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Abstract. We justify the WKB analysis for generalized nonlinear Schrödinger equations (NLS), including the hyperbolic NLS and the Davey-Stewartson II system. Since the leading order system in this analysis is not hyperbolic, we work with analytic regularity, with a radius of analyticity decaying with time, in order to obtain better energy estimates. This provides qualitative information regarding equations for which global well-posedness in Sobolev spaces is widely open.

1. Introduction

1.1. **Motivation.** The two-dimensional "hyperbolic" nonlinear Schrödinger equation,

$$i\partial_t\psi + \frac{1}{2}\partial_1^2\psi - \frac{1}{2}\partial_2^2\psi \pm |\psi|^2\psi = 0, \quad (x_1,x_2)\in\mathbb{R}^2, \tag{1.1}$$

appears in nonlinear optics (see e.g. [7, 20]), but remains quite mysterious as far as analysis is concerned: it is locally well-posed in $H^s(\mathbb{R}^2)$ for any $s>0$, it is L^2-critical, hence locally well-posed in $L^2(\mathbb{R}^2)$ (with a suitable definition of local well-posedness in this critical case), but apart from the small data case, the global existence issue remains a delicate issue in such spaces, even though refined Strichartz estimates are available [18], because the conserved energy is not a positive functional,

$$E = \|\partial_1\psi\|^2_{L^2(\mathbb{R}^2)} - \|\partial_2\psi\|^2_{L^2(\mathbb{R}^2)} \mp \|\psi\|^4_{L^4(\mathbb{R}^2)}.$$

Note that the sign of the nonlinearity is rather irrelevant, since we may exchange the roles of x_1 and x_2. However, global existence in $H^s(\mathbb{R}^2)$ for $s>0$ is obtained through modulation approximation in [22]. On the other hand, global solutions under the form of spatial standing waves have been studied in [6, 12], along with their stability.

Similarly, the Davey–Stewartson system

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\partial_1^2\psi - \frac{1}{2}\partial_2^2\psi = \left(\chi|\psi|^2 + \omega\partial_1\phi\right)\psi, \quad (x_1,x_2)\in\mathbb{R}^2, \quad \chi,\omega\in\mathbb{R},\\ \partial_1^2\phi + \partial_2^2\phi = \partial_1|\psi|^2, \end{cases} \tag{1.2}$$

is locally well-posed in the same spaces, L^2-critical, and enjoys a Hamiltonian structure with an energy whose sign is indefinite. Indeed, (1.2) can be rewritten

$$i\partial_t\psi + \frac{1}{2}\partial_1^2\psi - \frac{1}{2}\partial_2^2\psi = \left(\chi|\psi|^2 + \omega K * |\psi|^2\right)\psi,$$

where the symmetric kernel is such that

$$\widehat{K}(\xi) = \frac{\xi_1^2}{\xi_1^2 + \xi_2^2}.$$

On the other hand, for a suitable combination of the coefficients χ and ω, that is $2\chi + \omega = 0$, (1.2) is completely integrable (see e.g. [9, 13]). Global well-posedness and scattering in L^2 for the defocusing case were recently established in this specific case thanks to inverse scattering and harmonic analysis techniques, see [17]. In this note, we justify the approximation of such equations in a high frequency regime, known as semi-classical limit, this giving some extra information concerning the dynamics associated to these equations.

1.2. **Setting.** We consider the equation, including both (1.1) and (1.2),

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2} D^2 u^\varepsilon + i\varepsilon \left\langle \beta, \nabla \left[g\left(|u^\varepsilon|^2\right) u^\varepsilon \right] \right\rangle = V u^\varepsilon + \sum_{j=1}^{J} \left(K_j * |u^\varepsilon|^{2\sigma_j} \right) u^\varepsilon, \tag{1.3}$$

in the semi-classical limit $\varepsilon \to 0$, where $\varepsilon > 0$, $T > 0$, $d \geqslant 1$ is the spatial dimension, $J \geqslant 1$, and $(t, x) \in [0, T] \times \mathbb{E}^d$. More specifically,

- $u^\varepsilon(t, x) \in \mathbb{C}$ is the wave function
- $\mathbb{E}^d$ can be either $\mathbb{R}^d$ or to the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$,
- $$D^2 = \sum_{j,k=1}^{d} \eta_{j,k} \partial_j \partial_k = \langle \nabla, H\nabla \rangle,$$
  where $H = (\eta_{j,k})_{1 \leqslant j,k \leqslant d}$ is a symmetric (not necessarily positive or invertible) real matrix, and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$,
- $\beta = (\beta_j)_{1 \leqslant j \leqslant d} \in \mathbb{R}^d$ and $g(s) = \alpha s^\gamma$, where $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{N}\backslash\{0\}$. We consider such a function g in order to simplify the notations, but our method also works if g is not a monomial.
- For $j \in \{1, \ldots, J\}$, $\sigma_j \in \mathbb{N}\backslash\{0\}$ is an integer, and K_j denotes a tempered distribution with a bounded Fourier transform $\widehat{K_j} \in L^\infty(\mathbb{E}^d)$. This covers the case where $K = \delta$, typically as in (1.1).
- $V = V(t, x)$ is a potential. V is supposed to be analytic in the x variable. More precisely, we assume that V belongs to the space $L^2_{T_0}\mathcal{H}^{\ell+1/2}_{w_0}$ for some $T_0 > 0$, $w_0 > 0$, $\ell > (d+1)/2$, a space that will be defined below.

Our motivation for considering the case $\mathbb{E}^d = \mathbb{T}^d$ lies in the fact that numerical simulations are often performed in a periodic box: unless suitable absorbing boundary conditions are imposed, the observed dynamics is that of (1.3) on $\mathbb{T}^d$, which is fairly different from the one on $\mathbb{R}^d$.

Remark 1.1. In view of the assumption that is usually made on V in order to get H^s solutions (namely, $\partial^\alpha V \in L^\infty$ for $2 \leqslant |\alpha| \leqslant s$, see e.g. [3]), it is reasonable to ask for analyticity of V in order to get analytic solutions.

Remark 1.2. In a similar fashion as we consider an external potential, our analysis exports to the magnetic case, where

$$D^2 = \sum_{j,k=1}^{d} \eta_{j,k} \left(\partial_j - iA_j\right)\left(\partial_k - iA_k\right),$$

provided that the magnetic potentials A_j are analytic (in the same sense as for V).

The initial data that we consider are WKB states:

$$u^\varepsilon(0,x) = a_0^\varepsilon(x) e^{i\phi_0^\varepsilon(x)/\varepsilon} =: u_0^\varepsilon(x), \tag{1.4}$$

where $\phi_0^\varepsilon : \mathbb{E}^d \to \mathbb{R}$ is a real-valued phase, and $a_0^\varepsilon : \mathbb{E}^d \to \mathbb{C}$ is a possibly complex-valued amplitude. We emphasize that our approach is distinct from the polar decomposition known as Madelung transform, hence the possibility for the amplitude to be (or become) complex. Our goal is to understand the semi-classical limit of equation (1.3), that is to describe the behavior in the limit $\varepsilon \to 0$ of the solutions to (1.3) with initial data (1.4). Generalizing the idea of [11], we remark that if $(\phi^\varepsilon, a^\varepsilon)$ solves the system

$$\begin{cases} \partial_t \phi^\varepsilon + \dfrac{1}{2}\left\langle \nabla\phi^\varepsilon, H\nabla\phi^\varepsilon\right\rangle + g(|a^\varepsilon|^2)\left\langle \beta, \nabla\phi^\varepsilon\right\rangle + \displaystyle\sum_{j=1}^J K_j * |a^\varepsilon|^{2\sigma_j} + V = 0, \\ \qquad \phi^\varepsilon_{|t=0} = \phi_0^\varepsilon, \\ \partial_t a^\varepsilon + \left\langle \nabla\phi^\varepsilon, H\nabla a^\varepsilon\right\rangle + \dfrac{1}{2} a^\varepsilon D^2\phi^\varepsilon + \left\langle \beta, \nabla\left[g(|a^\varepsilon|^2)a^\varepsilon\right]\right\rangle = \dfrac{i\varepsilon}{2} D^2 a^\varepsilon, \\ \qquad a^\varepsilon_{|t=0} = a_0^\varepsilon, \end{cases} \tag{1.5}$$

then

$$u^\varepsilon(t,x) = a^\varepsilon(t,x) e^{i\phi^\varepsilon(t,x)/\varepsilon}$$

solves (1.3)-(1.4). Therefore, we focus on (1.5). Note that ϕ^ε, there, remains real-valued, while a^ε will be complex-valued (even if a_0^ε is real), due to the term $i\varepsilon D^2 a^\varepsilon$, which is a remain of dispersive effects in the initial Schrödinger equation.

