem that can be represented by a surface in  $\mathbf{R}^3$  Subsec. 9.1-9.2, the other one by building explicitly, given a PRAM, a set of algebraic surfaces such that the points accepted by the machine are exactly the integer points enclosed by the set of surfaces.

Finally, the proof is concluded by a purely geometrical theorem (Thm. 75)<sup>3</sup> expressing a tension between the two geometrizations. Our work focuses here only on the construction of a set algebraic surfaces representing the computation of a PRAM; the remaining part of our proof follows Mulmuley’s original technique closely.

**Building surfaces** The first step in Mulmuley’s proof is to use the parametric complexity results of Carstensen [6] to represent an instance of the decision problem associated to `maxflow` so that it induces naturally a partition of  $\mathbf{Z}^3$  that can then be represented by a particular surface.

The second step is to represent any partition of  $\mathbf{Z}^3$  induced by the run of a machine by a set of surfaces in  $\mathbf{R}^3$ , in order to be able to use geometric methods.

Let  $K$  be a compact of  $\mathbf{R}^3$  and  $P = (P_1, \dots, P_m)$  be a partition of  $\mathbf{Z}^3 \cap K$ .  $P$  can be extended to a partition of the whole of  $K$  in a number of ways, as pictured in Fig. 1. In particular,  $P$  can always be extended to a partition  $P_{\text{alg}}$  (resp.  $P_{\text{smooth}}$ ,  $P_{\text{ana}}$ ) of  $K$  such that all the cells are compact, and the boundaries of the cells are all algebraic (resp. smooth, analytic) surfaces.

In general, such surfaces have no reason to be easy to compute and the more they are endowed with structure, the more complicated to compute they are to be. In the specific case of PRAMS, the decomposition can naturally be represented with algebraic surfaces whose degree is moreover bounded. This choice of representation might not hold for any other model of computation, for which it might be more interesting to consider surfaces of a different kind.

The method for building such a set of algebraic surfaces is reminiscent of the technique we used for Ben-Or’s result: build a tree summarizing the computation of a specific PRAM and build, along this tree a system of polynomial equations on a larger space than the space of variables actually used by the machine, this larger space allowing to consider full-fledged division. This system of integer polynomials of bounded degree then defines surfaces exactly matching our needs.

## 2.6 The main result

Interestingly, this allows to use Ben-Or’s technique of adding new variables to handle operations such as division and square root to PRAMS, which is a mild improvement over Mulmuley’s proof (and indeed, as noted in his article, the method is able of handling additional instructions as long as arbitrary bits are not easy to compute: in our model, bits of low orders are easy to compute – parity is just the remainder of a

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<sup>3</sup>We would like to stress here that this separation in three movement, with a geometrical tour-de-force, is not explicit in the original article. We nonetheless believe it greatly improves the exposition.

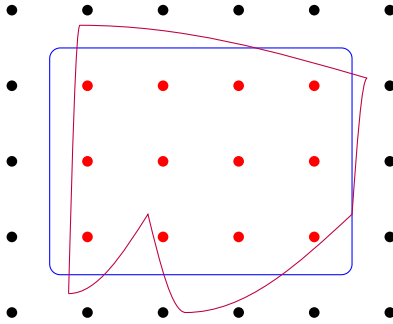


Figure 1: Two curves that define the same partition of  $\mathbf{Z}^2$

division – but computing the middle order bits of a number is difficult, see Prop. ??). By considering that the length of an input is be the minimal length of a binary word representing it, we get a realistic cost model for the PRAMs, for which we can prove:

**Theorem 63.** *Let  $G$  be a PRAM without bit operations with  $2^{O((\log N)^c)}$  processors, where  $N$  is the length of the inputs and  $c$  any positive integer.*

*$G$  does not decide `maxflow` in  $O((\log N)^c)$  steps.*

If we call  $\text{NC}_{\mathbf{Z}}$  the class of computation problems that can be decided by a PRAM over integers in time logarithmic in the length of the inputs and a number of processors polylogarithmic in the length of the inputs, we have proved that

$$\text{NC}_{\mathbf{Z}} \neq \text{PTIME}$$

## 2.7 Conclusion

This work not only provides a strengthened lower bound results, but shows how the semantic techniques based on abstract models of computation and graphings can shed new light on some lower bound techniques. In particular, it establishes some relationship between the lower bounds and the notion of entropy which, although arguably still superficial in this work, could potentially become deeper and provide new insights and finer techniques.

Showing that the interpretation of programs as graphings can translate, and even refine, such strong lower bounds results is also important from another perspective. Indeed, the techniques of Ben-Or and Mulmuley (as well as other results of e.g. Cucker [10], Yao [33]) seem at first sight restricted to algebraic models of computation due to their use of the Milnor-Thom theorem which holds only for real semi-algebraic sets. However, the second author’s characterisations of Boolean complexity classes in terms of graphings acting on algebraic spaces [28] opens the possibility of using such algebraic methods to provide lower bounds for boolean models of computation.

## 3 Programs as Dynamical systems

### 3.1 Abstract models of computation and graphings

We consider computations as a dynamical process, hence model them as a dynamical systems with two main components: a space  $\mathbf{X}$  that abstracts the notion of configuration space and a monoid acting on this space that represents the different operations allowed in the model of computation. Although the notion of *space* considered can vary (one could consider e.g. topological spaces, measure spaces, topological vector spaces), we restrict ourselves to topological spaces in this work.

**Definition 1.** An *abstract model of computation* (AMC) is a monoid action  $\alpha : M \curvearrowright \mathbf{X}$ , i.e. a monoid morphism from  $M$  to the group of endomorphisms of  $\mathbf{X}$ . The monoid  $M$  is often given by a set  $G$  of generators and a set of relations  $R$ . We denote such an AMC as  $\alpha : \langle G, R \rangle \curvearrowright \mathbf{X}$ .

Programs in an AMC  $\alpha : \langle G, R \rangle \curvearrowright \mathbf{X}$  is then defined as *graphings*, i.e. graphs whose vertices are subspaces of the space  $\mathbf{X}$  (representing sets of configurations on which the program act in the same way) and edges are labelled by elements of  $M \langle G, R \rangle$ , together with a global control state. More precisely, we use here the notion of *topological graphings*<sup>4</sup> [27].

**Definition 2.** An  $\alpha$ -graphing representative  $G$  w.r.t. a monoid action  $\alpha : M \curvearrowright \mathbf{X}$  is defined as a set of *edges*  $E^G$  together with a map that assigns to each element  $e \in E^G$  a pair  $(S_e^G, m_e^G)$  of a subspace  $S_e^G$  of  $\mathbf{X}$  – the *source* of  $e$  – and an element  $m_e^G \in M$  – the *realiser* of  $e$ .

While graphing representatives are convenient to manipulate, they do provide too much information about the programs. Indeed, if one is to study programs as dynamical systems, the focus should be on the *dynamics*, i.e. on how the object acts on the underlying space. The following notion of *refinement* captures this idea that the same dynamics may have different graph-like representations.

**Definition 3** (Refinement). An  $\alpha$ -graphing representative  $F$  is a refinement of an  $\alpha$ -graphing representative  $G$ , noted  $F \leq G$ , if there exists a partition  $(E_e^F)_{e \in E^G}$  of  $E^F$  such that  $\forall e \in E^G$ :

$$\begin{aligned} \left( \bigcup_{f \in E_e^F} S_f^F \right) \triangle S_e^G &= \emptyset; & \forall f \neq f' \in E_e^F, & S_f^F \triangle S_{f'}^F = \emptyset; \\ \forall f \in E_e^F, & m_f^F &= m_e^G. \end{aligned}$$

This induces an equivalence relation defined as

$$F \sim_{\text{ref}} G \Leftrightarrow \exists H, H \leq F \wedge H \leq G.$$

The notion of *graphing* is therefore obtained by considering the quotient of the set of graphing representatives w.r.t.  $\sim_{\text{ref}}$ . Intuitively, this corresponds to identifying graphings whose *actions on the underlying space are equal*.

**Definition 4.** An  $\alpha$ -*graphing* is an equivalence class of  $\alpha$ -graphing representatives w.r.t. the equivalence relation  $\sim_{\text{ref}}$ .

We can now define the notion of abstract program. These are defined as graphings

**Definition 5.** Given an AMC  $\alpha : M \curvearrowright \mathbf{X}$ , an  $\alpha$ -*program*  $A$  is a  $\bar{\alpha}$ -graphing  $G^A$  w.r.t. the monoid action  $\bar{\alpha} = \alpha \times \mathfrak{S}_k \curvearrowright \mathbf{X} \times \mathbf{S}^A$ , where  $\mathbf{S}^A$  is a finite set of *control states* of cardinality  $k$  and  $\mathfrak{S}_k$  is the group of permutations of  $k$  elements.

Now, as a sanity check, we will show how the notion of graphing do capture the dynamics as expected. For this, we restrict to *deterministic graphings*, and show the notion relates to the usual notion of dynamical system.

**Definition 6.** An  $\alpha$ -graphing representative  $G$  is *deterministic* if for all  $x \in \mathbf{X}$  there is at most one  $e \in E^G$  such that  $x \in S_e^G$ . An  $\alpha$ -graphing is *deterministic* if its representatives are deterministic. An abstract program is *deterministic* if its underlying graphing is deterministic.

**Lemma 7.** *There is a one-to-one correspondence between the set of deterministic graphings w.r.t. the action  $M \curvearrowright \mathbf{X}$  and the set of partial dynamical systems  $f : \mathbf{X} \hookrightarrow \mathbf{X}$  whose graph is contained in the preorder<sup>5</sup>  $\{(x, y) \mid \exists m \in M, \alpha(m)(x) = y\}$ .*

<sup>4</sup>While “measured” graphings were already considered [27], the definition adapts in a straightforward manner to allow for other notions such as graphings over topological vector spaces – which would be objects akin to the notion of quiver used in representation theory.

<sup>5</sup>When  $\alpha$  is a group action acting by measure-preserving transformations, this is a *Borel equivalence relation*  $\mathcal{R}$ , and the condition stated here boils down to requiring that  $f$  belongs to the *full group* of  $\alpha$ .

Lastly, we define some restrictions of  $\alpha$ -programs that will be important later. First, we will restrict the possible subspaces considered as sources of the edges, as unrestricted  $\alpha$ -programs could compute even undecidable problems by, e.g. encoding it into a subspace used as the source of an edge. Given an integer  $k \in \omega$ , we define the following subspaces of  $\mathbf{R}^\omega$ , for  $\star \in \{>, \geq, =, \neq, \leq, <\}$ :

$$\mathbf{R}_{k\star 0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k \star 0\}.$$

**Definition 8** (Computational graphings). Let  $\alpha : \langle G, \mathbf{R} \rangle \curvearrowright \mathbf{X}$  be an AMC. A *computational  $\alpha$ -graphing* is an  $\alpha$ -graphing  $T$  with distinguished states  $\top, \perp$  which admits a finite representative such that each edge  $e$  has its source equal to one among  $\mathbf{R}^\omega, \mathbf{R}_{k \geq 0}^\omega, \mathbf{R}_{k \leq 0}^\omega, \mathbf{R}_{k > 0}^\omega, \mathbf{R}_{k < 0}^\omega, \mathbf{R}_{k=0}^\omega$ , and  $\mathbf{R}_{k \neq 0}^\omega$ .

**Definition 9** (treeings). Let  $\alpha : \langle G, \mathbf{R} \rangle \curvearrowright \mathbf{X}$  be an AMC. An  *$\alpha$ -treeing* is an acyclic and finite  $\alpha$ -graphing, i.e. an  $\alpha$ -graphing  $F$  for which there exists a finite  $\alpha$ -graphing representative  $T$  whose set of control states  $\mathbf{S}^T = \{0, \dots, s\}$  can be endowed with an order  $<$  such that every edge of  $T$  is state-increasing, i.e. for each edge  $e$  of source  $S_e$ , for all  $x \in S_e$ ,

$$\pi_{\mathbf{S}^T}(\alpha(m_e)(x)) > \pi_{\mathbf{S}^T}(x),$$

where  $\pi_{\mathbf{S}^T}$  denotes the projection onto the control states space.

A *computational  $\alpha$ -treeing* is an  $\alpha$ -treeing  $T$  which is a computational  $\alpha$ -graphing with the distinguished states  $\top, \perp$  being incomparable maximal elements of the state space.

## 3.2 Quantitative Soundness

As mentioned in the introduction, we will use in this paper the property of *quantitative soundness* of the dynamic semantics just introduced. This result is essential, as it connects the time complexity of programs in the model considered (e.g. PRAMS, algebraic computation trees) with the length of the orbits of the considered dynamical system. We here only state quantitative soundness for *computational graphings*, i.e. graphings that have distinguished states  $\top$  and  $\perp$  representing acceptance and rejection respectively. In other words, we consider graphings which compute *decision problems*.

Quantitative soundness is expressed with respect to a translation of machines as graphings, together with a translation of inputs as points of the configuration space. In the following section, these operations are defined for each model of computation considered in this paper. In all these cases, the representation of inputs is straightforward.

**Definition 10.** Let AMC  $\alpha$  be an abstract model of computation, and  $\mathbb{M}$  a model of computation. A *translation* of  $\mathbb{M}$  w.r.t.  $\alpha$  is a pair of maps  $\llbracket \cdot \rrbracket$  which associate to each machine  $M$  in  $\mathbb{M}$  computing a decision problem a computational  $\alpha$ -graphing  $\llbracket M \rrbracket$  and to each input  $\iota$  a point  $\llbracket \iota \rrbracket$  in  $\mathbf{X} \times \mathbf{S}$ .

**Definition 11.** Let AMC  $\alpha$  be an abstract model of computation,  $\mathbb{M}$  a model of computation. The AMC  $\alpha$  is *quantitatively sound* for  $\mathbb{M}$  w.r.t. a translation  $\llbracket \cdot \rrbracket$  if for all machine  $M$  computing a decision problem and input  $\iota$ ,  $M$  accepts  $\iota$  (resp. rejects  $\iota$ ) in  $k$  steps if and only if  $\llbracket M \rrbracket^k(\llbracket \iota \rrbracket) = \top$  (resp.  $\llbracket M \rrbracket^k(\llbracket \iota \rrbracket) = \perp$ ).

## 3.3 The algebraic AMCs

We now define the actions  $\alpha_{\text{full}}$  and  $\alpha_{\mathbf{R}\text{full}}$ . Those will capture all algebraic models of computation considered in this paper, and the main theorem will be stated for this monoid action.

As we intend to consider PRAMS at some point, we consider from the beginning the memory of our machines to be separated in two infinite blocks  $\mathbf{Z}^\omega$ , intended to represent both *shared* and a *private* memory cells<sup>6</sup>.

**Definition 12.** The underlying space of  $\alpha_{\text{full}}$  is  $\mathbf{X} = \mathbf{Z}^{\mathbf{Z}} \cong \mathbf{Z}^\omega \times \mathbf{Z}^\omega$ . The set of generators is defined by their action on the underlying space, writing  $k//n$  the floor  $\lfloor k/n \rfloor$  of  $k/n$  with the convention that  $k//n = 0$  when  $n = 0$ :

<sup>6</sup>Obviously, this could be done without any explicit separation of the underlying space, but this will ease the constructions of the next section.

