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# PRAMs over integers do not compute maxflow efficiently

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Finding lower bounds in complexity theory has proven to be an extremely difficult task. In this article, we analyze two proofs of complexity lower bound: Ben-Or’s proof of minimal height of algebraic computational trees deciding certain problems and Mulmuley’s proof that restricted Parallel Random Access Machines (PRAMs) over integers can not decide P-complete problems efficiently. We present the aforementioned models of computation in a framework inspired by dynamical systems and models of linear logic : graphings.

This interpretation allows to connect the classical proofs to topological entropy, an invariant of these systems; to devise an algebraic formulation of parallelism of computational models; and finally to strengthen Mulmuley’s result by separating the geometrical insights of the proof from the ones related to the computation and blending these with Ben-Or’s proof. Looking forward, the interpretation of algebraic complexity theory as dynamical system might shed a new light on research programs such as Geometric Complexity Theory.

Additional Key Words and Phrases: Semantics, Computational Complexity, Dynamical Systems, Algebraic Geometry

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## 1 INTRODUCTION

While the general theory of computability focused on studying what a computable function is, computer scientists quickly realised that this notion was not meaningful in practice. Indeed, one can always define a computable function  $f$  such that no current computer could compute the values of  $f$  for two-digits inputs within the next, say, ten years.

This lead researchers to work on the definition and understanding of the notion of *feasible* computation, i.e. characterise a set of functions which can be effectively computed. Within the span of a single year three different papers [13, 18, 25] tackled this question, and all of them provided the same answer, namely feasible functions are those functions for which there exists a program whose running time is asymptotically bounded by a polynomial in the input. This is how the first complexity class was born: the class of polynomial-time computable functions.

Very quickly, other classes were defined, some of them considering constraints on *space* rather than time. The question of *classifying* the complexity classes became one of the main question in the field, and a number of important results were obtained within the first years.

*Lower bounds.* As part of the classification problem, complexity theory has traditionally been concerned with proving *separation results*. Among the numerous open separation problems lies the much advertised PTIME vs. NPTIME problem of showing that some problems considered hard to solve but efficient to verify do not have a polynomial time algorithm solving them.

Proving that two classes  $B \subset A$  are not equal can be reduced to finding lower bounds for problems in  $A$ : by proving that certain problems cannot be solved with less than certain resources on a specific model of computation, one can show that two classes are not equal. Conversely, proving a separation result  $B \subsetneq A$  provides a lower bound for the problems that are *A-complete* [15] – i.e. problems that are in some way *universal* for the class  $A$ .

Alas, the proven lower bound results are very few, and most separation problems remain as generally accepted conjectures. For instance, a proof that the class of non-deterministic exponential problems is not included in what is thought of as a very small class of circuits was not achieved until very recently [48].

The failure of most techniques of proof has been studied in itself, which lead to the proof of the existence of negative results that are commonly called *barriers*. Altogether, these results show that all proof methods we know are ineffective with respect to proving interesting lower bounds. Indeed, there are three barriers: relativisation [8], natural proofs [37] and algebrization [1], and every known proof method hits at least one of them. This shows the need for new methods<sup>1</sup>. However, to this day, only one research program aimed at proving new separation results is commonly believed to have the ability to bypass all barriers: Mulmuley and Sohoni’s Geometric Complexity Theory (GCT) program [35].

*Geometric Complexity Theory.* is widely considered to be a promising research program that might lead to interesting results. It is also widely believed to necessitate new and

<sup>1</sup>In the words of S. Aaronson and A. Wigderson [1], “We speculate that going beyond this limit [algebrization] will require fundamentally new methods.”

91 extremely sophisticated pieces of mathematics in order to achieve its goal. The research  
92 program aims to prove the  $P_{TIME} \neq NP_{TIME}$  lower bound by showing that certain algebraic  
93 surfaces (representing the permanent and the discriminant, which are believed [47] to  
94 have different complexity if  $P_{TIME} \neq NP_{TIME}$ ) cannot be embedded one into the other.  
95 Although this program has lead to interesting developments as far as pure mathematics  
96 is concerned, it has not enhanced our understanding of complexity lower bounds for the  
97 time being (actually, even for Mulmuley himself, such understanding will not be achieved  
98 in our lifetimes [20]). Recently, some negative results [28] have closed the easiest path  
99 towards it promised by GCT.

100 The GCT program was inspired, according to its creators, by a lower bound result ob-  
101 tained by Mulmuley [34]. Specifically, it was proved that the maxflow problem (deciding  
102 whether a certain quantity can flow from a source to a target in a weighted graph) is not  
103 solvable efficiently in a specific parallel model (the PRAM without bit operations). The  
104 maxflow problem is quite interesting as it is known to be in  $P_{TIME}$  (by reduction to linear  
105 programming, or the Ford-Fulkerson algorithm [19]), but there are no known efficiently  
106 parallel algorithm solving it. This lower bound proof, despite being the main inspiration of  
107 the well-known GCT research program, remains seldom cited and has not led to variations  
108 applied to other problems. At first sight it relies a lot on algebraic geometric techniques  
109 and results, such as the Milnor-Thom theorem<sup>2</sup>.

110 *Implicit Computational Complexity.* Another approach to complexity theory that emerged  
111 in the recent years is Implicit Computational Complexity (ICC). Related to logical approaches  
112 of computational complexity such as Descriptive Complexity, the aim of ICC is to study  
113 algorithmic complexity only in terms of restrictions of languages and computational  
114 principles. It has been established since Bellantoni and Cook' landmark paper [9], and  
115 following work by Leivant and Marion [31, 32].

116 As part of ICC techniques, some approaches derive from the proofs-as-programs (or  
117 Curry-Howard) correspondence. At its core, this correspondence allows one to view the  
118 execution of a program as the cut-elimination procedure of a corresponding proof in a  
119 formal deductive system (e.g. sequent calculus). Initially stated for intuitionistic logic  
120 [27], the correspondence extends to resource-aware logics such as linear logic (LL), which  
121 is well-suited to study computation. This approach to ICC therefore relies on restrictions on  
122 the deductive system considered to characterise complexity classes. In particular, several  
123 variants of LL were shown to characterise  $FP_{TIME}^3$ : BLL [23], SLL [30], DLAL [7] and LLL [22].  
124

125 *Dynamic Semantics.* The geometry of interaction program was proposed by Girard  
126 [21] shortly after the inception of linear logic. In opposition to traditional denotational  
127 semantics – e.g. domains –, the GOI program aims at giving an account of the proofs and  
128 programs which also interprets their dynamical features, i.e. cut-elimination/execution.  
129 This program is well-suited for tackling problems involving computational complexity, and  
130 indeed, geometry of interaction's first model was used to prove the optimality of Lamping's

131 <sup>2</sup> Let us here notice that, even though this is not mentioned by Mulmuley, the Milnor-Thom theorem was  
132 already used to prove lower bounds, c.f. papers by Dobkin and Lipton [17], Steele and Yao [46], Ben-Or [10],  
133 and references therein.

134 <sup>3</sup> $FP_{TIME}$  is a variant of  $P_{TIME}$  that computes a function and not just a boolean predicate.  
135

reduction in  $\lambda$ -calculus [24]. More recently, a series of characterisations of complexity classes were obtained using GOI techniques [3–6].

Among the most recent and full-fledged embodiment of this program lie the second author's Interaction Graphs models [38, 40–42]. These models, in which proofs/programs are interpreted as *graphings* – generalisations of dynamical systems –, encompass all previous GOI models introduced by Girard [42]. In particular, Interaction Graphs allow for modelling quantitative features of programs/proofs [41].

*Semantic Approach to Complexity.* Based on a study of several Interaction Graphs models characterising complexity classes [44, 45], the second author has proposed to use graphings to develop a semantic approach to complexity theory [39]. The basic idea behind this program is to model and study programs as dynamical systems that acts on a space – thought of as the space of configurations. As dynamical systems are inherently deterministic, the use of graphings is needed to extend the approach to probabilistic and/or non-deterministic programs. One can then study a program through the geometry of the associated graphing (for instance, a configuration caught in a loop is represented as a point of the space of finite orbit).

The second author conjectures that advanced methods developed within the theory of dynamical systems, in particular methods specific to the study of ergodic theory using techniques from operator algebras, could enable new proof techniques for separation. It can be argued that such techniques should be able to bypass barriers [43].

## 2 CONTENTS OF THE PAPER

### 2.1 Computation models as graphings.

The present work reports on the first investigations into how the interpretation of programs as graphings could lead to separation techniques, by rephrasing two well-known lower bound proofs. The interpretation of programs rely on two ingredients:

- the interpretation of models of computation as monoid actions. In our setting, we view the computational principles of a computational model as elements that act on a configuration space. As these actions can be composed, but are not necessarily reversible, it is natural to interpret them as composing a monoid acting on a configuration space. As, moreover, we are intersted in having control in our computations (knowing whether it is finished, failed, succeeded, . . .), we consider actions that can be decomposed as a part that computes using the principles of computation and a part that just modifies a control state;
- the realization of programs as graphings. We abstract programs as graphs whose vertices are subspaces of the product of the configuration space and the control states and edges are labelled by elements of the acting monoid, acting on subspaces of vertices.

The basic intuitions here can be summarised by the following slogan: "Computation, as a dynamical process, can be modelled as a dynamical system". Of course, the above affirmation cannot be true of all computational processes; for instance the traditional notion of dynamical system is deterministic. In practice, one works with a generalisation of

dynamical systems named *graphings*; introduced as part of a family of models of linear logic, graphings have been shown to model non-deterministic and probabilistic computation.

To do so, we consider that a computation model is given by a set of generators (that correspond to computation principles) and its actions on a space (representing the configuration space). So, in other words, we define a computation model as an action of a monoid (presented by its generators and relations) on a space  $\alpha : M \curvearrowright X$ . This action can then be specified to be continuous, measurable, ... depending on the properties we are interested in.

A program in such a model of computation is then viewed as a graph, whose vertices are subspace of the configuration space and edges are generators of the monoid: in this way, both the partiality of certain operations and branching is allowed. This point of view is very general, as it can allow to study, as special model of computations, models that can be discrete or continuous, algebraic, rewriting-based,...

## 2.2 Entropy

We fix an action  $\alpha : M \curvearrowright X$  for the following discussion. One important aspect of the representation of abstract programs as graphings is that restrictions of graphings correspond to known notions from mathematics. In a very natural way, a deterministic  $\alpha$ -graphing defines a partial dynamical system. Conversely, a partial dynamical system whose graph is contained in the *measured preorder*  $\{(x, y) \in X^2 \mid \exists m \in M, \alpha(m)(x) = y\}$  [39] can be associated to an  $\alpha$ -graphing.

The study of deterministic models of computations can thus profit from the methods of the theory of dynamical systems. In particular, the methods employed in this paper relate to the classical notion of *topological entropy*. The topological entropy of a dynamical system is a value representing the average exponential growth rate of the number of orbit segments distinguishable with a finite (but arbitrarily fine) precision. The definition is based on the notion of open covers: for each finite open cover  $C$ , one can compute the entropy of a map w.r.t.  $C$ , and the entropy of the map is then the supremum of these values when  $C$  ranges over the set of all finite covers. As we are considering graphings and those correspond to partial maps, we explain how the techniques adapt to this more general setting and define the entropy  $h(G, C)$  of a graphing  $G$  w.r.t. a cover  $C$ , as well as the topological entropy  $h(G)$  defined as the supremum of the values  $h(G, C)$  where  $C$  ranges over all finite open covers.

While the precise results described in this paper use the entropy  $h_0(G)$  w.r.t. a specific cover (similar bounds could be obtained from the topological entropy, but would lack precision), the authors believe entropy could play a much more prominent role in future proofs of lower bound. Indeed, while  $h_0(G)$  somehow quantifies over one aspect of the computation, namely the branchings, the topological entropy computed by considering all possible covers provides a much more precise picture of the dynamics involved. In particular, it provides information about the computational principles described by the AMC; this information may lead to more precise bounds based on how some principles are much more complex than some others, providing some lower bounds on possible simulations of the former with the latter.

All the while only the entropy w.r.t. a given cover will be essential in this work, the overall techniques related to entropy provide a much clearer picture of the techniques. In

particular, the definition of *entropic co-trees* (Theorem 37) are quite natural from this point of view and clarifies the methods employed by e.g. Ben-Or and Mulmuley.

### 2.3 Ben-Or's proof

One lower bounds result related to Mulmuley's techniques is the bounds obtained by Steele and Yao [46] on *Algebraic Decision Trees*. Algebraic decision trees are defined as finite ternary trees describing a program deciding a subset of  $\mathbf{R}^n$ : each node verifies whether a chosen polynomial, say  $P$ , takes a positive, negative, or null value at the point considered. A  $d$ -th order algebraic decision tree is an algebraic decision tree in which all polynomials are of degree bounded by  $d$ .

In a very natural manner, an algebraic decision tree can be represented as an  $\iota$ -graphings, when  $\iota$  is the trivial action on the space  $\mathbf{R}^n$ . We use entropy to provide a bound on the number of connected components of subsets decided by  $\iota$ -graphings. These bounds are obtained by combining a bound in terms of entropy and a variant of the Milnor-Thom theorem due to Ben-Or. The latter, which we recall below (Theorem 26) bounds the number of connected components of a semi-algebraic set in terms of the number of polynomial inequalities, their maximal degree, and the dimension of the space considered.

**Theorem 29.** *Let  $T$  be a  $d$ -th order algebraic decision tree deciding a subset  $W \subseteq \mathbf{R}^n$ . Then the number of connected components of  $W$  is bounded by  $2^h d(2d - 1)^{n+h-1}$ , where  $h$  is the height of  $T$ .*

This result of Steele and Yao adapts in a straightforward manner to a notion of algebraic computation trees describing the construction of the polynomials to be tested by mean of multiplications and additions of the coordinates. The authors remarked this result uses techniques quite similar to that of Mulmuley's lower bounds for the model of PRAMS *without bit operations*. It is also strongly similar to the techniques used by Cucker in proving that  $\text{NCR} \neq \text{PTIME}_{\mathbf{R}}$  [16].

However, a refinement of Steele and Yao's method was quickly obtained by Ben-Or so as to obtain a similar result for an extended notion of algebraic computation trees allowing for computing divisions and taking square roots. We here adapt Ben-Or techniques within the framework of graphings, in order to apply this refined approach to Mulmuley's framework, leading to a strengthened lower bounds result.

Adapting Ben-Or's method, we obtain a proof of the following result on *computational graphings* in the AMC of algebraic computational trees. The class of computational graphings contains the interpretation of algebraic computational trees and the result generalises that of Ben-Or by giving a bound on the number of connected components of the subset decided by a computational graphing. This bound depends on the number of edges of the computational graphing, as well as its algebraic degree (Theorem 42).

**Theorem 43.** *Let  $G$  be a computational graphing representative,  $\text{Card}(E)$  its number of edges, and  $D$  its algebraic degree. Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps, i.e. for each element of  $\mathbf{R}^n$ ,  $\pi_{\mathcal{S}}(G^k(x)) = \top$  if and only if  $x \in W$ . Then  $W$  has at most  $2^{h_0(|G|)+1} 3^{2kD+n+1}$  connected components.*



271 This reformulation of Ben-Or techniques is then applied to strengthen a lower bound  
272 obtained by Mulmuley [34]. While Mulmuley’s model of “PRAMs without bit operations”  
273 is a restriction of the usual notion of algebraic PRAMs over the integers, we obtain here  
274 similar lower bounds for the non-restricted model. For this purpose, we first need to show  
275 how parallelism can be accomodated within the framework of AMCs and graphings.  
276

## 277 2.4 PRAMs and the crew

278 We are able to introduce PRAMs acting over integers in this setting. They can be described  
279 as having a finite number of processors, each having access to a private memory on top  
280 of the shared memory, and able to perform the operations  $+$ ,  $-$ ,  $\times$ ,  $/$  as well as branching  
281 and indirect addressing. Interestingly, we can represent these machines in the graphings  
282 framework in two steps: first, by defining the SRAM model, with just one processor; and then  
283 by performing an algebraic operation at the level of the algebraic models of computation.