Example 1.3. In the case of the hyperbolic NLS (1.1), $d = 2$ and the equations in (1.5) read

$$\begin{cases} \partial_t \phi^\varepsilon + \dfrac{1}{2}\left(|\partial_1\phi^\varepsilon|^2 - |\partial_2\phi^\varepsilon|^2\right) \mp |a^\varepsilon|^2 = 0, \\ \partial_t a^\varepsilon + \partial_1\phi^\varepsilon\partial_1 a^\varepsilon - \partial_2\phi^\varepsilon\partial_2 a^\varepsilon + \dfrac{1}{2} a^\varepsilon\left(\partial_1^2\phi^\varepsilon - \partial_2^2\phi^\varepsilon\right) = \dfrac{i\varepsilon}{2}\left(\partial_1^2 a^\varepsilon - \partial_2^2 a^\varepsilon\right). \end{cases}$$

Passing formally to the limit $\varepsilon \to 0$, and setting $\rho = |a|^2$, $v = \nabla\phi$, and $\tilde{v} = (\partial_1\phi, -\partial_2\phi)^T$, we find, respectively,

$$\begin{cases} \partial_t v_j + v_1\partial_1 v_j - v_2\partial_2 v_j \mp \partial_j\rho = 0, \\ \partial_t\rho + \partial_1(\rho v_1) - \partial_2(\rho v_2) = 0, \end{cases} \qquad \begin{cases} \partial_t\tilde{v} + \left\langle \tilde{v}, \nabla\right\rangle \tilde{v} \mp \begin{pmatrix} \partial_1\rho \\ -\partial_2\rho \end{pmatrix} = 0, \\ \partial_t\rho + \left\langle \nabla, \rho\tilde{v}\right\rangle = 0. \end{cases}$$

No special structure such as symmetry (ensuring the hyperbolicity of the system) seems to be available here. Because of this, we work with analytic regularity, since Sobolev regularity is hopeless in such a case (see [14, 15]).

In [5], we have already addressed the issue of the semi-classical limit of (1.3) in the case where $d = 1$, $\mathbb{E}^d = \mathbb{R}$, $D^2 = \partial_x^2$, where the nonlinearity is local (that is, $K = \delta$) and where $V = 0$. We show that the method that was used in [5] can be generalized

- To any dimension of the space variable,
- To the torus $\mathbb{T}^d$,
- To a second order operator D^2 which is not necessarily elliptic,
- To nonlocal nonlinearities,
- To non-zero potentials V.

Note that at least for the first three aspects evoked above, our approach provides results which can be established by following the same strategy as in [8] (case $x \in \mathbb{T}^d$) and [21] (case $x \in \mathbb{R}^d$), based on the notion of analytic symbols, as developed by J. Sjöstrand [19]. On the other hand, incorporating nonlocal nonlinearities seems to be easier when relying on a notion of (time dependent) analyticity based on Fourier analysis, as in [10] (and [16] for the same idea in a different context); see the next subsection for more details.

1.3. **The functional framework.** For $w \geqslant 0$ and $\ell \geqslant 0$, we consider the space

$$\mathcal{H}_w^\ell = \{\psi \in L^2(\mathbb{E}^d), \quad \|\psi\|_{\mathcal{H}_w^\ell} < \infty\},$$

where

$$\|\psi\|_{\mathcal{H}_w^\ell}^2 := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \langle\xi\rangle^{2\ell} e^{2w\langle\xi\rangle} |\widehat{\psi}(\xi)|^2 d\xi & \text{if} \quad \mathbb{E}^d = \mathbb{R}^d, \\ \displaystyle\sum_{m\in\mathbb{Z}^d} \langle m\rangle^{2\ell} e^{2w\langle m\rangle} |\widehat{\psi}(m)|^2 & \text{if} \quad \mathbb{E}^d = \mathbb{T}^d, \end{cases}$$

with $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, and where the Fourier transform and series are defined by

$$\widehat{\psi}(\xi) = \mathcal{F}\psi(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \psi(y) e^{-i\langle\xi,y\rangle} dy \quad \text{if} \quad \mathbb{E}^d = \mathbb{R}^d,$$

$$\widehat{\psi}(m) = \mathcal{F}\psi(m) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} \psi(y) e^{-i\langle m,y\rangle} dy \quad \text{if} \quad \mathbb{E}^d = \mathbb{T}^d.$$

We obviously have the monotonicity property,

$$0 \leqslant w_1 \leqslant w_2 \quad \Longrightarrow \quad \|\psi\|_{\mathcal{H}_{w_1}^\ell} \leqslant \|\psi\|_{\mathcal{H}_{w_2}^\ell}. \tag{1.6}$$

The interest of considering a time-dependent, decreasing, weight w is that energy estimates become similar to parabolic estimates, since

$$\frac{d}{dt}\|\psi\|_{\mathcal{H}_w^\ell}^2 = 2\,\mathrm{Re}\,(\psi, \partial_t\psi)_{\mathcal{H}_w^\ell} + 2\dot{w}\|\psi\|_{\mathcal{H}_w^{\ell+1/2}}^2, \tag{1.7}$$

where $(\cdot,\cdot)_{\mathcal{H}_w^\ell}$ denotes the natural inner product stemming from the above definition. We choose a weight $w = w(t) = w_0 - Mt$, where $w_0 > 0$ and $M > 0$ are fixed. For $T > 0$, we work in spaces such as

$$C([0,T],\mathcal{H}_w^\ell) = \left\{\psi \mid \mathcal{F}^{-1}\left(e^{w(t)\langle\xi\rangle}\widehat{\psi}\right) \in C([0,T],\mathcal{H}_0^\ell) = C([0,T],H^\ell)\right\},$$

where $H^\ell = H^\ell(\mathbb{E}^d)$ is the standard Sobolev space, or

$$L^2([0,T],\mathcal{H}_w^\ell) = L_T^2\mathcal{H}_w^\ell = \left\{\psi \mid \int_0^T \|\psi(t)\|_{\mathcal{H}_{w(t)}^\ell}^2 dt < \infty\right\}.$$

Phases and amplitudes belong to spaces

$$Y_{w,T}^\ell = C([0,T],\mathcal{H}_w^\ell) \cap L_T^2\mathcal{H}_w^{\ell+1/2},$$

and the fact that phase and amplitude do not have exactly the same regularity shows up in the introduction of the space

$$X_{w,T}^\ell = Y_{w,T}^{\ell+1} \times Y_{w,T}^\ell,$$

which is reminiscent of the fact that in the case where the operator on the left hand side of (1.5) is hyperbolic (typically, starting from a defocusing cubic Schrödinger equation with the standard Laplacian), the good unknown is $(\nabla\phi^\varepsilon, a^\varepsilon)$ rather than $(\phi^\varepsilon, a^\varepsilon)$ (see [11]). The space $X^\ell_{w,T}$ is endowed with the norm

$$\|(\phi,a)\|_{X^\ell_{w,T}} = \|\!|\phi|\!\|_{\ell+1,T} + \|\!|a|\!\|_{\ell,T},$$

where

$$\|\!|\psi|\!\|^2_{\ell,t} = \max\left(\sup_{0\leqslant s\leqslant t}\|\psi(s)\|^2_{\mathcal{H}^\ell_{w(s)}}, 2M\int_0^t \|\psi(s)\|^2_{\mathcal{H}^{\ell+1/2}_{w(s)}}ds\right). \tag{1.8}$$

1.4. **Main results.** Our first result states local well-posedness for (1.5) in this functional framework.

Theorem 1.4. *Let $w_0>0$, $\ell>(d+1)/2$, $T_0>0$, $V\in L^2_{T_0}\mathcal{H}^{\ell+1/2}_{w_0}$ and $(\phi_0^\varepsilon, a_0^\varepsilon)_{\varepsilon\in[0,1]}$ be a bounded family in $\mathcal{H}^{\ell+1}_{w_0}\times\mathcal{H}^\ell_{w_0}$. Then, provided $M=M(\ell)>0$ is chosen sufficiently large, for all $\varepsilon\in[0,1]$, there is a unique solution $(\phi^\varepsilon, a^\varepsilon)\in X^\ell_{w,T}$ to (1.5), where $w(t)=w_0-Mt$ and $T=T(\ell)<\min(w_0/M, T_0)$. Moreover, up to the choice of a possibly larger value for M (and consequently a smaller one for T), we have the estimates*

$$\|\!|\phi^\varepsilon|\!\|^2_{\ell+1,T} \leqslant 4\|\phi_0^\varepsilon\|^2_{\mathcal{H}^{\ell+1}_{w_0}} + \max_{1\leqslant j\leqslant J}\|a_0^\varepsilon\|^{4\sigma_j}_{\mathcal{H}^\ell_{w_0}} + \|V\|^2_{L^2_{T_0}\mathcal{H}^{\ell+1/2}_{w_0}}, \qquad \|\!|a^\varepsilon|\!\|^2_{\ell,T}\leqslant 2\|a_0^\varepsilon\|^2_{\mathcal{H}^\ell_{w_0}}.$$

An important aspect in the above statement is the fact that the local existence time T is uniform in $\varepsilon\in[0,1]$. In view of the discussion in Subsection 1.2, this yields a uniform time of existence for the solution of (1.3). We emphasize that this property is not a consequence of the standard local well-posedness argument (based on a fixed point), which would yield a local existence time $T^\varepsilon=\mathcal{O}(\varepsilon^\alpha)$ for some $\alpha\geqslant 1$, while we recall that the a priori estimates do not make it possible to extend the local solution to much larger time. In other words, the formulation (1.5) is already helpful at the level of the life-span of the solution to (1.3).

Our second result states the convergence of the phase and of the complex amplitude as $\varepsilon\to 0$.

