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New predictions from the logotropic model

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In a previous paper [P.H. Chavanis, Eur. Phys. J. Plus **130**, 130 (2015)] we have introduced a new cosmological model that we called the logotropic model. This model involves a fundamental constant Λ which is the counterpart of Einstein's cosmological constant in the Λ CDM model. The logotropic model is able to account, without free parameter, for the constant surface density of the dark matter halos, for their mass-radius relation, and for the Tully-Fisher relation. In this paper, we explore other consequences of this model. By advocating a form of "strong cosmic coincidence" we predict that the present proportion of dark energy in the Universe is $\Omega_{de,0} = e/(1+e) \simeq 0.731$ which is close to the observed value. We also remark that the surface density of dark matter halos and the surface density of the Universe are of the same order as the surface density of the electron. This makes a curious connection between cosmological and atomic scales. Using these coincidences, we can relate the Hubble constant, the electron mass and the electron charge to the cosmological constant. We also suggest that the famous numbers 137 (fine-structure constant) and 123 (logotropic constant) may actually represent the same thing. This could unify microphysics and cosmophysics. We study the thermodynamics of the logotropic model and find a connection with the Bekenstein-Hawking entropy of black holes if we assume that the logotropic dark fluid is made of particles of mass $m_\Lambda \sim \hbar\sqrt{\Lambda}/c^2 = 2.08 \times 10^{-33} \text{ eV}/c^2$ (cosmons). In that case, the universality of the surface density of the dark matter halos may be related to a form of holographic principle (the fact that their entropy scales like their area). We use similar arguments to explain why the surface density of the electron and the surface density of the Universe are of the same order and justify the empirical Weinberg relation. Finally, we combine the results of our approach with the quantum Jeans instability theory to predict the order of magnitude of the mass of ultralight axions $m \sim 10^{-23} \text{ eV}/c^2$ in the Bose-Einstein condensate dark matter paradigm.

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I. INTRODUCTION

The nature of dark matter and dark energy remains one of the greatest mysteries of modern cosmology. Dark matter is responsible for the flat rotation curves of the galaxies and dark energy is responsible for the accelerated expansion of the Universe. It is found that dark energy represents about 70% of the energy content of the present Universe while the proportions of dark matter and baryonic matter are 25% and 5% respectively.

In a previous paper [1] (see also [2, 3]) we have introduced a new cosmological model that we called the *logotropic model*. In this model, there is no dark matter and no dark energy. There is just a single dark fluid. What we call "dark matter" actually corresponds to its rest-mass energy and what we call "dark energy" corresponds to its internal energy.¹

Our model does not contain any arbitrary parameter so that it is totally constrained. It involves a fundamental constant Λ which is the counterpart of Einstein's cosmological constant [4] in the Λ CDM (cold dark matter) model and which turns out to have the same value. Still the logotropic model is fundamentally different from the

Λ CDM model.

On the large (cosmological) scales, the logotropic model is indistinguishable from the Λ CDM model up to the present epoch [1–3]. The two models will differ in the far future, in about 25 Gyrs years, after which the logotropic model will become phantom (the energy density will increase as the Universe expands) and present a Little Rip (the energy density and the scale factor will become infinite in infinite time) contrary to the Λ CDM model in which the energy density tends towards a constant (de Sitter era).

On the small (galactic) scales, the logotropic model is able to solve some of the problems encountered by the Λ CDM model [1, 2]. In particular, it is able to account, without free parameter, for the constant surface density of the dark matter halos, for their mass-radius relation, and for the Tully-Fisher relation.

In this paper, we explore other consequences of this model. By advocating a form of "strong cosmic coincidence", stating that the present value of the dark energy density $\rho_{de,0}$ is equal to the fundamental constant ρ_Λ appearing in the logotropic model, we predict that the present proportion of dark energy in the Universe is $\Omega_{de,0} = e/(1+e) = 0.731$ which is close to the observed value 0.691 [5]. The consequences of this result, which implies that our epoch is very special in the history of the Universe, are intriguing and related to a form of anthropic cosmological principle [6].