284 So, in a way, parallel computation is modelled per se, at the level of models. As usual,  
285 one is bound to chose a mode of interaction between the different processes when dealing  
286 with shared memory. We will consider here only the case of *Concurrent Read Exclusive*  
287 *Write* (CREW), i.e. all processes can read the shared memory concurrently, but if several  
288 processes try to write in the shared memory only the process with the smallest index is  
289 allowed to do so.

290 The heart of our approach of parallelism is based on commutation. Among all the  
291 instructions, the ones affecting only the private memory of distinct processors can commute,  
292 while it is not the case of two instructions affecting the central memory. We do so by  
293 considering a notion of product for monoids that generalizes both the direct product and  
294 the free product: we specify, through a conflict relation, which of the generators can and  
295 can not commute, allowing us to build a monoid representing the simultaneous action.  
296

## 297 2.5 Mulmuley’s geometrization

298 Contrarily to Ben-Or’s model, the PRAM machines do not decide sets of reals but of integers,  
299 making the use of algebraico-geometric results to uncover their geometry much less  
300 obvious. The mechanisms of Mulmuley’s proof rely on twin geometrizations: one of a  
301 special optimization problem that can be represented by a surface in  $\mathbf{R}^3$  Subsec. 8.1-8.2,  
302 the other one by building explicitly, given a PRAM, a set of algebraic surfaces such that  
303 the points accepted by the machine are exactly the integer points enclosed by the set of  
304 surfaces.

305 Finally, the proof is concluded by a purely geometrical theorem (Thm. 61)<sup>4</sup> expressing a  
306 tension between the two geometrizations. Our work focuses here only on the construction  
307 of a set algebraic surfaces representing the computation of a PRAM; the remaining part of  
308 our proof follows Mulmuley’s original technique closely.  
309

310 *Building surfaces.* The first step in Mulmuley’s proof is to use the parametric complexity  
311 results of Carstensen [12] to represent an instance of the decision problem associated to  
312

---

313 <sup>4</sup>We would like to stress here that this separation in three movement, with a geometrical tour-de-force, is not  
314 explicit in the original article. We nonetheless believe it greatly improves the exposition.  
315





361 **Theorem 63.** Let  $G$  be a PRAM without bit operations with  $2^{O((\log N)^c)}$  processors, where  $N$   
 362 is the length of the inputs and  $c$  any positive integer.

363  $G$  does not decide maxflow in  $O((\log N)^c)$  steps.

364 If we call  $\text{NC}_Z$  the class of computation problems that can be decided by a PRAM over  
 365 integers in time logarithmic in the length of the inputs and a number of processors  
 366 polylogarithmic in the length of the inputs, we have proved that  
 367

$$368 \quad \text{NC}_Z \neq \text{PTIME}$$

## 369 2.7 Conclusion

370 This work not only provides a strengthened lower bound results, but shows how the  
 371 semantic techniques based on abstract models of computation and graphings can shed new  
 372 light on some lower bound techniques. In particular, it establishes some relationship  
 373 between the lower bounds and the notion of entropy which, although arguably still  
 374 superficial in this work, could potentially become deeper and provide new insights and  
 375 finer techniques.  
 376

377 Showing that the interpretation of programs as graphings can translate, and even refine,  
 378 such strong lower bounds results is also important from another perspective. Indeed, the  
 379 techniques of Ben-Or and Mulmuley (as well as other results of e.g. Cucker [16], Yao [49])  
 380 seem at first sight restricted to algebraic models of computation due to their use of the  
 381 Milnor-Thom theorem which holds only for real semi-algebraic sets. However, the second  
 382 author's characterisations of Boolean complexity classes in terms of graphings acting on  
 383 algebraic spaces [44] opens the possibility of using such algebraic methods to provide  
 384 lower bounds for boolean models of computation.

## 385 3 ABSTRACT MODELS OF COMPUTATION, ABSTRACT PROGAMS

386 The basic intuitions here can be summarised by the following slogan: "Computation,  
 387 as a dynamical process, can be modelled as a dynamical system". Of course, the above  
 388 affirmation cannot be true of all computational processes; for instance the traditional  
 389 notion of dynamical system is deterministic. In practice, one works with a generalisation of  
 390 dynamical systems named *graphings*; introduced as part of a family of models of linear logic,  
 391 graphings have been shown to model non-deterministic and probabilistic computation.  
 392

393 Given a set  $G$ , we denote by  $M\langle G \rangle$  the free monoid on  $G$ , i.e. the set of finite sequences  
 394 of elements of  $G$ .

395 **Definition 1.** We recall that a *presentation*  $\langle G, R \rangle$  of a monoid  $M$  is given by a set  $G$  of  
 396 *generators* and a set  $R$  of *relations* such that  $M$  is isomorphic to  $M\langle G \rangle/R$ .

397 **Definition 2.** Let  $M$  be a monoid and  $X$  be a space. An action of  $M$  on  $X$  is a monoid  
 398 morphism  $\alpha$  from  $M$  to the set of endomorphisms on  $X$ . We denote actions by  $\alpha : M \curvearrowright X$ ,  
 399 sometimes omitting the morphism  $\alpha$ .  
 400

401 In this definition, we purposely chose to not specify the kind of space considered. As  
 402 a consequence, if one considers a discrete space  $X$  (i.e. sets), the set of endomorphisms  
 403 will simply be the set of functions  $X \rightarrow X$ . Similarly, if  $X$  is a topological space, the set of  
 404 endomorphisms will be continuous maps (hence  $\alpha$  will be a *continuous action*). Etc.  
 405

**Definition 3.** An *abstract model of computation* (AMC) is defined as a triple  $(G, R, \alpha)$ , where  $\langle G, R \rangle$  is a presentation of a monoid  $M\langle G, R \rangle$  and  $\alpha$  is a monoid action  $M\langle G, R \rangle \curvearrowright X$ . We denote an AMC as  $\alpha : \langle G, R \rangle \curvearrowright X$ .

*Remark.* Although it might seem enough to define an abstract model of computation solely as a monoid action, the choice of a presentation of the monoid by generators and relations is important. First, when considering several models of computation, one wants to consider a notion of *compilation*: an element  $f \in \text{End}(Y)$  is compilable in the action  $M \curvearrowright X$  when there is an automorphism  $\Psi : X \rightarrow Y$  such that one can write  $X = \uplus_{i=1, \dots, n} X_i$  and the restriction of  $\Psi^{-1} \circ f \circ \Psi$  to  $X_i$  is the restriction of an element  $g_i$  of  $M$  to  $X_i$ . To use this notion of compilation in a meaningful way, one would want to quantify the *complexity* of compilation. This can be done only by considering a definition of the monoid  $M$  as generators and relations  $\langle G, R \rangle$ , allowing one to consider the *degree*<sup>5</sup> of  $g_i$  – the length of the smallest word in  $G^*$  representing  $g_i$  – and therefore the degree of the compilation of  $f$  into  $\alpha : \langle G, R \rangle \curvearrowright X$ .

Although we will not consider the notion of compilation in this work, it is remarkable that the representation of the monoid as generators and relations is needed in the definition of the *parallelisation of actions* – defined as the CREW operation.

**Definition 4.** A graphing representative  $G$  w.r.t. a monoid action  $M \curvearrowright X$  is defined as a set of *edges*  $E^G$  and for each element  $e \in E^G$  a pair  $(S_e^G, m_e^G)$  of a subspace  $S_e^G$  of  $X$  – the *source* of  $e$  – and an element  $m_e^G \in M$  – the *realiser* of  $e$ .

Graphings come in different flavours (discrete, topological, measurable), depending on the type of space  $X$  one wishes to consider. If  $X$  is a topological space, the action will be *continuous*, if  $X$  is a measure space, the action will be *measurable*. While the notion of graphing representative does not depend on this choice, the notion of *graphing* is defined as a quotient of the space of graphing representative w.r.t. an adequate notion of equivalence. We will here consider the notion of *topological graphing* [42], which we will simply call *graphings*. In this case, the notion of equivalence is easier to define than in the case of *measurable graphings* as the latter requires one to consider almost-everywhere equality.

**Definition 5** (Refinement). A graphing representative  $F$  is a refinement of a graphing representative  $G$ , noted  $F \leq G$ , if there exists a partition<sup>6</sup>  $(E_e^F)_{e \in E^G}$  of  $E^F$  such that  $\forall e \in E^G$ :

$$\left( \bigcup_{f \in E_e^F} S_f^F \right) \Delta S_e^G = \emptyset; \quad \forall f \neq f' \in E_e^F, \quad S_f^F \Delta S_{f'}^F = \emptyset;$$

$$\forall f \in E_e^F, \quad m_f^F = m_e^G$$

This notion defines an equivalence relation defined by  $F \sim G$  if and only if there exists  $H$  with  $H \leq F$  and  $H \leq G$ .

**Definition 6.** A *graphing* is an equivalence class of graphing representatives w.r.t. the equivalence relation generated by refinements.

<sup>5</sup>Let us notice that this notion has already been considered in relation to dynamical systems, used to define what is called the *algebraic entropy* and the *fundamental group entropy* [29, Section 3.1]

<sup>6</sup>We allow the sets  $E_e^F$  to be empty.

A graphing is *deterministic* if its representatives are deterministic, i.e. if any representative  $G$  is such that for all  $x \in \mathbf{X}$  there is at most one  $e \in E^G$  such that  $x \in S_e^G$ .

**Definition 7.** An *abstract program*  $A$  within an AMC  $\alpha : \langle G, R \rangle \curvearrowright \mathbf{X}$  is defined as a finite set  $S^A$  of *control states* and a graphing  $G^A$  w.r.t. the monoid action  $M\langle G, R \rangle \times \mathfrak{S}_k \curvearrowright \mathbf{X} \times S^A$ .

An abstract program is *deterministic* if its underlying graphing is deterministic.

## 4 THE CREW

In this section, we explain how the abstract framework described in the last section can be used to model *parallel* computation. As usual, one is bound to chose a mode of interaction between the different processes when dealing with shared memory. We will consider here only the case of *Concurrent Read Exclusive Write* (CREW), i.e. all processes can read the shared memory concurrently, but if several processes try to write in the shared memory only the process with the smallest index is allowed to do so.

We abstract the CREW mode of interaction at the level of monoid, by performing an operation reminiscent (in that it also generalizes the free product) of the amalgamated sum [11, A, I, §7, 3], but chosen relatively to monoid actions. For this, we suppose that we have two monoid actions  $M\langle G, R \rangle \curvearrowright \mathbf{X} \times \mathbf{Y}$  and  $M\langle H, Q \rangle \curvearrowright \mathbf{X} \times \mathbf{Z}$ , where  $\mathbf{X}$  represents the shared memory. Among the generators of each monoid, we will separate those that potentially conflict with the generator of the other monoid (typically a write) from the other and perform a sum over those generators.

**Definition 8** (Conflicted sum). Let  $M\langle G, R \rangle, M\langle G', R' \rangle$  be two monoids and  $\# \subseteq G \times G'$  a relation between the generators of  $G$  and  $G'$ , called the *conflict* relation, we define the *conflicted sum* of  $M\langle G, R \rangle$  and  $M\langle G', R' \rangle$  over  $\#$ , noted  $M\langle G, R \rangle *_\# M\langle G', R' \rangle$ , as the monoid  $M\langle (\{1\} \times G) \cup (\{2\} \times G'), Q \rangle$  where  $Q$  is defined as:

$$Q = (\{1\} \times G) \cup (\{2\} \times G') \cup \{(1, g)(2, g'), (2, g')(1, g)\}, (g, g') \notin \# \cup \{(1, e)\} \cup \{(1, e')\}$$

where  $1, e$  and  $e'$  are the neutral elements of  $M\langle G, R \rangle *_\# M\langle G', R' \rangle$  and its two components.

In the particular case where  $\# = (G \times H') \cup (H \times G')$ , with  $H, H'$  respectively subsets of  $G$  and  $G'$ , we will write the sum  $M\langle G, R \rangle_{H *_{H'}} M\langle G', R' \rangle$ .

*Remark.* When the conflict relation  $\#$  is empty, this defines the usual direct product of monoids. This was to be expected. Indeed, one should think of this relation as representing the elements that do not commute because they interact with the shared memory. As a consequence, when it is empty no conflicts can arise w.r.t. the shared memory. In other words, the direct product of monoids corresponds to the parallelisation of processes *without shared memory*.

Dually, when the relation full ( $\# = G \times G'$ ), it defines the free product of the monoids, so the free product corresponds to the parallelisation of processes where all instructions interact with the shared memory.

**Definition 9.** Let  $\alpha : M \curvearrowright \mathbf{X} \times \mathbf{Y}$  be a monoid action. We say that an element  $m \in M$  is *central relatively to  $\alpha$*  (or just *central*) if the action of  $m$  commutes with the first projection  $\pi_X : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{X}$ , i.e.  $\alpha(m); \pi_X = \alpha(m)$ ; in other words  $m$  acts as the identity on  $\mathbf{X}$ .

Intuitively, central elements are those that will not affect the shared memory. As such, they do not raise any issues when the processes are put in parallel. On the other hand, *non-central elements* need to be dealt with care.

**Definition 10.** Let  $M\langle G, R \rangle \curvearrowright \mathbf{X} \times \mathbf{Y}$  be an AMC. We note  $Z_\alpha$  the set of central elements and  $\bar{Z}_\alpha(G)$  the set  $\{m \in G \mid n \notin Z_\alpha\}$ .

**Definition 11** (The CREW operation). Let  $\alpha : M\langle G, R \rangle \curvearrowright \mathbf{X} \times \mathbf{Y}$  and  $\beta : M\langle H, Q \rangle \curvearrowright \mathbf{X} \times \mathbf{Z}$  be AMCs. We define the AMC

$$\text{CREW}(\alpha, \beta) : M\langle G, R \rangle_{\bar{Z}_\alpha(G)} *_{\bar{Z}_\beta(G')} M\langle G', R' \rangle \curvearrowright \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$$

by letting  $\text{CREW}(\alpha, \beta)(m, m') = \alpha(m) * \beta(m')$  on elements of  $G \times G'$ , where  $\alpha(m) * \beta(m')$  is defined as:

$$\begin{aligned} \alpha(m) * \beta(m') &: \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \rightarrow \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \\ &= \begin{cases} \Delta; [\alpha(m); \pi_Y, \beta(m')] & \text{if } m \notin \bar{Z}_\alpha(G), m' \in \bar{Z}_\beta(G'); \\ \Delta; [\alpha(m), \beta(m'); \pi_Z] & \text{otherwise} \end{cases} \end{aligned}$$

with  $\Delta : (x, y, z) \mapsto (x, y, x, z) : \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} \rightarrow \mathbf{X} \times \mathbf{Y} \times \mathbf{X} \times \mathbf{Z}$ .

We now need to check that we defined the operation on monoids and the action coherently. In other words, that the previous operation is compatible with the quotient by the adequate relations, i.e. it does define a monoid action.

LEMMA 12. *The CREW operation on AMCs is well-defined.*

## 5 ENTROPY AND CELLS

### 5.1 Topological Entropy

Topological Entropy was introduced in the context of dynamical systems in an attempt to classify the latter w.r.t. conjugacy. The topological entropy of a dynamical system is a value representing the average exponential growth rate of the number of orbit segments distinguishable with a finite (but arbitrarily fine) precision. The definition is based on the notion of open covers.

*Open covers.* Given a topological space  $\mathbf{X}$ , an *open cover* of  $\mathbf{X}$  is a family  $\mathcal{U} = (U_i)_{i \in I}$  of open subsets of  $\mathbf{X}$  such that  $\cup_{i \in I} U_i = \mathbf{X}$ . A finite cover  $\mathcal{U}$  is a cover whose indexing set is finite. A *subcover* of a cover  $\mathcal{U} = (U_i)_{i \in I}$  is a sub-family  $\mathcal{S} = (U_j)_{j \in J}$  for  $J \subseteq I$  such that  $\mathcal{S}$  is a cover, i.e. such that  $\cup_{j \in J} U_j = \mathbf{X}$ .

We will denote by  $\text{Cov}(\mathbf{X})$  (resp.  $\text{FCov}(\mathbf{X})$ ) the set of all open covers (resp. all finite open covers) of the space  $\mathbf{X}$ .