Theorem 1.5. *Let $w_0>0$, $\ell>(d+1)/2$, $T_0>0$, $V\in L^2_{T_0}\mathcal{H}^{\ell+3/2}_{w_0}$, $(\phi_0,a_0)\in\mathcal{H}^{\ell+2}_{w_0}\times\mathcal{H}^{\ell+1}_{w_0}$ and $(\phi_0^\varepsilon,a_0^\varepsilon)_{\varepsilon\in(0,1]}$ bounded in $\mathcal{H}^{\ell+1}_{w_0}\times\mathcal{H}^\ell_{w_0}$ such that*

$$r_0^\varepsilon := \|\phi_0^\varepsilon-\phi_0\|_{\mathcal{H}^{\ell+1}_{w_0}} + \|a_0^\varepsilon-a_0\|_{\mathcal{H}^\ell_{w_0}} \underset{\varepsilon\to0}{\longrightarrow} 0.$$

Let $M=M(\ell+1)$ and $T=T(\ell+1)$, as defined as in Theorem 1.4. Then there is an ε-independent $C>0$ such that for all $\varepsilon\in(0,1]$,

$$\|\!|\phi^\varepsilon-\phi|\!\|_{\ell+1,T} + \|\!|a^\varepsilon-a|\!\|_{\ell,T}\leqslant C\left(r_0^\varepsilon+\varepsilon\right),$$

where $(\phi^\varepsilon,a^\varepsilon)$ denotes the solution to (1.5) and (ϕ,a) is the solution to the formal limit of (1.5) as $\varepsilon\to0$

$$\begin{cases}\partial_t\phi + \frac12\langle\nabla\phi, H\nabla\phi\rangle + g\left(|a|^2\right)\langle\beta,\nabla\phi\rangle + \sum_{j=1}^J K_j * |a|^{2\sigma_j} + V = 0,\\ \qquad \phi_{|t=0}=\phi_0,\\ \partial_t a + \langle\nabla\phi, H\nabla a\rangle + \frac12 a D^2\phi + \left\langle\beta, \nabla\left(g\left(|a|^2\right)a\right)\right\rangle = 0,\\ \qquad a_{|t=0}=a_0,\end{cases} \tag{1.9}$$

whose existence and uniqueness stem from Theorem 1.4.

However, regarding convergence of the wave function u^ε, the previous result is not sufficient. Indeed, as fast as the initial data ϕ_0^ε and a_0^ε may converge as $\varepsilon \to 0$, Theorem 1.5 at most guarantees that $\phi^\varepsilon - \phi = \mathcal{O}(\varepsilon)$, which only ensures that $a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - a e^{i\phi/\varepsilon} = \mathcal{O}(1)$, due to the rapid oscillations. However, the above convergence result suffices to infer the convergence of quadratic observables:

Corollary 1.6. *Under the assumptions of Theorem 1.5, the position and momentum densities converge:*

$$|u^\varepsilon|^2 \underset{\varepsilon\to 0}{\longrightarrow} |a|^2, \quad \textit{and} \quad \operatorname{Im}\left(\varepsilon \bar{u}^\varepsilon \partial u^\varepsilon\right) \underset{\varepsilon\to 0}{\longrightarrow} |a|^2 \partial\phi, \quad \textit{in } L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{E}^d)),$$

where $\partial = \partial_j$, for any $j \in \{1, \cdots, d\}$.

In order to get a good approximation of the wave function $a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$, we have to approximate ϕ^ε up to an error which is small compared to ε. It will be done by adding a corrective term to (ϕ, a). For this purpose, we consider the system obtained by linearizing (1.5) about (ϕ, a), solution to (1.9),

$$\tag{1.10} \begin{cases} \partial_t \phi_1 + \langle \nabla\phi, H\nabla\phi_1\rangle + g\left(|a|^2\right)\langle \beta, \nabla\phi_1\rangle + 2g'\left(|a|^2\right)\langle\beta,\nabla\phi\rangle \operatorname{Re}\left(\bar{a}a_1\right) \\ \qquad + 2\displaystyle\sum_{j=1}^J \sigma_j K_j * \left(|a|^{2(\sigma_j-1)}\operatorname{Re}\left(\bar{a}a_1\right)\right) = 0, \\ \qquad \phi_{1|t=0} = \phi_{10}, \\ \partial_t a_1 + \langle\nabla\phi, H\nabla a_1\rangle + \dfrac{1}{2}a_1 D^2\phi + \langle\nabla a, H\nabla\phi_1\rangle + \dfrac{1}{2} a D^2\phi_1 \\ \qquad + \left\langle \beta, \nabla\left(g\left(|a|^2\right)a_1\right)\right\rangle + 2\left\langle\beta, \nabla\left(a g'\left(|a|^2\right)\operatorname{Re}\left(\bar{a}a_1\right)\right)\right\rangle = \dfrac{i}{2}D^2 a, \\ \qquad a_{1|t=0} = a_{10}. \end{cases}$$

Provided $(\phi_0, a_0) \in \mathcal{H}^{\ell+3}_{w_0} \times \mathcal{H}^{\ell+2}_{w_0}$ (which implies $(\phi, a) \in X^{\ell+2}_{w,T}$ according to Theorem 1.4) and $(\phi_{10}, a_{10}) \in \mathcal{H}^{\ell+2}_{w_0} \times \mathcal{H}^{\ell+1}_{w_0}$, we will see that the solution to (1.10) belongs to $X^{\ell+1}_{w,T}$, and our final result is the following.

Theorem 1.7. *Let $w_0 > 0$, $\ell > (d+1)/2$, $T_0 > 0$, $V \in L^2_{T_0}\mathcal{H}^{\ell+5/2}_{w_0}$, $(\phi_0, a_0) \in \mathcal{H}^{\ell+3}_{w_0} \times \mathcal{H}^{\ell+2}_{w_0}$, $(\phi_{10}, a_{10}) \in \mathcal{H}^{\ell+2}_{w_0} \times \mathcal{H}^{\ell+1}_{w_0}$ and $(\phi_0^\varepsilon, a_0^\varepsilon)_{\varepsilon\in(0,1]}$ bounded in $\mathcal{H}^{\ell+1}_{w_0} \times \mathcal{H}^{\ell}_{w_0}$ such that*

$$r_1^\varepsilon := \|\phi_0^\varepsilon - \phi_0 - \varepsilon\phi_{10}\|_{\mathcal{H}^{\ell+1}_{w_0}} + \|a_0^\varepsilon - a_0 - \varepsilon a_{10}\|_{\mathcal{H}^{\ell}_{w_0}} = o(\varepsilon) \ \textit{as } \varepsilon \to 0.$$

Then, for $M = M(\ell+2)$ and $T = T(\ell+2)$ as in Theorem 1.4, there is an ε-independent $C > 0$ such that for all $\varepsilon \in (0,1]$,

$$\tag{1.11} |||\phi^\varepsilon - \phi - \varepsilon\phi_1|||_{\ell+1,T} + |||a^\varepsilon - a - \varepsilon a_1|||_{\ell,T} \leqslant C\left(r_1^\varepsilon + \varepsilon^2\right),$$

where $(\phi^\varepsilon, a^\varepsilon)$ denotes the solution to (1.5), (ϕ, a) is the solution to (1.9), and (ϕ_1, a_1) is the solution to (1.10). In particular,

$$\left\| u^\varepsilon - a e^{i\phi_1} e^{i\phi/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty(\mathbb{E}^d))} = \mathcal{O}\left(\frac{r_1^\varepsilon}{\varepsilon} + \varepsilon\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Outline. In Section 2, we prove Theorem 1.4, by starting with a generalization of key estimates established in [10] to the periodic setting $\mathbb{E}^d = \mathbb{T}^d$. Theorem 1.5 is proved in Section 3, and the proof of Theorem 1.7 is sketched in the final Section 4.

2. Well-posedness

2.1. A key bilinear estimate. The following proposition is proved in [10] in the case $\mathbb{E}^d = \mathbb{R}^d$ in the context of long range scattering. We have used it in [4, 5] in the context of semi-classical analysis. We extend it here to the case $\mathbb{E}^d = \mathbb{T}^d$.

Proposition 2.1. *Let $\ell \geqslant 0$ and $s > d/2$. Then, for every $\psi_1, \psi_2 \in \mathcal{H}_w^{\max(\ell,s)}$,*

$$\|\psi_1\psi_2\|_{\mathcal{H}_w^\ell} \leqslant C_{\ell,s}\left(\|\psi_1\|_{\mathcal{H}_w^\ell}\|\psi_2\|_{\mathcal{H}_w^s} + \|\psi_1\|_{\mathcal{H}_w^s}\|\psi_2\|_{\mathcal{H}_w^\ell}\right), \tag{2.1}$$

where

$$C_{\ell,s} = \begin{cases} \frac{2^\ell}{(2\pi)^{d/2}}\left\|\frac{1}{\langle\cdot\rangle^s}\right\|_{L^2(\mathbb{R}^d)} & \text{if} \quad \mathbb{E}^d = \mathbb{R}^d, \\ \frac{2^\ell}{(2\pi)^{d/2}}\left\|\frac{1}{\langle\cdot\rangle^s}\right\|_{\ell^2(\mathbb{Z}^d)} & \text{if} \quad \mathbb{E}^d = \mathbb{T}^d. \end{cases}$$

We detail the proof only in the case $\mathbb{E}^d = \mathbb{T}^d$. The proof is analogous in the case $\mathbb{E}^d = \mathbb{R}^d$, and can be found in [10] (with different notations, though). Proposition 2.1 in the case $\mathbb{E}^d = \mathbb{T}^d$ stems from the following sequence of lemmas. We skip the proofs of the most classical ones.