- $\text{const}_i(c)$  initialises the register  $i$  with the constant  $c \in \mathbf{Z}$ :  $\alpha_{\text{full}}(\text{const}_i(c))(\vec{x}) = (\vec{x}\{x_i := c\})$ ;
- $\star_i(j, k)$  ( $\star \in \{+, -, \times, /\}$ ) performs the algebraic operation  $\star$  on the values in registers  $j$  and  $k$  and store the result in register  $i$ :  $\alpha_{\text{full}}(\star_i(j, k))(\vec{x}) = (\vec{x}\{x_i := x_j \star x_k\})$ ;
- $\star_i^c(j)$  ( $\star \in \{+, -, \times, /\}$ ) performs the algebraic operation  $\star$  on the value in register  $j$  and the constant  $c \in \mathbf{Z}$  and store the result in register  $i$ :  $\alpha_{\text{full}}(\star_i^c(j))(\vec{x}) = (\vec{x}\{x_i := c \star x_j\})$ ;
- $\text{copy}(i, j)$  copies the value stored in register  $j$  in register  $i$ :  $\alpha_{\text{full}}(\text{copy}(i, j))(\vec{x}) = (\vec{x}\{x_i := x_j\})$ ;
- $\text{copy}(\#i, j)$  copies the value stored in register  $j$  in the register whose index is the value stored in register  $i$ :  $\alpha_{\text{full}}(\text{copy}(\#i, j))(\vec{x}) = (\vec{x}\{x_{x_i} := x_j\})$ ;
- $\text{copy}(i, \#j)$  copies the value stored in the register whose index is the value stored in register  $j$  in register  $i$ :  $\alpha_{\text{full}}(\text{copy}(i, \#j))(\vec{x}) = (\vec{x}\{x_i := x_{x_j}\})$ ;
- $\sqrt[n]{i}(j)$  computes the floor of the  $n$ -th root of the value stored in register  $j$  and store the result in register  $i$ :  $\alpha_{\text{full}}(\sqrt[n]{i}(j))(\vec{x}) = (\vec{x}\{x_i := \lfloor \sqrt[n]{x_j} \rfloor\})$ .

We also define the real-valued equivalent, which will be essential for the proof of lower bounds. The corresponding AMC  $\alpha_{\text{Rram}}$  is defined in the same way than the integer-valued one, but with underlying space  $\mathbf{X} = \mathbf{R}^{\mathbf{Z}}$  and with instructions adapted accordingly:

- the division and  $n$ -th root operations are the usual operations on the reals;
- the three copy operators are only effective on integers.

**Definition 13.** The underlying space of  $\alpha_{\text{Rfull}}$  is  $\mathbf{X} = \mathbf{R}^{\mathbf{Z}} \cong \mathbf{R}^{\omega} \times \mathbf{R}^{\omega}$ . The set of generators is defined by their action on the underlying space, with the convention that  $k/n = 0$  when  $n = 0$ :

- $\text{const}_i(c)$  initialises the register  $i$  with the constant  $c \in \mathbf{R}$ :  $\alpha_{\text{Rfull}}(\text{const}_i(c))(\vec{x}) = (\vec{x}\{x_i := c\})$ ;
- $\star_i(j, k)$  ( $\star \in \{+, -, \times, /\}$ ) performs the algebraic operation  $\star$  on the values in registers  $j$  and  $k$  and store the result in register  $i$ :  $\alpha_{\text{Rfull}}(\star_i(j, k))(\vec{x}) = (\vec{x}\{x_i := x_j \star x_k\})$ ;
- $\star_i^c(j)$  ( $\star \in \{+, -, \times, /\}$ ) performs the algebraic operation  $\star$  on the value in register  $j$  and the constant  $c \in \mathbf{R}$  and store the result in register  $i$ :  $\alpha_{\text{Rfull}}(\star_i^c(j))(\vec{x}) = (\vec{x}\{x_i := c \star x_j\})$ ;
- $\text{copy}(i, j)$  copies the value stored in register  $j$  in register  $i$ :  $\alpha_{\text{Rfull}}(\text{copy}(i, j))(\vec{x}) = (\vec{x}\{x_i := x_j\})$ ;
- $\text{copy}(\#i, j)$  copies the value stored in register  $j$  in the register whose index is the floor of the value stored in register  $i$ :  $\alpha_{\text{Rfull}}(\text{copy}(\#i, j))(\vec{x}) = (\vec{x}\{x_{\lfloor x_i \rfloor} := x_j\})$ ;
- $\text{copy}(i, \#j)$  copies the value stored in the register whose index is the floor of the value stored in register  $j$  in register  $i$ :  $\alpha_{\text{Rfull}}(\text{copy}(i, \#j))(\vec{x}) = (\vec{x}\{x_i := x_{\lfloor x_j \rfloor}\})$ ;
- $\sqrt[n]{i}(j)$  computes the  $n$ -th real root of the value stored in register  $j$  and store the result in register  $i$ :  $\alpha_{\text{Rfull}}(\sqrt[n]{i}(j))(\vec{x}) = (\vec{x}\{x_i := \sqrt[n]{x_j}\})$ .

## 4 Algebraic models of computations as AMCs

### 4.1 Algebraic computation trees

The first model considered here will be that of *algebraic computation tree* as defined by Ben-Or [4]. Let us note this model refines the *algebraic decision trees* model of Steele and Yao [30], a model of computation consisting in binary trees for which each branching performs a test w.r.t. a polynomial and each leaf is labelled YES or NO. Algebraic computation trees only allow tests w.r.t. 0, while additional vertices corresponding to algebraic operations can be used to construct polynomials.

**Definition 14** (algebraic computation trees, [4]). An *algebraic computation tree* on  $\mathbf{R}^n$  is a binary tree  $T$  with an function assigning:

- to any vertex  $v$  with only one child (simple vertex) an operational instruction of the form  $f_v = f_{v_i} \star f_{v_j}$ ,  $f_v = c \star f_{v_i}$ , or  $f_v = \sqrt{f_{v_i}}$ , where  $\star \in \{+, -, \times, /\}$ ,  $v_i, v_j$  are ancestors of  $v$  and  $c \in \mathbf{R}$  is a constant;
- to any vertex  $v$  with two children a test instruction of the form  $f_{v_i} \star 0$ , where  $\star \in \{>, =, \geq\}$ , and  $v_i$  is an ancestor of  $v$  or  $f_{v_i} \in \{x_1, \dots, x_n\}$ ;
- to any leaf an output YES or NO.

Let  $W \subseteq \mathbf{R}^n$  be any set and  $T$  be an algebraic computation tree. We say that  $T$  computes the membership problem for  $W$  if for all  $x \in \mathbf{R}^n$ , the traversal of  $T$  following  $x$  ends on a leaf labelled YES if and only if  $x \in W$ .

As algebraic computation trees are *trees*, they will be represented by treeings, i.e.  $\alpha_{\mathbf{R}^{\text{full}}}$ -programs whose set of control states can be ordered so that any edge in the graphing is strictly increasing on its control states component.

**Definition 15.** Let  $T$  be a computational  $\alpha_{\mathbf{R}^{\text{full}}}$ -treeing. The set of inputs  $\text{In}(T)$  (resp. outputs  $\text{Out}(T)$ ) is the set of integers  $k$  (resp.  $i$ ) such that there exists an edge  $e$  in  $T$  satisfying that:

- either  $e$  is realised by one of  $+_i(j, k)$ ,  $+_i(k, j)$ ,  $-_i(j, k)$ ,  $-_i(k, j)$ ,  $\times_i(j, k)$ ,  $\times_i(k, j)$ ,  $/_i(j, k)$ ,  $/_i(k, j)$ ,  $+_i^c(k)$ ,  $-_i^c(k)$ ,  $\times_i^c(k)$ ,  $/_i^c(k)$ ,  $\sqrt[i]{\cdot}(k)$ ;
- or the source of  $e$  is one among  $\mathbf{R}_{k \geq 0}^\omega$ ,  $\mathbf{R}_{k \leq 0}^\omega$ ,  $\mathbf{R}_{k > 0}^\omega$ ,  $\mathbf{R}_{k < 0}^\omega$ ,  $\mathbf{R}_{k=0}^\omega$ , and  $\mathbf{R}_{k \neq 0}^\omega$ .

The *effective input space*  $\mathbf{In}^E(T)$  of an  $\alpha_{\text{act}}$ -treeing  $T$  is defined as the set of indices  $k \in \omega$  belonging to  $\text{In}(T)$  but not to  $\text{Out}(T)$ . The *implicit input space*  $\mathbf{In}^I(T)$  of an  $\alpha_{\text{act}}$ -treeing  $T$  is defined as the set of indices  $k \in \omega$  such that  $k \notin \text{Out}(T)$ .

**Definition 16.** Let  $T$  be an  $\alpha_{\mathbf{R}^{\text{full}}}$ -treeing, and assume that  $1, 2, \dots, n \in \mathbf{In}^I(T)$ . We say that  $T$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps if  $k$  successive iterations of  $T$  restricted to  $\{(x_i)_{i \in \omega} \in \mathbf{R}^\omega \mid \forall 1 \leq i \leq n, x_i = y_i\} \times \{0\}$  reach state  $\top$  if and only if  $(y_1, y_2, \dots, y_n) \in W$ .

*Remark.* Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  be an element of  $\mathbf{R}^n$  and consider two elements  $a, b$  in the subspace  $\{(y_1, \dots, y_n, \dots) \in \mathbf{R}^\omega \mid \forall 1 \geq i \geq n, y_i = x_i\} \times \{0\}$ . One easily checks that  $\pi_{\mathbf{S}}(T^k(a)) = \top$  if and only if  $\pi_{\mathbf{S}}(T^k(b)) = \top$ , where  $\pi_{\mathbf{S}}$  is the projection onto the state space and  $T^k(a)$  represents the  $k$ -th iteration of  $T$  on  $a$ . It is therefore possible to consider only a standard representative  $\llbracket \vec{x} \rrbracket$  of  $\vec{x} \in \mathbf{R}^n$ , for instance  $(x_1, \dots, x_n, 0, 0, \dots) \in \mathbf{R}^\omega$ , to decide whether  $\vec{x}$  is accepted by  $T$ .

**Definition 17.** Let  $T$  be an algebraic computation tree on  $\mathbf{R}^n$ , and  $T^\circ$  be the associated directed acyclic graph, built from  $T$  by merging all the leaves tagged YES in one leaf  $\top$  and all the leaves tagged NO in one leaf  $\perp$ . Suppose the internal vertices are numbered  $\{n+1, \dots, n+\ell\}$ ; the numbers  $1, \dots, n$  being reserved for the input.

We define  $\llbracket T \rrbracket$  as the  $\alpha_{\text{act}}$ -graphing with control states  $\{n+1, \dots, n+\ell, \top, \perp\}$  and where each internal vertex  $i$  of  $T^\circ$  defines either:

- a single edge of source  $\mathbf{R}^\omega$  realized by:
  - $(\star_i(j, k), i \mapsto t)$  ( $\star \in \{+, -, \times\}$ ) if  $i$  is associated to  $f_{v_i} = f_{v_j} \star f_{v_k}$  and  $t$  is the child of  $i$ ;
  - $(\star_i^c(j), i \mapsto t)$  ( $\star \in \{+, -, \times\}$ ) if  $i$  is associated to  $f_{v_i} = c \star f_{v_k}$  and  $t$  is the child of  $i$ ;
- a single edge of source  $\mathbf{R}_{k \neq 0}^\omega$  realized by:
  - $(/_i(j, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = f_{v_j} / f_{v_k}$  and  $t$  is the child of  $i$ ;
  - $(/_i^c(k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = c / f_{v_k}$  and  $t$  is the child of  $i$ ;

- a single edge of source  $\mathbf{R}_{k \geq 0}^\omega \times \{i\}$  realized by  $(\sqrt[2]{i}(k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = \sqrt{f_{v_k}}$  and  $t$  is the child of  $i$ ;
- two edges if  $i$  is associated to  $f_{v_i} \star 0$  (where  $\star$  ranges in  $>, \geq$ ) and its two sons are  $j$  and  $k$ . Those are of respective sources  $\mathbf{R}_{k \star 0}^\omega \times \{i\}$  and  $\mathbf{R}_{k \bar{\star} 0}^\omega \times \{i\}$  (where  $\bar{\star} = '<'$  if  $\star = '>'$ ,  $\bar{\star} = '<'$  if  $\star = '\geq'$ , and  $\bar{\star} = '\neq'$  if  $\star = '='.$ ), respectively realized by  $(\text{Id}, i \mapsto j)$  and  $(\text{Id}, i \mapsto k)$

**Proposition 18.** *Any algebraic computation tree  $T$  of depth  $k$  is faithfully and quantitatively interpreted as the  $\alpha_{\mathbf{R}^{\text{full}}}$ -program  $\llbracket T \rrbracket$ . I.e.  $T$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  if and only if  $\llbracket T \rrbracket$  computes the membership problem for  $W$  in  $k$  steps – that is  $\pi_{\mathbf{S}}(\llbracket T \rrbracket^k(\llbracket \vec{x} \rrbracket)) = \top$ .*

As a corollary of this proposition, we get quantitative soundness.

**Theorem 19.** *The representation of ACTS as  $\alpha_{\mathbf{R}^{\text{full}}}$ -programs is quantitatively sound.*

## 4.2 Algebraic circuits

As we will recover Cucker’s proof that  $\text{NC}_{\mathbf{R}} \neq \text{PTIME}_{\mathbf{R}}$ , we introduce the model of *algebraic circuits* and their representation as  $\alpha_{\mathbf{R}^{\text{full}}}$ -programs.

**Definition 20.** An algebraic circuit over the reals with inputs in  $\mathbf{R}^n$  is a finite directed graph whose vertices have labels in  $\mathbf{N} \times \mathbf{N}$ , that satisfies the following conditions:

- There are exactly  $n$  vertices  $v_{0,1}, v_{0,2}, \dots, v_{0,n}$  with first index 0, and they have no incoming edges;
- all the other vertices  $v_{i,j}$  are of one of the following types:
  1. arithmetic vertex: they have an associated arithmetic operation  $\{+, -, \times, /\}$  and there exist natural numbers  $l, k, r, m$  with  $l, k < i$  such that their two incoming edges are of sources  $v_{l,r}$  and  $v_{k,m}$ ;
  2. constant vertex: they have an associated real number  $y$  and no incoming edges;
  3. sign vertex: they have a unique incoming edge of source  $v_{k,m}$  with  $k < i$ .

We call *depth* of the circuit the largest  $m$  such that there exist a vertex  $v_{m,r}$ , and *size* of the circuit the total number of vertices. A circuit of depth  $d$  is *decisional* if there is only one vertex  $v_{d,r}$  at level  $d$ , and it is a sign vertex; we call  $v_{d,r}$  the *end vertex* of the decisional circuit.

To each vertex  $v$  one inductively associates a function  $f_v$  of the input variables in the usual way, where a sign node with input  $x$  returns 1 if  $x > 0$  and 0 otherwise. The accepted set of a decisional circuit  $C$  is defined as the set  $S \subseteq \mathbf{R}^n$  of points whose image by the associated function is 1, i.e.  $S = f_v^{-1}(\{1\})$  where  $v$  is the end vertex of  $C$ .

We represent algebraic circuit as computational  $\alpha_{\mathbf{R}^{\text{full}}}$ -treeings as follows. The first index in the pairs  $(i, j) \in \mathbf{N} \times \mathbf{N}$  are represented as states, the second index is represented as an index in the infinite product  $\mathbf{R}^\omega$ , and vertices are represented as edges.

**Definition 21.** Let  $C$  be an algebraic circuit, defined as a finite directed graph  $(V, E, s, t, \ell)$  where  $V \subset \mathbf{N} \times \mathbf{N}$ , and  $\ell : V \rightarrow \{\text{init}, +, -, \times, /, \text{sgn}\} \cup \{\text{const}_c \mid c \in \mathbf{R}\}$  is a vertex labelling map. We suppose without loss of generality that for each  $j \in \mathbf{N}$ , there is at most one  $i \in \mathbf{N}$  such that  $(i, j) \in V$ . We define  $N$  as  $\max\{j \in \mathbf{N} \mid \exists i \in \mathbf{N}, (i, j) \in V\}$ .