We also remark that the universal surface density of dark matter halos (found from the observations [7] and

¹ Many models try to unify dark matter and dark energy. They are called unified dark energy and dark matter (UDE/M) models. However, the interpretation of dark matter and dark energy that we give in Refs. [1, 2] is new and original.

predicted by our model [1, 2]) and the surface density of the Universe are of the same order of magnitude as the surface density of the electron. This makes a curious connection between cosmological and atomic scales. Exploiting this coincidence, we can relate the Hubble constant, the electron mass and the electron charge to the cosmological constant Λ . We also argue that the famous numbers 137 (fine-structure constant) and 123 (logotropic constant) may actually represent the same thing. This may be a hint for a theory of unification of microphysics and cosmophysics. Speculations are made in the Appendices to try to relate these interconnections to a form of holographic principle [8] stating that the entropy of the electron, the entropy of dark matter halos, and the entropy of the Universe scales like their area as in the case of the entropy of black holes [9, 10].

II. THE LOGOTROPIC MODEL

A. Unification of dark matter and dark energy

The Friedmann equations for a flat universe without cosmological constant are [11]:

$$\frac{d\epsilon}{dt} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0, \quad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon, \quad (1)$$

where $\epsilon(t)$ is the energy density of the Universe, $P(t)$ is the pressure, $a(t)$ is the scale factor, and $H = \dot{a}/a$ is the Hubble parameter.

For a relativistic fluid experiencing an adiabatic evolution such that $Td(s/\rho) = 0$, the first law of thermodynamics reduces to [11]:

$$d\epsilon = \frac{P + \epsilon}{\rho}d\rho, \quad (2)$$

where $\rho(t)$ is the rest-mass density of the Universe. Combined with the equation of continuity (1), we get

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho = 0 \Rightarrow \rho = \frac{\rho_0}{a^3}, \quad (3)$$

where ρ_0 is the present value of the rest-mass density (the present value of the scale factor is taken to be $a_0 = 1$). This equation, which expresses the conservation of the rest-mass, is valid for an arbitrary equation of state.

For an equation of state specified under the form $P = P(\rho)$, Eq. (2) can be integrated to obtain the relation between the energy density ϵ and the rest-mass density ρ . We obtain [1]:

$$\epsilon = \rho c^2 + \rho \int^\rho \frac{P(\rho')}{\rho'^2} d\rho' = \rho c^2 + u(\rho). \quad (4)$$

We note that $u(\rho)$ can be interpreted as an internal energy density [1]. Therefore, the energy density ϵ is the sum of the rest-mass energy ρc^2 and the internal energy $u(\rho)$.

B. The logotropic dark fluid

We assume that the Universe is filled with a single dark fluid described by the logotropic equation of state [1]:

$$P = A \ln\left(\frac{\rho}{\rho_P}\right), \quad (5)$$

where $\rho_P = c^5/\hbar G^2 = 5.16 \times 10^{99} \text{ g m}^{-3}$ is the Planck density and A is a new fundamental constant of physics, with the dimension of an energy density, which is the counterpart of the cosmological constant Λ in the Λ CDM model (see below). Using Eqs. (4) and (5), the relation between the energy density and the rest-mass density is

$$\epsilon = \rho c^2 - A \ln\left(\frac{\rho}{\rho_P}\right) - A = \rho c^2 + u(\rho). \quad (6)$$

The energy density is the sum of two terms: a rest-mass energy term $\rho c^2 = \rho_0 c^2/a^3$ that mimics the energy density ϵ_m of dark matter and an internal energy term $u(\rho) = -A \ln(\rho/\rho_P) - A = -P(\rho) - A = 3A \ln a - A \ln(\rho_0/\rho_P) - A$ that mimics the energy density ϵ_{de} of dark energy. This decomposition leads to a natural, and physical, unification of dark matter and dark energy and elucidates their mysterious nature.