Lemma 2.2. *For all $m, n \in \mathbb{R}^d$,* $\quad \langle m+n\rangle \leqslant \langle m\rangle + \langle n\rangle$.

Lemma 2.3. *If $\psi_1, \psi_2 \in L^2(\mathbb{T}^d)$, for all $m \in \mathbb{Z}^d$,*

$$\widehat{\psi_1\psi_2}(m) = \frac{1}{(2\pi)^{d/2}}\sum_{k\in\mathbb{Z}^d}\widehat{\psi_1}(k)\widehat{\psi_2}(m-k) =: \frac{1}{(2\pi)^{d/2}}\widehat{\psi_1} * \widehat{\psi_2}(m).$$

Lemma 2.4. *For $\ell, w \geqslant 0$, if $\psi_1, \psi_2 \in \mathcal{H}_w^\ell$, then*

$$\|\psi_1\psi_2\|_{\mathcal{H}_w^\ell} \leqslant \frac{2^\ell}{(2\pi)^{d/2}}\left[\left\|\left(\langle\cdot\rangle^\ell e^{w\langle\cdot\rangle}|\widehat{\psi_1}|\right) * \left(e^{w\langle\cdot\rangle}|\widehat{\psi_2}|\right)\right\|_{\ell^2(\mathbb{Z}^d)} + \left\|\left(e^{w\langle\cdot\rangle}|\widehat{\psi_1}|\right) * \left(\langle\cdot\rangle^\ell e^{w\langle\cdot\rangle}|\widehat{\psi_2}|\right)\right\|_{\ell^2(\mathbb{Z}^d)}\right]$$

Proof. From Lemma 2.3,

$$\|\psi_1\psi_2\|_{\mathcal{H}_w^\ell}^2 \leqslant \frac{1}{(2\pi)^d}\sum_{m\in\mathbb{Z}^d}\left(\sum_{n\in\mathbb{Z}^d}\langle m\rangle^\ell e^{w\langle m\rangle}|\widehat{\psi_1}(n)||\widehat{\psi_2}(m-n)|\right)^2.$$

From Lemma 2.2 and because for any $m, n \in \mathbb{Z}^d$, we have either $\langle n\rangle \geqslant \langle m\rangle/2$ or $\langle m-n\rangle \geqslant \langle m\rangle/2$, we deduce

$$\begin{aligned}\|\psi_1\psi_2\|_{\mathcal{H}_w^\ell}^2 \leqslant \frac{1}{(2\pi)^d}\sum_{m\in\mathbb{Z}^d}\Bigg(&\sum_{n\in\mathbb{Z}^d,\langle n\rangle\geqslant\langle m\rangle/2}\langle m\rangle^\ell e^{w\langle n\rangle}|\widehat{\psi_1}(n)|e^{w\langle m-n\rangle}|\widehat{\psi_2}(m-n)| \\ &+ \sum_{n\in\mathbb{Z}^d,\langle m-n\rangle\geqslant\langle m\rangle/2}\langle m\rangle^\ell e^{w\langle n\rangle}|\widehat{\psi_1}(n)|e^{w\langle m-n\rangle}|\widehat{\psi_2}(m-n)|\Bigg)^2 \\ \leqslant \frac{2^{2\ell}}{(2\pi)^d}\sum_{m\in\mathbb{Z}^d}\Bigg(&\sum_{n\in\mathbb{Z}^d}\langle n\rangle^\ell e^{w\langle n\rangle}|\widehat{\psi_1}(n)|e^{w\langle m-n\rangle}|\widehat{\psi_2}(m-n)| \\ &+ \sum_{n\in\mathbb{Z}^d}e^{w\langle n\rangle}|\widehat{\psi_1}(n)|\langle m-n\rangle^\ell e^{w\langle m-n\rangle}|\widehat{\psi_2}(m-n)|\Bigg)^2.\end{aligned}$$

The result follows thanks to the triangle inequality in $\ell^2(\mathbb{Z}^d)$. □

Lemma 2.5. *If $u \in \ell^2(\mathbb{Z}^d)$ and $v \in \ell^1(\mathbb{Z}^d)$, then $u * v \in \ell^2(\mathbb{Z}^d)$, and*

$$\|u * v\|_{\ell^2(\mathbb{Z}^d)} \leqslant \|u\|_{\ell^2(\mathbb{Z}^d)} \|v\|_{\ell^1(\mathbb{Z}^d)}$$

Proof of Proposition 2.1. Let us estimate the first term in the bracket of the right hand side of the inequality in Lemma 2.4. The other term is treated similarly. According to Lemma 2.5, we have

$$\begin{aligned}
&\left\| \left(\langle\cdot\rangle^\ell e^{w\langle\cdot\rangle} |\widehat{\psi_1}| \right) * \left(e^{w\langle\cdot\rangle} |\widehat{\psi_2}| \right) \right\|_{\ell^2(\mathbb{Z}^d)} \\
&\quad \leqslant \left\| \langle\cdot\rangle^\ell e^{w\langle\cdot\rangle} |\widehat{\psi_1}| \right\|_{\ell^2(\mathbb{Z}^d)} \left\| e^{w\langle\cdot\rangle} |\widehat{\psi_2}| \right\|_{\ell^1(\mathbb{Z}^d)} \\
&\quad \leqslant \left\| \langle\cdot\rangle^\ell e^{w\langle\cdot\rangle} |\widehat{\psi_1}| \right\|_{\ell^2(\mathbb{Z}^d)} \left\| \frac{1}{\langle\cdot\rangle^s} \right\|_{\ell^2(\mathbb{Z}^d)} \left\| \langle\cdot\rangle^s e^{w\langle\cdot\rangle} |\widehat{\psi_2}| \right\|_{\ell^2(\mathbb{Z}^d)} \\
&\quad \leqslant \left\| \frac{1}{\langle\cdot\rangle^s} \right\|_{\ell^2(\mathbb{Z}^d)} \|\psi_1\|_{\mathcal{H}^\ell_w} \|\psi_2\|_{\mathcal{H}^s_w},
\end{aligned}$$

where we have also used the Cauchy-Schwarz inequality. □

2.2. **The iterative scheme.** In this section, $\varepsilon \in [0,1]$ is fixed. To lighten the notations, we consider the case $J = 1$ (only one Fourier multiplier), and leave out the corresponding index: the proof shows that considering finitely many such terms is straightforward. Solutions to (1.5) are constructed as limits of the solutions of the iterative scheme

$$\begin{cases}
\partial_t \phi^\varepsilon_{j+1} + \dfrac{1}{2} \left\langle \nabla\phi^\varepsilon_j, H\nabla\phi^\varepsilon_{j+1} \right\rangle + g(|a^\varepsilon_j|^2) \left\langle \beta, \nabla\phi^\varepsilon_{j+1} \right\rangle = -K * |a^\varepsilon_j|^{2\sigma} - V, \\
\qquad \phi^\varepsilon_{j+1|t=0} = \phi^\varepsilon_0, \\
\partial_t a^\varepsilon_{j+1} + \left\langle \nabla\phi^\varepsilon_j, H\nabla a^\varepsilon_{j+1} \right\rangle + \dfrac{1}{2}(D^2\phi^\varepsilon_j) a^\varepsilon_{j+1} \\
\quad + \left\langle \beta, \nabla \left(g(|a^\varepsilon_j|^2) \right) \right\rangle a^\varepsilon_{j+1} + h(|a^\varepsilon_j|^2) \overline{a^\varepsilon_j} \left\langle \beta, \nabla a^\varepsilon_j \right\rangle a^\varepsilon_{j+1} = \dfrac{i\varepsilon}{2} D^2 a^\varepsilon_{j+1}, \\
\qquad a^\varepsilon_{j+1|t=0} = a^\varepsilon_0,
\end{cases} \tag{2.2}$$

where $h(s) = g(s)/s$. The scheme is initialized with the time-independent pair $(\phi^\varepsilon_0, a^\varepsilon_0) \in \mathcal{H}^{\ell+1}_{w_0} \times \mathcal{H}^\ell_{w_0} \subset X^\ell_{w,T}$ for any $T > 0$.