We define the  $\alpha_{\mathbf{R}^{\text{full}}}$ -program  $\llbracket C \rrbracket$  by choosing as set of control states  $\{i \in \mathbf{N} \mid \exists j \in \mathbf{N}, (i, j) \in V\}$  and the collection of edges  $\{e_{(i,j)} \mid i \in \mathbf{N}^*, j \in \mathbf{N}, (i, j) \in V\} \cup \{e_{(i,j)}^+ \mid i \in \mathbf{N}^*, j \in \mathbf{N}, (i, j) \in V, \ell(v) = \text{sgn}\}$  realised as follows:

- if  $\ell(v) = \text{const}_c$ , the edge  $e_{(i,j)}$  is realised as  $(+^n_j(c), 0 \mapsto i)$  of source  $\mathbf{R}_{n_v=0}^\omega \times \{0\}$ ;

- if  $\ell(v) = \star$  ( $\star \in \{+, -, \times\}$ ) of incoming edges  $(k, l)$  and  $(k', l')$ , the edge  $e_{(i,j)}$  is of source  $\mathbf{R}^\omega \times \{\max(k, k')\}$  and realised by  $(\star_j(l, l'), \max(k, k') \mapsto i)$ ;
- if  $\ell(v) = /$  of incoming edges  $(k, l)$  and  $(k', l')$ , the edge  $e_{(i,j)}$  is of source  $\mathbf{R}_{\ell' \neq 0}^\omega \times \{\max(k, k')\}$  and realised by  $(/j(l, l'), \max(k, k') \mapsto i)$ ;
- if  $\ell(v) = \text{sgn}$  of incoming edge  $(k, l)$ , the edges  $e_{(i,j)}$  and  $e_{(i,j)}^+$  are of respective sources  $\mathbf{R}_{n_v=0 \wedge x_i \leq 0}^\omega \times \{k\}$  and  $\mathbf{R}_{n_v=0 \wedge x_i > 0}^\omega \times \{k\}$  realised by  $(\text{Id}, k \mapsto i)$  and  $(+_j(n_v, 1), k \mapsto i)$  respectively.

As each step of computation in the algebraic circuit is translated as going through a single edge in the corresponding  $\alpha_{\mathbf{R}\text{full}}\text{-program}$ , the following result is straightforward.

**Theorem 22.** *The representation of ALGCIRC as  $\alpha_{\mathbf{R}\text{full}}\text{-programs}$  is quantitatively sound.*

### 4.3 Algebraic RAMS

In this paper, we will consider algebraic parallel random access machines, that act not on strings of bits, but on integers. In order to define those properly, we first define the notion of (sequential) random access machine (RAM) before considering their parallelisation.

A RAM *command* is a pair  $(\ell, I)$  of a *line*  $\ell \in \mathbf{N}^*$  and an *instruction*  $I$  among the following, where  $i, j \in \mathbf{N}$ ,  $\star \in \{+, -, \times, /\}$ ,  $c \in \mathbf{Z}$  is a constant and  $\ell, \ell' \in \mathbf{N}^*$  are lines:

$$\begin{aligned} & \text{skip}; \quad \mathbf{X}_i := c; \quad \mathbf{X}_i := \mathbf{X}_j \star \mathbf{X}_k; \quad \mathbf{X}_i := \mathbf{X}_j; \\ & \mathbf{X}_i := \# \mathbf{X}_j; \quad \# \mathbf{X}_i := \mathbf{X}_j; \quad \text{if } \mathbf{X}_i = 0 \text{ goto } \ell \text{ else } \ell'. \end{aligned}$$

A RAM *machine*  $M$  is then a finite set of commands such that the set of lines is  $\{1, 2, \dots, |M|\}$ , with  $|M|$  the *length* of  $M$ . We will denote the commands in  $M$  by  $(i, \text{Inst}_M(i))$ , i.e.  $\text{Inst}_M(i)$  denotes the line  $i$  instruction.

Following Mulmuley [20], we will here make the assumption that the input in the RAM (and in the PRAM model defined in the next section) is split into *numeric* and *nonnumeric* data – e.g. in the `maxflow` problem the nonnumeric data would specify the network and the numeric data would specify the edge-capacities – and that indirect references use pointers depending only on nonnumeric data<sup>7</sup>. We refer the reader to Mulmuley’s article for more details.

Machines in the RAM model can be represented as graphings w.r.t. the action  $\alpha_{\mathbf{R}\text{full}}$ . Intuitively the encoding works as follows. The notion of *control state* allows to represent the notion of *line* in the program. Then, the action just defined allows for the representation of all commands but the conditionals. The conditionals are represented as follows: depending on the value of  $X_i$  one wants to jump either to the line  $\ell$  or to the line  $\ell'$ ; this is easily modelled by two different edges of respective sources  $\mathbb{H}(i) = \{\vec{x} \mid x_i = 0\}$  and  $\mathbb{H}(i)^c = \{\vec{x} \mid x_i \neq 0\}$ .

**Definition 23.** Let  $M$  be a RAM machine. We define the translation  $\llbracket M \rrbracket$  as the  $\alpha_{\mathbf{R}\text{ram}}\text{-program}$  with set of control states  $\{0, 1, \dots, L, L+1\}$  where each line  $\ell$  defines (in the following,  $\star \in \{+, -, \times\}$  and we write  $\ell++$  the map  $\ell \mapsto \ell + 1$ ):

- a single edge  $e$  of source  $\mathbf{X} \times \{\ell\}$  and realised by:
  - $(\text{Id}, \ell++)$  if  $\text{Inst}_M(\ell) = \text{skip}$ ;
  - $(\text{const}_i(c), \ell++)$  if  $\text{Inst}_M(\ell) = \mathbf{X}_i := c$ ;
  - $(\star_j(j, k), \ell++)$  if  $\text{Inst}_M(\ell) = \mathbf{X}_i := \mathbf{X}_j \star \mathbf{X}_k$ ;
  - $(\text{copy}(i, j), \ell++)$  if  $\text{Inst}_M(\ell) = \mathbf{X}_i := \mathbf{X}_j$ ;
  - $(\text{copy}(i, \#j), \ell++)$  if  $\text{Inst}_M(\ell) = \mathbf{X}_i := \# \mathbf{X}_j$ ;

<sup>7</sup>Quoting Mulmuley: "We assume that the pointer involved in an indirect reference is not some numeric argument in the input or a quantity that depends on it. For example, in the max-flow problem the algorithm should not use an edge-capacity as a pointer—which is a reasonable condition. To enforce this restriction, one initially puts an invalid-pointer tag on every numeric argument in the input. During the execution of an arithmetic instruction, the same tag is also propagated to the result if any operand has that tag. Trying to use a memory value with invalid-pointer tag results in error." [20, Page 1468].



- (copy( $\#i, j$ ),  $\ell_{++}$ ) if  $\text{Inst}_M(\ell) = \#X_i := X_j$ .
- an edge  $e$  of source  $\mathbb{H}(k)^c \times \{\ell\}$  realised by  $(//_i(j, k), \ell_{++})$  if  $\text{Inst}_M(\ell)$  is  $X_i := X_j/X_k$ ;
- a pair of edges  $e, e^c$  of respective sources  $\mathbb{H}(i) \times \{\ell\}$  and  $\mathbb{H}(i)^c \times \{\ell\}$  and realised by respectively  $(\text{Id}, \ell \mapsto \ell^0)$  and  $(\text{Id}, \ell \mapsto \ell^1)$ , if the line is a conditional **if**  $X_i = 0$  **goto**  $\ell^0$  **else**  $\ell^1$ .

The translation  $\llbracket \iota \rrbracket$  of an input  $\iota \in \mathbf{Z}^d$  is the point  $(\bar{\iota}, 0)$  where  $\bar{\iota}$  is the sequence  $(\iota_1, \iota_2, \dots, \iota_k, 0, 0, \dots)$ .

Now, the main result for the representation of RAMs is the following. The proof is straightforward, as each instruction corresponds to exactly one edge, except for the conditional case (but given a configuration, it lies in the source of at most one of the two edges translating the conditional).

**Theorem 24.** *The representation of RAMs as  $\alpha_{\text{full}}$ -programs is quantitatively sound w.r.t. the translation just defined.*

#### 4.4 The Crew operation and PRAMS

Based on the notion of RAM, we are now able to consider their parallelisation, namely PRAMS. A PRAM  $M$  is given as a finite sequence of RAM machines  $M_1, \dots, M_p$ , where  $p$  is the number of *processors* of  $M$ . Each processor  $M_i$  has access to its own, private, set of registers  $(X_k^i)_{k \geq 0}$  and a *shared memory* represented as a set of registers  $(X_k^0)_{k \geq 0}$ .

One has to deal with conflicts when several processors try to access the shared memory simultaneously. We here chose to work with the *Concurrent Read, Exclusive Write* (CREW) discipline: at a given step at which several processors try to write in the shared memory, only the processor with the smallest index will be allowed to do so. In order to model such parallel computations, we abstract the CREW at the level of monoids. For this, we suppose that we have two monoid actions  $M\langle G, R \rangle \curvearrowright \mathbf{X} \times \mathbf{Y}$  and  $M\langle H, Q \rangle \curvearrowright \mathbf{X} \times \mathbf{Z}$ , where  $\mathbf{X}$  represents the shared memory. We then consider the subset  $\# \subset G \times H$  of pairs of generators that potentially conflict with one another – the conflict relation.

**Definition 25** (Conflicted sum). Let  $M\langle G, R \rangle, M\langle G', R' \rangle$  be two monoids and  $\# \subseteq G \times G'$ . The *conflicted sum* of  $M\langle G, R \rangle$  and  $M\langle G', R' \rangle$  over  $\#$ , noted  $M\langle G, R \rangle *_{\#} M\langle G', R' \rangle$ , is defined as the monoid with generators  $(\{1\} \times G) \cup (\{2\} \times G')$  and relations

$$\begin{aligned} & (\{1\} \times R) \cup (\{2\} \times R') \cup \{(\mathbf{1}, e)\} \cup \{(\mathbf{1}, e')\} \\ & \cup \{((1, g)(2, g'), (2, g')(1, g)) \mid (g, g') \notin \#\} \end{aligned}$$

where  $\mathbf{1}, e, e'$  are the units of  $M\langle G, R \rangle *_{\#} M\langle G', R' \rangle, M\langle G, R \rangle$  and  $M\langle G', R' \rangle$  respectively.

In the particular case where  $\# = (G \times H') \cup (H \times G')$ , with  $H, H'$  respectively subsets of  $G$  and  $G'$ , we will write the sum  $M\langle G, R \rangle_{H *_{H'} M\langle G', R' \rangle}$ .

*Remark.* When the conflict relation  $\#$  is empty, this defines the usual direct product of monoids. This corresponds to the case in which no conflicts can arise w.r.t. the shared memory. In other words, the direct product of monoids corresponds to the parallelisation of processes *without shared memory*.

Dually, when the conflict relation is full ( $\# = G \times G'$ ), this defines the free product of the monoids.

**Definition 26.** Let  $\alpha : M \curvearrowright \mathbf{X} \times \mathbf{Y}$  be a monoid action. We say that an element  $m \in M$  is *central relatively to  $\alpha$*  (or just *central*) if the action of  $m$  commutes with the first projection  $\pi_X : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{X}$ , i.e.<sup>8</sup>  $\alpha(m); \pi_X = \alpha(m)$ ; in other words  $m$  acts as the identity on  $\mathbf{X}$ .

Intuitively, central elements are those that will not affect the shared memory. As such, only *non-central elements* require care when putting processes in parallel.

**Definition 27.** Let  $M\langle G, R \rangle \curvearrowright \mathbf{X} \times \mathbf{Y}$  be an AMC. We note  $Z_\alpha$  the set of central elements and  $\bar{Z}_\alpha(G) = \{m \in G \mid n \notin Z_\alpha\}$ .

<sup>8</sup>Here and in the following, we denote by  $;$  the sequential composition of functions. I.e.  $f;g$  denotes what is usually written  $g \circ f$ .

**Definition 28** (The CREW of AMCs). Let  $\alpha : M\langle G, R \rangle \curvearrowright \mathbf{X} \times \mathbf{Y}$  and  $\beta : M\langle H, Q \rangle \curvearrowright \mathbf{X} \times \mathbf{Z}$  be AMCs. We define the AMC  $\text{CREW}(\alpha, \beta) : M\langle G, R \rangle \bar{Z}_\alpha(G) * \bar{Z}_\beta(G') \curvearrowright \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$  by letting  $\text{CREW}(\alpha, \beta)(m, m') = \alpha(m) * \beta(m')$  on elements of  $G \times G'$ , where:

$$\alpha(m) * \beta(m') = \begin{cases} \Delta; [\alpha(m); \pi_Y, \beta(m')] & \text{if } m \notin \bar{Z}_\alpha(G), m' \in \bar{Z}_\beta(G'), \\ \Delta; [\alpha(m), \beta(m'); \pi_Z] & \text{otherwise,} \end{cases}$$

with  $\Delta : \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \rightarrow \mathbf{X} \times \mathbf{Y} \times \mathbf{X} \times \mathbf{Z}; (x, y, z) \mapsto (x, y, x, z)$ .

We can now define AMC of PRAMS and thus the interpretations of PRAMS as abstract programs. For each integer  $p$ , we define the AMC  $\text{CREW}^p(\alpha_{\text{full}})$ . This allows the consideration of up to  $p$  parallel RAMs: the translation of such a RAM with  $p$  processors is defined by extending the translation of RAMs by considering a set of states equal to  $L_1 \times L_2 \times \dots \times L_p$  where for all  $i$  the set  $L_i$  is the set of lines of the  $i$ -th processor.

Now, to deal with arbitrary large PRAMS, i.e. with arbitrarily large number of processors, one considers the following AMC defined as a *direct limit*.

**Definition 29** (The AMC of PRAMS). Let  $\alpha : M \curvearrowright \mathbf{X} \times \mathbf{X}$  be the AMC  $\alpha_{\text{full}}$ . The AMC of PRAMS is defined as  $\alpha_{\text{pram}} = \varinjlim \text{CREW}^k(\alpha)$ , where  $\text{CREW}^{k-1}(\alpha)$  is identified with a restriction of  $\text{CREW}^k(\alpha)$  through  $\text{CREW}^{k-1}(\alpha)(m_1, \dots, m_{k-1}) \mapsto \text{CREW}^k(\alpha)(m_1, \dots, m_{k-1}, 1)$ .

Remark that the underlying space of the PRAM AMC  $\alpha_{\text{pram}}$  is defined as the union  $\cup_{n \in \omega} \mathbf{Z}^\omega \times (\mathbf{Z}^\omega)^n$  which we will write  $\mathbf{Z}^\omega \times (\mathbf{Z}^\omega)^\omega$ . In practise a given  $\alpha_{\text{pram}}$ -program admitting a finite  $\alpha_{\text{pram}}$  representative will only use elements in  $\text{CREW}^p(\alpha_{\text{full}})$ , and can therefore be understood as a  $\text{CREW}^p(\alpha)$ -program.

**Theorem 30.** *The representation of PRAMS as  $\alpha_{\text{pram}}$ -programs is quantitatively sound.*

## 4.5 Real PRAMS

These definitions and results stated for integer-valued PRAMS can be adapted to define *real-valued* PRAMS and their translation as  $\alpha_{\mathbf{R}\text{full}}$ -programs.