We now define two operations on open covers that are essential to the definition of entropy. An open cover  $\mathcal{U} = (U_i)_{i \in I}$ , together with a continuous function  $f : \mathbf{X} \rightarrow \mathbf{X}$ , defines the inverse image open cover  $f^{-1}(\mathcal{U}) = (f^{-1}(U_i))_{i \in I}$ . Note that if  $\mathcal{U}$  is finite,  $f^{-1}(\mathcal{U})$  is finite as well. Given two open covers  $\mathcal{U} = (U_i)_{i \in I}$  and  $\mathcal{V} = (V_j)_{j \in J}$ , we define their join  $\mathcal{U} \vee \mathcal{V}$  as the family  $(U_i \cap V_j)_{(i,j) \in I \times J}$ . Once again, if both initial covers are finite, their join is finite.

541 *Entropy.* Usually, entropy is defined for continuous maps on a compact set, following the  
 542 original definition by Adler, Konheim and McAndrews [2]. Using the fact that arbitrary  
 543 open covers have a finite subcover, this allows one to ensure that the smallest subcover  
 544 of any cover is finite. I.e. given an arbitrary cover  $\mathcal{U}$ , one can consider the smallest – in  
 545 terms of cardinality – subcover  $\mathcal{S}$  and associate to  $\mathcal{U}$  the finite quantity  $\log_2(\text{Card}(\mathcal{S}))$ .  
 546 This quantity, obviously, need not be finite in the general case of an arbitrary cover on a  
 547 non-compact set.

548 However, a generalisation of entropy to non-compact sets can easily be defined by  
 549 restricting the usual definition to *finite* covers<sup>7</sup>. This is the definition we will use here.

550 **Definition 13.** Let  $X$  be a topological space, and  $\mathcal{U} = (U_i)_{i \in I}$  be a finite cover of  $X$ . We  
 551 define the quantity  $H_X^0(\mathcal{U})$  as

$$552 \min\{\log_2(\text{Card}(J)) \mid J \subset I, \cup_{j \in J} U_j = X\}.$$

553  
 554 In other words, if  $k$  is the cardinality of the smallest subcover of  $\mathcal{U}$ ,  $H^0(\mathcal{O}) = \log_2(k)$ .

555 **Definition 14.** Let  $X$  be a topological space and  $f : X \rightarrow X$  be a continuous map. For any  
 556 finite open cover  $\mathcal{U}$  of  $X$ , we define:

$$557 H_X^k(f, \mathcal{U}) = \frac{1}{k} H_X^0(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \dots \vee f^{-(k-1)}(\mathcal{U})).$$

558 One can show that the limit  $\lim_{n \rightarrow \infty} H_X^n(f, \mathcal{U})$  exists and is finite; it will be noted  $h(f, \mathcal{U})$ .  
 559 The topological entropy of  $f$  is then defined as the supremum of these values, when  $\mathcal{U}$   
 560 ranges over the set of all finite covers  $\text{FCov}(X)$ .

561 **Definition 15.** Let  $X$  be a topological space and  $f : X \rightarrow X$  be a continuous map. The  
 562 *topological entropy* of  $f$  is defined as  $h(f) = \sup_{\mathcal{U} \in \text{FCov}(X)} h(f, \mathcal{U})$ .

## 566 5.2 Graphings and Entropy

567 We now need to define the entropy of *deterministic graphing*. As mentioned briefly already,  
 568 deterministic graphings on a space  $X$  are in one-to-one correspondence with partial  
 569 dynamical systems on  $X$ . To convince oneself of this, it suffices to notice that any partial  
 570 dynamical system can be represented as a graphing with a single edge, and that if the  
 571 graphing  $G$  is deterministic its edges can be glued together to define a partial continuous  
 572 function  $[G]$ . Thus, we only need to extend the notion of entropy to partial maps, and we  
 573 can then define the entropy of a graphing  $G$  as the entropy of its corresponding map  $[G]$ .

574 Given a finite cover  $\mathcal{U}$ , the only issue with partial continuous maps is that  $f^{-1}(\mathcal{U})$  is not  
 575 in general a cover. Indeed,  $\{f^{-1}(U) \mid U \in \mathcal{U}\}$  is a family of open sets by continuity of  $f$   
 576 but the union  $\cup_{U \in \mathcal{U}} f^{-1}(U)$  is a strict subspace of  $X$  (namely, the domain of  $f$ ). It turns out  
 577 the solution to this problem is quite simple: we notice that  $f^{-1}(\mathcal{U})$  is a cover of  $f^{-1}(X)$  and  
 578 now work with covers of subspaces of  $X$ . Indeed,  $\mathcal{U} \vee f^{-1}(\mathcal{U})$  is itself a cover of  $f^{-1}(X)$   
 579 and therefore the quantity  $H_X^2(f, \mathcal{U})$  can be defined as  $(1/2)H_{f^{-1}(X)}^0(\mathcal{U} \vee f^{-1}(\mathcal{U}))$ .

580 We now generalise this definition to arbitrary iterations of  $f$  by extending Definitions  
 581 14 and 15 to partial maps as follows.

582  
 583 <sup>7</sup>This is discussed by Hofer [26] together with another generalisation based on the Stone-Ćech compactification  
 584 of the underlying space.



**Definition 16.** Let  $\mathbf{X}$  be a topological space and  $f : \mathbf{X} \rightarrow \mathbf{X}$  be a continuous partial map. For any finite open cover  $\mathcal{U}$  of  $\mathbf{X}$ , we define:

$$H_{\mathbf{X}}^k(f, \mathcal{U}) = \frac{1}{k} H_{f^{-k+1}(\mathbf{X})}^0(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \dots \vee f^{-(k-1)}(\mathcal{U})).$$

The *entropy* of  $f$  is then defined as  $h(f) = \sup_{\mathcal{U} \in \text{FCov}(\mathbf{X})} h(f, \mathcal{U})$ , where  $h(f, \mathcal{U})$  is again defined as the limit  $\lim_{n \rightarrow \infty} H_{\mathbf{X}}^n(f, \mathcal{U})$ .

Now, let us consider the special case of a graphing  $G$  with set of control states  $S^G$ . For an intuitive understanding, one can think of  $G$  as the representation of a PRAM machine. We focus on the specific open cover indexed by the set of control states, i.e.  $\mathcal{S} = (\mathbf{X} \times \{s\}_{s \in S^G})$ , and call it *the states cover*. We will now show how the partial entropy  $H^k(G, \mathcal{S})$  is related to the set of *admissible sequence of states*. Let us define those first.

**Definition 17.** Let  $G$  be a graphing, with set of control states  $S^G$ . An *admissible sequence of states* is a sequence  $\mathbf{s} = s_1 s_2 \dots s_n$  of elements of  $S^G$  such that for all  $i \in \{1, 2, \dots, n-1\}$  there exists a subset  $C$  of  $\mathbf{X}$  – i.e. a set of configurations – such that  $G$  contains an edge from  $C \times \{s_i\}$  to a subspace of  $\mathbf{X} \times \{s_{i+1}\}$ .

*Example 18.* As an example, let us consider the very simple graphing with four control states  $a, b, c, d$  and edges from  $\mathbf{X} \times \{a\}$  to  $\mathbf{X} \times \{b\}$ , from  $\mathbf{X} \times \{b\}$  to  $\mathbf{X} \times \{c\}$ , from  $\mathbf{X} \times \{c\}$  to  $\mathbf{X} \times \{b\}$  and from  $\mathbf{X} \times \{c\}$  to  $\mathbf{X} \times \{d\}$ . Then the sequences  $abcd$  and  $abc b c b c$  are admissible, but the sequences  $aba$ ,  $abc d d$ , and  $abc b a$  are not.

**LEMMA 19.** *Let  $G$  be a graphing, and  $\mathcal{S}$  its states cover. Then for all integer  $k$ , the set  $\text{Adm}_k(G)$  of admissible sequences of states of length  $k > 1$  is of cardinality  $2^{k \cdot H^k(G, \mathcal{S})}$ .*

**PROOF.** We show that the set  $\text{Adm}_k(G)$  of admissible sequences of states of length  $k$  has the same cardinality as the smallest subcover of  $\mathcal{S} \vee [G]^{-1}(\mathcal{S}) \vee \dots \vee [G]^{-(k-1)}(\mathcal{S})$ . Hence  $H^k(G, \mathcal{S}) = \frac{1}{k} \log_2(\text{Card}(\text{Adm}_k(G)))$ , which implies the result.

The proof is done by induction. As a base case, let us consider the set of  $\text{Adm}_2(G)$  of admissible sequences of states of length 2 and the open cover  $\mathcal{V} = \mathcal{S} \vee [G]^{-1}(\mathcal{S})$  of  $D = [G]^{-1}(\mathbf{X})$ . An element of  $\mathcal{V}$  is an intersection  $\mathbf{X} \times \{s_1\} \cap [G]^{-1}(\mathbf{X} \times \{s_2\})$ , and it is therefore equal to  $C[s_1, s_2] \times \{s_1\}$  where  $C[s_1, s_2] \subset \mathbf{X}$  is the set  $\{x \in \mathbf{X} \mid [G](x, s_1) \in \mathbf{X} \times \{s_2\}\}$ . This set is empty if and only if the sequence  $s_1 s_2$  belongs to  $\text{Adm}_2(G)$ . Moreover, given another sequence of states  $s'_1 s'_2$  (not necessarily admissible), the sets  $C[s_1, s_2]$  and  $C[s'_1, s'_2]$  are disjoint. Hence a set  $C[s_1, s_2]$  is *removable from the cover  $\mathcal{V}$*  if and only if the sequence  $s_1 s_2$  is not admissible. This implies the result for  $k = 2$ .

The step for the induction is similar to the base case. It suffices to consider the partition  $\mathcal{S}_k = \mathcal{S} \vee [G]^{-1}(\mathcal{S}) \vee \dots \vee [G]^{-(k-1)}(\mathcal{S})$  as  $\mathcal{S}_{k-1} \vee [G]^{-(k-1)}(\mathcal{S})$ . By the same argument, one can show that elements of  $\mathcal{S}_{k-1} \vee [G]^{-(k-1)}(\mathcal{S})$  are of the form  $C[\mathbf{s} = (s_0 s_1 \dots s_{k-1}), s_k] \times \{s_1\}$  where  $C[\mathbf{s}, s_k] \subset \mathbf{X}$  is the set  $\{x \in \mathbf{X} \mid \forall i = 2, \dots, k, [G]^{i-1}(x, s_1) \in \mathbf{X} \times \{s_i\}\}$ . Again, these sets  $C[\mathbf{s}, s_k]$  are pairwise disjoint and empty if and only if the sequence  $s_0 s_1 \dots s_{k-1}, s_k$  is not admissible.  $\square$

A tractable bound on the number of admissible sequences of states can be obtained by noticing that the sequence  $H^k(G, \mathcal{S})$  is *sub-additive*, i.e.  $H^{k+k'}(G, \mathcal{S}) \leq H^k(G, \mathcal{S}) +$

631  $H^k(G, \mathcal{S})$ . A consequence of this is that  $H^k(G, \mathcal{S}) \leq kH^1(G, \mathcal{S})$ . Thus the number of  
 632 admissible sequences of states of length  $k$  is bounded by  $2^{k^2 H^1(G, \mathcal{S})}$ . We now study how  
 633 the cardinality of admissible sequences can be related to the entropy of  $G$ .

634 LEMMA 20. For all  $\epsilon > 0$ , there exists an integer  $N$  such that for all  $k \geq N$ ,  $H_k(\mathcal{U})/k <$   
 635  $h([G]) + \epsilon$ .

636 PROOF. Let us fix some  $\epsilon > 0$ . Notice that if we let  $H_k(\mathcal{U}) = H^0(\mathcal{U} \vee [G]^{-1}(\mathcal{U}) \vee \dots \vee$   
 637  $[G]^{-(k-1)}(\mathcal{U}))$ , the sequence  $H_k(\mathcal{U})$  satisfies  $H_{k+l}(\mathcal{U}) \leq H_k(\mathcal{U}) + H_l(\mathcal{U})$ . By Fekete's  
 638 lemma on subadditive sequences, this implies that  $\lim_{k \rightarrow \infty} H_k/k$  exists and is equal to  
 639  $\inf_k H_k/k$ . Thus  $h([G], \mathcal{U}) = \inf_k H_k/k$ .

640 Now, the entropy  $h([G])$  is defined as  $\sup_{\mathcal{U}} \lim_{k \rightarrow \infty} H_k(\mathcal{U})/k$ . This then rewrites as  
 641  $\sup_{\mathcal{U}} \inf_k H_k(\mathcal{U})/k$ . We can conclude that  $h([G]) \geq \inf_k H_k(\mathcal{U})/k$  for all finite open cover  
 642  $\mathcal{U}$ .

643 Since  $\inf_k H_k(\mathcal{U})/k$  is the limit of the sequence  $H_k/k$ , there exists an integer  $N$  such  
 644 that for all  $k \geq N$  the following inequality holds:  $|H_k(\mathcal{U})/k - \inf_k H_k(\mathcal{U})/k| < \epsilon$ , which  
 645 rewrites as  $H_k(\mathcal{U})/k - \inf_k H_k(\mathcal{U})/k < \epsilon$ . From this we deduce  $H_k(\mathcal{U})/k < h([G]) + \epsilon$ .  $\square$

646 LEMMA 21. Let  $G$  be a graphing, and let  $c : k \mapsto \text{Card}(\text{Adm}_k(G))$ . Then  $c(k) = O(2^k 2^{h([G])})$   
 647 as  $k$  goes to infinity.

### 648 5.3 Cells Decomposition

649 Now, let us consider a deterministic graphing  $G$ , with its state cover  $\mathcal{S}$ . We fix a length  $k > 2$   
 650 and reconsider the sets  $C[\mathbf{s}] = C[(s_1 s_2 \dots s_{k-1}, s_k)]$  (for a sequence of states  $\mathbf{s} = s_1 s_2 \dots s_k$ )  
 651 that appear in the proof of Lemma 19. The set  $(C[\mathbf{s}])_{\mathbf{s} \in \text{Adm}_k(G)}$  is a partition of the space  
 652  $[G]^{-k+1}(\mathbf{X})$ .

653 This decomposition splits the set of initial configurations into cells satisfying the follow-  
 654 ing property: for any two initial configurations contained in the same cell  $C[\mathbf{s}]$ , the  $k$ -th first  
 655 iterations of  $G$  goes through the same admissible sequence of states  $\mathbf{s}$ .

656 **Definition 22.** Let  $G$  be a deterministic graphing, with its state cover  $\mathcal{S}$ . Given an integer  
 657  $k$ , we define the  $k$ -fold decomposition of  $\mathbf{X}$  along  $G$  as the partition  $\{C[\mathbf{s}] \mid \mathbf{s} \in \text{Adm}_k(G)\}$ .

658 Then Lemma 19 provides a bound on the cardinality of the  $k$ -th cell decomposition.  
 659 Using the results in the previous section, we can then obtain the following proposition.

660 **Proposition 23.** Let  $G$  be a deterministic graphing, with entropy  $h(G)$ . The cardinality of  
 661 the  $k$ -th cell decomposition of  $\mathbf{X}$  w.r.t.  $G$ , as a function  $c(k)$  of  $k$ , is asymptotically bounded by  
 662  $g(k) = 2^k 2^{h([G])}$ , i.e.  $c(k) = O(g(k))$ .

663 We also state another bound on the number of cells of the  $k$ -th cell decomposition, based  
 664 on the state cover entropy, i.e. the entropy with respect to the state cover rather than the  
 665 usual entropy which takes the supremum of cover entropies when the cover ranges over  
 666 all finite covers of the space. This result is a simple consequence of Theorem 19.

667 **Proposition 24.** Let  $G$  be a deterministic graphing. We consider the state cover entropy  
 668  $h_0([G]) = \lim_{n \rightarrow \infty} H_{\mathbf{X}}^n([G], \mathcal{S})$  where  $\mathcal{S}$  is the state cover. The cardinality of the  $k$ -th cell  
 669 decomposition of  $\mathbf{X}$  w.r.t.  $G$ , as a function  $c(k)$  of  $k$ , is asymptotically bounded by  $g(k) =$   
 670  $2^k 2^{h_0([G])}$ , i.e.  $c(k) = O(g(k))$ .