Since, in our model, the rest-mass energy of the dark fluid mimics dark matter, we identify $\rho_0 c^2$ with the present energy density of dark matter. We thus set $\rho_0 c^2 = \Omega_{m,0} \epsilon_0$, where $\epsilon_0/c^2 = 3H_0^2/8\pi G$ is the present energy density of the Universe and $\Omega_{m,0}$ is the present fraction of dark matter (we also include baryonic matter). As a result, the present internal energy of the dark fluid, $u_0 = \epsilon_0 - \rho_0 c^2$, is identified with the present dark energy density $\epsilon_{de,0} = \Omega_{de,0} \epsilon_0$ where $\Omega_{de,0} = 1 - \Omega_{m,0}$ is the present fraction of dark energy. Applying Eq. (6) at the present epoch ($a_0 = 1$), we obtain the identity

$$A = \frac{\epsilon_{de,0}}{\ln\left(\frac{\rho_P c^2}{\epsilon_{de,0}}\right) + \ln\left(\frac{\Omega_{de,0}}{1 - \Omega_{de,0}}\right) - 1}. \quad (7)$$

At that stage, we can have two points of view. We can consider that this equation determines the constant A as a function of ϵ_0 and $\Omega_{de,0}$ that are both obtained from the observations [5]. This allows us to determine the value of A . This is the point of view that we have adopted in our previous papers [1, 2] and that we adopt in Sec. IID below. However, in the following section, we present another point of view leading to an intriguing result.

C. Strong cosmic coincidence and prediction of $\Omega_{de,0}$

Let us recall that, in our model, A is considered as a fundamental constant whose value is fixed by Nature. As a result, Eq. (7) relates $\Omega_{de,0}$ to ϵ_0 for a given value of A . A priori, we have two unknowns for just one equation. However, we can obtain the value of $\Omega_{de,0}$ by the following argument.

We can always write the constant A under the form

$$A = \frac{\rho_\Lambda c^2}{\ln\left(\frac{\rho_P}{\rho_\Lambda}\right)}. \quad (8)$$

This is just a change of notation. Eq. (8) defines a new constant, the cosmological density ρ_Λ , in place of A . From the cosmological density ρ_Λ , we can define an effective cosmological constant Λ by²

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}. \quad (9)$$

Again this is just a change of notation. Therefore, the fundamental constant of our model is either A , ρ_Λ or Λ (equivalently). We now advocate a form of “strong cosmic coincidence”. We assume that the present value of the dark energy density is equal to $\rho_\Lambda c^2$, i.e.,

$$\epsilon_{\text{de},0} = \rho_\Lambda c^2. \quad (10)$$

Since, in the Λ CDM model, ϵ_{de} is a constant usually measured at the present epoch our postulate implies that $\rho_\Lambda c^2$ coincides with the cosmological density in the Λ CDM model and that Λ , as defined by Eq. (9), coincides with the ordinary cosmological constant. This is why we have used the same notations. Now, comparing Eqs. (7), (8) and (10) we obtain $\ln[\Omega_{\text{de},0}/(1 - \Omega_{\text{de},0})] - 1 = 0$ which determines $\Omega_{\text{de},0}$. We find that

$$\Omega_{\text{de},0}^{\text{th}} = \frac{e}{1+e} \simeq 0.731 \quad (11)$$

which is close to the observed value $\Omega_{\text{de},0}^{\text{obs}} = 0.691$ [5]. This agreement is puzzling. It relies on the “strong cosmic coincidence” of Eq. (10) implying that our epoch is very special. This is a form of anthropic cosmological principle [6]. This may also correspond to a fixed point of our model. In order to avoid philosophical issues, in the following, we adopt the more conventional point of view discussed at the end of Sec. II B.

D. The logotropic constant B

We can rewrite Eq. (8) as

$$A = B\rho_\Lambda c^2 \quad \text{with} \quad B = \frac{1}{\ln(\rho_P/\rho_\Lambda)}. \quad (12)$$

Again, this is just a change of notation defining the dimensionless number B . We shall call it the logotropic

constant since it is equal to the inverse of the logarithm of the cosmological density normalized by the Planck density (see Appendix A). We note that A can be expressed in terms of B (see below) so that the fundamental constant of our model is either A , ρ_Λ , Λ , or B . In the following, we shall express all the results in terms of B . For example, the relation (6) between the energy density and the scale factor can be rewritten as

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\text{m},0}}{a^3} + (1 - \Omega_{\text{m},0})(1 + 3B \ln a). \quad (13)$$

Combined with the Friedmann equation (1) this equation determines the evolution of the scale factor $a(t)$ of the Universe in the logotropic model. This evolution has been studied in detail in [1–3].