The scheme is well-defined: if $\ell > (d+1)/2$, for a given $(\phi^\varepsilon_j, a^\varepsilon_j) \in X^\ell_{w,T}$, (2.2) defines $(\phi^\varepsilon_{j+1}, a^\varepsilon_{j+1})$. Indeed, in the first equation, ϕ^ε_{j+1} solves a linear transport equation with smooth coefficients, which guarantees the existence of a solution $\phi^\varepsilon_{j+1} \in L^\infty_T L^2$ (see e.g [2, Section 3], or [1, Section II.C, Proposition 1.2]). The same argument provides a solution $a^\varepsilon_{j+1} \in L^\infty_T L^2$ to the second equation in the case $\varepsilon = 0$. On the other hand, if $\varepsilon > 0$, the second equation is equivalent through the relation $v^\varepsilon_{j+1} = a^\varepsilon_{j+1} e^{i\phi^\varepsilon_j/\varepsilon}$ to the equation

$$i\varepsilon\partial_t v^\varepsilon_{j+1} + \frac{\varepsilon^2}{2} D^2 v^\varepsilon_{j+1} + W(t,x) v^\varepsilon_{j+1} = 0, \tag{2.3}$$

with initial condition

$$v^\varepsilon_{j+1|t=0} = v^\varepsilon_0 = a^\varepsilon_0 e^{i\phi^\varepsilon_0/\varepsilon},$$

where

$$W=\partial_t\phi_j^\varepsilon+\frac{1}{2}\left\langle\nabla\phi_j^\varepsilon,H\nabla\phi_j^\varepsilon\right\rangle+i\varepsilon\left\langle\beta,\nabla\left(g\left(|a_j^\varepsilon|^2\right)\right)\right\rangle+i\varepsilon h(|a_j^\varepsilon|^2)\overline{a_j^\varepsilon}\left\langle\beta,\nabla a_j^\varepsilon\right\rangle.$$

This is a linear Schrödinger like equation, with a second order operator D^2 which is not necessarily elliptic, and with a smooth and bounded external time-dependent potential $W(t,x)$. Note that this external potential is complex-valued, so the existence of a solution for (2.3) is not quite standard. On the other hand, a fixed point argument applied to the map

$$\Psi:\left\{\begin{array}{rcl} C([0,T^\varepsilon],L^2) & \longrightarrow & C([0,T^\varepsilon],L^2)\\ u & \mapsto & e^{i\varepsilon tD^2/2}v_0^\varepsilon+\frac{i}{\varepsilon}\int_0^t e^{i\varepsilon(t-s)D^2/2}\left[W(s)u(s)\right]ds,\end{array}\right.$$

where $0<T^\varepsilon\leqslant T$, provides the existence of a $C([0,T^\varepsilon],L^2)$ solution to (2.3), for some $T^\varepsilon>0$. We actually have $v_{j+1}^\varepsilon\in C([0,T],L^2)$, since $W\in L^\infty([0,T]\times\mathbb{E}^d)$.

The following lemma gives the estimates that will ensure that $(\phi_{j+1}^\varepsilon,a_{j+1}^\varepsilon)\in X_{w,T}^\ell$ provided $(\phi_j^\varepsilon,a_j^\varepsilon)\in X_{w,T}^\ell$. It is almost identical to Lemma 2.2 in [5].

Lemma 2.6. *Let $\ell>(d+1)/2$ and $T>0$. Let $(\phi,a)\in X_{w,T}^\ell$, $\tilde a\in Y_{w,T}^{\ell+1}$ and $(F,G)\in L^2([0,T],\mathcal{H}_w^{\ell+1/2}\times\mathcal{H}_w^{\ell-1/2})$ such that*

$$\partial_t\phi=F,\qquad \phi(0)\in\mathcal{H}_{w_0}^{\ell+1},\tag{2.4}$$

$$\partial_t a=G+i\theta_1D^2a+i\theta_2D^2\tilde a,\qquad a(0)\in\mathcal{H}_{w_0}^{\ell},\tag{2.5}$$

where $\theta_1,\theta_2\in\mathbb{R}$. Then

$$|||\phi|||_{\ell+1,T}^2\leqslant\|\phi(0)\|_{\mathcal{H}_{w_0}^{\ell+1}}^2+\frac{1}{M}|||\phi|||_{\ell+1,T}\sqrt{2M}\|F\|_{L_T^2\mathcal{H}_w^{\ell+1/2}},\tag{2.6}$$

$$|||a|||_{\ell,T}^2\leqslant\|a(0)\|_{\mathcal{H}_{w_0}^{\ell}}^2+\frac{1}{M}|||a|||_{\ell,T}\sqrt{2M}\|G\|_{L_T^2\mathcal{H}_w^{\ell-1/2}}+\frac{|\theta_2|}{2M}|||a|||_{\ell,T}|||\tilde a|||_{\ell+1,T}.\tag{2.7}$$

Moreover, there exists $C>0$ (that depends only on ℓ, not on w) such that

- *If $F=\partial_j\psi_1\partial_k\psi_2$ with $\psi_1,\psi_2\in Y_{\ell+1,T}$, then*

$$\sqrt{2M}\|F\|_{L_T^2\mathcal{H}_w^{\ell+1/2}}\leqslant C|||\psi_1|||_{\ell+1,T}|||\psi_2|||_{\ell+1,T}.\tag{2.8}$$

- *If $F=\left(\prod_{j=1}^{2n}b_j\right)\partial_k\psi$ with $n\geqslant 1$, $\psi\in Y_{\ell+1,T}$ and $b_j\in Y_{\ell,T}$ for all j, then*

$$\sqrt{2M}\|F\|_{L_T^2\mathcal{H}_w^{\ell+1/2}}\leqslant C\left(\prod_{j=1}^{2n}|||b_j|||_{\ell,T}\right)|||\psi|||_{\ell+1,T}.\tag{2.9}$$

- *If $F=K*\left(\prod_{j=1}^{2n}b_j\right)$ with $n\geqslant 1$, $b_j\in Y_{\ell,T}$ for all j and $\widehat{K}$ uniformly bounded, then*

$$\sqrt{2M}\|F\|_{L_T^2\mathcal{H}_w^{\ell+1/2}}\leqslant C\left(\prod_{j=1}^{2n}|||b_j|||_{\ell,T}\right).\tag{2.10}$$

- *If $G=\partial_j\psi\partial_k b$ with $\psi\in Y_{\ell+1,T}$ and $b\in Y_{\ell,T}$, then*

$$\sqrt{2M}\|G\|_{L_T^2\mathcal{H}_w^{\ell-1/2}}\leqslant C|||\psi|||_{\ell+1,T}|||b|||_{\ell,T}.\tag{2.11}$$

- *If $G = bD^2\psi$ with $\psi \in Y_{\ell+1,T}$ and $b \in Y_{\ell,T}$, then*

$$\sqrt{2M}\|G\|_{L^2_T\mathcal{H}^{\ell-1/2}_w} \leqslant C|||\psi|||_{\ell+1,T}|||b|||_{\ell,T}. \tag{2.12}$$

- *If $G = \left(\prod_{j=1}^{2n} b_j\right)\partial_k b$ with $n \geqslant 1$, $b, b_j \in Y_{\ell,T}$ for all j, then*

$$\sqrt{2M}\|G\|_{L^2_T\mathcal{H}^{\ell-1/2}_w} \leqslant C\left(\prod_{j=1}^{2n}|||b_j|||_{\ell,T}\right)|||b|||_{\ell,T}. \tag{2.13}$$

Proof. The proof of (2.6) and (2.7) is identical to the one given in [5]. The new constraint $\ell > (d+1)/2$ plays no role here. Inequalities similar to (2.8)-(2.13) were proved in [5]. The only differences with [5] are the presence of the kernel K in (2.10) and the constraint on s in Proposition 2.1, which is $s > d/2$ whereas it was $s > 1/2$ in [5] (where we had $d = 1$). It is actually sufficient to assume $\ell > d/2$ for the proof of (2.8)-(2.10), as we use several times (2.1) with $m = \ell + 1/2$ and $s = \ell > d/2$, or with $m = s = \ell$. For instance (2.10) follows from

$$\begin{aligned}\left\|K * \left(\prod_{j=1}^{2n} b_j\right)\right\|_{\mathcal{H}^{\ell+1/2}_w} &\leqslant \|\widehat{K}\|_{L^\infty}\left\|\prod_{j=1}^{2n} b_j\right\|_{\mathcal{H}^{\ell+1/2}_w}\\ &\leqslant C\sum_{j=1}^{2n}\prod_{k=1}^{j-1}\|b_k\|_{\mathcal{H}^\ell_w}\|b_j\|_{\mathcal{H}^{\ell+1/2}_w}\prod_{k=j+1}^{2n}\|b_k\|_{\mathcal{H}^\ell_w}.\end{aligned}$$

In order to prove (2.11)-(2.13), we use (2.1) with $m = s = \ell - 1/2 > d/2$ which is possible thanks to the assumption $\ell > (d+1)/2$. Actually, even (2.11)-(2.13) can be proved under the condition $\ell > d/2$, thanks to a refined version of Lemma 2.1 (see [4]). However, since it is not useful in the sequel to sharpen this assumption, we choose to make the stronger assumption $\ell > (d+1)/2$ for the sake of conciseness. □

First step: boundedness of the sequence. In view of the equation satisfied by ϕ^ε_{j+1} in (2.2), Lemma 2.6 yields

$$\begin{aligned}|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T} \leqslant \|\phi^\varepsilon_0\|^2_{\mathcal{H}^{\ell+1}_{w_0}} &+ \frac{C}{M}|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T}|||\phi^\varepsilon_j|||_{\ell+1,T} + \frac{C}{M}|||\phi^\varepsilon_{j+1}|||_{\ell+1,T}|||a^\varepsilon_j|||^{2\sigma}_{\ell,T}\\ &+ \frac{C}{M}|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T}|||a^\varepsilon_j|||^{2\gamma}_{\ell,T} + \frac{C}{M}|||\phi^\varepsilon_{j+1}|||_{\ell+1,T}\sqrt{2M}\|V\|_{L^2_T\mathcal{H}^{\ell+1/2}_w}.\end{aligned}$$