A real-valued RAM *command* is a pair  $(\ell, I)$  of a *line*  $\ell \in \mathbf{N}^*$  and an *instruction*  $I$  among the following, where  $i, j \in \mathbf{N}$ ,  $\star \in \{+, -, \times, /\}$ ,  $c \in \mathbf{Z}$  is a constant and  $\ell, \ell' \in \mathbf{N}^*$  are lines:

$$\begin{aligned} & \text{skip}; \quad X_i := c; \quad X_i := X_j \star X_k; \quad X_i := X_j; \\ X_i := \sharp X_j; \quad \sharp X_i := X_j; \quad \text{if } X_i = 0 \text{ goto } \ell \text{ else } \ell'. \end{aligned}$$

We consider a restriction for pointers similar to that considered in the case of integer-valued RAMs. A real-valued RAM *machine*  $M$  is then a finite set of commands such that the set of lines is  $\{1, 2, \dots, |M|\}$ , with  $|M|$  the *length* of  $M$ . We will denote the commands in  $M$  by  $(i, \text{Inst}_M(i))$ , i.e.  $\text{Inst}_M(i)$  denotes the line  $i$  instruction.

A real-valued PRAM  $M$  is given as a finite sequence of real-valued RAM machines  $M_1, \dots, M_p$ , where  $p$  is the number of *processors* of  $M$ . Each processor  $M_i$  has access to its own, private, set of registers  $(X_k^i)_{k \geq 0}$  and a *shared memory* represented as a set of registers  $(X_k^0)_{k \geq 0}$ . Again, we chose to work with the *Concurrent Read, Exclusive Write* (CREW) discipline as it is well translated through the CREW operation of AMCs.

**Definition 31.** Let  $M$  be a real-valued RAM machine. We define the translation  $\llbracket M \rrbracket$  as the  $\alpha_{\mathbf{R}\text{full}}$ -program with set of control states  $\{0, 1, \dots, L, L+1\}$  where each line  $\ell$  defines (in the following,  $\star \in \{+, -, \times\}$  and we write  $\ell++$  the map  $\ell \mapsto \ell+1$ ):

- a single edge  $e$  of source  $\mathbf{X} \times \{\ell\}$  and realised by:
  - $(\text{Id}, \ell++)$  if  $\text{Inst}_M(\ell) = \text{skip}$ ;
  - $(\text{const}_i(c), \ell++)$  if  $\text{Inst}_M(\ell) = X_i := c$ ;
  - $(\star_i(j, k), \ell++)$  if  $\text{Inst}_M(\ell) = X_i := X_j \star X_k$ ;
  - $(\text{copy}(i, j), \ell++)$  if  $\text{Inst}_M(\ell) = X_i := X_j$ ;

- (copy( $i, \#j$ ),  $\ell_{++}$ ) if  $\text{Inst}_M(\ell) = \mathbf{X}_i := \# \mathbf{X}_j$ ;
- (copy( $\#i, j$ ),  $\ell_{++}$ ) if  $\text{Inst}_M(\ell) = \# \mathbf{X}_i := \mathbf{X}_j$ .
- an edge  $e$  of source  $\mathbf{R}_{k \neq 0} \times \{\ell\}$  realised by  $(/_{i(j, k)}, \ell_{++})$  if  $\text{Inst}_M(\ell)$  is  $\mathbf{X}_i := \mathbf{X}_j / \mathbf{X}_k$ ;
- a pair of edges  $e, e^c$  of respective sources  $\mathbf{R}_{i=0} \times \{\ell\}$  and  $\mathbf{R}_{i \neq 0} \times \{\ell\}$  and realised by respectively  $(\text{Id}, \ell \mapsto \ell^0)$  and  $(\text{Id}, \ell \mapsto \ell^1)$ , if the line is a conditional **if**  $\mathbf{X}_i = 0$  **goto**  $\ell^0$  **else**  $\ell^1$ .

The translation  $\llbracket \iota \rrbracket$  of an input  $\iota \in \mathbf{Z}^d$  is the point  $(\bar{\iota}, 0)$  where  $\bar{\iota}$  is the sequence  $(\iota_1, \iota_2, \dots, \iota_k, 0, 0, \dots)$ .

For each integer  $p$ , we then define the AMC  $\text{CREW}^p(\alpha_{\mathbf{R}^{\text{full}}})$ . This allows the consideration of up to  $p$  parallel real-valued RAMs: the translation of such a RAM with  $p$  processors is defined by extending the translation of real-valued RAMs just defined by considering a set of states equal to  $L_1 \times L_2 \times \dots \times L_p$  where for all  $i$  the set  $L_i$  is the set of lines of the  $i$ -th processor.

Since we need to translate arbitrary large real-valued PRAMs, i.e. with arbitrarily large number of processors, one considers the following AMC defined as a *direct limit*.

**Definition 32** (The AMC of real-valued PRAMs). Let  $\alpha : M \curvearrowright \mathbf{X} \times \mathbf{X}$  be the AMC  $\alpha_{\mathbf{R}^{\text{full}}}$ . The AMC of real-valued PRAMs is defined as  $\alpha_{\mathbf{R}^{\text{pram}}} = \varinjlim \text{CREW}^k(\alpha)$ , where  $\text{CREW}^{k-1}(\alpha)$  is identified with a restriction of  $\text{CREW}^k(\alpha)$  through  $\text{CREW}^{k-1}(\alpha)(m_1, \dots, m_{k-1}) \mapsto \text{CREW}^k(\alpha)(m_1, \dots, m_{k-1}, 1)$ .

Then the following results are quite straightforward.

**Theorem 33.** *The representation of real-valued RAMs as  $\alpha_{\mathbf{R}^{\text{full}}}$ -programs is quantitatively sound. The representation of real-valued PRAMs as  $\alpha_{\mathbf{R}^{\text{pram}}}$ -programs is quantitatively sound.*

## 5 Entropy and Cells

### 5.1 Topological Entropy

Topological Entropy was introduced in the context of dynamical systems in an attempt to classify the latter w.r.t. conjugacy. The topological entropy of a dynamical system is a value representing the average exponential growth rate of the number of orbit segments distinguishable with a finite (but arbitrarily fine) precision. The definition is based on the notion of open covers.

**Open covers.** Given a topological space  $\mathbf{X}$ , an *open cover* of  $\mathbf{X}$  is a family  $\mathcal{U} = (U_i)_{i \in I}$  of open subsets of  $\mathbf{X}$  such that  $\cup_{i \in I} U_i = \mathbf{X}$ . A finite cover  $\mathcal{U}$  is a cover whose indexing set is finite. A *subcover* of a cover  $\mathcal{U} = (U_i)_{i \in I}$  is a sub-family  $\mathcal{S} = (U_j)_{j \in J}$  for  $J \subseteq I$  such that  $\mathcal{S}$  is a cover, i.e. such that  $\cup_{j \in J} U_j = \mathbf{X}$ .

We will denote by  $\text{Cov}(\mathbf{X})$  (resp.  $\text{FCov}(\mathbf{X})$ ) the set of all open covers (resp. all finite open covers) of the space  $\mathbf{X}$ .

We now define two operations on open covers that are essential to the definition of entropy. An open cover  $\mathcal{U} = (U_i)_{i \in I}$ , together with a continuous function  $f : \mathbf{X} \rightarrow \mathbf{X}$ , defines the inverse image open cover  $f^{-1}(\mathcal{U}) = (f^{-1}(U_i))_{i \in I}$ . Note that if  $\mathcal{U}$  is finite,  $f^{-1}(\mathcal{U})$  is finite as well. Given two open covers  $\mathcal{U} = (U_i)_{i \in I}$  and  $\mathcal{V} = (V_j)_{j \in J}$ , we define their join  $\mathcal{U} \vee \mathcal{V}$  as the family  $(U_i \cap V_j)_{(i,j) \in I \times J}$ . Once again, if both initial covers are finite, their join is finite.

**Entropy.** Usually, entropy is defined for continuous maps on a compact set, following the original definition by Adler, Konheim and McAndrews [2]. Using the fact that arbitrary open covers have a finite subcover, this allows one to ensure that the smallest subcover of any cover is finite. I.e. given an arbitrary cover  $\mathcal{U}$ , one can consider the smallest – in terms of cardinality – subcover  $\mathcal{S}$  and associate to  $\mathcal{U}$  the finite quantity  $\log_2(\text{Card}(\mathcal{S}))$ . This quantity, obviously, need not be finite in the general case of an arbitrary cover on a non-compact set.

However, a generalisation of entropy to non-compact sets can easily be defined by restricting the usual definition to *finite covers*<sup>9</sup>. This is the definition we will use here.

<sup>9</sup>This is discussed by Hofer [16] together with another generalisation based on the Stone-Ćech compactification of the underlying space.

**Definition 34.** Let  $\mathbf{X}$  be a topological space, and  $\mathcal{U} = (U_i)_{i \in I}$  be a finite cover of  $\mathbf{X}$ . We define the quantity  $H_{\mathbf{X}}^0(\mathcal{U})$  as

$$\min\{\log_2(\text{Card}(J)) \mid J \subset I, \cup_{j \in J} U_j = \mathbf{X}\}.$$

In other words, if  $k$  is the cardinality of the smallest subcover of  $\mathcal{U}$ ,  $H^0(\mathcal{O}) = \log_2(k)$ .

**Definition 35.** Let  $\mathbf{X}$  be a topological space and  $f : \mathbf{X} \rightarrow \mathbf{X}$  be a continuous map. For any finite open cover  $\mathcal{U}$  of  $\mathbf{X}$ , we define:

$$H_{\mathbf{X}}^k(f, \mathcal{U}) = \frac{1}{k} H_{\mathbf{X}}^0(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \dots \vee f^{-(k-1)}(\mathcal{U})).$$

One can show that the limit  $\lim_{n \rightarrow \infty} H_{\mathbf{X}}^n(f, \mathcal{U})$  exists and is finite; it will be noted  $h(f, \mathcal{U})$ . The topological entropy of  $f$  is then defined as the supremum of these values, when  $\mathcal{U}$  ranges over the set of all finite covers  $\text{FCov}(\mathbf{X})$ .

**Definition 36.** Let  $\mathbf{X}$  be a topological space and  $f : \mathbf{X} \rightarrow \mathbf{X}$  be a continuous map. The *topological entropy* of  $f$  is defined as  $h(f) = \sup_{\mathcal{U} \in \text{FCov}(\mathbf{X})} h(f, \mathcal{U})$ .

## 5.2 Graphings and Entropy

We now need to define the entropy of *deterministic graphing*. As mentioned briefly already, deterministic graphings on a space  $\mathbf{X}$  are in one-to-one correspondence with partial dynamical systems on  $\mathbf{X}$ . To convince oneself of this, it suffices to notice that any partial dynamical system can be represented as a graphing with a single edge, and that if the graphing  $G$  is deterministic its edges can be glued together to define a partial continuous function  $[G]$ . Thus, we only need to extend the notion of entropy to partial maps, and we can then define the entropy of a graphing  $G$  as the entropy of its corresponding map  $[G]$ .

Given a finite cover  $\mathcal{U}$ , the only issue with partial continuous maps is that  $f^{-1}(\mathcal{U})$  is not in general a cover. Indeed,  $\{f^{-1}(U) \mid U \in \mathcal{U}\}$  is a family of open sets by continuity of  $f$  but the union  $\cup_{U \in \mathcal{U}} f^{-1}(U)$  is a strict subspace of  $\mathbf{X}$  (namely, the domain of  $f$ ). It turns out the solution to this problem is quite simple: we notice that  $f^{-1}(\mathcal{U})$  is a cover of  $f^{-1}(\mathbf{X})$  and now work with covers of subspaces of  $\mathbf{X}$ . Indeed,  $\mathcal{U} \vee f^{-1}(\mathcal{U})$  is itself a cover of  $f^{-1}(\mathbf{X})$  and therefore the quantity  $H_{\mathbf{X}}^2(f, \mathcal{U})$  can be defined as  $(1/2)H_{f^{-1}(\mathbf{X})}^0(\mathcal{U} \vee f^{-1}(\mathcal{U}))$ .

We now generalise this definition to arbitrary iterations of  $f$  by extending Definitions 35 and 36 to partial maps as follows.

**Definition 37.** Let  $\mathbf{X}$  be a topological space and  $f : \mathbf{X} \rightarrow \mathbf{X}$  be a continuous partial map. For any finite open cover  $\mathcal{U}$  of  $\mathbf{X}$ , we define:

$$H_{\mathbf{X}}^k(f, \mathcal{U}) = \frac{1}{k} H_{f^{-k+1}(\mathbf{X})}^0(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \dots \vee f^{-(k-1)}(\mathcal{U})).$$

The *entropy* of  $f$  is then defined as  $h(f) = \sup_{\mathcal{U} \in \text{FCov}(\mathbf{X})} h(f, \mathcal{U})$ , where  $h(f, \mathcal{U})$  is again defined as the limit  $\lim_{n \rightarrow \infty} H_{\mathbf{X}}^n(f, \mathcal{U})$ .

Now, let us consider the special case of a graphing  $G$  with set of control states  $S^G$ . For an intuitive understanding, one can think of  $G$  as the representation of a PRAM machine. We focus on the specific open cover indexed by the set of control states, i.e.  $\mathcal{S} = (\mathbf{X} \times \{s\}_{s \in S^G})$ , and call it *the states cover*. We will now show how the partial entropy  $H^k(G, \mathcal{S})$  is related to the set of *admissible sequence of states*. Let us define those first.

**Definition 38.** Let  $G$  be a graphing, with set of control states  $S^G$ . An admissible sequence of states is a sequence  $\mathbf{s} = s_1 s_2 \dots s_n$  of elements of  $S^G$  such that for all  $i \in \{1, 2, \dots, n-1\}$  there exists a subset  $C$  of  $\mathbf{X}$  – i.e. a set of configurations – such that  $G$  contains an edge from  $C \times \{s_i\}$  to a subspace of  $\mathbf{X} \times \{s_{i+1}\}$ .

*Example 39.* As an example, let us consider the very simple graphing with four control states  $a, b, c, d$  and edges from  $\mathbf{X} \times \{a\}$  to  $\mathbf{X} \times \{b\}$ , from  $\mathbf{X} \times \{b\}$  to  $\mathbf{X} \times \{c\}$ , from  $\mathbf{X} \times \{c\}$  to  $\mathbf{X} \times \{b\}$  and from  $\mathbf{X} \times \{c\}$  to  $\mathbf{X} \times \{d\}$ . Then the sequences  $abcd$  and  $abcbbc$  are admissible, but the sequences  $aba$ ,  $abcd$ , and  $abca$  are not.

**Lemma 40.** *Let  $G$  be a graphing, and  $\mathcal{S}$  its states cover. Then for all integer  $k$ , the set  $\text{Adm}_k(G)$  of admissible sequences of states of length  $k > 1$  is of cardinality  $2^{k \cdot H^k(G, \mathcal{S})}$ .*

*Proof.* We show that the set  $\text{Adm}_k(G)$  of admissible sequences of states of length  $k$  has the same cardinality as the smallest subcover of  $\mathcal{S} \vee [G]^{-1}(\mathcal{S}) \vee \dots \vee [G]^{-(k-1)}(\mathcal{S})$ . Hence  $H^k(G, \mathcal{S}) = \frac{1}{k} \log_2(\text{Card}(\text{Adm}_k(G)))$ , which implies the result.