671

## 6 ALGEBRAIC COMPUTATION TREES AND BEN-OR'S TECHNIQUE

We will now explain how to obtain lower bounds for algebraic models of computation based on the interpretation of programs as graphings and entropic bounds. These results make use of the Milnor-Thom theorem which bounds the sum of the Betti numbers of algebraic varieties. In fact, we will use a version due to Ben-Or of this theorem.

### 6.1 Milnor-Thom theorem

Let us first recall the classic Milnor-Thom theorem.

**Theorem 25** ([33, Theorem 3]). *If  $X \subseteq \mathbf{R}^m$  is defined by polynomial identities of the form*

$$f_1 \geq 0, \dots, f_p \geq 0$$

*with total degree  $d = \deg f_1 + \dots + \deg f_p$ , then*

$$\text{rank } H^* X \leq \frac{1}{2}(2+d)(1+d)^{m-1}.$$

We will use in the proof the following variant of the Milnor-Thom bounds, stated and proved by Ben-Or.

**Theorem 26.** *Let  $d, n, h \in \mathbf{N}$ .*

*Let  $\beta_d(n, h)$  be the maximal number of connected components of sets  $V \subseteq \mathbf{R}^n$  be a set defined by the following polynomial equations:*

$$\left\{ \begin{array}{l} q_1(x_1, \dots, x_n) = 0 \\ \vdots \\ q_m(x_1, \dots, x_n) = 0 \\ p_1(x_1, \dots, x_n) > 0 \\ \vdots \\ p_s(x_1, \dots, x_n) > 0 \\ p_{s+1}(x_1, \dots, x_n) \geq 0 \\ \vdots \\ p_h(x_1, \dots, x_n) \geq 0 \end{array} \right.$$

*for  $p_i, q_i \in \mathbf{R}[X_1, \dots, X_n]$  of degree lesser than  $d$ .*

*If  $d \geq 2$ , we have:*

$$\beta_d(n, h) \leq d(2d-1)^{n+h-1}$$

First, we will write composition of functions as  $f;g$  instead of  $g \circ f$ .

### 6.2 Algebraic decision trees

One lower bounds result related to Mulmuley's techniques is the bounds obtained by Steele and Yao [46] on *Algebraic Decision Trees*. Algebraic decision trees are defined as finite ternary trees describing a program deciding a subset of  $\mathbf{R}^n$ : each node verifies whether a chosen polynomial, say  $P$ , takes a positive, negative, or null value at the point considered.

**Definition 27** ([46]). Let  $n \in \mathbb{N}$ .

A  $d$ -th order algebraic decision tree for  $\mathbb{R}^n$  is a ternary tree where

- each internal node contains a test of the form  $p(x_1, x_2, \dots, x_n) : 0$ , where  $p$  is a polynomial of degree at most  $d$ ;
- each leaf is labelled by yes or no.

We say that the son of an internal node labeled by a polynomial  $p$  is *consistent* for  $(x_1, \dots, x_n) \in \mathbb{R}^n$  if it is the right son and  $p(x_1, \dots, x_n) > 0$ , the middle son and  $p(x_1, \dots, x_n) = 0$ , or the left son and  $p(x_1, \dots, x_n) < 0$ . A branch is *consistent* for  $(x_1, \dots, x_n) \in \mathbb{R}^n$  if all the sons of the internal nodes in the branch are consistent for  $(x_1, \dots, x_n)$ .

An algebraic decision tree *decides* a set  $W \subseteq \mathbb{R}^n$  if, for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $(x_1, \dots, x_n) \in W$  if and only if the unique maximal branch consistent with  $(x_1, \dots, x_n)$  ends on a leaf labelled by yes.

We now define an AMC of algebraic decision trees. In a very peculiar way, the underlying space of algebraic decision trees is  $\mathbb{R}^n$ , and the set of generators and relations of the monoid is empty (which means that the monoid is  $\{\star\}$ ), so the AMC is  $1 : \langle \emptyset, \emptyset \rangle \curvearrowright \mathbf{X}$  where  $1$  denotes the trivial action. Intuitively, this is to be expected as algebraic decision trees do not act on the space of configuration.

Let  $T$  be an algebraic decision tree. It can be described as a finite set  $\mathbf{S}^T = \{\top, \perp, p_1, \dots, p_n\}$  where the  $(p_i)_{1 \leq i \leq n}$  are polynomials on  $\mathbb{R}$ , together with a relation between the elements of the control states.

**Definition 28.** Let  $T$  be an algebraic decision tree. We define  $[T]$  as the graphing with set of control states  $\{\top, \perp, p_1, \dots, p_n\}$  where the  $(p_i)_{1 \leq i \leq n}$  are the polynomials of  $T$ , and each internal node with label  $p$  and sons  $(a, b, c)$  defines three edges:

- one of source  $\{\vec{x} \in \mathbb{R}^n \mid p(x_1, \dots, x_n) > 0\} \times \{p\}$  realized by  $(\text{Id}, p \mapsto a)$ ;
- one of source  $\{\vec{x} \in \mathbb{R}^n \mid p(x_1, \dots, x_n) = 0\} \times \{p\}$  realized by  $(\text{Id}, p \mapsto b)$ ;
- one of source  $\{\vec{x} \in \mathbb{R}^n \mid p(x_1, \dots, x_n) < 0\} \times \{p\}$  realized by  $(\text{Id}, p \mapsto c)$ .

From Theorem 23, one obtains easily the following theorem.

**Theorem 29.** Let  $T$  be a  $d$ -th order algebraic decision tree deciding a subset  $W \subseteq \mathbb{R}^n$ . Then the number of connected components of  $W$  is bounded by  $2^h d(2d - 1)^{n+h-1}$ , where  $h$  is the height of  $T$ .

**PROOF.** We let  $h$  be the height of  $T$ , and  $d$  be the maximal degree of the polynomials appearing in  $T$ . Then the  $h$ -th cell decomposition of  $[T]$  defines a family of semi-algebraic sets defined by  $h$  polynomials equalities and inequalities of degree at most  $d$ . Moreover, Theorem 24 states that this family has cardinality bounded by  $2^h 2_0^h([T])$ ; since  $h_0([T]) = 0$  because each state has at most one antecedent state, this bound becomes  $2^h$ . Thus, the  $h$ -th cell decomposition defines at most  $2^h$  algebraic sets which have at most  $d(2d - 1)^{n+h-1}$  connected components. Since the set  $W$  decided by  $T$  is obtained as a union of the semi-algebraic sets in the  $h$ -th cell decomposition, it has at most  $2^h d(2d - 1)^{n+h-1}$  connected components.  $\square$

**COROLLARY 30** (STEELE AND YAO [46]). A  $d$ -th order algebraic decision tree deciding a subset  $W \subseteq \mathbb{R}^n$  with  $N$  connected components has height  $\Omega(\log N)$ .

766 This result of Steele and Yao adapts in a straightforward manner to a notion of algebraic  
 767 computation trees describing the construction of the polynomials to be tested by mean of  
 768 multiplications and additions of the coordinates. The authors remarked this result uses  
 769 techniques quite similar to that of Mulmuley's lower bounds for the model of PRAMS *without*  
 770 *bit operations*. It is also strongly similar to the techniques used by Cucker in proving that  
 771  $\text{NC}_R \subsetneq \text{PTIME}_R$  [16].

772 However, a refinement of Steele and Yao's method was quickly obtained by Ben-Or so  
 773 as to allow for computing divisions and taking square roots in this notion of algebraic  
 774 computation trees. We will now explain Ben-Or techniques from within the framework of  
 775 graphings. We will later adapt this refinement of Steele and Yao's method to Mulmuley's  
 776 PRAMS *without bit operations*, in order to obtain the main theorem of this paper.

### 778 6.3 Algebraic Computational Trees

779 Algebraic computational trees follow the same principles as algebraic decision trees, but  
 780 they allow for the representation of computations as part of the tree. I.e. one consider  
 781 nodes for every algebraic operation on the set of polynomials.

782 More formally, an algebraic computational tree is defined from the nodes  $\times$ ,  $+$ ,  $-$ ,  $/$ ,  $\sqrt{\quad}$   
 783 and a test node with three sons corresponding to  $< 0$ ,  $= 0$  and  $> 0$  as in the algebraic  
 784 decision trees case.

785 The difference is thus that algebraic computation trees only perform tests on expression  
 786 that are first defined by means of algebraic operations. If one restricts to the fragment  
 787 without division and square root, the overall computational power, i.e. the sets decided,  
 788 of computational trees and decision trees are the same. However, while testing wether a  
 789 given polynomial is greater than 0 need only one node in an algebraic decision tree, in  
 790 general it requires more in algebraic computational trees since one needs to compute the  
 791 polynomial explicitly from basic algebraic operations.

792 It is not a surprise then that similar bounds to that of algebraic decisions trees can be  
 793 computed using similar methods in the restricted fragment without division and square  
 794 roots. An improvement on this is the result of Ben-Or generalising the technique to  
 795 algebraic computational trees with division and square root nodes. The principle is quite  
 796 simple: one simply adds additional variables to avoid using the square root or division,  
 797 obtaining in this way a system of polynomial equations. For instance, instead of writing  
 798 the equation  $p/q < 0$ , one defines a fresh variable  $r$  and considers the system

$$800 \quad p = qr; r < 0$$

801 This method seems different from the direct entropy bound obtained in the case of  
 802 algebraic decision trees. However, we will see how it can be adapted directly to graphings.

803 Given an integer  $k \in \omega$ , we define the following subspaces of  $\mathbf{R}^\omega$ :

- 804 •  $\mathbf{R}_{k \geq 0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k \geq 0\}$ ;
- 805 •  $\mathbf{R}_{k \leq 0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k \leq 0\}$ ;
- 806 •  $\mathbf{R}_{k > 0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k > 0\}$ ;
- 807 •  $\mathbf{R}_{k < 0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k < 0\}$ ;
- 808 •  $\mathbf{R}_{k=0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k = 0\}$ ;

810

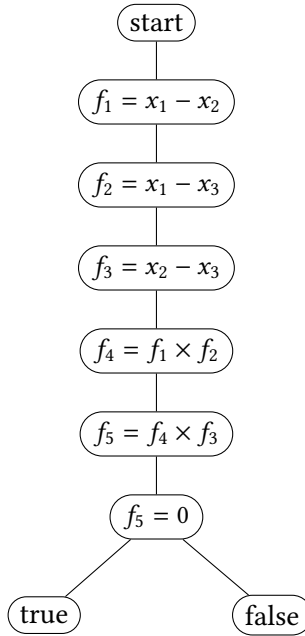


Fig. 2. An algebraic computation tree

- $\mathbf{R}_{k \neq 0}^\omega = \{(x_1, \dots, x_k, \dots) \in \mathbf{R}^\omega \mid x_k \neq 0\}$ .

**Definition 31** (treeings). A *treeing* is an acyclic and finite graphing, i.e. a graphing  $F$  for which there exists a finite graphing representative  $T$  with set of control states  $\mathbf{S}^\top = \{0, \dots, s\}$  and such that every edge of  $T$  is state-increasing, i.e. for each edge  $e$  of source  $S_e$ , for all  $x \in S_e$ ,

$$\pi_{\mathbf{S}^\top}(\alpha(m_e)(x)) > \pi_{\mathbf{S}^\top}(x),$$

where  $\pi_{\mathbf{S}^\top}$  denotes the projection onto the control states space.

A *computational graphing* is a graphing  $T$  with distinguished states  $\top, \perp$  which admits a finite representative such that each edge  $e$  has its source equal to one among  $\mathbf{R}^\omega$ ,  $\mathbf{R}_{k \geq 0}^\omega$ ,  $\mathbf{R}_{k \leq 0}^\omega$ ,  $\mathbf{R}_{k > 0}^\omega$ ,  $\mathbf{R}_{k < 0}^\omega$ ,  $\mathbf{R}_{k=0}^\omega$ , and  $\mathbf{R}_{k \neq 0}^\omega$ .

A *computational treeing* is a treeing  $T$  which is a computational graphing with the distinguished states  $\top, \perp$  being incomparable maximal elements of the state space.

**Definition 32** (Algebraic computation trees, [10]). An *algebraic computation tree* on  $\mathbf{R}^n$  is a binary tree  $T$  with a function that assigns:

- to any vertex  $v$  with only one son (simple vertex) an operational instruction of the form

$$f_v = f_{v_i} \circ f_{v_j}$$

$$f_v = c \circ f_{v_i}$$

$$f_v = \sqrt{f_{v_i}}$$



where  $\circ \in \{+, -, \times, /\}$ ,  $v_i, v_j$  are ancestors of  $v$  and  $c \in \mathbf{R}$  is a constant;

- to any vertex  $v$  with two sons a test instruction of the form

$$f_{v_i} > 0$$

$$f_{v_i} = 0$$

$$f_{v_i} \geq 0$$

where  $v_i$  is an ancestor of  $v$  or  $f_{v_i} \in \{x_1, \dots, x_n\}$ ;

- to any leaf an output YES or NO.

Let  $W \subseteq \mathbf{R}^n$  be any set and  $T$  be an algebraic computation tree. We say that  $T$  computes the membership problem for  $W$  if for all  $x \in \mathbf{R}^n$ , the traversal of  $T$  following  $x$  ends on a leaf labelled YES if and only if  $x \in W$ .

We can define the AMC of algebraic computation trees. The underlying space is  $\mathbf{R}^\omega$  and the acting monoid is generated by  $\text{add}_i(j, k)$ ,  $\text{sub}_i(j, k)$ ,  $\text{mult}_i(j, k)$ ,  $\text{div}_i(j, k)$ ,  $\text{add}_i(c, j)$ ,  $\text{sub}_i(c, j)$ ,  $\text{mult}_i(c, j)$ ,  $\text{div}_i(c, j)$ ,  $\text{sqrt}_i(j)$ , for  $i, j, k \in \omega$  and  $c \in \mathbf{R}$  acting on  $\mathbf{R}^\omega$  as follows:

- $\alpha(\text{add}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j + x_k, x_{i+1}, \dots)$ ;
- $\alpha(\text{sub}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j - x_k, x_{i+1}, \dots)$ ;
- $\alpha(\text{mult}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j \times x_k, x_{i+1}, \dots)$ ;
- $\alpha(\text{div}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j / x_k, x_{i+1}, \dots)$  if  $x_k \neq 0$ ;
- $\alpha(\text{add}_i(c, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, c + x_k, x_{i+1}, \dots)$ ;
- $\alpha(\text{sub}_i(c, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, c - x_k, x_{i+1}, \dots)$ ;
- $\alpha(\text{mult}_i(c, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, c \times x_k, x_{i+1}, \dots)$ ;
- $\alpha(\text{div}_i(c, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, c / x_k, x_{i+1}, \dots)$  if  $x_k \neq 0$ ;
- $\alpha(\text{sqrt}_i(k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, \sqrt{x_k}, x_{i+1}, \dots)$  if  $x_k \geq 0$ .

**Definition 33.** Let  $T$  be a computational treeing on the AMC of algebraic computational trees. The set of inputs  $\text{In}(T)$  (resp. outputs  $\text{Out}(T)$ ) is the set of integers  $k$  (resp.  $i$ ) such that there exists an edge  $e$  in  $T$ :

- either  $e$  is realised by one of  $\text{add}_i(j, k)$ ,  $\text{add}_i(k, j)$ ,  $\text{sub}_i(j, k)$ ,  $\text{sub}_i(k, j)$ ,  $\text{mult}_i(j, k)$ ,  $\text{mult}_i(k, j)$ ,  $\text{div}_i(j, k)$ ,  $\text{div}_i(k, j)$ ,  $\text{add}_i(c, k)$ ,  $\text{sub}_i(c, k)$ ,  $\text{mult}_i(c, k)$ ,  $\text{div}_i(c, k)$ ,  $\text{sqrt}_i(k)$ ;
- or the source of  $e$  is one among  $\mathbf{R}_{k \geq 0}^\omega$ ,  $\mathbf{R}_{k < 0}^\omega$ ,  $\mathbf{R}_{k > 0}^\omega$ ,  $\mathbf{R}_{k < 0}^\omega$ ,  $\mathbf{R}_{k=0}^\omega$ , and  $\mathbf{R}_{k \neq 0}^\omega$ .