Remark: Considering Eq. (13), we see that the Λ CDM model is recovered for $B = 0$. According to Eq. (12) this implies that $\rho_P \rightarrow +\infty$, i.e., $\hbar \rightarrow 0$. Therefore, the Λ CDM model corresponds to the semiclassical limit of the logotropic model. The fact that B is intrinsically nonzero implies that quantum mechanics ($\hbar \neq 0$) plays some role in our model in addition to general relativity. This may suggest a link with a theory of quantum gravity.

E. The value of B from the observations

The fundamental constant (A , ρ_Λ , Λ , or B) appearing in our model can be determined from the observations by using Eq. (7). We take $\Omega_{\text{de},0} = 0.6911$ and $H_0 = 2.195 \times 10^{-18} \text{ s}^{-1}$ [5]. This implies $\epsilon_0/c^2 = 3H_0^2/8\pi G = 8.62 \times 10^{-24} \text{ g m}^{-3}$ and $\epsilon_{\text{de},0}/c^2 = \Omega_{\text{de},0}\epsilon_0/c^2 = 5.96 \times 10^{-24} \text{ g m}^{-3}$. Since $\ln[\Omega_{\text{de},0}/(1 - \Omega_{\text{de},0})] - 1 = -0.195$ is small as compared to $\ln(\rho_P c^2/\epsilon_{\text{de},0}) = 283$, we can write in very good approximation A as in Eq. (8) with $\rho_\Lambda \simeq \epsilon_{\text{de},0}/c^2$ as in Eq. (10). Therefore,

$$\rho_\Lambda = \frac{3\Omega_{\text{de},0}H_0^2}{8\pi G} = 5.96 \times 10^{-24} \text{ g m}^{-3} \quad (14)$$

and

$$\Lambda = 3\Omega_{\text{de},0}H_0^2 = 1.00 \times 10^{-35} \text{ s}^{-2} \quad (15)$$

are approximately equal to the cosmological density and to the cosmological constant in the Λ CDM model. From Eq. (12) we get

$$B = \frac{1}{\ln(\rho_P/\rho_\Lambda)} \simeq \frac{1}{123 \ln(10)} \simeq 3.53 \times 10^{-3}. \quad (16)$$

As discussed in our previous papers [1–3], B is essentially the inverse of the famous number 123 (see Appendix A). Finally,

$$A = B\rho_\Lambda c^2 = 1.89 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}. \quad (17)$$

From now on, we shall view B given by Eq. (16) as the fundamental constant of the theory. Therefore, everything should be expressed in terms of B and the other

² We stress that our model is different from the Λ CDM model so that Λ is fundamentally different from Einstein’s cosmological constant [4]. However, it is always possible to introduce from the constant A an effective cosmological density ρ_Λ and an effective cosmological constant Λ by Eqs. (8) and (9).

fundamental constants of physics defining the Planck scales. First, we have

$$\frac{\rho_\Lambda}{\rho_P} = \frac{G\hbar\Lambda}{8\pi c^5} = e^{-1/B} = 1.16 \times 10^{-123}. \quad (18)$$

Then,

$$\frac{A}{\rho_P c^2} = B e^{-1/B} = 4.08 \times 10^{-126}. \quad (19)$$

The logotropic equation of state (5) can be written as $P/\rho_P c^2 = B e^{-1/B} \ln(\rho/\rho_P)$. Using Eq. (10) and $\epsilon_{\text{de},0} = \Omega_{\text{de},0}\epsilon_0$, we get

$$\frac{\epsilon_0}{\rho_P c^2} = \frac{1}{\Omega_{\text{de},0}} e^{-1/B} = 1.67 \times 10^{-123}. \quad (20)$$

Finally, using Eq. (1),

$$t_P H_0 = \left(\frac{8\pi}{3\Omega_{\text{de},0}} \right)^{1/2} e^{-1/2B} = 1.18 \times 10^{-61}, \quad (21)$$

where $t_P = (\hbar G/c^5)^{1/2} = 5.391 \times 10^{-44}$ s is the Planck time. In the last two expressions, we can either consider that $\Omega_{\text{de},0}$ is ‘‘predicted’’ by Eq. (11) or take its measured value. To the order of accuracy that we consider, this does not change the numerical values.