The proof is done by induction. As a base case, let us consider the set of  $\text{Adm}_2(G)$  of admissible sequences of states of length 2 and the open cover  $\mathcal{V} = \mathcal{S} \vee [G]^{-1}(\mathcal{S})$  of  $D = [G]^{-1}(\mathbf{X})$ . An element of  $\mathcal{V}$  is an intersection  $\mathbf{X} \times \{s_1\} \cap [G]^{-1}(\mathbf{X} \times \{s_2\})$ , and it is therefore equal to  $C[s_1, s_2] \times \{s_1\}$  where  $C[s_1, s_2] \subset \mathbf{X}$  is the set  $\{x \in \mathbf{X} \mid [G](x, s_1) \in \mathbf{X} \times \{s_2\}\}$ . This set is empty if and only if the sequence  $s_1 s_2$  belongs to  $\text{Adm}_2(G)$ . Moreover, given another sequence of states  $s'_1 s'_2$  (not necessarily admissible), the sets  $C[s_1, s_2]$  and  $C[s'_1, s'_2]$  are disjoint. Hence a set  $C[s_1, s_2]$  is *removable from the cover*  $\mathcal{V}$  if and only if the sequence  $s_1 s_2$  is not admissible. This implies the result for  $k = 2$ .

The step for the induction is similar to the base case. It suffices to consider the partition  $\mathcal{S}_k = \mathcal{S} \vee [G]^{-1}(\mathcal{S}) \vee \dots \vee [G]^{-(k-1)}(\mathcal{S})$  as  $\mathcal{S}_{k-1} \vee [G]^{-(k-1)}(\mathcal{S})$ . By the same argument, one can show that elements of  $\mathcal{S}_{k-1} \vee [G]^{-(k-1)}(\mathcal{S})$  are of the form  $C[\mathbf{s} = (s_0 s_1 \dots s_{k-1}), s_k] \times \{s_1\}$  where  $C[\mathbf{s}, s_k] \subset \mathbf{X}$  is the set  $\{x \in \mathbf{X} \mid \forall i = 2, \dots, k, [G]^{i-1}(x, s_1) \in \mathbf{X} \times \{s_i\}\}$ . Again, these sets  $C[\mathbf{s}, s_k]$  are pairwise disjoint and empty if and only if the sequence  $s_0 s_1 \dots s_{k-1}, s_k$  is not admissible.  $\square$

A tractable bound on the number of admissible sequences of states can be obtained by noticing that the sequence  $H^k(G, \mathcal{S})$  is *sub-additive*, i.e.  $H^{k+k'}(G, \mathcal{S}) \leq H^k(G, \mathcal{S}) + H^{k'}(G, \mathcal{S})$ . A consequence of this is that  $H^k(G, \mathcal{S}) \leq k H^1(G, \mathcal{S})$ . Thus the number of admissible sequences of states of length  $k$  is bounded by  $2^{k^2 H^1(G, \mathcal{S})}$ . We now study how the cardinality of admissible sequences can be related to the entropy of  $G$ .

**Lemma 41.** *For all  $\epsilon > 0$ , there exists an integer  $N$  such that for all  $k \geq N$ ,  $H^k(G, \mathcal{U}) < h([G]) + \epsilon$ .*

*Proof.* Let us fix some  $\epsilon > 0$ . Notice that if we let  $H_k(G, \mathcal{U}) = H^0(\mathcal{U} \vee [G]^{-1}(\mathcal{U}) \vee \dots \vee [G]^{-(k-1)}(\mathcal{U}))$ , the sequence  $H_k(\mathcal{U})$  satisfies  $H_{k+l}(\mathcal{U}) \leq H_k(\mathcal{U}) + H_l(\mathcal{U})$ . By Fekete's lemma on subadditive sequences, this implies that  $\lim_{k \rightarrow \infty} H_k/k$  exists and is equal to  $\inf_k H_k/k$ . Thus  $h([G], \mathcal{U}) = \inf_k H_k/k$ .

Now, the entropy  $h([G])$  is defined as  $\sup_{\mathcal{U}} \lim_{k \rightarrow \infty} H_k(\mathcal{U})/k$ . This then rewrites as  $\sup_{\mathcal{U}} \inf_k H_k(\mathcal{U})/k$ . We can conclude that  $h([G]) \geq \inf_k H_k(\mathcal{U})/k$  for all finite open cover  $\mathcal{U}$ .

Since  $\inf_k H_k(\mathcal{U})/k$  is the limit of the sequence  $H_k/k$ , there exists an integer  $N$  such that for all  $k \geq N$  the following inequality holds:  $|H_k(\mathcal{U})/k - \inf_k H_k(\mathcal{U})/k| < \epsilon$ , which rewrites as  $H_k(\mathcal{U})/k - \inf_k H_k(\mathcal{U})/k < \epsilon$ . From this we deduce  $H_k(\mathcal{U})/k < h([G]) + \epsilon$ , hence  $H^k(G, \mathcal{U}) < h([G]) + \epsilon$  since  $H^k(G, \mathcal{U}) = H_k(G, \mathcal{U})$ .  $\square$

**Lemma 42.** *Let  $G$  be a graphing, and let  $c : k \mapsto \text{Card}(\text{Adm}_k(G))$ . Then  $c(k) = O(2^{k \cdot h([G])})$  as  $k$  goes to infinity.*

### 5.3 Cells Decomposition

Now, let us consider a deterministic graphing  $G$ , with its state cover  $\mathcal{S}$ . We fix a length  $k > 2$  and reconsider the sets  $C[\mathbf{s}] = C[(s_1 s_2 \dots s_{k-1}), s_k]$  (for a sequence of states  $\mathbf{s} = s_1 s_2 \dots s_k$ ) that appear in the proof of Lemma 40. The set  $(C[\mathbf{s}])_{\mathbf{s} \in \text{Adm}_k(G)}$  is a partition of the space  $[G]^{-k+1}(\mathbf{X})$ .

This decomposition splits the set of initial configurations into cells satisfying the following property: *for any two initial configurations contained in the same cell  $C[\mathbf{s}]$ , the  $k$ -th first iterations of  $G$  goes through the same admissible sequence of states  $\mathbf{s}$ .*

**Definition 43.** Let  $G$  be a deterministic graphing, with its state cover  $\mathcal{S}$ . Given an integer  $k$ , we define the  $k$ -fold decomposition of  $\mathbf{X}$  along  $G$  as the partition  $\{C[\mathbf{s}] \mid \mathbf{s} \in \text{Adm}_k(G)\}$ .

Then Lemma 40 provides a bound on the cardinality of the  $k$ -th cell decomposition. Using the results in the previous section, we can then obtain the following proposition.

**Proposition 44.** *Let  $G$  be a deterministic graphing, with entropy  $h(G)$ . The cardinality of the  $k$ -th cell decomposition of  $\mathbf{X}$  w.r.t.  $G$ , as a function  $c(k)$  of  $k$ , is asymptotically bounded by  $g(k) = 2^{k \cdot h(G)}$ , i.e.  $c(k) = O(g(k))$ .*

We also state another bound on the number of cells of the  $k$ -th cell decomposition, based on the state cover entropy, i.e. the entropy with respect to the state cover rather than the usual entropy which takes the supremum of cover entropies when the cover ranges over all finite covers of the space. This result is a simple consequence of Theorem 40.

**Proposition 45.** *Let  $G$  be a deterministic graphing. We consider the state cover entropy  $h_0([G]) = \lim_{n \rightarrow \infty} H_{\mathbf{X}}^n([G], \mathcal{S})$  where  $\mathcal{S}$  is the state cover. The cardinality of the  $k$ -th cell decomposition of  $\mathbf{X}$  w.r.t.  $G$ , as a function  $c(k)$  of  $k$ , is asymptotically bounded by  $g(k) = 2^{k \cdot h_0([G])}$ , i.e.  $c(k) = O(g(k))$ .*

## 6 First lower bounds

We will now explain how to obtain lower bounds for algebraic models of computation based on the interpretation of programs as graphings and entropic bounds. These results make use of the Milnor-Thom theorem which bounds the sum of the Betti numbers of algebraic varieties. In fact, we will use a version due to Ben-Or of this theorem.

### 6.1 Milnor-Thom theorem

Let us first recall the classic Milnor-Thom theorem.

**Theorem 46** ([19, Theorem 3]). *If  $X \subseteq \mathbf{R}^m$  is defined by polynomial identities of the form*

$$f_1 \geq 0, \dots, f_p \geq 0$$

*with total degree  $d = \deg f_1 + \dots + \deg f_p$ , then*

$$\text{rank } H^* X \leq \frac{1}{2}(2+d)(1+d)^{m-1}.$$

We will use in the proof the following variant of the Milnor-Thom bounds, stated and proved by Ben-Or.

**Theorem 47.** *Let  $V \subseteq \mathbf{R}^n$  be a set defined by polynomial in-equations ( $n, m, h \in \mathbf{N}$ ):*

$$\left\{ \begin{array}{l} q_1(x_1, \dots, x_n) = 0 \\ \vdots \\ q_m(x_1, \dots, x_n) = 0 \\ p_1(x_1, \dots, x_n) > 0 \\ \vdots \\ p_s(x_1, \dots, x_n) > 0 \\ p_{s+1}(x_1, \dots, x_n) \geq 0 \\ \vdots \\ p_h(x_1, \dots, x_n) \geq 0 \end{array} \right.$$

*for  $p_i, q_i \in \mathbf{R}[X_1, \dots, X_n]$  of degree lesser than  $d$ .*

*Then  $\beta_0(V)$  is at most  $d(2d-1)^{n+h-1}$ , where  $d = \max\{2, \deg(q_i), \deg(p_j)\}$ .*

## 6.2 Algebraic decision trees

From Theorem 44, one obtains easily the following theorem.

**Theorem 48.** *Let  $T$  be a  $d$ -th order algebraic decision tree deciding a subset  $W \subseteq \mathbf{R}^n$ . Then the number of connected components of  $W$  is bounded by  $2^h d(2d-1)^{n+h-1}$ , where  $h$  is the height of  $T$ .*

*Proof.* We let  $h$  be the height of  $T$ , and  $d$  be the maximal degree of the polynomials appearing in  $T$ . Then the  $h$ -th cell decomposition of  $[T]$  defines a family of semi-algebraic sets defined by  $h$  polynomials equalities and inequalities of degree at most  $d$ . By theorem 47, each of the cells have at most  $d(2d-1)^{n+h-1}$  connected components. Moreover, Theorem 45 states that this family has cardinality bounded by  $2^{h \cdot h_0([T])}$ ; since  $h_0([T]) = 1$  because each state has at most one antecedent state, this bound becomes  $2^h$ . Thus, the  $h$ -th cell decomposition defines at most  $2^h$  algebraic sets which have at most  $d(2d-1)^{n+h-1}$  connected components. Since the set  $W$  decided by  $T$  is obtained as a union of the semi-algebraic sets in the  $h$ -th cell decomposition, it has at most  $2^h d(2d-1)^{n+h-1}$  connected components.  $\square$

**Corollary 49** (Steele and Yao [30]). *A  $d$ -th order algebraic decision tree deciding a subset  $W \subseteq \mathbf{R}^n$  with  $N$  connected components has height  $\Omega(\log N)$ .*

This result of Steele and Yao adapts in a straightforward manner to a notion of algebraic computation trees describing the construction of the polynomials to be tested by mean of multiplications and additions of the coordinates. The authors remarked this result uses techniques quite similar to that of Mulmuley's lower bounds for the model of PRAMS *without bit operations*. It is also strongly similar to the techniques used by Cucker in proving that  $\text{NC}_{\mathbf{R}} \subsetneq \text{PTIME}_{\mathbf{R}}$  [10].

However, a refinement of Steele and Yao's method was quickly obtained by Ben-Or so as to allow for computing divisions and taking square roots in this notion of algebraic computation trees. In the next section, we explain Ben-Or techniques from within the framework of graphings through the introduction of *entropic co-trees*. We first explain how the present approach already captures Mulmuley's proof of lower bounds for PRAMS *without bit operations*, which we will later strengthen using entropic co-trees.

## 6.3 Mulmuley's result

At this point, we are already capable of recovering Mulmuley's proof of lower bounds for PRAMS *without bit operations* [20]. The gist of the proof is to notice that given a PRAM  $M$  over integers *not using* the instructions  $//(\cdot, \cdot)$  or  $\sqrt{\cdot}(\cdot)$ , one can define a real-valued PRAM  $\tilde{M}$  *not using* the instructions  $/(\cdot, \cdot)$  or  $\sqrt{\cdot}(\cdot)$  such that:

$M$  accepts an integer-valued point  $\vec{x}$  in  $k$  steps if and only if  $\tilde{M}$  accepts  $\vec{x}$  in  $k$  steps.

In particular, the subset  $W \subseteq \mathbf{Z}^n$  decided by  $M$  is contained in the subset  $\bar{W} \subseteq \mathbf{R}^n$  decided by  $\bar{M}$ , and  $W^c \subseteq \bar{W}^c$  – where the complement of  $W$  is taken in  $\mathbf{Z}^n$  and the complement of  $\bar{W}$  is taken in  $\mathbf{R}^n$ . Moreover, the number of steps needed by  $\bar{M}$  to decide if  $\vec{x}$  belongs to  $\bar{W}$  is equal to the number of steps needed by  $M$  to decide if  $\vec{x}$  belongs to  $W$ .

The proof of this is straightforward, as each instructions available in the integer-valued PRAMS model coincide with the restriction to integer values of an instruction available in the integer-valued PRAMS model. We can then prove the following result.

**Theorem 50.** *A set  $V$  decided by a PRAM without bit operations  $M$  with  $p$  processors in  $k$  steps is described by a set of in-equations of total degree<sup>10</sup> bounded by  $pk \cdot 2^{O(hp)}$ .*

*Proof.* Now, one supposes that  $M$  computes `maxflow` in  $k$  steps. Then it is computed by  $\bar{M}$  in  $k$  steps. Applying proposition 44 on the translation of  $\bar{M}$  we obtain that the  $k$ -th cell decomposition of  $[[\bar{M}]]$  contains at most  $2^{h \cdot h_0([T])}$  cells. Moreover, each cell is described by a system of at most  $pk$  polynomial in-equations of degree at most 2. Thus, the whole set decided by  $\bar{M}$  is described by at most  $pk \cdot 2^{h \cdot h_0([T])}$  in-equations

<sup>10</sup>The sum of the degrees of the polynomials defining  $V$ .

of degree at most 2. Lastly, one can check that  $h_0([T])$  grows linearly w.r.t. the number of processors, following the product rule for topological entropy which states that the entropy of a product  $f \times g$  satisfies  $h(f \times g) \leq h(f) + h(g)$  [14].  $\square$

The second part of the proof, which does not differ from Mulmuley's argument, is detailed and reformulated in Section 9. We only provide a quick sketch here.

**Theorem 51.** *Let  $M$  be a PRAM without bit operations, with at most  $2^{O((\log N)^c)}$  processors, where  $N$  is the length of the inputs and  $c$  any positive integer.*

*Then  $M$  does not decide `maxflow` in  $O((\log N)^c)$  steps.*

*Sketch.* The crux is the obtention of Theorem 75 which, combined with Theorem 68, implies that there exists a polynomial  $P$  such that no surface of total degree  $\delta$  can separate the integer points defined by `maxflow` as long as  $2^\Omega(n) > P(\delta)$ . Hence, if we suppose that  $M$  is such that both  $k$  and  $p$  are  $O((\log^k(n))^c)$  (for any positive integers  $k$  and  $c$ ), the previous result implies that the total degree  $\delta$  of the set decided by  $\bar{M}$  is at most  $O(2^{pk})$ . Hence  $2^\Omega(n) > P(\delta)$  for any polynomial  $P$ , and we conclude that  $\bar{M}$  does not compute a set separating the integer points defined by `maxflow`, hence  $M$  does not decide `maxflow`.  $\square$

## 7 Refining the method

It is not a surprise then that similar bounds to that of algebraic decisions trees can be computed using similar methods in the restricted fragment without division and square roots. An improvement on this is the result of Ben-Or generalising the technique to algebraic computation trees with division and square root nodes. The principle is quite simple: one simply adds additional variables to avoid using the square root or division, obtaining in this way a system of polynomial equations. For instance, instead of writing the equation  $p/q < 0$ , one defines a fresh variable  $r$  and considers the system

$$p = qr; r < 0$$

This method seems different from the direct entropy bound obtained in the case of algebraic decision trees. However, we will see how it can be adapted directly to graphings.