The *input space*  $\text{In}(T)$  of a treeing  $T$  on the AMC of algebraic computational trees is defined as the set of indices  $k \in \omega$  belonging to  $\text{In}(T)$  but not to  $\text{Out}(T)$ .

**Definition 34.** Let  $T$  be a treeing on the AMC of computational trees, and let  $n$  be an integer larger than the maximal element in  $\text{In}(T)$ . We say that  $T$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  if for all  $(x_1, \dots, x_n) \in \mathbf{R}^n$ , the successful iterations of  $T$  on the subspace  $\{(y_1, \dots, y_n, \dots) \in \mathbf{R}^\omega \mid \forall 1 \geq i \geq n, y_i = x_i\} \times \{0\}$  reach the state  $\top$  if and only if  $x \in W$ .

*Remark.* Consider two elements  $a, b$ , in  $\{(y_1, \dots, y_n, \dots) \in \mathbf{R}^\omega \mid \forall 1 \geq i \geq n, y_i = x_i\} \times \{0\}$ . One can easily check that  $\pi_S(T^n(a)) = \top$  if and only if  $\pi_S(T^n(b)) = \top$ , where  $\pi_S$  is the projection onto the state space and  $T^n(a)$  represents the  $n$ -th iteration of  $T$  on  $a$ . It is therefore possible to consider only a standard representative of  $(x_1, \dots, x_n) \in \mathbf{R}^n$ , for instance  $(x_1, \dots, x_n, 0, 0, \dots) \in \mathbf{R}^\omega$ , to decide whether  $(x_1, \dots, x_n) \in \mathbf{R}^n$  is accepted by  $T$ .

**Definition 35.** Let  $T$  be an algebraic computation tree on  $\mathbf{R}^n$ , and  $T^\circ$  be the associated directed acyclic graph, built from  $T$  by merging all the leaves tagged YES in one leaf  $\top$  and all the leaves tagged NO in one leaf  $\perp$ . Suppose the internal vertices are numbered  $\{n+1, \dots, n+\ell\}$ ; the numbers  $1, \dots, n$  being reserved for the input.

We define  $[T]$  as the graphing with control states  $\{n+1, \dots, n+\ell, \top, \perp\}$  and where each internal vertex  $i$  of  $T^\circ$  defines either:

- a single edge of source  $\mathbf{R}^\omega$  realized by:
  - $(\text{add}_i(j, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = f_{v_j} + f_{v_k}$  and  $t$  is the son of  $i$ ;
  - $(\text{sub}_i(j, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = f_{v_j} - f_{v_k}$  and  $t$  is the son of  $i$ ;
  - $(\text{mult}_i(j, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = f_{v_j} \times f_{v_k}$  and  $t$  is the son of  $i$ ;
  - $(\text{add}_i(c, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = c + f_{v_k}$  and  $t$  is the son of  $i$ ;
  - $(\text{sub}_i(c, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = c - f_{v_k}$  and  $t$  is the son of  $i$ ;
  - $(\text{mult}_i(c, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = c \times f_{v_k}$  and  $t$  is the son of  $i$ ;
- a single edge of source  $\{(x_1, \dots, x_{n+\ell}, \dots) \in \mathbf{R}^\omega \mid x_k \neq 0\}$  realized by:
  - $(\text{div}_i(j, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = f_{v_j}/f_{v_k}$  and  $t$  is the son of  $i$ ;
  - $(\text{div}_i(c, k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = c/f_{v_k}$  and  $t$  is the son of  $i$ ;
- a single edge of source  $\mathbf{R}_{k \neq 0}^\omega \times \{i\}$  realized by  $(\text{sqrt}_i(k), i \mapsto t)$  if  $i$  is associated to  $f_{v_i} = \sqrt{f_{v_k}}$  and  $t$  is the son of  $i$ ;
- a pair of edges:
  - one of source  $\mathbf{R}_{k > 0}^\omega \times \{i\}$  realized by  $(\text{Id}, i \mapsto j)$  and one of source  $\mathbf{R}_{k \leq 0}^\omega \times \{i\}$  realized by  $(\text{Id}, i \mapsto k)$  if  $i$  is associated to  $f_{v_i} > 0$  and its two sons are  $j$  and  $k$ ;
  - one of source  $\mathbf{R}_{k \geq 0}^\omega \times \{i\}$  realized by  $(\text{Id}, i \mapsto j)$  and one of source  $\mathbf{R}_{k < 0}^\omega \times \{i\}$  realized by  $(\text{Id}, i \mapsto k)$  if  $i$  is associated to  $f_{v_i} \geq 0$  and its two sons are  $j$  and  $k$ ;
  - one of source  $\mathbf{R}_{k=0}^\omega \times \{i\}$  realized by  $(\text{Id}, i \mapsto j)$  and one of source  $\mathbf{R}_{k \neq 0}^\omega \times \{i\}$  realized by  $(\text{Id}, i \mapsto k)$  if  $i$  is associated to  $f_{v_i} > 0$  and its two sons are  $j$  and  $k$ ;

**Proposition 36.** Let  $\alpha$  be the AMC of algebraic computation trees. Then any algebraic computation tree  $T$  is faithfully interpreted as an  $\alpha$ -graphing  $[T]$ . I.e.  $T$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  if and only if  $[T]$  computes the membership problem for  $W$ .

Moreover,  $[T]$  is a computational treeing, and for any computational treeing  $G$  computing the membership problem for  $W \subseteq \mathbf{R}^n$ , there exists an algebraic computation tree  $T_W$  computing the membership problem for  $W$ .

**PROOF.** A computation tree defines a graphing  $[T]$ , and the natural graphing representative obtained from the inductive definition of  $[T]$  is clearly a treeing because  $T$  is a tree. That this treeing represents faithfully the computational tree  $T$  raises no difficulty.

Let us now show that the membership problem of a subset  $W \subseteq \mathbf{R}^n$  that can be decided by a computational treeing is also decided by an algebraic computational tree  $T$ . We prove the result by induction on the number of states of the computational treeing. The initial case is when  $T$  the set of states is exactly  $\{1, \top, \perp\}$  with the order defined by  $1 < \top$  and  $1 < \perp$  and no other relations. This computational treeing has at most 2 edges, since it is deterministic and the source of each edge is a subset among  $\mathbf{R}^\omega$ ,  $\mathbf{R}_{k \geq 0}^\omega$ ,  $\mathbf{R}_{k \leq 0}^\omega$ ,  $\mathbf{R}_{k > 0}^\omega$ ,  $\mathbf{R}_{k < 0}^\omega$ ,  $\mathbf{R}_{k=0}^\omega$ , and  $\mathbf{R}_{k \neq 0}^\omega$ .

We first treat the case when there is only one edge of source  $\mathbf{R}^n$ . An element  $(x_1, \dots, x_n) \in \mathbf{R}^n$  is decided by  $T$  if the main representative  $((x_1, \dots, x_n, 0, \dots), 1)$  is mapped to  $\top$ . Since there is only one edge of source the whole space, either this edge maps into the state  $\top$  and the decided subset  $W$  is equal to  $\mathbf{R}^n$ , or it maps into  $\perp$  and the subset  $W$  is empty. In both cases, there exists an algebraic computational tree deciding  $W$ . For the purpose of the proof, we will however construct a specific algebraic computation tree, namely the one that first computes the right expression and then accepts or rejects. I.e. if the only edge is mapping into  $\top$  (resp.  $\perp$ ) is realised by an element  $m$  in the AMC of algebraic computation trees which can be written as a product of generators  $g_1, \dots, g_k$ , we construct the tree of height  $k + 1$  that performs (in that order) the operations corresponding to  $g_1, g_2$ , etc., and then answers "yes" (resp. "no").

Now, the case where there is one edge of source a strict subspace, e.g.  $\mathbf{R}_{k \geq 0}^\omega$  (all other cases are treated in a similar manner) and mapping into  $\top$  (the other case is treated by symmetry). First, let us remark that if there is no other edge, one could very well add an edge to  $T$  mapping into  $\perp$  and realised by the identity with source the complementary subspace  $\mathbf{R}_{k < 0}^\omega$ . We build a tree as follows. First, we test whether the variable  $x_k$  is greater or equal to zero; this node has two sons corresponding to whether the answer to the test is "yes" or "no". We now construct the two subtrees corresponding to these two sons. The branch corresponding to "yes" is described by the edge of source  $\mathbf{R}_{k \geq 0}^\omega$ : we construct the tree of height  $k + 1$  performing the operations corresponding to the generators  $g_1, g_2$ , etc. whose product defined the realiser  $m$  of  $e$ , and then answers "yes" (resp. "no") if the edge  $e$  maps into the state  $\top$  (resp.  $\perp$ ). Similarly, the other subtree is described by the realiser of the edge of source  $\mathbf{R}_{k < 0}^\omega$ .

The result then follows by induction, plugging small subtrees as described above in place of the leaves of smaller subtrees.  $\square$

## 6.4 Entropic co-trees and $k$ -th computational forests

**Definition 37** ( $k$ -th entropic co-tree). Consider a deterministic graphing representative  $T$ , and fix an element  $\top$  of the set of control states. We can define the  $k$ -th entropic co-tree of  $T$  along  $\top$  and the state cover inductively:

- $k = 0$ , the co-tree  $\text{coT}_0(T)$  is simply the root  $n^\epsilon = \mathbf{R}^n \times \{\top\}$ ;
- $k = 1$ , one considers the preimage of  $n^\epsilon$  through  $T$ , i.e.  $T^{-1}(\mathbf{R}^n \times \{\top\})$  the set of all non-empty sets  $\alpha(m_e)^{-1}(\mathbf{R}^n \times \{\top\})$  and intersects it pairwise with the state cover, leading to a finite family (of cardinality bounded by the number of states multiplied by the number of edges of  $T$ )  $(n^i)_i$  defined as  $n^i = T^{-1}(n^\epsilon) \cap \mathbf{R}^n \times \{i\}$ . The first entropic co-tree  $\text{coT}_1(T)$  of  $T$  is then the tree defined by linking each  $n^i_e$  to  $n^\epsilon$  with an edge labelled by  $m_e$ ;
- $k + 1$ , suppose defined the  $k$ -th entropic co-tree of  $T$ , defined as a family of elements  $n_e^\pi$  where  $\pi$  is a finite sequence of states of length at most  $k$  and  $e$  a sequence of edges of  $T$  of the same length, and where  $n_e^\pi$  and  $n_{e'}^{\pi'}$  are linked by an edge labelled  $f$  if and only if  $\pi' = \pi.s$  and  $e' = f.e$  where  $s$  is a state and  $f$  an edge of  $T$ . We consider the subset of elements  $n_e^\pi$  where  $\pi$  is exactly of length  $k$ , and for each such element we define new nodes  $n_{e.e'}^{\pi.s}$  defined as  $\alpha(m_e)^{-1}(n_e^\pi) \cap \mathbf{R}^n \times \{s\}$  when it is non-empty. The

$k + 1$ -th entropic co-tree  $\text{coT}_{k+1}(T)$  is defined by extending the  $k$ -th entropic co-tree  $\text{coT}_k(T)$ , adding the nodes  $n_{e.e}^{\pi.s}$  and linking them to  $n_e^\pi$  with an edge labelled by  $e$ .

*Remark.* The co-tree can alternatively be defined non-inductively in the following way: the  $n_e^\pi$  for  $\pi$  is a finite sequence of states and  $e$  a sequence of edges of  $T$  of the same length by  $n_e^\epsilon = \mathbf{R}^n \times \{\top\}$  and

$$n_{e.e}^{\pi.s} = [\alpha(m_e)^{-1}(n_e^\pi)] \cap [\mathbf{R}^n \times \{s\}]$$

The  $k$ -th entropic co-tree of  $T$  along  $\top$  has as vertices the non-empty sets  $n_e^\pi$  for  $\pi$  and  $e$  of length at most  $k$  and as only edges, links  $n_{e.e}^{\pi.s} \rightarrow n_e^\pi$  labelled by  $m_e$ .

This definition formalises a notion that appears more or less clearly in the work of Lipton and Steele, and of Ben-Or, as well as in the proof by Mulmuley. The nodes for paths of length  $k$  in the  $k$ -th co-tree corresponds to the  $k$ -th cell decomposition, and the corresponding path defines the polynomials describing the semi-algebraic set decided by a computational tree. The co-tree can be used to reconstruct the algebraic computation tree  $T$  from the graphing representative  $[T]$ , or constructs *some* algebraic computation tree (actually a forest) that approximates the computation of the graphing  $F$  under study when the latter is not equal to  $[T]$  for some tree  $T$ .

**Definition 38** ( $k$ -th computational forest). Consider a deterministic graphing  $T$ , and fix an element  $\top$  of the set of control states. We define the  $k$ -th computational forest of  $T$  along  $\top$  and the state cover as follows. Let  $\text{coT}_k(T)$  be the  $k$ -th entropic co-tree of  $T$ . The  $k$ -th computational forest of  $T$  is defined by regrouping all elements  $n_{e.e'}^\pi$  of length  $m$ : if the set  $N_e^m = \{n_{e.e'}^\pi \in \text{coT}_k(T) \mid \text{len}(\pi) = m\}$  is non-empty it defines a new node  $N_e^m$ . Then one writes down an edge from  $N_e^m$  to  $N_{e'}^{m-1}$ , labelled by  $e$ , if and only if there exists  $n_{e.e'.\bar{f}}^{s.\pi} \in N_e^m$  such that  $n_{e'.\bar{f}}^\pi \in N_{e'}^{m-1}$ .

One checks easily that the  $k$ -th computational forest is indeed a forest: an edge can exist between  $N_e^m$  and  $N_f^n$  only when  $n = m + 1$ , a property that forbids cycles. The following proposition shows how the  $k$ -th computational forest is linked to computational trees.

**Proposition 39.** *If  $T$  is a computational tree of depth  $k$ , the  $k$ -th computational forest of  $[T]$  is a tree which defines straightforwardly a graphing (treeing) representative of  $T$ .*

We now state and prove an easy bound on the size of the entropic co-trees.

**Proposition 40** (Size of the entropic co-trees). *Let  $T$  be a graphing representative,  $\text{Card}(E)$  its number of edges. The number of nodes of its  $k$ -th entropic co-tree  $\text{coT}_k(T)$ , as a function  $n(k)$  of  $k$ , is asymptotically bounded by  $2^{k+1}2^{\text{Card}(E) \cdot h([G])}$ , i.e.  $n(k) = O(2^{k+1})$ .*

PROOF. The number of elements  $n_{e.e'}^\pi$  of length  $m$  in  $\text{coT}_k(T)$  is equal to the number of elements in the  $m$ -th cell decomposition of  $T$ , and is therefore bounded by  $g(m) = 2^m 2^{h([T])}$  by Theorem 23. The size of  $\text{coT}_k(T)$  is thus bounded by  $2^{k+1} 2^{h([T])}$ .  $\square$

## 6.5 The theorem Ben-Or

We now use Ben-Or proof technique to obtain a bound on the number of connected components of the subsets  $W \subseteq \mathbf{R}^n$  whose membership problem is computed by a graphing in less than a given number of iterations.

This theorem specialises to the original theorem by Ben-Or relating the number of connected components of a set  $W$  and the depth of the algebraic computational trees that compute the membership problem for  $W$ .

**Theorem 41.** *Let  $G$  be a computational graphing representative with edges realised only by generators of the AMC of algebraic computational trees, and  $\text{Card}(E)$  its number of edges. Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$  in  $k$  steps, i.e. for each element of  $\mathbf{R}^n$ ,  $\pi_S(G^k(x)) = \top$  if and only if  $x \in W$ . Then  $W$  has at most  $2^{h_0(|G|)+1} 3^{2k+n+1}$  connected components.*

**PROOF.** If  $G$  computes the membership problem for  $W$  in  $k$  steps, it means  $W$  can be described as the union of the subspaces corresponding to the nodes  $n_e^\pi$  with  $\pi$  of length  $k$  in  $\text{coT}_k(T)$ . Now, each such subspace is an algebraic set, as it can be described by a set of polynomials as follows.