F. Generalized logotropic models

Recently, the original logotropic model corresponding to a purely logarithmic equation of state $P = A \ln(\rho/\rho_*)$ [1–3] has been generalized to an expression of the form $P = A(\rho/\rho_*)^{-n} \ln(\rho/\rho_*)$ where n is a free parameter [12–14]. Interestingly, this equation of state is similar to the Anton-Schmidt [15] equation of state for crystalline solids in the Debye approximation [16]. In that case, the index n can be written as $n = -1/6 - \gamma_G$ where γ_G is the so-called Grüneisen [17] parameter. The original logotropic model is recovered for $n = 0$. The authors of [12] compared the predictions of the generalized logotropic model with cosmological observations and found that $B \simeq 3.54 \times 10^{-3}$ and $n = -0.147_{-0.107}^{+0.113}$. This confirms the robustness of the value of the fundamental constant $B = 3.53 \times 10^{-3}$ introduced in [1, 2]. On the other hand, up to the error bars, the value of n is close to $n = 0$, corresponding to the logotropic model (see also [3]). This suggests that the logotropic model tends to be selected among more general families of models containing additional parameters $\{n\}$. What is crucial in the logotropic model is the logarithmic factor $\ln(\rho/\rho_*)$. Very generally, we could consider an equation of state of the form $P = A_n(\rho) \ln(\rho/\rho_*)$ with $A_n(\rho) \rightarrow \text{cte}$ when $n \rightarrow 0$. In the present paper, we stick to the original logotropic model [1–3] but it would be interesting to see how the results are modified in more general circumstances [12–14].

III. PREVIOUS PREDICTIONS OF THE LOGOTROPIC MODEL

The interest of the logotropic model becomes apparent when it is applied to dark matter halos [1, 2]. We assume that dark matter halos are described by the logotropic equation of state of Eq. (5) with $A = 1.89 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}$ (or $B = 3.53 \times 10^{-3}$). At the galactic scale, we can use Newtonian gravity.

A. Surface density of dark matter halos

It is an empirical evidence that the surface density of galaxies has the same value

$$\Sigma_0^{\text{obs}} \equiv \rho_0 r_h \simeq 295 \text{ g m}^{-2} \simeq 141 M_\odot/\text{pc}^2 \quad (22)$$

even if their sizes and masses vary by several orders of magnitude (up to 14 orders of magnitude in luminosity) [7]. Here ρ_0 is the central density and r_h is the halo radius at which the density has decreased by a factor of 4. The logotropic model predicts that the surface density of the dark matter halos is the same for all the halos (because A is a universal constant) and that it is given by [1, 2]

$$\Sigma_0^{\text{th}} = \left(\frac{A}{4\pi G} \right)^{1/2} \xi_h = \left(\frac{B}{32} \right)^{1/2} \frac{\xi_h c\sqrt{\Lambda}}{\pi G}, \quad (23)$$

where $\xi_h = 5.8458\dots$ is a pure number arising from the Lane-Emden equation of index $n = -1$ expressing the condition of hydrostatic equilibrium of logotropic spheres.³ Numerically,

$$\Sigma_0^{\text{th}} = 278 \text{ g m}^{-2} \simeq 133 M_\odot/\text{pc}^2, \quad (24)$$

which is very close to the observational value (22). The fact that the surface density of dark matter halos is determined by the effective cosmological constant Λ (usually related to the dark energy) tends to confirm that dark matter and dark energy are just two manifestations of the *same* dark fluid, as we have assumed in our model.