As for a^ε_{j+1}, we obtain in a similar way

$$|||a^\varepsilon_{j+1}|||^2_{\ell,T} \leqslant \|a^\varepsilon_0\|^2_{\mathcal{H}^\ell_{w_0}} + \frac{C}{M}|||a^\varepsilon_{j+1}|||^2_{\ell,T}|||\phi^\varepsilon_j|||_{\ell+1,T} + \frac{C}{M}|||a^\varepsilon_{j+1}|||^2_{\ell,T}|||a^\varepsilon_j|||^{2\gamma}_{\ell,T}.$$

Up to the term with V in the first one, the last two estimates are exactly the ones we had in [5]. The proof of the boundedness of the sequence $(\phi^\varepsilon_j, a^\varepsilon_j)$ in $X^\ell_{w,T}$ is quite similar to what was done in [5]. Indeed, under the assumption

$$\frac{C}{M}|||\phi^\varepsilon_j|||_{\ell+1,T} \leqslant \frac{1}{4}, \qquad \frac{C}{M}|||a^\varepsilon_j|||^{2\gamma}_{\ell,T} \leqslant \frac{1}{4}, \tag{2.14}$$

we have

$$\frac{1}{4}|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T} \leqslant \|\phi^\varepsilon_0\|^2_{\mathcal{H}^{\ell+1}_{w_0}} + \frac{2C^2}{M^2}|||a^\varepsilon_j|||^{4\sigma}_{\ell,T} + \frac{4C^2}{M}\|V\|^2_{L^2_T\mathcal{H}^{\ell+1/2}_w} \tag{2.15}$$

and

$$\frac{1}{2}\|\!|a_{j+1}^\varepsilon|\!\|_{\ell,T}^2 \leqslant \|a_0^\varepsilon\|_{\mathcal{H}_{w_0}^\ell}^2. \tag{2.16}$$

Note that the monotonicity property (1.6) implies

$$\|V\|_{L_T^2\mathcal{H}_w^{\ell+1/2}} \leqslant \|V\|_{L_{T_0}^2\mathcal{H}_{w_0}^{\ell+1/2}}.$$

We next show by induction that, provided M is sufficiently large, we can construct a sequence $(\phi_j^\varepsilon, a_j^\varepsilon)_{j\in\mathbb{N}}$ such that for every $j \in \mathbb{N}$,

$$\|\!|\phi_j^\varepsilon|\!\|_{\ell+1,T}^2 \leqslant 4\|\phi_0^\varepsilon\|_{\mathcal{H}_{w_0}^{\ell+1}}^2 + \frac{8C^2}{M^2}\left(2\|a_0^\varepsilon\|_{\mathcal{H}_{w_0}^\ell}^2\right)^{2\sigma} + \frac{16C^2}{M}\|V\|_{L_{T_0}^2\mathcal{H}_{w_0}^{\ell+1/2}}^2, \tag{2.17}$$

$$\|\!|a_j^\varepsilon|\!\|_{\ell,T}^2 \leqslant 2\|a_0^\varepsilon\|_{\mathcal{H}_{w_0}^\ell}^2. \tag{2.18}$$

For that purpose, we choose M sufficiently large such that (2.14) holds for $j = 0$ and such that

$$4\|\phi_0^\varepsilon\|_{\mathcal{H}_{w_0}^{\ell+1}}^2 + \frac{8C^2}{M^2}\left(2\|a_0^\varepsilon\|_{\mathcal{H}_{w_0}^\ell}^2\right)^{2\sigma} + \frac{16C^2}{M}\|V\|_{L_{T_0}^2\mathcal{H}_{w_0}^{\ell+1/2}}^2 \leqslant \frac{M^2}{16C^2}, \tag{2.19}$$

and

$$(2\|a_0^\varepsilon\|_{\mathcal{H}_{w_0}^\ell}^2)^\gamma \leqslant \frac{M}{4C}. \tag{2.20}$$

Then, (2.17)-(2.18) hold for $j = 0$, since with $(\phi_0^\varepsilon, a_0^\varepsilon)(t,x) = (\phi_0^\varepsilon, a_0^\varepsilon)(x)$ independent of time, it is easy to check that $\|\!|\phi_0^\varepsilon|\!\|_{\ell+1,T} = \|\phi_0^\varepsilon\|_{\mathcal{H}_{w_0}^{\ell+1}}$ and $\|\!|a_0^\varepsilon|\!\|_{\ell,T} = \|a_0^\varepsilon\|_{\mathcal{H}_{w_0}^\ell}$. Let $j \geqslant 0$ and assume that (2.17)-(2.18) hold. Then (2.17)-(2.18) and (2.19)-(2.20) ensure that the condition (2.14) is satisfied, and therefore (2.15)-(2.16) hold, from which we infer easily that (2.17)-(2.18) are true for j replaced by $j+1$ (for the estimate on the norm of V, we use here the fact that $T \leqslant T_0$ and $w \leqslant w_0$).

Second step: convergence. For $j \geqslant 1$, we set $\delta\phi_j^\varepsilon = \phi_j^\varepsilon - \phi_{j-1}^\varepsilon$, and $\delta a_j^\varepsilon = a_j^\varepsilon - a_{j-1}^\varepsilon$. Then, for every $j \geqslant 1$, we have

$$\begin{aligned}\partial_t\delta\phi_{j+1}^\varepsilon &+ \frac{1}{2}\left(\left\langle\nabla\phi_j^\varepsilon, H\nabla\delta\phi_{j+1}^\varepsilon\right\rangle + \left\langle\nabla\delta\phi_j^\varepsilon, H\nabla\phi_j^\varepsilon\right\rangle\right) + g\left(|a_j^\varepsilon|^2\right)\left\langle\beta,\nabla\delta\phi_{j+1}^\varepsilon\right\rangle \\ &+ \left(g\left(|a_j^\varepsilon|^2\right) - g\left(|a_{j-1}^\varepsilon|^2\right)\right)\left\langle\beta,\nabla\phi_j^\varepsilon\right\rangle + K*\left(|a_j^\varepsilon|^{2\sigma} - |a_{j-1}^\varepsilon|^{2\sigma}\right) = 0.\end{aligned}$$

and

$$\begin{aligned}&\partial_t\delta a_{j+1}^\varepsilon + \left\langle\nabla\phi_j^\varepsilon, H\nabla\delta a_{j+1}^\varepsilon\right\rangle + \left\langle\nabla\delta\phi_j^\varepsilon, H\nabla a_j^\varepsilon\right\rangle + \frac{1}{2}\delta a_{j+1}^\varepsilon D^2\phi_j^\varepsilon + \frac{1}{2}a_j^\varepsilon D^2\delta\phi_j^\varepsilon \\ &\quad + \left\langle\beta,\nabla\left(g\left(|a_j^\varepsilon|^2\right)\right)\right\rangle\delta a_{j+1}^\varepsilon + \left\langle\beta,\nabla\left(g\left(|a_j^\varepsilon|^2\right) - g\left(|a_{j-1}^\varepsilon|^2\right)\right)\right\rangle a_j^\varepsilon \\ &\quad + h\left(|a_j^\varepsilon|^2\right)\left\langle\beta,\nabla a_j^\varepsilon\right\rangle\overline{a_j^\varepsilon}\delta a_{j+1}^\varepsilon + h\left(|a_j^\varepsilon|^2\right)\left\langle\beta,\nabla a_j^\varepsilon\right\rangle\overline{\delta a_j^\varepsilon}a_j^\varepsilon \\ &\quad + h\left(|a_j^\varepsilon|^2\right)\left\langle\beta,\nabla\delta a_j^\varepsilon\right\rangle\overline{a_{j-1}^\varepsilon}a_j^\varepsilon + \left(h\left(|a_j^\varepsilon|^2\right) - h\left(|a_{j-1}^\varepsilon|^2\right)\right)\left\langle\beta,\nabla a_{j-1}^\varepsilon\right\rangle\overline{a_{j-1}^\varepsilon}a_j^\varepsilon \\ &\quad = i\frac{\varepsilon}{2}D^2\delta a_{j+1}^\varepsilon,\end{aligned}$$

Lemma 2.6 and the boundedness of $(\phi_j^\varepsilon, a_j^\varepsilon)$ in $X_{w,T}^\ell$ imply like in [5] that for M large enough,

$$\max\left(\|\!|\delta\phi_{j+1}^\varepsilon|\!\|_{\ell+1,T}^2, \|\!|\delta a_{j+1}^\varepsilon|\!\|_{\ell,T}^2\right) \leqslant \frac{K}{M}\left(\|\!|\delta\phi_j^\varepsilon|\!\|_{\ell+1,T}^2 + \|\!|\delta a_j^\varepsilon|\!\|_{\ell,T}^2\right)$$

for some $K > 0$ which does not depend on ε provided $(\phi_0^\varepsilon, a_0^\varepsilon)_{\varepsilon\in[0,1]}$ is uniformly bounded in $\mathcal{H}_{w_0}^{\ell+1}\times\mathcal{H}_{w_0}^\ell$. We conclude as in [5] that provided $\ell > (d+1)/2$, possibly increasing M, $(\phi_j^\varepsilon, a_j^\varepsilon)$ converges geometrically in $X^\ell_{w,T}$ as $j\to\infty$. Uniqueness of the solution $(\phi^\varepsilon, a^\varepsilon)$ to (1.5) follows from the same kind of estimates as the ones which prove the convergence.