We define a system of equations  $(E_i^e)_i$  for each node  $n_e^\pi$  of the entropic co-tree  $\text{coT}_k(T)$ . This is done inductively on the size of the path  $\vec{e}$ , keeping track of the last modifications of each register. I.e. we define both the system of equations  $(E_i^e)_i$  and a function  $h(\mathbf{e}) : \mathbf{R}^\omega + \perp \rightarrow \omega$  (which is almost everywhere null)<sup>8</sup>. For an empty sequence, the system of equations is empty, and the function  $h(\epsilon)$  is constant, equal to 0.

Suppose now that  $\vec{e}' = (e_1, \dots, e_m, e_m + 1)$ , with  $\vec{e} = (e_1, \dots, e_m)$ , and that one already computed  $(E_i^e)_{i \geq m}$  and the function  $h(\mathbf{e})$ . We now consider the edge  $e_{m+1}$  and let  $(r, r')$  be its realizer. We extend the system of equations  $(E_i^e)_{i \geq m}$  by a new equation  $E_{m+1}$  and define the function  $h(\mathbf{e}')$  as follows:

- if  $r = \text{add}_i(j, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = x_j^{h(\mathbf{e}')(j)} + x_k^{h(\mathbf{e}')(k)}$ ;
- if  $r = \text{sub}_i(j, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = x_j^{h(\mathbf{e}')(j)} - x_k^{h(\mathbf{e}')(k)}$ ;
- if  $r = \text{mult}_i(j, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = x_j^{h(\mathbf{e}')(j)} \times x_k^{h(\mathbf{e}')(k)}$ ;
- if  $r = \text{div}_i(j, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = x_j^{h(\mathbf{e}')(j)} / x_k^{h(\mathbf{e}')(k)}$ ;
- if  $r = \text{add}_i(c, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = c + x_k^{h(\mathbf{e}')(k)}$ ;
- if  $r = \text{sub}_i(c, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = c - x_k^{h(\mathbf{e}')(k)}$ ;
- if  $r = \text{mult}_i(c, k)$ ,  $h(\mathbf{e}')(x) = h(\mathbf{e})(x) + 1$  if  $x = i$ , and  $h(\mathbf{e}')(x) = h(\mathbf{e})(x)$  otherwise; then  $E_{m+1}$  is  $x_i^{h(\mathbf{e}')(i)} = c \times x_k^{h(\mathbf{e}')(k)}$ ;

<sup>8</sup>The use of  $\perp$  is to allow for the creation of fresh variables not related to a register.

- 1081 • if  $r = \text{div}_i(c, k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  
1082  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = c/x_k^{\mathfrak{h}(\mathbf{e}')(k)}$ ;
- 1083 • if  $r = \text{sqr}_i(k)$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = i$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise; then  
1084  $E_{m+1}$  is  $x_i^{\mathfrak{h}(\mathbf{e}')(i)} = \sqrt{x_k^{\mathfrak{h}(\mathbf{e}')(k)}}$ ;
- 1085 • if  $r = \text{Id}$ , the source of the edge  $e_q$  is of the form  $\{(x_1, \dots, x_{n+\ell}) \in \mathbf{R}^{n+\ell} \mid P(x_k)\} \times \{i\}$ .  
1086 – If  $P(x_k)$  is  $x_k \neq 0$ ,  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x) + 1$  if  $x = \perp$ , and  $\mathfrak{h}(\mathbf{e}')(x) = \mathfrak{h}(\mathbf{e})(x)$  otherwise  
1087 then  $E_{m+1}$  is  $x_{\perp}^{\mathfrak{h}(\mathbf{e}')(i)} x_k^{\mathfrak{h}(\mathbf{e}')(k)} - 1 = 0$ ;  
1088 – otherwise we set  $\mathfrak{h}(\mathbf{e}') = \mathfrak{h}(\mathbf{e})$  and  $E_{m+1}$  equal to  $P$ .  
1089

1090 We now consider the system of equations  $(E_i)_{i=1}^k$  defined from the path  $\mathbf{e}$  of length  $k$   
1091 corresponding to a node  $n_{\mathbf{e}}^{\pi}$  of the  $k$ -th entropic co-tree of  $G$ . This system consists in  $k$   
1092 equations of degree at most 2 and containing at most  $k + n$  variables, counting the variables  
1093  $x_1^0, \dots, x_n^0$  corresponding to the initial registers, and adding at most  $k$  additional variables  
1094 since an edge of  $\vec{e}$  introduces at most one fresh variable. By Theorem 26 one obtains  
1095 that the defined semi-algebraic set has at most  $2 \cdot 3^{2k+n-1}$  connected components. Since  
1096 the number of vertices  $n_{\mathbf{e}}^{\pi}$  is bounded by  $2^{h_0(|G|)}$  by Theorem 24, we have that the set  
1097  $W$  whose membership problem is decided by  $G$  has at most  $2^{h_0(|G|)+1} 3^{2k+n+1}$  connected  
1098 components.  $\square$   
1099

1100 This theorem extends to the case of general computational graphings by considering  
1101 the *algebraic degree* of the graphing.

1102 **Definition 42** (Algebraic degree). The algebraic degree of an element of the AMC is the  
1103 minimal number of generators needed to express it.

1104 The algebraic degree of a graphing is the maximum of the algebraic degrees of the  
1105 realisers of its edges.

1106 If an edge is realised by an element  $m$  of algebraic degree  $D$ , then the method above  
1107 applies by introducing the  $D$  new equations corresponding to the  $D$  generators used to  
1108 define  $m$ . The general result then follows.  
1109

1110 **Theorem 43.** *Let  $G$  be a computational graphing representative,  $\text{Card}(E)$  its number of  
1111 edges, and  $D$  its algebraic degree. Suppose  $G$  computes the membership problem for  $W \subseteq \mathbf{R}^n$   
1112 in  $k$  steps, i.e. for each element of  $\mathbf{R}^n$ ,  $\pi_S(G^k(x)) = \top$  if and only if  $x \in W$ . Then  $W$  has at  
1113 most  $2^{h_0(|G|)+1} 3^{2kD+n+1}$  connected components.*

1114 **COROLLARY 44** ([10, THEOREM 5]). *Let  $W \subseteq \mathbf{R}^n$  be any set, and let  $N$  be the maximum of  
1115 the number of connected components of  $W$  and  $\mathbf{R}^n \setminus W$ .*

1116 *An algebraic computation tree computing the membership problem for  $W$  has height  
1117  $\Omega(\log N)$ .*  
1118

1119 **PROOF.** Let  $T$  be an algebraic computation tree computing the membership problem for  
1120  $W$ , and consider the computational treeing  $[T]$ . Let  $d$  be the height of  $T$ ; by definition of  
1121  $[T]$  the membership problem for  $W$  is computed in exactly  $d$  steps. Thus, by the previous  
1122 theorem,  $W$  has at most  $2^{h_0([T])+1} 3^{2d+n+1}$  connected components. As the interpretation of  
1123 an algebraic computational tree,  $h_0([T])$  is at most equal to 2. Hence  $N \leq 2^3 3^{n+1} 3^{2d}$ , i.e.  
1124  $d = \Omega(\log N)$ .  $\square$   
1125



1126 We immediately deduce an application that will be useful to us in the remainder. Let  
 1127  $m \in \mathbf{N}$  and  $0 < x < 2^m$ . Let  $k \in \mathbf{N}$  be such that  $1 \leq k \leq m$ . We call  $\left\lfloor \frac{x}{2^{k-1}} \right\rfloor - 2 \left\lfloor \frac{x}{2^k} \right\rfloor$  the  
 1128  $k$ -th bit of  $x$ .  
 1129

1130 LEMMA 45. *An algebraic computation tree computing the  $k$ -th bit of  $x$  has height  $\Omega(\log(m -$   
 1131  $k))$ .*

1132 PROOF. Let

$$1133 \quad W = \left\{ x \in \mathbf{R} \mid \left\lfloor \frac{x}{2^{k-1}} \right\rfloor - 2 \left\lfloor \frac{x}{2^k} \right\rfloor = 1 \right\}$$

1136  $W$  is the disjoint union of  $m - k + 1$  intervals, and so is its complement in  $]0; 2^m[$ . So, by  
 1137 Theorem 44, any algebraic computation tree computing the  $k$ -th bit has height  $\Omega(\log m -$   
 1138  $k)$ .  $\square$   
 1139

## 1140 7 SRAMS AND PRAMS

### 1141 7.1 Random Access Machines (SRAMS)

1142 In this paper, we will consider algebraic parallel random access machines, that act not on  
 1143 strings of bits, but on integers. In order to define those properly, we first define the notion  
 1144 of (sequential) random access machine (SRAM) before considering their parallelisation.  
 1145

1146 A SRAM command is a pair  $(\ell, I)$  of a label (or line)  $\ell \in \mathbf{N}^*$  and a command  $I$  among the  
 1147 following:

1148 Commands := skip;  $X_i := c$ ;  $X_i := X_j \circ X_k$ ;  
 1149  $X_i := X_j$ ;  $X_i := \#X_j$ ;  $\#X_i := X_j$ ;  
 1150 if  $X_i = 0$  goto  $\ell$  else  $\ell'$ ;

1151 where  $i, j \in \mathbf{N}$ ,  $\circ \in \{+, -, \times, /\}$ ,  $c \in \mathbf{Z}$  is a constant and  $\ell, \ell' \in \mathbf{N}^*$  are labels.

1152 A SRAM machine  $M$  is then a finite set of commands such that the set of labels is  
 1153  $\{1, 2, \dots, |M|\}$ , with  $|M|$  the length of  $M$ . We will denote the commands in  $M$  by  $(i, \text{Inst}_M(i))$ ,  
 1154 i.e.  $\text{Inst}_M(i)$  denotes the  $i$ -labelled command.

1155 The semantics of SRAM machines is quite straightforward: a configuration is represented  
 1156 as a pair  $(i, \sigma)$  where  $i$  is the current label and  $\sigma : \mathbf{N} \rightarrow \mathbf{Z}$  – an eventually null function –  
 1157 represent the state of the registers. Then the commands above are easily interpreted as a  
 1158 transition to the label  $i + 1$  (except in case of the conditional) and an updated function  $\sigma$ .  
 1159 E.g. a  $\#X_i := X_j$  command induces the following transition:  $(i, \sigma) \rightarrow (i + 1, \sigma[\sigma(j)/\sigma(i)])$ ,  
 1160 while a  $X_i := \#X_j$  command induces the transition  $(i, \sigma) \rightarrow (i + 1, \sigma[\sigma(j)/i])$ .  
 1161

1162 *Parallel Random Access Machines (PRAMS)*. Based on the notion of SRAM, we are now  
 1163 able to consider their parallelisation, namely PRAMS. A PRAM machine  $M$  is simply given  
 1164 as a finite sequence of SRAM machines  $M_1, \dots, M_p$ , where  $p$  is the number of processors of  
 1165  $M$ . Each processor  $M_i$  has access to its own, private, set of registers  $(X_k^i)_{k \geq 0}$  and a shared  
 1166 memory represented as a set of registers  $(X_k^0)_{k \geq 0}$ .

1167 As usual, one has to deal with conflicts when several processors try to access the shared  
 1168 memory simultaneously. We here chose to work with the CREW discipline, i.e. *Concurrent*  
 1169 *Read, Exclusive Write*, implemented as follows: at a given step at which several processors  
 1170

1171 try to write in the shared memory, only the processor with the smallest index will be  
1172 allowed to do so.

1173

## 1174 7.2 The SRAM action

1175 We now define the SRAM action. As we intend to consider PRAMs, we consider from the  
1176 beginning the memory of a SRAM to be separated in two infinite blocks  $Z^\omega$ , intended to  
1177 represent both *shared* and a *private* memory cells. The underlying space is  $\mathbf{X} = Z^\omega \times Z^\omega$ .  
1178 The set of generators is defined following the possible actions of an SRAM on the memory:  
1179  $\text{const}_i(c)$ ,  $\text{add}_i(j, k)$ ,  $\text{sub}_i(j, k)$ ,  $\text{mult}_i(j, k)$ ,  $\text{eucdiv}_i(j, k)$ ,  $\text{copy}(i, j)$ ,  $\text{copy}(\#i, j)$ ,  $\text{copy}(i, \#j)$ .  
1180 Each of the generator acts as follows (we do not distinguish the two kinds of memory and  
1181 suppose the indices unique here):

- 1182 •  $\alpha(\text{const}_i(c))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, c, x_{i+1}, \dots)$ ;
- 1183 •  $\alpha(\text{add}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j + x_k, x_{i+1}, \dots)$ ;
- 1184 •  $\alpha(\text{sub}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j - x_k, x_{i+1}, \dots)$ ;
- 1185 •  $\alpha(\text{mult}_i(j, k))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j \times x_k, x_{i+1}, \dots)$
- 1186 • for  $\alpha(\text{eucdiv}_i(j, k))$ ,

$$1187 \quad (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) \mapsto \begin{cases} (x_1, \dots, x_{i-1}, b, x_{i+1}, \dots) & \text{such that } x_j = bx_k + r \\ & \text{and } 0 \leq r < x_k \\ & \text{if } x_k \neq 0 \\ (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots) & \text{else} \end{cases}$$

- 1192 •  $\alpha(\text{copy}(i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots)$ ;
- 1193 •  $\alpha(\text{copy}(\#i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_{x_j}, x_{i+1}, \dots)$ ;
- 1194 •  $\alpha(\text{copy}(i, \#j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{x_i-1}, x_j, x_{x_i+1}, \dots)$ .

1195 SRAM machines can be represented as graphings w.r.t. this action; intuitively the encoding  
1196 works as follows. The notion of *control state* allows to represent the notion of *line* in the  
1197 program. Then, the action just defined allows for the representation of all commands but  
1198 the conditionals. The conditionals are represented as follows: depending on the value of  
1199  $X_i$  one wants to jumps either to the line  $\ell$  or to the line  $\ell'$ ; this is easily modelled by two  
1200 different edges of respective sources  $\mathbb{H}(i) = \{\vec{x} \mid x_i = 0\}$  and  $\mathbb{H}(i)^c = \{\vec{x} \mid x_i \neq 0\}$ .

1201

1202 **Definition 46.** Let  $M$  be a SRAM machine. We define  $[M]$  as the graphing with set of  
1203 control states  $\{0, 1, \dots, L, L + 1\}$  where each line  $\ell$  defines:

- 1204 • either a single edge  $e$  of source  $\mathbf{X} \times \{\ell\}$  and realised by:
  - 1205 –  $(\text{Id}, \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is skip;
  - 1206 –  $(\text{const}_i(c), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $X_i := c$ ;
  - 1207 –  $(\text{add}_i(j, k), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $X_i := X_j + X_k$ ;
  - 1208 –  $(\text{sub}_i(j, k), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $X_i := X_j - X_k$ ;
  - 1209 –  $(\text{mult}_i(j, k), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $X_i := X_j \times X_k$ ;
  - 1210 –  $(\text{copy}(i, j), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $X_i := X_j$ ;
  - 1211 –  $(\text{copy}(i, \#j), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $X_i := \#X_j$ ;
  - 1212 –  $(\text{copy}(\#i, j), \ell \mapsto \ell + 1)$  if  $\text{Inst}_M(\ell)$  is  $\#X_i := X_j$ ;
- 1213 • a single edge  $e$  of source  $\mathbb{H}(k) \times \{\ell\}$  and realised by  $(\text{eucdiv}_i(j, k), \ell \mapsto \ell + 1)$  if  
1214  $\text{Inst}_M(\ell)$  is  $X_i := X_j / X_k$ ;

1215

- or, if the line is a conditional  $\text{if } \chi_i = 0 \text{ goto } \ell^0 \text{ else } \ell^1$ , a pair  $e, e^c$  of edges of respective sources  $\mathbb{H}(i) \times \{\ell\}$  and  $\mathbb{H}(i)^c \times \{\ell\}$  and realised by respectively  $(\text{Id}, \ell \mapsto \ell^0)$  and  $(\text{Id}, \ell \mapsto \ell^1)$ .

If a graphing  $T$  in the AMC of PRAMS has a distinguished control state  $\top$ , we say that  $T$  accepts  $x \in \mathbf{Z}^d$  if  $T$  accepts  $((d, x_1, \dots, x_d, 0, \dots), (0, \dots), (0, \dots), \dots)$ .

One can now easily check that the representation of the machine  $M$  as a graphing is sound.