### 7.1 Entropic co-trees and $k$ -th computational forests

**Definition 52** ( $k$ -th entropic co-tree). Consider a deterministic graphing representative  $T$ , and fix an element  $\top$  of the set of control states. We can define the  $k$ -th entropic co-tree of  $T$  along  $\top$  and the state cover inductively:

- $k = 0$ , the co-tree  $\text{coT}_0(T)$  is simply the root  $n^\epsilon = \mathbf{R}^n \times \{\top\}$ ;
- $k = 1$ , one considers the preimage of  $n^\epsilon$  through  $T$ , i.e.  $T^{-1}(\mathbf{R}^n \times \{\top\})$  the set of all non-empty sets  $\alpha(m_e)^{-1}(\mathbf{R}^n \times \{\top\})$  and intersects it pairwise with the state cover, leading to a finite family (of cardinality bounded by the number of states multiplied by the number of edges of  $T$ )  $(n_e^i)_i$  defined as  $n^i = T^{-1}(n^\epsilon) \cap \mathbf{R}^n \times \{i\}$ . The first entropic co-tree  $\text{coT}_1(T)$  of  $T$  is then the tree defined by linking each  $n_e^i$  to  $n^\epsilon$  with an edge labelled by  $m_e$ ;
- $k + 1$ , suppose defined the  $k$ -th entropic co-tree of  $T$ , defined as a family of elements  $n_e^\pi$  where  $\pi$  is a finite sequence of states of length at most  $k$  and  $\mathbf{e}$  a sequence of edges of  $T$  of the same length, and where  $n_e^\pi$  and  $n_e^{\pi'}$  are linked by an edge labelled  $f$  if and only if  $\pi' = \pi.s$  and  $\mathbf{e}' = f.\mathbf{e}$  where  $s$  is a state and  $f$  an edge of  $T$ . We consider the subset of elements  $n_e^\pi$ , where  $\pi$  is exactly of length  $k$ , and for each such element we define new nodes  $n_{e,\mathbf{e}'}^{\pi,s}$  defined as  $\alpha(m_e)^{-1}(n_e^\pi) \cap \mathbf{R}^n \times \{s\}$  when it is non-empty. The  $k + 1$ -th entropic co-tree  $\text{coT}_{k+1}(T)$  is defined by extending the  $k$ -th entropic co-tree  $\text{coT}_k(T)$ , adding the nodes  $n_{e,\mathbf{e}'}^{\pi,s}$  and linking them to  $n_e^\pi$  with an edge labelled by  $e$ .



*Remark.* The co-tree can alternatively be defined non-inductively in the following way: the  $n_{\mathbf{e}}^{\pi}$  for  $\pi$  is a finite sequence of states and  $\mathbf{e}$  a sequence of edges of  $T$  of the same length by  $n_{\mathbf{e}}^{\epsilon} = \mathbf{R}^n \times \{\top\}$  and

$$n_{\mathbf{e}.e}^{\pi.s} = [\alpha(m_e)^{-1}(n_{\mathbf{e}}^{\pi})] \cap [\mathbf{R}^n \times \{s\}]$$

The  $k$ -th entropic co-tree of  $T$  along  $\top$  has as vertices the non-empty sets  $n_{\mathbf{e}}^{\pi}$  for  $\pi$  and  $\mathbf{e}$  of length at most  $k$  and as only edges, links  $n_{\mathbf{e}.e}^{\pi.s} \rightarrow n_{\mathbf{e}}^{\pi}$  labelled by  $m_e$ .

This definition formalises a notion that appears more or less clearly in the work of Lipton and Steele, and of Ben-Or, as well as in the proof by Mulmuley. The nodes for paths of length  $k$  in the  $k$ -th co-tree corresponds to the  $k$ -th cell decomposition, and the corresponding path defines the polynomials describing the semi-algebraic set decided by a computational tree. The co-tree can be used to reconstruct the algebraic computation tree  $T$  from the graphing representative  $[T]$ , or constructs *some* algebraic computation tree (actually a forest) that approximates the computation of the graphing  $F$  under study when the latter is not equal to  $[T]$  for some tree  $T$ .

**Definition 53** ( $k$ -th computational forest). Consider a deterministic graphing  $T$ , and fix an element  $\top$  of the set of control states. We define the  $k$ -th computational forest of  $T$  along  $\top$  and the state cover as follows. Let  $\text{coT}_k(T)$  be the  $k$ -th entropic co-tree of  $T$ . The  $k$ -th computational forest of  $T$  is defined by regrouping all elements  $n_{\mathbf{e}.e}^{\pi}$  of length  $m$ : if the set  $N_e^m = \{n_{\mathbf{e}.e}^{\pi} \in \text{coT}_k(T) \mid \text{len}(\pi) = m\}$  is non-empty it defines a new node  $N_e^m$ . Then one writes down an edge from  $N_e^m$  to  $N_{e'}^{m-1}$ , labelled by  $e$ , if and only if there exists  $n_{\mathbf{e}.e'.f}^{s.\pi} \in N_e^m$  such that  $n_{\mathbf{e}'.f}^{\pi} \in N_{e'}^{m-1}$ .

One checks easily that the  $k$ -th computational forest is indeed a forest: an edge can exist between  $N_e^m$  and  $N_f^n$  only when  $n = m + 1$ , a property that forbids cycles. The following proposition shows how the  $k$ -th computational forest is linked to computational trees.

**Proposition 54.** *If  $T$  is a computational tree of depth  $k$ , the  $k$ -th computational forest of  $[T]$  is a tree which defines straightforwardly a graphing (treeing) representative of  $T$ .*

We now state and prove an easy bound on the size of the entropic co-trees.

**Proposition 55** (Size of the entropic co-trees). *Let  $T$  be a graphing representative,  $E$  its set of edges, and  $\text{Seq}_k(E)$  the set of length  $k$  sequences of edges in  $T$ . The number of nodes of its  $k$ -th entropic co-tree  $\text{coT}_k(T)$ , as a function  $n(k)$  of  $k$ , is asymptotically bounded by  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{(k+1) \cdot h([G])}$ , itself bounded by  $2^{\text{Card}(E)} \cdot 2^{(k+1) \cdot h([G])}$ .*

*Proof.* For a fixed sequence  $\vec{e}$ , the number of elements  $n_{\vec{e}}^{\pi}$  of length  $m$  in  $\text{coT}_k(T)$  is bounded by the number of elements in the  $m$ -th cell decomposition of  $T$ , and is therefore bounded by  $g(m) = 2^{m \cdot h([T])}$  by Theorem 44. The number of sequences  $\vec{e}$  is bounded by  $\text{Card}(\text{Seq}_k(E))$  and therefore the size of  $\text{coT}_k(T)$  is thus bounded by  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{(k+1) \cdot h([T])}$ .  $\square$

From the proof, one sees that the following variant of Theorem 45 holds.

**Proposition 56.** *Let  $G$  be a deterministic graphing with a finite set of edges  $E$ , and  $\text{Seq}_k(E)$  the set of length  $k$  sequences of edges in  $G$ . We consider the state cover entropy  $h_0([G]) = \lim_{n \rightarrow \infty} H_{\mathbf{X}}^n([G], \mathcal{S})$  where  $\mathbf{S}$  is the state cover. The cardinality of the length  $k$  nodes of the entropic co-tree of  $G$ , as a function  $c(k)$  of  $k$ , is asymptotically bounded by  $g(k) = \text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0([G])}$ , which is itself bounded by  $2^{\text{Card}(E)} \cdot 2^{k \cdot h_0([G])}$ .*

## 7.2 The technical lemma

This definition formalises a notion that appears more or less clearly in the work of Steele and Yao, and of Ben-Or, as well as in the proof by Mulmuley. The vertices for paths of length  $k$  in the  $k$ -th co-tree corresponds to the  $k$ -th cell decomposition, and the corresponding path defines the polynomials describing the semi-algebraic set decided by a computational tree. While in Steele and Yao and Mulmuley's proofs, one

obtain directly a polynomial for each cell, we here need to construct a system of equations for each branch of the co-tree.

Given a  $\text{CREW}^p(\alpha_{\mathbf{R}_{\text{full}}})$ -graphing representative  $G$  we will write  $\sqrt[p]{G}$  the maximal value of  $n$  for which an instruction  $\sqrt[p]{i}(j)$  appears in the realiser of an edge of  $G$ .

The proof of this theorem is long but simple to understand as it follows Ben-Or's method. We define, for each vertex of the  $k$ -th entropic co-tree, a system of algebraic equations (each of degree at most 2). The system is defined by induction on  $k$ , and uses the information of the specific instruction used to extend the sequence indexing the vertex at each step. For instance, the case of division follows Ben-Or's method, introducing a fresh variable and writing down two equations as explained in Section ??.

**Lemma 57.** *Let  $G$  be a computational graphing representative with edges realised only by generators of the AMC  $\text{CREW}^p(\alpha_{\mathbf{R}_{\text{full}}})$ , and  $\text{Seq}_k(E)$  the set of length  $k$  sequences of edges in  $G$ . Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps, i.e. for each element of  $\mathbf{R}^n$ ,  $\pi_{\mathbf{S}}(G^k(x)) = \top$  if and only if  $x \in W$ . Then  $W$  is a semi-algebraic set defined by at most  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot \text{ho}([G])}$  systems of  $pk$  equations of degree at most  $\max(2, \sqrt[p]{G})$  and involving at most  $pk + n$  variables.*

*Proof.* If  $G$  computes the membership problem for  $W$  in  $k$  steps, it means  $W$  can be described as the union of the subspaces corresponding to the nodes  $n_{\mathbf{e}}^{\pi}$  with  $\pi$  of length  $k$  in  $\text{COT}_k(T)$ . Now, each such subspace is an algebraic set, as it can be described by a set of polynomials as follows.

Finally let us note that, as in Mulmuley's work [20], since in our model the memory pointers are allowed to depend only on the nonnumeric parameters, indirect memory instructions can be treated as standard – direct – memory instructions. In other words, whenever an instruction involving a memory pointer is encountered during the course of execution, the value of the pointer is completely determined by nonnumerical data, and the index of the involved registers is completely determined, independently of the numerical inputs.

We define a system of equations  $(E_i^{\mathbf{e}})_i$  for each node  $n_{\mathbf{e}}^{\pi}$  of the entropic co-tree  $\text{COT}_k(T)$ . We explicit the construction for the case  $p = 1$ , i.e. for the AMC  $\text{CREW}^1(\alpha_{\mathbf{R}_{\text{full}}}) = \alpha_{\mathbf{R}_{\text{full}}}$ ; the case for arbitrary  $p$  is then dealt with by following the construction and introducing  $p$  equations at each step (one for each of the  $p$  instructions in  $\alpha_{\mathbf{R}_{\text{full}}}$  corresponding to an element of  $\text{CREW}^p(\alpha_{\mathbf{R}_{\text{full}}})$ ). This is done inductively on the size of the path  $\vec{e}$ , keeping track of the last modifications of each register. I.e. we define both the system of equations  $(E_i^{\mathbf{e}})_i$  and a function  $\mathfrak{h}(\mathbf{e}) : \mathbf{R}^{\omega} + \perp \rightarrow \omega$  (which is almost everywhere null)<sup>11</sup>. For an empty sequence, the system of equations is empty, and the function  $\mathfrak{h}(\epsilon)$  is constant, equal to 0.

Suppose now that  $\vec{e}' = (e_1, \dots, e_m, e_m + 1)$ , with  $\vec{e} = (e_1, \dots, e_m)$ , and that one already computed  $(E_i^{\mathbf{e}})_{i \geq m}$  and the function  $\mathfrak{h}(\mathbf{e})$ . We now consider the edge  $e_{m+1}$  and let  $(r, r')$  be its realizer. We extend the system of equations  $(E_i^{\mathbf{e}})_{i \geq m}$  by a new equation  $E_{m+1}$  and define the function  $\mathfrak{h}(\mathbf{e}')$  as follows:

- if  $r = +_i(j, k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = x_j^{\mathfrak{h}(\mathbf{e}')(j)} + x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = -_i(j, k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = x_j^{\mathfrak{h}(\mathbf{e}')(j)} - x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = \times_i(j, k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = x_j^{\mathfrak{h}(\mathbf{e}')(j)} \times x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = /_i(j, k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = x_j^{\mathfrak{h}(\mathbf{e}')(j)} / x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = +_i^c(k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = c + x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = -_i^c(k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = c - x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;

<sup>11</sup>The use of  $\perp$  is to allow for the creation of fresh variables not related to a register.

- if  $r = \times_i^c(k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = c \times x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = /_i^c(k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = c/x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- if  $r = \sqrt[n]{i}(k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = \sqrt[n]{x_k^{\mathfrak{h}(\mathbf{e}')(k)}}$ ;
- if  $r = \text{Id}$ , the source of the edge  $e_q$  is of the form  $\{(x_1, \dots, x_{n+\ell}) \in \mathbf{R}^{n+\ell} \mid P(x_k)\} \times \{i\}$  where  $P$  compares the variable  $x_k$  with 0:
  - if  $P(x_k)$  is  $x_k \neq 0$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = \perp$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise then  $E_{m+1}$  is  $x_{\perp}^{\mathfrak{h}(\mathbf{e}')(i)} x_k^{\mathfrak{h}(\mathbf{e}')(k)} - 1 = 0$ ;
  - otherwise we set  $\mathfrak{h}(\mathbf{e}') = \mathfrak{h}(\mathbf{e})$  and  $E_{m+1}$  equal to  $P$ .

We now consider the system of equations  $(E_i)_{i=1}^k$  defined from the path  $\mathbf{e}$  of length  $k$  corresponding to a node  $n_{\mathbf{e}}^{\pi}$  of the  $k$ -th entropic co-tree of  $G$ . This system consists in  $k$  equations of degree at most  $\max(2, \sqrt[p]{G})$  and containing at most  $k+n$  variables, counting the variables  $x_1^0, \dots, x_n^0$  corresponding to the initial registers, and adding at most  $k$  additional variables since an edge of  $\vec{e}$  introduces at most one fresh variable. Since the number of vertices  $n_{\mathbf{e}}^{\pi}$  is bounded by  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0(|G|)}$  by Theorem 56, we obtained the stated result in the case  $p = 1$ .

The case for arbitrary  $p$  is then deduced by noticing that each step in the induction would introduce at most  $p$  new equations and  $p$  new variables. The resulting system thus contains at most  $pk$  equations of degree at most  $\max(2, \sqrt[p]{G})$  and containing at most  $pk + n$  variables.  $\square$

This theorem extends to the case of general computational graphings by considering the *algebraic degree* of the graphing.

**Definition 58** (Algebraic degree). Let  $\alpha : \langle G, \mathbf{R} \rangle \curvearrowright \mathbf{X}$  be an AMC. The algebraic degree of an element of  $\mathbf{M}\langle G, \mathbf{R} \rangle$  is the minimal number of generators needed to express it. The algebraic degree of an  $\alpha$ -graphing is the maximum of the algebraic degrees of the realisers of its edges.

If an edge is realised by an element  $m$  of algebraic degree  $D$ , then the method above applies by introducing the  $D$  new equations corresponding to the  $D$  generators used to define  $m$ . The general result then follows.