3. First order approximation

As in the previous section, we assume $J=1$ for the sake of conciseness.

Proof of Theorem 1.5. Next, assume that $(\phi_0,a_0)\in \mathcal{H}^{\ell+2}_{w_0}\times \mathcal{H}^{\ell+1}_{w_0}$. Then, in view of Theorem 1.4, the solution (ϕ,a) to (1.9) belongs to $X^{\ell+1}_{w,T}$. Given $\varepsilon>0$, if $(\phi_0^\varepsilon,a_0^\varepsilon)\in \mathcal{H}^{\ell+1}_{w_0}\times \mathcal{H}^{\ell}_{w_0}$, we denote by $(\phi^\varepsilon,a^\varepsilon)$ the solution to (1.5). We also denote $(\delta\phi^\varepsilon,\delta a^\varepsilon)=(\phi^\varepsilon-\phi,a^\varepsilon-a)$. Then, in the same fashion as above, we have

$$\begin{aligned}\partial_t\delta\phi^\varepsilon &+\frac12\left(\langle\nabla\delta\phi^\varepsilon,H\nabla\phi^\varepsilon\rangle+\langle\nabla\phi,H\nabla\delta\phi^\varepsilon\rangle\right)+g\left(|a^\varepsilon|^2\right)\langle\beta,\nabla\delta\phi^\varepsilon\rangle\\ &+\left(g\left(|a^\varepsilon|^2\right)-g\left(|a|^2\right)\right)\langle\beta,\nabla\phi\rangle+K*\left(|a^\varepsilon|^{2\sigma}-|a|^{2\sigma}\right)=0\end{aligned}$$

and

$$\begin{aligned}&\partial_t\delta a^\varepsilon+\langle\nabla\delta\phi^\varepsilon,H\nabla a^\varepsilon\rangle+\langle\nabla\phi,H\nabla\delta a^\varepsilon\rangle+\frac12\delta a^\varepsilon D^2\phi^\varepsilon+\frac12 aD^2\delta\phi^\varepsilon\\ &\quad+\left\langle\beta,\nabla\left(g\left(|a^\varepsilon|^2\right)\right)\right\rangle\delta a^\varepsilon+\left\langle\beta,\nabla\left(g\left(|a^\varepsilon|^2\right)-g\left(|a|^2\right)\right)\right\rangle a\\ &\quad+h\left(|a^\varepsilon|^2\right)\langle\beta,\nabla a^\varepsilon\rangle\overline{a^\varepsilon}\delta a^\varepsilon+h\left(|a^\varepsilon|^2\right)\langle\beta,\nabla a^\varepsilon\rangle\overline{\delta a^\varepsilon}a\\ &\quad+h\left(|a^\varepsilon|^2\right)\langle\beta,\nabla\delta a^\varepsilon\rangle|a|^2+\left(h\left(|a^\varepsilon|^2\right)-h\left(|a|^2\right)\right)\langle\beta,\nabla a\rangle|a|^2=i\frac{\varepsilon}{2}D^2\delta a^\varepsilon+i\frac{\varepsilon}{2}D^2a.\end{aligned}$$

Like in [5], for some new constant k, Lemma 2.6 and Theorem 1.4 imply, for M large enough,

$$|||\delta\phi^\varepsilon|||^2_{\ell+1,T}\leqslant k\|\phi_0^\varepsilon-\phi_0\|^2_{\mathcal{H}^{\ell+1}_{w_0}}+\frac{k}{M}|||\delta a^\varepsilon|||^2_{\ell,T},$$

and

$$|||\delta a^\varepsilon|||^2_{\ell,T}\leqslant k\|a_0^\varepsilon-a_0\|^2_{\mathcal{H}^{\ell}_{w_0}}+\frac{k}{M}|||\delta\phi^\varepsilon|||^2_{\ell+1,T}+\frac{k}{M}\varepsilon|||\delta a^\varepsilon|||_{\ell,T}|||a|||_{\ell+1,T}.$$

Possibly increasing the value of M and adding the last two inequalities, we deduce

$$|||\delta\phi^\varepsilon|||^2_{\ell+1,T}+|||\delta a^\varepsilon|||^2_{\ell,T}\leqslant C\|\phi_0^\varepsilon-\phi_0\|^2_{\mathcal{H}^{\ell+1}_{w_0}}+C\|a_0^\varepsilon-a_0\|^2_{\mathcal{H}^{\ell}_{w_0}}+C\varepsilon^2,$$

hence Theorem 1.5. As for the choice of M, a careful examination of the previous inequalities shows that aside from the assumption $M\geqslant M(\ell+1)$, which enables to estimate the source term, M can be chosen as in Theorem 1.4, namely such that $M\geqslant M(\ell)$. □

Proof of Corollary 1.6. Notice that, provided $w\geqslant 0$,

$$\|\psi\|_{H^\ell(\mathbb{E}^d)}\leqslant\|\psi\|_{\mathcal{H}^\ell_w}. \tag{3.1}$$

In particular, Sobolev embedding yields, for $\ell>(d+1)/2\geqslant 1$,

$$\|\psi\|_{L^\infty(\mathbb{E}^d)}\leqslant C\|\psi\|_{\mathcal{H}^\ell_w},$$

where C is independent of $w\geqslant 0$. With these remarks in mind, the L^1 estimates of Corollary 1.6 follow from Theorem 1.5 and Cauchy-Schwarz inequality, since

$$\left\||u^\varepsilon|^2-|a|^2\right\|_{L^\infty_TL^1}=\left\||a^\varepsilon|^2-|a|^2\right\|_{L^\infty_TL^1}\leqslant\|a^\varepsilon+a\|_{L^\infty_TL^2}\|\delta a^\varepsilon\|_{L^\infty_TL^2},$$

and

$$\begin{aligned}\| \operatorname{Im}(\varepsilon \bar{u}^\varepsilon \partial u^\varepsilon) - |a|^2 \partial\phi\|_{L^\infty_T L^1} &\leqslant \varepsilon \| \operatorname{Im} \bar{a}^\varepsilon \partial a^\varepsilon \|_{L^\infty_T L^1} + \| |a^\varepsilon|^2 \partial \phi^\varepsilon - |a|^2 \partial \phi\|_{L^\infty_T L^1} \\ &\leqslant \varepsilon \|a^\varepsilon\|^2_{L^\infty_T H^1} + \|a^\varepsilon + a\|_{L^\infty_T L^2} \|\delta a^\varepsilon\|_{L^\infty_T L^2} \|\partial \phi\|_{L^\infty_T L^\infty} \\ &\quad + \|a^\varepsilon\|_{L^\infty_T L^\infty} \|a^\varepsilon\|_{L^\infty_T L^2} \|\delta\phi^\varepsilon\|_{L^\infty_T H^1}.\end{aligned}$$

The L^∞ estimates in space follow by replacing L^1 and L^2 by L^∞ in the above inequalities, and using Sobolev embedding again. □

4. Convergence of the wave function

Again, we assume $J = 1$ for the sake of conciseness.

Proof of Theorem 1.7. Let $\ell > (d+1)/2$, and $(\phi_0, a_0) \in \mathcal{H}^{\ell+2}_{w_0} \times \mathcal{H}^{\ell+1}_{w_0}$. Theorem 1.4 yields a unique solution $(\phi, a) \in X^{\ell+1}_{w,T}$ to (1.9).

Let $(\phi_{10}, a_{10}) \in \mathcal{H}^{\ell+1}_{w_0} \times \mathcal{H}^{\ell}_{w_0}$. Note that (1.10) is a system of linear transport equations in the unknown $(\nabla\phi_1, a_1)$, whose coefficients are smooth functions. The general theory of transport equations (see e.g. [2, Section 3]) then shows that (1.10) has a unique solution $(\phi_1, a_1) \in C([0,T], L^2 \times L^2)$. We already know by this argument that the solution is actually more regular (in terms of Sobolev regularity), but we shall directly use a priori estimates in $\mathcal{H}^\ell_w$ spaces. Indeed, Lemma 2.6 implies that $(\phi_1, a_1) \in X^\ell_{w,T}$ with, exactly as in [5],

$$\begin{aligned}\|\!|\phi_1|\!\|^2_{\ell+1,T} &\leqslant \|\phi_{10}\|^2_{\mathcal{H}^{\ell+1}_{w_0}} + \frac{C}{M} \|\!|\phi_1|\!\|^2_{\ell+1,T} \|\!|\phi|\!\|_{\ell+1,T} + \frac{C}{M} \|\!|\phi_1|\!\|^2_{\ell+1,T} \|\!|a|\!\|^{2\gamma}_{\ell,T} \\ &\quad + \frac{C}{M} \|\!|\phi_1|\!\|_{\ell+1,T} \|\!|\phi|\!\|_{\ell+1,T} \|\!|a|\!\|^{2\gamma-1}_{\ell,T} \|\!|a_1|\!\|_{\ell,T} + \frac{C}{M} \|\!|\phi_1|\!\|_{\ell+1,T} \|\!|a|\!\|^{2\sigma-1}_{\ell,T} \|\!|a_1|\!\|_{\ell,T},\end{aligned}$$

along with

$$\begin{aligned}\|\!|a_1|\!\|^2_{\ell,T} &\leqslant \|a_{10}\|^2_{\mathcal{H}^\ell_{w_0}} + \frac{C}{M} \|\!|a_1|\!\|_{\ell,T} \|\!|a|\!\|_{\ell,T} \|\!|\phi_1|\!\|_{\ell+1,T} + \frac{C}{M} \|\!|a_1|\!\|^2_{\ell,T} \|\!|\phi|\!\|_{\ell+1,T} \\ &\quad + \frac{C}{M} \|\!|a_1|\!\|^2_{\ell,T} \|\!|a|\!\|^{2\gamma}_{\ell,T} + \frac{C}{M} \|\!|a_1|\!\|_{\ell,T} \|\!|a|\!\|_{\ell+1,T},\end{aligned}$$

for some $C > 0$.