**Theorem 47.** *The representation of SRAMS as graphings is sound.*

PROOF. The proof is an easy verification that the interpretation of commands in the graphing do coincide with the semantics of these commands.  $\square$

### 7.3 The crew of the SRAM action: PRAMS

We can now define AMC of PRAMS and thus the interpretations of PRAMS as abstract programs. For each integer  $p$ , we define the AMC  $\text{CREW}^k(\alpha)$  for  $\alpha$  is the AMC for SRAMS defined in the previous section. This allows the consideration of up to  $p$  parallel SRAMS. The interpretation of such a SRAM with  $p$  processors is then defined by considering a set of states equal to  $L_1 \times L_2 \times \dots \times L_p$  where for all  $i$  the set  $L_i$  is the set of labels of the  $i$ -th processor.

Now, to deal with arbitrary large PRAMS, i.e. with arbitrarily large number of processors, one considers the following AMC defined as a *direct limit*.

**Definition 48** (The AMC of PRAMS). Let  $\alpha : M \curvearrowright \mathbf{X} \times \mathbf{X}$  be the SRAM AMC. The AMC of PRAMS is defined as  $\varinjlim \text{CREW}^k(\alpha)$ , where  $\text{CREW}^{k-1}(\alpha)$  is identified with a restriction of  $\text{CREW}^k(\alpha)$  through  $\text{CREW}^{k-1}(\alpha)(m_1, \dots, m_{k-1}) \mapsto \text{CREW}^k(\alpha)(m_1, \dots, m_{k-1}, 1)$ .

Remark that the underlying space of the PRAM AMC is defined as the union  $\bigcup_{n \in \omega} \mathbf{Z}^\omega \times (\mathbf{Z}^\omega)^n$  which we will write  $\mathbf{Z}^\omega \times (\mathbf{Z}^\omega)^{(\omega)}$ .

The fact that the CREW operation on actions does define the semantics of the CREW discipline for concurrent read/write should be clear from the definitions. As a consequence, Theorem 47 the interpretation of PRAMS as graphings is sound.

**Theorem 49.** *The representation of PRAMS as graphings is sound.*

### 7.4 The entropic co-trees of a PRAM

A PRAM defines a family of entropic co-trees by using Def. 37. These co-trees are in the AMC of PRAMS, which we designed to be similar to the AMC of algebraic computation trees. The three main differences are:

- the AMC of PRAMS acts on  $\mathbf{Z}^\omega \times (\mathbf{Z}^\omega)^{(\omega)}$ , while that algebraic computational trees acts on  $\mathbf{R}^d$ ;
- an action of the AMC of PRAMS with  $p$  processors is a tuple of  $p$  actions in the AMC of SRAMS;
- the euclidian division in the AMC of PRAMS can not be translated straightforwardly.

We will handle this difficulties separately, by introducing an AMC of PRAMS over  $\mathbf{R}$  and translating a treeing in the AMC of PRAMS into a treeing in the AMC of PRAMS over  $\mathbf{R}$ , and then by adapting the proofs of Section above to this AMC.

1261 PRAMs over  $\mathbf{R}$ . The AMC of SRAMs over  $\mathbf{R}$  is defined as the AMC of SRAMs, but the underlying  
 1262 space is  $\mathbf{R}^\omega \times \mathbf{R}^\omega \times \mathbf{R}^\omega$ , where we interpret the first two copies as public and the third  
 1263 as private, and the generator corresponding to euclidian division is replaced by the real  
 1264 division. we introduce the second public copy so as to be able to have fresh variables:  
 1265 indeed, we cannot define the set of memory cells that a program can use, due to the indirect  
 1266 addressing  $\#x_i := x_j$ .

1267 Explicitly the set of generators is given by:  $\text{const}_i(c)$ ,  $\text{add}_i(i, j)$ ,  $\text{sub}_i(i, j)$ ,  $\text{mult}_i(i, j)$ ,  
 1268  $\text{div}_i(i, j)$ ,  $\text{copy}(i, j)$ ,  $\text{copy}(\#i, j)$ ,  $\text{copy}(i, \#j)$  that act by:

- 1269 •  $\alpha(\text{const}_i(c))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, c, x_{i+1}, \dots)$ ;
- 1270 •  $\alpha(\text{add}_i(i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_i + x_j, x_{i+1}, \dots)$ ;
- 1271 •  $\alpha(\text{sub}_i(i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_i - x_j, x_{i+1}, \dots)$ ;
- 1272 •  $\alpha(\text{mult}_i(i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_i \times x_j, x_{i+1}, \dots)$
- 1273 •  $\alpha(\text{div}_i(i, j)) = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_i/x_j, x_{i+1}, \dots)$  if  $x_j \neq 0$ ;
- 1274 •  $\alpha(\text{copy}(i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots)$ ;
- 1275 •  $\alpha(\text{copy}(\#i, j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, x_{x_j}, x_{i+1}, \dots)$  if  $x_j \in \omega$ ;
- 1276 •  $\alpha(\text{copy}(i, \#j))(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{x_i-1}, x_j, x_{x_i+1}, \dots)$  if  $x_i \in \omega$ .

1277 The AMC of PRAM over  $\mathbf{R}$  is then defined as the limit of the crew of this AMC, just as in the  
 1278  $\mathbf{Z}$  case, and so acts on  $\mathbf{R}^\omega \times \mathbf{R}^\omega \times (\mathbf{R}^\omega)^{(\omega)}$ .

1279 We do not define the PRAMs in this AMC, as we are just interested in it to translate treeings.  
 1280 If a graphing  $T$  in the AMC of PRAMs over  $\mathbf{R}$  has a distinguished control state  $\top$ , we say that  
 1281  $T$  accepts  $x \in \mathbf{R}^d$  if  $T$  accepts  $((d, x_1, \dots, x_d, 0, \dots), (0, \dots), (0, \dots), \dots)$ .

1282 *From a treeing of PRAM to a treeing of PRAM over  $\mathbf{R}$ .* We can associate to every treeing  
 1283 in the AMC of PRAMs a treeing in the AMC of PRAMs over  $\mathbf{R}$ , that, limited to the integers,  
 1284 decide the same set.

1285 In what follow, we will distinguish the copies on which a PRAM acts by writing  $\vec{x} =$   
 1286  $((x_1, \dots), (y_1, \dots), (z_i^j)_{0 \leq i}) \in \mathbf{R}^\omega \times \mathbf{R}^\omega \times (\mathbf{R}^\omega)^{(\omega)}$  the different elements.

1287 **Definition 50.** Let  $T$  be a treeing in the AMC of PRAMs with control states  $S$ , and suppose  
 1288 that its edges labelled with  $\text{eucdiv}_i(j, k)$  are numbered  $1, \dots, t$ .

1289 We define the *real mate*  $\mathbf{R}(T)$  as the treeing with control states  $S \times \{1, 2, 3\}$  where the  
 1290  $n$ -th vertex labelled  $\text{eucdiv}_i(j, k) : a \rightarrow b$  is replaced by the vertices:

- 1291 • one of source  $a \cap \{\vec{x} \mid y_{2n} \geq 0\}$  and labelled by identity;
- 1292 • one of source  $a \cap \{\vec{x} \mid y_{2n} < x_k\}$  and labelled by the identity;
- 1293 • one of source  $a$  labelled by  $\text{sub}_{y_{2n+1}}(j, y_{2n})$
- 1294 • one of source  $a$  labelled by  $\text{div}_i(y_{2n+1}, k)$

1295 where the state moves in all these operations in its new component.

1296 The real mate allows to compute the euclidian division when restricted to integers, by  
 1297 adding new variables and relations between them.

1300 **Theorem 51.** Let  $T$  be a PRAM with  $p$  processors, that ends in  $k$  steps.

1301 The real mate of its  $k$ -th entropic co-tree is a treeing  $Q$  in the AMC of PRAMs over  $\mathbf{R}$  such  
 1302 that:

- 1303 •  $Q$  is of height at most  $4k$ ;

1304

1305

- $\forall x \in \mathbf{Z}^d$ ,  $Q$  accepts  $x$  if and only if  $T$  accepts  $x$ ;
- the subspace accepted by this treeing can be defined by a set of  $(2p)^{4k}$  polynomial equations of degree at most  $2^{4k}$ .

PROOF. The only point that needs to be developed is the last one: we traverse the tree and add equations as for the algebraic computation trees.  $\square$

We remark here that we could add a square root instruction to our PRAMS, just as in the algebraic computation trees.

## 8 ALGEBRAIC SURFACES FOR AN OPTIMIZATION PROBLEM

### 8.1 Geometric Interpretation of Optimization Problems

We start by showing how decision problems of a particular form induce a binary partition of the space  $\mathbf{Z}^d$ : the points that are accepted and those that are rejected. Intuitively, the machine decides the problem if the partition it induces refines the one of the problem.

We will consider problems of a very specific form: decisions problems in  $\mathbf{Z}^3$  associated to optimization problems. Let  $\mathcal{P}_{\text{opt}}$  be an optimization problem on  $\mathbf{R}^d$ . Solving  $\mathcal{P}_{\text{opt}}$  on an instance  $t$  amounts to optimizing a function  $f_t(\cdot)$  over a space of parameters. We note  $\text{Max}\mathcal{P}_{\text{opt}}(t)$  this optimal value. An affine function  $\text{Param} : [p; q] \rightarrow \mathbf{R}^d$  is called a *parametrization* of  $\mathcal{P}_{\text{opt}}$ . Such a parametrization defines naturally a decision problem  $\mathcal{P}_{\text{dec}}$ : for all  $(x, y, z) \in \mathbf{Z}^3$ ,  $(x, y, z) \in \mathcal{P}_{\text{dec}}$  iff  $z > 0$ ,  $x/z \in [p; q]$  and  $y/z \leq \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x/z)$ .

In order to study the geometry of  $\mathcal{P}_{\text{dec}}$  in a way that makes its connection with  $\mathcal{P}_{\text{opt}}$  clear, we consider the ambient space to be  $\mathbf{R}^3$ , and we define the *ray*  $[p]$  of a point  $p$  as the half-line starting at the origin and containing  $p$ . The projection  $\Pi(p)$  of a point  $p$  on a plane is the intersection of  $[p]$  and the affine plane  $\mathcal{A}_1$  of equation  $z = 1$ . For any point  $p \in \mathcal{A}_1$ , and all  $p_1 \in [p]$ ,  $\Pi(p_1) = p$ . It is clear that for  $(p, p', q) \in \mathbf{Z}^2 \times \mathbf{N}^+$ ,  $\Pi((p, p', q)) = (p/q, p'/q, 1)$ .

The *cone*  $[C]$  of a curve  $C$  is the set of rays of points of the curve. The projection  $\Pi(C)$  of a surface or a curve  $C$  is the set of projections of points in  $C$ . We note  $\text{Front}$  the frontier set

$$\text{Front} = \{(x, y, 1) \in \mathbf{R}^3 \mid y = \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x)\}.$$

and we remark that

$$[\text{Front}] = \{(x, y, z) \in \mathbf{R}^2 \times \mathbf{R}^+ \mid y/z = \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x/z)\}.$$

Finally, a machine  $M$  decides the problem  $\mathcal{P}_{\text{dec}}$  if the sub-partition of accepting cells in  $\mathbf{Z}^3$  induced by the machine is finer than the one defined by the problem's frontier  $[\text{Front}]$  (which is defined by the equation  $y/z \leq \text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(x/z)$ ).

### 8.2 Parametric Complexity

We now further restrict the class of problems we are interested in: we will only consider  $\mathcal{P}_{\text{opt}}$  such that  $\text{Front}$  is simple enough. Precisely:

**Definition 52.** We say that  $\text{Param}$  is an *affine parametrization* of  $\mathcal{P}_{\text{opt}}$  if  $\text{Param}; \text{Max}\mathcal{P}_{\text{opt}}$  is

- convex
- piecewise linear, with breakpoints  $\lambda_1 < \dots < \lambda_\rho$

- such that the  $(\lambda_i)_i$  and the  $(\text{Max}\mathcal{P}_{\text{opt}} \circ \text{Param}(\lambda_i))_i$  are all rational.

The (*parametric*) *complexity*  $\rho(\text{Param})$  is defined as the number of breakpoints of  $\text{Param}; \text{Max}\mathcal{P}_{\text{opt}}$ .

An optimization problem that admits an affine parametrization of complexity  $\rho$  is thus represented by a surface [Front] that is quite simple: the cone of the graph of a piecewise affine function, constituted of  $\rho$  segments. We say that such a surface is a  $\rho$ -*fan*. This restriction seems quite serious when viewed geometrically. Nonetheless, many optimization problems admit such a parametrization. Before giving examples, we introduce another measure of the complexity of a parametrization.

**Definition 53.** Let  $\mathcal{P}_{\text{opt}}$  be an optimization problem and  $\text{Param}$  be an affine parametrization of it. The *bitsize* of the parametrization is the maximum of the bitsizes of the numerators and denominators of the coordinates of the breakpoints of  $\text{Param}; \text{Max}\mathcal{P}_{\text{opt}}$ .

In the same way, we say that a  $\rho$ -fan is of *bitsize*  $\beta$  if all its breakpoints are rational and the bitsize of their coordinates is lesser than  $\beta$ .

**Theorem 54** (Murty [36], Carstensen [12]).

- (1) *there exists an affine parametrization of bitsize  $O(n)$  and complexity  $2^{\Omega(n)}$  of combinatorial linear programming, where  $n$  is the total number of variables and constraints of the problem.*
- (2) *there exists an affine parametrization of bitsize  $O(n^2)$  and complexity  $2^{\Omega(n)}$  of the maxflow problem for directed and undirected networks, where  $n$  is the number of nodes in the network.*

We refer the reader to Mulmuley's paper [34, Thm. 3.1.3] for proofs, discussions and references.

### 8.3 Algebraic Surfaces

An algebraic surface in  $\mathbf{R}^3$  is a surface defined by an equation of the form  $p(x, y, z) = 0$  where  $p$  is a polynomial. If  $S$  is a set of surfaces, each defined by a polynomial, the *total degree* of  $S$  is defined as the sum of the degrees of polynomials defining the surfaces in  $S$ .

Let  $K$  be a compact of  $\mathbf{R}^3$  delimited by algebraic surfaces and  $S$  be a finite set of algebraic surfaces, of total degree  $\delta$ . We can assume that  $K$  is actually delimited by two affine planes of equation  $z = \mu$  and  $z = 2\mu_z$  and the cone of a rectangle  $\{(x, y, 1) \mid |x|, |y| \leq \mu_{x,y}\}$ , by taking any such compact containing  $K$  and adding the surfaces bounding  $K$  to  $S$ .  $S$  defines a partition of  $K$  by considering maximal compact subspaces of  $K$  whose boundaries are included in surfaces of  $S$ . Such elements are called the *cells* of the decomposition associated to  $S$ .

The cell of this partition can have complicated shapes: in particular, a cell can have a arbitrarily high number of surfaces of  $S$  as boundaries. We are going to refine this partition into a partition  $\text{Col}_S$  whose cells are all bounded by cones of curves and at most two surfaces in  $S$ .



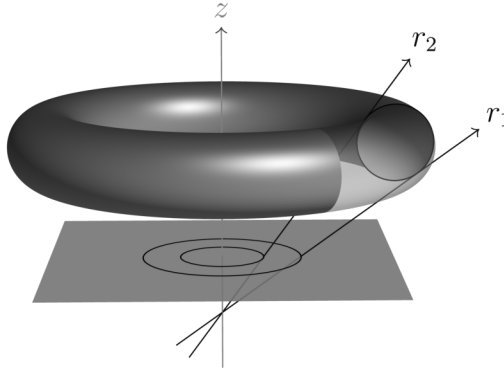


Fig. 3. A torus and the projection of its silhouette

#### 8.4 Collins' decomposition

We define the *silhouette* of a surface defined by the equation  $p(x, y, z) = 0$  by:

$$\begin{cases} p(x, y, z) = 0 \\ x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} + z \frac{\partial p}{\partial z} = 0. \end{cases}$$

The silhouette of a surface is the curve on the surface such that all points  $(x, y, z)$  of the silhouette are such that the ray  $[(x, y, z)]$  is orthogonal to the tangent plane of the surface on  $(x, y, z)$ .

Up to infinitesimal perturbation of the coefficients of the polynomials, we can assume that the surfaces of  $S$  have no integer points in  $K$ .