**Theorem 59.** *Let  $G$  be a  $\text{CREW}^p(\alpha_{\mathbf{R}^{\text{full}}})$ -computational graphing representative,  $\text{Seq}_k(E)$  the set of length  $k$  sequences of edges in  $G$ , and  $D$  its algebraic degree. Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps, i.e. for each element of  $\mathbf{R}^n$ ,  $\pi_{\mathbf{S}}(G^k(x)) = \top$  if and only if  $x \in W$ . Then  $W$  is a semi-algebraic set defined by at most  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0(|G|)}$  systems of  $pkD$  equations of degree at most  $\max(2, \sqrt[p]{G})$  and involving at most  $pkD + n$  variables.*

## 8 Recovering Ben Or and Cucker's theorems

### 8.1 Ben-Or

We now recover Ben-Or result by obtaining a bound on the number of connected components of the subsets  $W \subseteq \mathbf{R}^n$  whose membership problem is computed by a graphing in less than a given number of iterations. This theorem is obtained by applying the Milnor-Thom theorem on the obtained systems of equations to bound the number of connected components of each cell. Notice that in this case  $p = 1$  and  $\sqrt[p]{G} = 2$  since the model of algebraic computation trees use only square roots. A more general result holds for algebraic computation trees extended with arbitrary roots, but we here limit ourselves here to the original model.

**Theorem 60.** *Let  $G$  be a computational  $\alpha_{\mathbf{R}}^{\text{full}}$ -graphing representative translating an algebraic computational tree,  $\text{Seq}_k(E)$  the set of length  $k$  sequences of edges in  $G$ . Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps. Then  $W$  has at most  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0([G]) + 1} 3^{2k+n-1}$  connected components.*

*Proof.* By Lemma 57 (using the fact that  $p = 1$  and  $\sqrt[3]{G} = 2$ ), the problem  $W$  decided by  $G$  in  $k$  steps is described by at most  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0([G])}$  systems of  $k$  equations of degree 2 involving at most  $k + n$  variables. Applying Theorem 47, we deduce that each such system of in-equations (of  $k$  equations of degree 2 in  $\mathbf{R}^{k+n}$ ) describes a semi-algebraic variety  $S$  such that  $\beta_0(S) < 2.3^{(n+k)+k-1}$ . This begins true for each of the  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0([G])}$  cells, we have that  $\beta_0(W) < \text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0([G]) + 1} 3^{2k+n-1}$ .  $\square$

Since a subset computed by a tree  $T$  of depth  $k$  is computed by  $\llbracket T \rrbracket$  in  $k$  steps by Theorem 19, we get as a corollary the original theorem by Ben-Or relating the number of connected components of a set  $W$  and the depth of the algebraic computational trees that compute the membership problem for  $W$ .

**Corollary 61** ([4, Theorem 5]). *Let  $W \subseteq \mathbf{R}^n$  be any set, and let  $N$  be the maximum of the number of connected components of  $W$  and  $\mathbf{R}^n \setminus W$ . An algebraic computation tree computing the membership problem for  $W$  has height  $\Omega(\log N)$ .*

*Proof.* Let  $T$  be an algebraic computation tree computing the membership problem for  $W$ , and consider the computational treeing  $\llbracket T \rrbracket$ . Let  $d$  be the height of  $T$ ; by definition of  $\llbracket T \rrbracket$  the membership problem for  $W$  is computed in exactly  $d$  steps. Thus, by the previous theorem,  $W$  has at most  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{d \cdot h_0(\llbracket T \rrbracket) + 1} 3^{2d+n-1}$  connected components. As the interpretation of an algebraic computational tree,  $h_0(\llbracket T \rrbracket)$  is at most equal to 2, and  $\text{Card}(\text{Seq}_k(E))$  is bounded by  $2^d$ . Hence  $N \leq 2^d \cdot 2^{2d+1} 3^{n-1} 3^{2d}$ , i.e.  $d = \Omega(\log N)$ .  $\square$

We immediately deduce an application that will be useful to us in the remainder. Let  $m \in \mathbf{N}$  and  $0 < x < 2^m$ . Let  $k \in \mathbf{N}$  be such that  $1 \leq k \leq m$ . We call  $\lfloor \frac{x}{2^{k-1}} \rfloor - 2 \lfloor \frac{x}{2^k} \rfloor$  the  $k$ -th bit of  $x$ .

**Lemma 62.** *An algebraic computation tree computing the  $k$ -th bit of  $x$  has height  $\Omega(\log(m - k))$ .*

*Proof.* Let

$$W = \left\{ x \in \mathbf{R} \mid \left\lfloor \frac{x}{2^{k-1}} \right\rfloor - 2 \left\lfloor \frac{x}{2^k} \right\rfloor = 1 \right\}$$

$W$  is the disjoint union of  $m - k + 1$  intervals, and so is its complement in  $]0; 2^m[$ . So, by Theorem 61, any algebraic computation tree computing the  $k$ -th bit has height  $\Omega(\log(m - k))$ .  $\square$

We will see later that bit-extraction is also difficult for the PRAM model (cf. Prop. ??). This is an essential difference between the booleans and algebraic models.

*Remark.* In the case of algebraic PRAMS discussed in the next sections, the  $k$ -th entropic co-tree  $\text{COT}_k(T)[M]$  of a machine  $M$  defines an algebraic computation tree which follows the  $k$ -th first steps of computation of  $M$ . I.e. the algebraic computation tree  $\text{COT}_k(T)[M]$  approximate the computation of  $M$  in such a way that  $M$  and  $\text{COT}_k(T)[M]$  behave in the exact same manner in the first  $k$  steps.

## 8.2 Cucker's theorem

Cucker's proof considers the problem defined as the following algebraic set.

**Definition 63.** Define  $\mathfrak{S}_{\text{er}}$  to be the set:

$$\{x \in \mathbf{R}^\omega \mid |x| = n \Rightarrow x_1^{2^n} + x_2^{2^n} = 1\},$$

where  $|x| = \max\{n \in \omega \mid x_n \neq 0\}$ .

It can be shown to lie within  $\text{PTIME}_{\mathbf{R}}$ , i.e. it is decided by a real Turing machine [5] – i.e. working with real numbers and real operations –, running in polynomial time.

**Theorem 64** (Cucker ([10], Proposition 3)). *The problem  $\mathfrak{F}_{\text{er}}$  belongs to  $\text{PTIME}_{\mathbf{R}}$ .*

We now prove that  $\mathfrak{F}_{\text{er}}$  is not computable by an algebraic circuit of polylogarithmic depth. The proof follows Cucker's argument, but uses the lemma proved in the previous section.

**Theorem 65** (Cucker ([10], Theorem 3.2)). *No algebraic circuit of depth  $k = \log^i n$  and size  $kp$  compute  $\mathfrak{F}_{\text{er}}$ .*

*Proof.* For this, we will use the lower bounds result obtained in the previous section. Indeed, by Theorem 22 and Lemma 57, any problem decided by an algebraic circuit of depth  $k$  is a semi-algebraic set defined by at most  $\text{Card}(\text{Seq}_k(E)) \cdot 2^{k \cdot h_0(|G|)}$  systems of  $k$  equations of degree at most  $\max(2, \sqrt[k]{G}) = 2$  (since only square roots are allowed in the model) and involving at most  $k + n$  variables. But the curve  $\mathfrak{F}_{2^n}^{\mathbf{R}}$  defined as  $\{x_1^{2^n} + x_2^{2^n} - 1 = 0 \mid x_1, x_2 \in \mathbf{R}\}$  is infinite. As a consequence, one of the systems of equation must describe a set containing an infinite number of points of  $\mathfrak{F}_{2^n}^{\mathbf{R}}$ .

This set  $S$  is characterized, up to some transformations on the set of equations obtained from the entropic co-tree, by a finite system of inequalities of the form

$$\bigwedge_{i=1}^s F_i(X_1, X_2) = 0 \wedge \bigwedge_{j=1}^t G_j(X_1, X_2) < 0,$$

where  $t$  is bounded by  $kp$  and the degree of the polynomials  $F_i$  and  $G_j$  are bounded by  $2^k$ . Moreover, since  $\mathfrak{F}_{2^n}^{\mathbf{R}}$  is a curve and no points in  $S$  must lie outside of it, we must have  $s > 0$ .

Finally, the polynomials  $F_i$  vanish on that infinite subset of the curve and thus in a 1-dimensional component of the curve. Since the curve is an irreducible one, this implies that every  $F_i$  must vanish on the whole curve. Using the fact that the ideal  $(X_1^{2^n} + X_2^{2^n} - 1)$  is prime (and thus radical), we conclude that all the  $F_i$  are multiples of  $X_1^{2^n} + X_2^{2^n} - 1$  which is impossible if their degree is bounded by  $2^{\log^i n}$  as it is strictly smaller than  $2^n$ .  $\square$

## 9 Algebraic surfaces for an optimization problem

### 9.1 Geometric Interpretation of Optimization Problems

We start by showing how decision problems of a particular form induce a binary partition of the space  $\mathbf{Z}^d$ : the points that are accepted and those that are rejected. Intuitively, the machine decides the problem if the partition it induces refines the one of the problem.

We will consider problems of a very specific form: decisions problems in  $\mathbf{Z}^3$  associated to optimization problems. Let  $\mathcal{P}_{\text{opt}}$  be an optimization problem on  $\mathbf{R}^d$ . Solving  $\mathcal{P}_{\text{opt}}$  on an instance  $t$  amounts to optimizing a function  $f_t(\cdot)$  over a space of parameters. We note  $\text{Max}\mathcal{P}_{\text{opt}}(t)$  this optimal value. An affine function  $\text{Param} : [p; q] \rightarrow \mathbf{R}^d$  is called a *parametrization* of  $\mathcal{P}_{\text{opt}}$ . Such a parametrization defines naturally a decision problem  $\mathcal{P}_{\text{dec}}$ : for all  $(x, y, z) \in \mathbf{Z}^3$ ,  $(x, y, z) \in \mathcal{P}_{\text{dec}}$  iff  $z > 0$ ,  $x/z \in [p; q]$  and  $y/z \leq \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x/z)$ .

In order to study the geometry of  $\mathcal{P}_{\text{dec}}$  in a way that makes its connection with  $\mathcal{P}_{\text{opt}}$  clear, we consider the ambient space to be  $\mathbf{R}^3$ , and we define the *ray*  $[p]$  of a point  $p$  as the half-line starting at the origin and containing  $p$ . The projection  $\Pi(p)$  of a point  $p$  on a plane is the intersection of  $[p]$  and the affine plane  $\mathcal{A}_1$  of equation  $z = 1$ . For any point  $p \in \mathcal{A}_1$ , and all  $p_1 \in [p]$ ,  $\Pi(p_1) = p$ . It is clear that for  $(p, p', q) \in \mathbf{Z}^2 \times \mathbf{N}^+$ ,  $\Pi((p, p', q)) = (p/q, p'/q, 1)$ .

The *cone*  $[C]$  of a curve  $C$  is the set of rays of points of the curve. The projection  $\Pi(C)$  of a surface or a curve  $C$  is the set of projections of points in  $C$ . We note  $\text{Front}$  the frontier set

$$\text{Front} = \{(x, y, 1) \in \mathbf{R}^3 \mid y = \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x)\}.$$

and we remark that

$$[\text{Front}] = \{(x, y, z) \in \mathbf{R}^2 \times \mathbf{R}^+ \mid y/z = \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x/z)\}.$$

Finally, a machine  $M$  decides the problem  $\mathcal{P}_{\text{dec}}$  if the sub-partition of accepting cells in  $\mathbf{Z}^3$  induced by the machine is finer than the one defined by the problem's frontier [Front] (which is defined by the equation  $y/z \leq \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x/z)$ ).

## 9.2 Parametric Complexity

We now further restrict the class of problems we are interested in: we will only consider  $\mathcal{P}_{\text{opt}}$  such that Front is simple enough. Precisely:

**Definition 66.** We say that Param is an *affine parametrization* of  $\mathcal{P}_{\text{opt}}$  if  $\text{Param}; \text{Max}\mathcal{P}_{\text{opt}}$  is

- convex
- piecewise linear, with breakpoints  $\lambda_1 < \dots < \lambda_\rho$
- such that the  $(\lambda_i)_i$  and the  $(\text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(\lambda_i))_i$  are all rational.

The (*parametric*) *complexity*  $\rho(\text{Param})$  is defined as the number of breakpoints of  $\text{Param}; \text{Max}\mathcal{P}_{\text{opt}}$ .

An optimization problem that admits an affine parametrization of complexity  $\rho$  is thus represented by a surface [Front] that is quite simple: the cone of the graph of a piecewise affine function, constituted of  $\rho$  segments. We say that such a surface is a  $\rho$ -fan. This restriction seems quite serious when viewed geometrically. Nonetheless, many optimization problems admit such a parametrization. Before giving examples, we introduce another measure of the complexity of a parametrization.

**Definition 67.** Let  $\mathcal{P}_{\text{opt}}$  be an optimization problem and Param be an affine parametrization of it. The *bitsize* of the parametrization is the maximum of the bitsizes of the numerators and denominators of the coordinates of the breakpoints of  $\text{Param}; \text{Max}\mathcal{P}_{\text{opt}}$ .

In the same way, we say that a  $\rho$ -fan is of *bitsize*  $\beta$  if all its breakpoints are rational and the bitsize of their coordinates is lesser than  $\beta$ .

**Theorem 68** (Murty [22], Carstensen [6]).

1. *there exists an affine parametrization of bitsize  $O(n)$  and complexity  $2^{\Omega(n)}$  of combinatorial linear programming, where  $n$  is the total number of variables and constraints of the problem.*
2. *there exists an affine parametrization of bitsize  $O(n^2)$  and complexity  $2^{\Omega(n)}$  of the `maxflow` problem for directed and undirected networks, where  $n$  is the number of nodes in the network.*

We refer the reader to Mulmuley's paper [20, Thm. 3.1.3] for proofs, discussions and references.

## 9.3 Algebraic Surfaces

An algebraic surface in  $\mathbf{R}^3$  is a surface defined by an equation of the form  $p(x, y, z) = 0$  where  $p$  is a polynomial. If  $S$  is a set of surfaces, each defined by a polynomial, the *total degree* of  $S$  is defined as the sum of the degrees of polynomials defining the surfaces in  $S$ .

Let  $K$  be a compact of  $\mathbf{R}^3$  delimited by algebraic surfaces and  $S$  be a finite set of algebraic surfaces, of total degree  $\delta$ . We can assume that  $K$  is actually delimited by two affine planes of equation  $z = \mu$  and  $z = 2\mu_z$  and the cone of a rectangle  $\{(x, y, 1) \mid |x|, |y| \leq \mu_{x,y}\}$ , by taking any such compact containing  $K$  and adding the surfaces bounding  $K$  to  $S$ .  $S$  defines a partition of  $K$  by considering maximal compact subspaces of  $K$  whose boundaries are included in surfaces of  $S$ . Such elements are called the *cells* of the decomposition associated to  $S$ .

The cell of this partition can have complicated shapes: in particular, a cell can have a arbitrarily high number of surfaces of  $S$  as boundaries. We are going to refine this partition into a partition  $\text{Col}_S$  whose cells are all bounded by cones of curves and at most two surfaces in  $S$ .

## 9.4 Collins' decomposition

We define the *silhouette* of a surface defined by the equation  $p(x, y, z) = 0$  by:

$$\begin{cases} p(x, y, z) = 0 \\ x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} + z \frac{\partial p}{\partial z} = 0. \end{cases}$$

The silhouette of a surface is the curve on the surface such that all points  $(x, y, z)$  of the silhouette are such that the ray  $[(x, y, z)]$  is orthogonal to the tangent plane of the surface on  $(x, y, z)$ .

Up to infinitesimal perturbation of the coefficients of the polynomials, we can assume that the surfaces of  $S$  have no integer points in  $K$ .