Let $\ell > (d+1)/2$. For $(\phi_0, a_0) \in \mathcal{H}^{\ell+3}_{w_0} \times \mathcal{H}^{\ell+2}_{w_0}$, $(\phi_{10}, a_{10}) \in \mathcal{H}^{\ell+2}_{w_0} \times \mathcal{H}^{\ell+1}_{w_0}$ and $(\phi_0^\varepsilon, a_0^\varepsilon) \in \mathcal{H}^{\ell+1}_{w_0} \times \mathcal{H}^{\ell}_{w_0}$, we consider:

- $(\phi, a) \in X^{\ell+2}_{w,T}$ the solution to (1.9).
- $(\phi_1, a_1) \in X^{\ell+1}_{w,T}$ the solution to (1.10).
- $(\phi^\varepsilon_{\rm app}, a^\varepsilon_{\rm app}) = (\phi, a) + \varepsilon(\phi_1, a_1)$.
- $(\phi^\varepsilon, a^\varepsilon) \in X^\ell_{w,T}$ the solution to (1.5).

We assume that $\|\phi_0^\varepsilon - \phi_0 - \varepsilon\phi_{10}\|_{\mathcal{H}^{\ell+1}_{w_0}} = o(\varepsilon)$ and $\|a_0^\varepsilon - a_0 - \varepsilon a_{10}\|_{\mathcal{H}^\ell_{w_0}} = o(\varepsilon)$. Set

$$\begin{aligned}\delta\phi_1^\varepsilon &= \phi^\varepsilon - \phi^\varepsilon_{\rm app} = \phi^\varepsilon - \phi - \varepsilon\phi_1 = \delta\phi^\varepsilon - \varepsilon\phi_1, \\ \delta a_1^\varepsilon &= a^\varepsilon - a^\varepsilon_{\rm app} = a^\varepsilon - a - \varepsilon a_1 = \delta a^\varepsilon - \varepsilon a_1.\end{aligned}$$

The equation satisfied by $\delta\phi_1^\varepsilon$ writes

$$\partial_t \delta\phi_1^\varepsilon + \langle \nabla\phi, H\nabla\delta\phi_1^\varepsilon\rangle + \frac{1}{2}\langle \nabla\delta\phi^\varepsilon, H\nabla\delta\phi^\varepsilon\rangle + g(|a|^2)\langle \beta, \nabla\delta\phi_1^\varepsilon\rangle$$
$$+ \left(g(|a^\varepsilon|^2) - g(|a|^2)\right)\langle \beta, \nabla\delta\phi^\varepsilon\rangle + \left(g(|a^\varepsilon|^2) - g(|a|^2) - 2g'(|a|^2)\operatorname{Re}(\overline{a}\varepsilon a_1)\right)\langle\beta,\nabla\phi\rangle$$
$$+ K * \left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} - 2\sigma|a|^{2\sigma-2}\operatorname{Re}(\overline{a}\varepsilon a_1)\right) = 0.$$

Moreover, the Taylor formula yields

$$g(|a^\varepsilon|^2) - g(|a|^2) - 2g'(|a|^2)\operatorname{Re}(\overline{a}\varepsilon a_1) = \left(2\operatorname{Re}(\overline{a}\delta a_1^\varepsilon) + |\delta a^\varepsilon|^2\right) g'\left(|a|^2\right)$$
$$+ \left(|a^\varepsilon|^2 - |a|^2\right)^2 \int_0^1 (1-s) g''\left(|a|^2 + s\left(|a^\varepsilon|^2 - |a|^2\right)\right) ds, \tag{4.1}$$

and the same identity holds for g replaced by $f(r) = r^\sigma$. Thus, taking into account Theorem 1.4, which implies $\|\!|\phi^\varepsilon|\!\|_{\ell+1,T}, \|\!|a^\varepsilon|\!\|_{\ell,T} = \mathcal{O}(1)$, and Theorem 1.5, which implies $\|\!|\delta\phi^\varepsilon|\!\|_{\ell+1,T}, \|\!|\delta a^\varepsilon|\!\|_{\ell,T} = \mathcal{O}(\varepsilon)$, it follows from Lemma 2.6 that

$$\|\!|\delta\phi_1^\varepsilon|\!\|_{\ell+1,T}^2 \leqslant \|\phi_0^\varepsilon - \phi_0 - \varepsilon\phi_{1,0}\|^2_{\mathcal{H}^{\ell+1}_{w_0}} + \frac{C}{M}\|\!|\delta\phi_1^\varepsilon|\!\|_{\ell+1,T}\left[\|\!|\delta\phi_1^\varepsilon|\!\|_{\ell+1,T} + \varepsilon^2 + \|\!|\delta a_1^\varepsilon|\!\|_{\ell,T}\right].$$

We deduce, for M large enough,

$$\|\!|\delta\phi_1^\varepsilon|\!\|_{\ell+1,T}^2 \leqslant C\|\phi_0^\varepsilon - \phi_0 - \varepsilon\phi_{10}\|^2_{\mathcal{H}^{\ell+1}_{w_0}} + \frac{C}{M}\varepsilon^4 + \frac{C}{M}\|\!|\delta a_1^\varepsilon|\!\|_{\ell,T}^2. \tag{4.2}$$

Similarly, δa_1^ε solves

$$\partial_t\delta a_1^\varepsilon + \langle\nabla\phi, H\nabla\delta a_1^\varepsilon\rangle + \langle\nabla\delta\phi_1^\varepsilon, H\nabla a\rangle + \langle\nabla\delta\phi^\varepsilon, H\nabla\delta a^\varepsilon\rangle$$
$$+ \frac{1}{2}aD^2\delta\phi_1^\varepsilon + \frac{1}{2}\delta a_1^\varepsilon D^2\phi + \frac{1}{2}\delta a^\varepsilon D^2\delta\phi^\varepsilon$$
$$+ \left\langle \beta, \nabla\left[\left(g(|a^\varepsilon|^2) - g(|a|^2) - 2\varepsilon g'(|a|^2)\operatorname{Re}(\overline{a}a_1)\right)a\right]\right\rangle$$
$$+ \left\langle\beta, \nabla\left[\left(g(|a^\varepsilon|^2) - g(|a|^2)\right)\varepsilon a_1\right]\right\rangle + \left\langle\beta,\nabla\left[g(|a^\varepsilon|^2)\delta a_1^\varepsilon\right]\right\rangle = \frac{i\varepsilon^2}{2}D^2 a_1 + \frac{i\varepsilon}{2}D^2\delta a_1^\varepsilon.$$

From (4.1), Theorems 1.4 and 1.5, and Lemma 2.6, we deduce

$$\|\!|\delta a_1^\varepsilon|\!\|_{\ell,T}^2 \le C\|a_0^\varepsilon - a_0 - \varepsilon a_{10}\|^2_{\mathcal{H}^\ell_{w_0}} + \frac{C}{M}\varepsilon^4 + \frac{C}{M}\|\!|\delta\phi_1^\varepsilon|\!\|_{\ell+1,T}^2. \tag{4.3}$$

Adding (4.2) and (4.3), (1.11) follows. Like in the proof of Theorem 1.5, a careful examination of the inequalities that we have used shows that all the above estimates are valid provided that we assume $M \geqslant M(\ell)$, the constant provided by Theorem 1.4, and also $M \geqslant \max(M(\ell+1), M(\ell+2))$ in order to estimate the source terms.

To complete the proof of Theorem 1.7, consider the point-wise estimate

$$\begin{aligned}\left|a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - ae^{i\phi_1}e^{i\phi/\varepsilon}\right| &\leqslant |a^\varepsilon - a| + |a^\varepsilon|\left|e^{i\phi^\varepsilon/\varepsilon} - e^{i(\phi+\varepsilon\phi_1)/\varepsilon}\right| \\ &\leqslant |a^\varepsilon - a| + |a^\varepsilon|\left|2\sin\left(\frac{\phi^\varepsilon - \phi - \varepsilon\phi_1}{2\varepsilon}\right)\right| \\ &\leqslant |\delta a^\varepsilon| + \frac{1}{\varepsilon}|a^\varepsilon|\,|\delta\phi_1^\varepsilon|.\end{aligned}$$

We then conclude like in the proof of Corollary 1.6, by using Cauchy-Schwarz inequality, (3.1), and Sobolev embedding. □

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