$\Pi(K) = \{\Pi(x) \mid x \in K\}$  is a compact of the affine plane  $\mathcal{A}_1$ . Let us consider the set  $\Pi(S)$  of curves in  $\Pi(K)$  containing:

- the projection of the silhouettes of surfaces in  $S$ ;
- the projection of the intersections of surfaces in  $S$  and of the intersection of surfaces in  $S$  with the planes  $z = \mu(1 + \frac{n}{6\delta})$ ,  $n \in \{1, \dots, 6\delta - 1\}$ , where  $\delta$  is the total degree of  $S$ ;
- *vertical lines* of the form  $\{(x, a, 1) \mid |x| \leq 2^{\beta+1}\}$  for  $a$  a constant such that such lines pass through:
  - all intersections among the curves;
  - all singular points of the curves;
  - all critical points of the curves with a tangent supported by  $\vec{e}_y$ .

$\Pi(S)$  defines a Collins decomposition [14] of  $\Pi(K)$ . The intersection of any affine line supported by  $\vec{e}_y$  of the plane with a region of this decomposition is connected if nonempty.

Let  $c$  be a cell in  $\Pi(S)$ . It is enclosed by two curves in  $\Pi(K)$  and at most two vertical lines. The curves can be parametrized by  $c_{\max} : x \mapsto \max\{y \in \mathbf{R} \mid (x, y, 1) \in c\}$  and  $c_{\min} : x \mapsto \min\{y \in \mathbf{R} \mid (x, y, 1) \in c\}$ , which are both smooth functions. The *volatility* of  $c$  is defined as the number of extrema of the second derivatives  $c''_{\min}$  and  $c''_{\max}$  on their domains of definition.

This set of curves  $\Pi(S)$  can be lifted to a set of surfaces  $\text{Col}_5(K)$  of  $K$  that contains:

- 1441 • the surfaces of  $S$ ;
- 1442 • the cones  $[s]$  of every curve  $s$  in  $\Pi(S)$ ;
- 1443 • the planes bounding  $K$ ;
- 1444 •  $6\delta - 2$  dividing planes of equation  $z = \mu(1 + \frac{n}{6\delta})$ ,  $n \in \{1, \dots, 6\delta - 1\}$ .

1445 The projection of a cell of  $\text{Col}_S$  is a cell of  $\Pi(S)$ . We say that a cell of  $\text{Col}_S(K)$  is *flat* if none  
 1446 of its boundaries are included in surfaces of  $S$ .

1447 Let us call  $\mathbf{d}(S)$  the number of cells in  $\text{Col}_S(K)$ .

1448 Let  $c$  be a cell in  $\text{Col}_S(K)$ . Its *volatility* is defined as the volatility of its projection in  $\Pi(S)$ .

1449

## 1450 8.5 Volatility and Separation

1451 **Definition 55.** Let  $K$  be a compact of  $\mathbf{R}^3$ .

1452 A finite set of surfaces  $S$  on  $K$  *separates* a  $\rho$ -fan  $\text{Fan}$  on  $K$  if the partition on  $\mathbf{Z}^3 \cap K$   
 1453 induced by  $S$  is finer than the one induced by  $\text{Fan}$ .

1454 **Theorem 56.** *Let  $S$  be a finite set of algebraic surfaces of total degree  $\delta$ , and  $\text{Fan}$  a  $\rho$ -fan of*  
 1455 *bitsize  $\beta$ .*

1456 *If  $S$  separates  $\text{Fan}$ , there exists a compact  $K$  and a cell of  $\text{Col}_S(K)$  with volatility greater*  
 1457 *than  $\rho/\mathbf{d}(S)$ .*

1458

1459 In order to prove this theorem, we will build explicitly the compact  $K$  and this cell by  
 1460 considering sample points on  $\text{Fan}$  and show in Lemma 58 a bound on the volatility of this  
 1461 cell.

1462 Let  $K$  be a compact delimited by the cone of a rectangle  $\{(x, y, 1) \mid |x|, |y| \leq 2^{\beta+1}\}$  and  
 1463 two planes of equation  $z = \mu$  and  $z = 2\mu$ , with  $\mu > (6\delta + 1)2^\beta$ . We first remark that all  
 1464 affine segments of  $\text{Fan}$  are in the rectangle base of  $K$ .

1465 For each affine segment of  $\text{Fan}$  with endpoints  $(x_i, y_i, 1)$  and  $(x_{i+1}, y_{i+1}, 1)$  let, for  $0 <$   
 1466  $k < 10\mathbf{d}(S)$ ,  $y_i^k$  be such that  $(x_i^k, y_i^k, 1)$  is in the affine segment, where  $x_i^k = \frac{(10\mathbf{d}(S)-k)x_i + kx_{i+1}}{10\mathbf{d}(S)}$ .  
 1467 We remark that, as  $|x_i - x_{i+1}| > 2^{-\beta}$ , we have, for  $k, k'$ ,  $|x_i^k - x_i^{k'}| > 2^{-\beta}/10\mathbf{d}(S)$ .

1468

1469 **LEMMA 57.** *For all sample points  $(x_i^k, y_i^k, 1)$ , there exists a flat cell in  $\text{Col}_S$  that contains an*  
 1470 *integer point of  $[(x_i^k, y_i^k, 1)]$ .*

1471

1472 **PROOF.** Let  $(x_i^k, y_i^k, 1)$  be a sample point.  $[(x_i^k, y_i^k, 1)]$  is divided in  $N + 1$  intervals by the  
 1473 dividing planes. On the other hand,  $[(x_i^k, y_i^k, 1)]$  intersects surfaces of  $S$  in at most  $\delta$  points,  
 1474 by Bézout theorem. So, there exists an interval  $e$  of  $[(x_i^k, y_i^k, 1)]$  that is bounded by the  
 1475 dividing planes and that do not intersect any surface in  $S$ . By construction,  $e$  is included in  
 1476 a flat cell, and its projection on the  $z$ -axis has length  $\mu/(6\delta + 1)$ , so, as  $(x_i^k, y_i^k, 1)$  is of bitsize  
 1477  $\beta$ ,  $(n2^\beta x_i^k, n2^\beta y_i^k, n2^\beta)$  is, for all  $n \in \mathbf{N}$  an integer point of the ray, so, as  $\mu > (6\delta + 1)2^\beta$ ,  $e$   
 1478 contains an integer point.  $\square$

1479

1480 So, for each affine segment of  $\text{Fan}$ , there exists a flat cell in  $\text{Col}_S$  that contains integer  
 1481 points in the ray of at least 10 sample points of the affine segment. Going further, there  
 1482 exists a cell  $c$  of  $\text{Col}_S$  that contains integer points in the ray of at least 10 sample points of  
 1483  $\rho/\mathbf{d}(S)$  affine segments of  $\text{Fan}$ .

1484 **LEMMA 58.** *The volatility of  $c$  is at least  $\rho/\mathbf{d}(S)$ .*

1485

This is achieved by applying the mean value theorem on the function  $\Pi(c)'_{\max}$  on pairs of sample points. In particular, this proof uses no algebraic geometry.

PROOF. Let  $e$  be a segment of Fan such that the ray of 10 of its sample points contain an integer point in  $c$ . Let  $p = (x, y, z)$  be one of its integer point and  $\Pi(p) = (x_p, y_p, 1)$  its projection, which is a sample point in  $\Pi(c)$ . Let  $q = (x, y + 1, z)$ . As  $\Pi(p)$  is in Fan, and  $S$  separates Fan,  $q$  is not in  $c$ , and  $\Pi(q) = (x_q, y_q, 1)$  is not in  $\Pi(c)$ . By Thalès theorem,  $0 < y_q - y_p < \frac{1}{\mu}$ . So, as  $y_q > \Pi(c)_{\max}(x_p) > y_p$ , we have in particular that  $0 < \Pi(c)_{\max}(x_p) - y_p < \frac{1}{\mu}$ .

So, the 10 sample points have coordinates that approximate the graph of  $\Pi(c)_{\max}$  with an error bounded by  $\frac{1}{\mu}$ . Consider two of them  $p_1 = (x_1, y_1, 1)$  and  $p_2 = (x_2, y_2, 1)$ , such that  $x_1 < x_2$ . Let  $a$  be the slope of  $e$  (in particular  $a = (y_2 - y_1)/(x_2 - x_1)$ ). By the mean value theorem, there exists  $\alpha \in [x_1, x_2]$  such that  $\Pi(c)'_{\max}(\alpha) = \frac{\Pi(c)_{\max}(x_2) - \Pi(c)_{\max}(x_1)}{x_2 - x_1}$ . But  $|\Pi(c)_{\max}(x_2) - \Pi(c)_{\max}(x_1)| \leq |y_2 - y_1| + \frac{2}{\mu}$  and  $|x_2 - x_1| > \frac{1}{10d(S)2^\beta}$ . So,  $|\Pi(c)'_{\max}(\alpha) - a| \leq 2 \frac{10d(S)2^\beta}{\mu}$ .

So, the function  $\Pi(c)'_{\max}$  is close to the value  $a$ , with error bounded, between all the sample points. By applying the mean value theorem again, we get that there exists a point in the interval such that  $\Pi(c)''_{\max}$  is close to 0, with an error bounded by  $2 \frac{10d(S)2^\beta}{\mu}$ .

In the same way, let  $e'$  be another segment of Fan such that the ray of 10 of its sample points contain an integer point in  $c$ , of slope  $a'$ . Let two of them be  $p'_1 = (x'_1, y'_1, 1)$  and  $p'_2 = (x'_2, y'_2, 1)$ , and suppose  $x'_2 > x'_1 > x_2$ . By the same reasoning as above, there exists  $\alpha' \in [x'_1, x'_2]$  such that  $|\Pi(c)'_{\max}(\alpha') - a'| \leq 2 \frac{10d(S)2^\beta}{\mu}$ . By the mean value theorem, there exists  $\beta \in [\alpha, \alpha']$  such that  $\Pi(c)''_{\max}(\beta) = \frac{\Pi(c)'_{\max}(\alpha') - \Pi(c)'_{\max}(\alpha)}{\alpha' - \alpha} > \frac{1}{\mu} (|a - a'| - 2 \frac{10d(S)2^\beta}{\mu})$ .

So, for each of the  $\rho/d(S)$  segments of Fan, we can exhibit a point such that  $\Pi(c)''_{\max}$  is close to zero, and for each successive segment, a point such that it is far. So  $\Pi(c)''_{\max}$  has at least  $\rho/d(S)$  extrema.  $\square$

### 8.6 Volatility and Degree

Mulmuley's result follows from Thm. 56 and the two lemmas:

LEMMA 59. *Let  $S$  be a finite set of curves of total degree  $\delta$ , and  $K$  be a compact. The cells of the decomposition  $\text{Col}_S$  of  $K$  have a volatility bounded by a polynomial in  $\delta$ .*

PROOF. Let  $c$  be a cell in  $\text{Col}_S$  and  $g(x, y) = 0$  be the equation of one of the boundaries of  $\Pi(c)$  in the affine plane. The degree of  $g$  is bounded by the degree of the intersection of surfaces in  $S$ . Any extrema  $x$  of  $f''$ , where  $f$  is a parametrization  $y = f(x)$  of this boundary, can be represented as a point  $(x, y, y^{(1)}, y^{(2)}, y^{(3)})$  in the 5-dimensional phase space that satisfy polynomial equations of the form:

$$\begin{aligned} g(x, y) = 0, \quad g_1(x, y, y^{(1)}) = 0, \quad g_2(x, y, y^{(1)}, y^{(2)}) = 0 \\ g_3(x, y, y^{(1)}, y^{(2)}, y^{(3)}) = 0, \quad y^{(3)} = 0, \end{aligned}$$

1531 where all the polynomials' degrees are all bounded by the degree of the intersection of  
 1532 surfaces in  $S$  (as they are the derivatives of  $g$ ). So, by the Milnor–Thom theorem, such  
 1533 points are in number polynomial in the total degree of the surfaces of  $S$ .  $\square$

1534 LEMMA 60. *The number of cells  $\mathbf{d}(S)$  of the Collins decomposition of  $S$  is polynomial in  $\delta$ .*

1536 PROOF. The intersection of the surfaces in  $S$  are algebraic varieties of number bounded  
 1537 by  $\delta$ , by the Milnor–Thom theorem. Moreover, so are the silhouettes of the surfaces, as  
 1538 they are the intersection of two algebraic varieties of total degree smaller than  $\delta$ . So, the  
 1539 number of cells in  $\text{Col}_S$  is bounded by the number of cells of  $S$  times the number of dividing  
 1540 planes times the number of intersections, silhouettes and vertical lines they engender.  $\square$

1542 **Theorem 61** (Mulmuley). *Let  $S$  be a finite set of algebraic surfaces of total degree  $\delta$ .*  
 1543 *There exists a polynomial  $P$  such that, for all  $\rho > P(\delta)$ ,  $S$  does not separate  $\rho$ -fans.*

## 1545 9 THE RESULT ON PRAMS

1546 **Theorem 62.** *Let  $G$  be a deterministic graphing interpreting a PRAM with  $p$  processors.*  
 1547 *There exists a polynomial  $P$  such that, for all  $\rho > P(2^{k+1}p^k)$ ,  $G$  does not separate  $\rho$ -fans in*  
 1548  *$k$  steps.*

1550 The definition of PRAM we considered is quite ideal in that the complexities are stated  
 1551 without references to the size of the inputs. We can consider the *length* of an input to be  
 1552 the minimal length of a binary word representing it. To account for the size of the input,  
 1553 we can refine the model and consider that the memory locations do not contain natural  
 1554 numbers but binary words whose lengths can be a parameter of the model. The approach  
 1555 outlined here is very sensible to the fact that all computed quantities are polynomial in the  
 1556 inputs, which would be wrong if we allowed for access of any arbitrary bit of the memory.  
 1557 Mulmuley defines this model as the PRAM *model without bit operations*, by taking the point  
 1558 of view that the integers stored in the memory really are binary words, whose individual  
 1559 bits can not be accessed easily.

1560 Here, we are able to compute arbitrary quotients by adding new phantom variables.  
 1561 Our model is thus of PRAM without bit operations and arbitrary divisions, in Mulmuley's  
 1562 parlance.

1564 **Theorem 63.** *Let  $G$  be a deterministic graphing interpreting a PRAM with  $2^{O((\log N)^c)}$  pro-*  
 1565 *cessors, where  $N$  is the length of the inputs and  $c$  any positive integer.*  
 1566  *$G$  does not decide maxflow in  $O((\log N)^c)$  steps.*

1568 PROOF. Let  $n = N^{2c}$  and consider the problem 2 of Thm. 54 and its family of affine  
 1569 parametrizations of bitsize  $O(n^2) = O(N^{4c})$  and complexity  $\rho(n) = 2^{\Omega(n)} = 2^{\Omega(N^{2c})}$ .

1570 By Thm. 51, we know that the partition induced by  $G$  in  $O((\log N)^c)$  steps can be defined  
 1571 by a set of  $(2 \times 2^{O((\log N)^c)})^{4O((\log N)^c)}$  polynomial equations of degree at most  $2^{4O((\log N)^c)}$ ,  
 1572 that is a set of total degree  $2^{O((\log N)^{2c})}$ .

1573 Let  $P$  be the polynomial of Thm. 62. For large enough values of  $N$ ,  $\rho(n)$  is larger than  
 1574  $P(2^{O((\log N)^{2c})}) = 2^{O((\log N)^{2c})}$ . So,  $G$  does not decide maxflow in  $O((\log N)^c)$  steps.  $\square$

1575

1576 *The problem of bit extraction.* This result, just as Mulmuley's, does not translate to boolean  
 1577 languages, and thus does not prove  $\text{NC} \neq \text{P TIME}$ . Indeed, the different bits constituting  
 1578 a number cannot be accessed independently. As all the operations our PRAMS can do are  
 1579 polynomial, the proposition holds:

1580 **Proposition 64.** *Let  $k \in \mathbb{N}$ .*

1582 *The  $k$ -th bit of a  $k$ -bit operand can not be extracted in  $O(\sqrt{k})$  time using  $2^{O(\sqrt{k})}$  processors.*

1583 PROOF. See [34, Proposition 2.2.1] □

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