$\Pi(K) = \{\Pi(x) \mid x \in K\}$  is a compact of the affine plane  $\mathcal{A}_1$ . Let us consider the set  $\Pi(S)$  of curves in  $\Pi(K)$  containing:

- the projection of the silhouettes of surfaces in  $S$ ;
- the projection of the intersections of surfaces in  $S$  and of the intersection of surfaces in  $S$  with the planes  $z = \mu(1 + \frac{n}{6\delta})$ ,  $n \in \{1, \dots, 6\delta - 1\}$ , where  $\delta$  is the total degree of  $S$ ;
- *vertical lines* of the form  $\{(x, a, 1) \mid |x| \leq 2^{\beta+1}\}$  for  $a$  a constant such that such lines pass through:
  - all intersections among the curves;
  - all singular points of the curves;
  - all critical points of the curves with a tangent supported by  $\vec{e}_y$ .

$\Pi(S)$  defines a Collins decomposition [8] of  $\Pi(K)$ . The intersection of any affine line supported by  $\vec{e}_y$  of the plane with a region of this decomposition is connected if nonempty.

Let  $c$  be a cell in  $\Pi(S)$ . It is enclosed by two curves in  $\Pi(K)$  and at most two vertical lines. The curves can be parametrized by  $c_{\max} : x \mapsto \max\{y \in \mathbf{R} \mid (x, y, 1) \in c\}$  and  $c_{\min} : x \mapsto \min\{y \in \mathbf{R} \mid (x, y, 1) \in c\}$ , which are both smooth functions. The *volatility* of  $c$  is defined as the number of extrema of the second derivatives  $c''_{\min}$  and  $c''_{\max}$  on their domains of definition.

This set of curves  $\Pi(S)$  can be lifted to a set of surfaces  $\text{Col}_S(K)$  of  $K$  that contains:

- the surfaces of  $S$ ;
- the cones  $[s]$  of every curve  $s$  in  $\Pi(S)$ ;
- the planes bounding  $K$ ;
- $6\delta - 2$  *dividing planes* of equation  $z = \mu(1 + \frac{n}{6\delta})$ ,  $n \in \{1, \dots, 6\delta - 1\}$ .

The projection of a cell of  $\text{Col}_S$  is a cell of  $\Pi(S)$ . We say that a cell of  $\text{Col}_S(K)$  is *flat* if none of its boundaries are included in surfaces of  $S$ .

Let us call  $\mathbf{d}(S)$  the number of cells in  $\text{Col}_S(K)$ .

Let  $c$  be a cell in  $\text{Col}_S(K)$ . Its *volatility* is defined as the volatility of its projection in  $\Pi(S)$ .

## 9.5 Volatility and Separation

**Definition 69.** Let  $K$  be a compact of  $\mathbf{R}^3$ .

A finite set of surfaces  $S$  on  $K$  *separates* a  $\rho$ -fan  $\text{Fan}$  on  $K$  if the partition on  $\mathbf{Z}^3 \cap K$  induced by  $S$  is finer than the one induced by  $\text{Fan}$ .

**Theorem 70.** Let  $S$  be a finite set of algebraic surfaces of total degree  $\delta$ , and  $\text{Fan}$  a  $\rho$ -fan of bitsize  $\beta$ .

If  $S$  separates  $\text{Fan}$ , there exists a compact  $K$  and a cell of  $\text{Col}_S(K)$  with volatility greater than  $\rho/\mathbf{d}(S)$ .

In order to prove this theorem, we will build explicitly the compact  $K$  and this cell by considering sample points on  $\text{Fan}$  and show in Lemma 72 a bound on the volatility of this cell.

Let  $K$  be a compact delimited by the cone of a rectangle  $\{(x, y, 1) \mid |x|, |y| \leq 2^{\beta+1}\}$  and two planes of equation  $z = \mu$  and  $z = 2\mu$ , with  $\mu > (6\delta + 1)2^\beta$ . We first remark that all affine segments of  $\text{Fan}$  are in the rectangle base of  $K$ .

For each affine segment of  $\text{Fan}$  with endpoints  $(x_i, y_i, 1)$  and  $(x_{i+1}, y_{i+1}, 1)$  let, for  $0 < k < 10\mathbf{d}(S)$ ,  $y_i^k$  be such that  $(x_i^k, y_i^k, 1)$  is in the affine segment, where  $x_i^k = \frac{(10\mathbf{d}(S)-k)x_i + kx_{i+1}}{10\mathbf{d}(S)}$ . We remark that, as  $|x_i - x_{i+1}| > 2^{-\beta}$ , we have, for  $k, k'$ ,  $|x_i^k - x_i^{k'}| > 2^{-\beta}/10\mathbf{d}(S)$ .

**Lemma 71.** *For all sample points  $(x_i^k, y_i^k, 1)$ , there exists a flat cell in  $\text{Col}_S$  that contains an integer point of  $[(x_i^k, y_i^k, 1)]$ .*

*Proof.* Let  $(x_i^k, y_i^k, 1)$  be a sample point.  $[(x_i^k, y_i^k, 1)]$  is divided in  $N + 1$  intervals by the dividing planes. On the other hand,  $[(x_i^k, y_i^k, 1)]$  intersects surfaces of  $S$  in at most  $\delta$  points, by Bézout theorem. So, there exists an interval  $e$  of  $[(x_i^k, y_i^k, 1)]$  that is bounded by the dividing planes and that do not intersect any surface in  $S$ . By construction,  $e$  is included in a flat cell, and its projection on the  $z$ -axis has length  $\mu/(6\delta + 1)$ , so, as  $(x_i^k, y_i^k, 1)$  is of bitsize  $\beta$ ,  $(n2^\beta x_i^k, n2^\beta y_i^k, n2^\beta)$  is, for all  $n \in \mathbf{N}$  an integer point of the ray, so, as  $\mu > (6\delta + 1)2^\beta$ ,  $e$  contains an integer point.  $\square$

So, for each affine segment of  $\text{Fan}$ , there exists a flat cell in  $\text{Col}_S$  that contains integer points in the ray of at least 10 sample points of the affine segment. Going further, there exists a cell  $c$  of  $\text{Col}_S$  that contains integer points in the ray of at least 10 sample points of  $\rho/\mathbf{d}(S)$  affine segments of  $\text{Fan}$ .

**Lemma 72.** *The volatility of  $c$  is at least  $\rho/\mathbf{d}(S)$ .*

This is achieved by applying the mean value theorem on the function  $\Pi(c)'_{\max}$  on pairs of sample points. In particular, this proof uses no algebraic geometry.

*Proof.* Let  $e$  be a segment of  $\text{Fan}$  such that the ray of 10 of its sample points contain an integer point in  $c$ . Let  $p = (x, y, z)$  be one of its integer point and  $\Pi(p) = (x_p, y_p, 1)$  its projection, which is a sample point in  $\Pi(c)$ . Let  $q = (x, y + 1, z)$ . As  $\Pi(p)$  is in  $\text{Fan}$ , and  $S$  separates  $\text{Fan}$ ,  $q$  is not in  $c$ , and  $\Pi(q) = (x_q, y_q, 1)$  is not in  $\Pi(c)$ . By Thalès theorem,  $0 < y_q - y_p < \frac{1}{\mu}$ . So, as  $y_q > \Pi(c)_{\max}(x_p) > y_p$ , we have in particular that  $0 < \Pi(c)_{\max}(x_p) - y_p < \frac{1}{\mu}$ .

So, the 10 sample points have coordinates that approximate the graph of  $\Pi(c)_{\max}$  with an error bounded by  $\frac{1}{\mu}$ . Consider two of them  $p_1 = (x_1, y_1, 1)$  and  $p_2 = (x_2, y_2, 1)$ , such that  $x_1 < x_2$ . Let  $a$  be the slope of  $e$  (in particular  $a = (y_2 - y_1)/(x_2 - x_1)$ ). By the mean value theorem, there exists  $\alpha \in [x_1, x_2]$  such that  $\Pi(c)'_{\max}(\alpha) = \frac{\Pi(c)_{\max}(x_2) - \Pi(c)_{\max}(x_1)}{x_2 - x_1}$ . But  $|\Pi(c)_{\max}(x_2) - \Pi(c)_{\max}(x_1)| \leq |y_2 - y_1| + \frac{2}{\mu}$  and  $|x_2 - x_1| > \frac{1}{10\mathbf{d}(S)2^\beta}$ . So,  $|\Pi(c)'_{\max}(\alpha) - a| \leq 2\frac{10\mathbf{d}(S)2^\beta}{\mu}$ .

So, the function  $\Pi(c)'_{\max}$  is close to the value  $a$ , with error bounded, between all the sample points. By applying the mean value theorem again, we get that there exists a point in the interval such that  $\Pi(c)''_{\max}$  is close to 0, with an error bounded by  $2\frac{10\mathbf{d}(S)2^\beta}{\mu}$ .

In the same way, let  $e'$  be another segment of  $\text{Fan}$  such that the ray of 10 of its sample points contain an integer point in  $c$ , of slope  $a'$ . Let two of them be  $p'_1 = (x'_1, y'_1, 1)$  and  $p'_2 = (x'_2, y'_2, 1)$ , and suppose  $x'_2 > x'_1 > x_2$ . By the same reasoning as above, there exists  $\alpha' \in [x'_1, x'_2]$  such that  $|\Pi(c)'_{\max}(\alpha') - a'| \leq 2\frac{10\mathbf{d}(S)2^\beta}{\mu}$ . By the mean value theorem, there exists  $\beta \in [\alpha, \alpha']$  such that  $\Pi(c)''_{\max}(\beta) = \frac{\Pi(c)'_{\max}(\alpha') - \Pi(c)'_{\max}(\alpha)}{\alpha' - \alpha} > \frac{1}{\mu}(|a - a'| - 2\frac{10\mathbf{d}(S)2^\beta}{\mu})$ .

So, for each of the  $\rho/\mathbf{d}(S)$  segments of  $\text{Fan}$ , we can exhibit a point such that  $\Pi(c)''_{\max}$  is close to zero, and for each successive segment, a point such that it is far. So  $\Pi(c)''_{\max}$  has at least  $\rho/\mathbf{d}(S)$  extrema.  $\square$

## 9.6 Volatility and Degree

Mulmuley's result follows from Thm. 70 and the two lemmas:

**Lemma 73.** *Let  $S$  be a finite set of curves of total degree  $\delta$ , and  $K$  be a compact. The cells of the decomposition  $\text{Col}_S$  of  $K$  have a volatility bounded by a polynomial in  $\delta$ .*

*Proof.* Let  $c$  be a cell in  $\text{Col}_S$  and  $g(x, y) = 0$  be the equation of one of the boundaries of  $\Pi(c)$  in the affine plane. The degree of  $g$  is bounded by the degree of the intersection of surfaces in  $S$ . Any extrema  $x$  of  $f''$ ,



where  $f$  is a parametrization  $y = f(x)$  of this boundary, can be represented as a point  $(x, y, y^{(1)}, y^{(2)}, y^{(3)})$  in the 5-dimensional phase space that satisfy polynomial equations of the form:

$$\begin{aligned} g(x, y) = 0, \quad g_1(x, y, y^{(1)}) = 0, \quad g_2(x, y, y^{(1)}, y^{(2)}) = 0 \\ g_3(x, y, y^{(1)}, y^{(2)}, y^{(3)}) = 0, \quad y^{(3)} = 0, \end{aligned}$$

where all the polynomials' degrees are all bounded by the degree of the intersection of surfaces in  $S$  (as they are the derivatives of  $g$ ). So, by the Milnor–Thom theorem, such points are in number polynomial in the total degree of the surfaces of  $S$ .  $\square$

**Lemma 74.** *The number of cells  $\mathbf{d}(S)$  of the Collins decomposition of  $S$  is polynomial in  $\delta$ .*

*Proof.* The intersection of the surfaces in  $S$  are algebraic varieties of number bounded by  $\delta$ , by the Milnor–Thom theorem. Moreover, so are the silhouettes of the surfaces, as they are the intersection of two algebraic varieties of total degree smaller than  $\delta$ . So, the number of cells in  $\text{Col}_S$  is bounded by the number of cells of  $S$  times the number of dividing planes times the number of intersections, silhouettes and vertical lines they engender.  $\square$

**Theorem 75** (Mulmuley). *Let  $S$  be a finite set of algebraic surfaces of total degree  $\delta$ .*

*There exists a polynomial  $P$  such that, for all  $\rho > P(\delta)$ ,  $S$  does not separate  $\rho$ -fans.*

## 10 Improving Mulmuley’s result

We will now prove our strengthening of Mulmuley’s lower bounds for “PRAMs without bit operations” [20]. For this, we will combine the results from previous sections to establish the following result.

**Theorem 76.** *Let  $M$  be an integer-valued PRAM with at most  $2^{O((\log N)^c)}$  processors, where  $N$  is the length of the inputs and  $c$  any positive integer.*

*Then  $M$  does not decide  $\mathbf{maxflow}$  in  $O((\log N)^c)$  steps.*

So, let  $M$  be an integer-valued PRAM. We can associate to it a real-valued PRAM  $\tilde{M}$  such that  $M$  and  $\tilde{M}$  accept the same (integer) values, and the ratio between the running time of the two machines is a constant. Indeed:

**Proposition 77.** *Euclidian division can be computed by a constant time real-valued PRAM.*

*Proof of Theorem 76.* Suppose now that  $\llbracket M \rrbracket$  has a finite set of edges  $E$ . Then  $\llbracket \tilde{M} \rrbracket$  has too has a finite set of edge of cardinality  $O(\text{Card}(E))$ . Since the running time of the initial PRAM over integers is equal, up to a constant, to the computation time of the  $\text{CREW}^p(\alpha_{\mathbf{Rfull}})$ -program  $\llbracket \tilde{M} \rrbracket$ , we deduce that if  $M$  computes  $\mathbf{maxflow}$  in  $k$  steps, then  $\llbracket \tilde{M} \rrbracket$  computes  $\mathbf{maxflow}$  in at most  $Ck$  steps where  $C$  is a fixed constant.

By Lemma 57, the problem decided by  $\llbracket \tilde{M} \rrbracket$  in  $Ck$  steps defines a system of equations separating the integral inputs accepted by  $M$  from the ones rejected. I.e. if  $M$  computes  $\mathbf{maxflow}$  in  $Ck$  steps, then this system of equations defines a set of algebraic surfaces that separate the  $\rho$ -fan defined by  $\mathbf{maxflow}$ . Moreover, this system of equation has a total degree bounded by  $Ck \max(2, \sqrt[p]{G}) 2p \times 2^{O(\text{Card}(E))} \times 2^{k \cdot h_0(\llbracket \tilde{M} \rrbracket)}$ .

By Theorem 68 and Theorem 75, there exists a polynomial  $P$  such that a finite set of algebraic surfaces of total degree  $\delta$  cannot separate the  $2^{\Omega(n)}$ -fan defined by  $\mathbf{maxflow}$  as long as  $2^{\Omega(n)} > P(\delta)$ . But here the entropy of  $G$  is  $O(p)$ , as the entropy of a product  $f \times g$  satisfies  $h(f \times g) \leq h(f) + h(g)$  [14]. Hence  $\delta = O(2^p 2^k)$ , contradicting the hypotheses that  $p = 2^{O((\log N)^c)}$  and  $k = 2^{O((\log N)^c)}$ .  $\square$

This has Theorem 1 as a corollary, which shows that the class  $\text{NC}_{\mathbf{Z}}$  does not contain  $\mathbf{maxflow}$ , and hence is distinct from  $\text{P TIME}$ . The question of how this class relates to  $\text{NC}$  is open: indeed, while bit extractions cannot be performed in constant time by our machines (a consequence of Theorem 57), they can be simulated in logarithmic time.

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