Remark: The dimensional term $c\sqrt{\Lambda}/G$ in Eq. (23) can be interpreted as representing the surface density of the Universe (see Appendix B). We note that this term alone, $c\sqrt{\Lambda}/G = 14200 \text{ g m}^{-2} = 6800 M_\odot/\text{pc}^2$, is too large to

³ The logotropic spheres [1, 2], like the isothermal spheres and some polytropic spheres [18], have an infinite mass. This implies that the logotropic equation of state cannot describe dark matter halos at infinitely large distances. Nevertheless, it may describe the inner region of dark matter halos and this is sufficient to determine their surface density. The stability of bounded logotropic spheres has been studied in [19] by analogy with the stability of bounded isothermal and polytropic spheres [20–22] and similar results have been obtained. In particular, bounded logotropic spheres are stable provided that the density contrast is not too large.

account precisely for the surface density of dark matter halos so that the prefactor $(B/32)^{1/2}(\xi_h/\pi) = 0.01955$ is necessary to reduce this number. It is interesting to remark that the term $c\sqrt{\Lambda}/G$ arises from classical general relativity while the prefactor $\propto B^{1/2}$ has a quantum origin as discussed at the end of Sec. IID. Actually, we will see that it is related to the fine-structure constant α [see Eq. (33) below].

B. Mass-radius relation

There are interesting consequences of the preceding result. For logotropic halos, the mass of the halos calculated at the halo radius r_h is given by [1, 2]

$$M_h = 1.49 \Sigma_0 r_h^2. \quad (25)$$

This determines the mass-radius relation of dark matter halos. On the other hand, the circular velocity at the halo radius is $v_h^2 = GM_h/r_h = 1.49 \Sigma_0 G r_h$. Since the surface density of the dark matter halos is constant, we obtain

$$\frac{M_h}{M_\odot} = 198 \left(\frac{r_h}{\text{pc}} \right)^2, \quad \left(\frac{v_h}{\text{km s}^{-1}} \right)^2 = 0.852 \frac{r_h}{\text{pc}}. \quad (26)$$

The scalings $M_h \propto r_h^2$ and $v_h^2 \propto r_h$ (and also the prefactors) are consistent with the observations.

C. The Tully-Fisher relation

Combining the previous equations, the logotropic model leads to the Tully-Fisher [23] relation $v_h^4 \propto M_h$ or, more precisely,

$$\left(\frac{M_b}{v_h^4} \right)^{\text{th}} = \frac{f_b}{1.49 \Sigma_0^{\text{th}} G^2} = 46.4 M_\odot \text{km}^{-4} \text{s}^4, \quad (27)$$

where $f_b = M_b/M_h \sim 0.17$ is the cosmic baryon fraction [24]. The predicted value from Eq. (27) is close to the observed one $(M_b/v_h^4)^{\text{obs}} = 47 \pm 6 M_\odot \text{km}^{-4} \text{s}^4$ [24].

Remark: The Tully-Fisher relation is sometimes justified by the MOND (Modification of Newtonian dynamics) theory [25] which predicts a relation of the form $v_h^4 = Ga_0 M_b$ between the asymptotic circular velocity and the baryon mass, where a_0 is a critical acceleration. Our results imply $a_0^{\text{th}} = 1.62 \times 10^{-10} \text{ m s}^{-2}$ which is close to the value $a_0^{\text{obs}} = (1.3 \pm 0.3) \times 10^{-10} \text{ m s}^{-2}$ obtained from the observations [24]. Combining Eqs. (25) and (27), we first get $a_0^{\text{th}} = (1.49/f_b) G \Sigma_0^{\text{th}} = GM_h/(f_b r_h^2)$ which shows that a_0 can be interpreted as the surface gravity of the galaxies $G \Sigma_0$ (which corresponds to Newton's acceleration GM_h/r_h^2) or as the surface gravity of the Universe (see Appendix C). Then, using Eqs. (15) and (23), we obtain $a_0^{\text{th}} = (1.49/f_b)(B/32)^{1/2}(\xi_h/\pi)c\sqrt{\Lambda} \simeq H_0 c/4$ which explains why a_0 is of the order of $H_0 c$ [2]. We emphasize, however, that we do not use the MOND theory in our approach and that the logotropic model assumes the existence of a dark fluid.

D. The mass M_{300}

The logotropic equation of state also explains the observation of Strigari *et al.* [26] that all the dwarf spheroidals (dSphs) of the Milky Way have the same total dark matter mass M_{300} contained within a radius $r_u = 300 \text{ pc}$, namely $M_{300}^{\text{obs}} \simeq 10^7 M_\odot$. The logotropic model predicts the value [1, 2]

$$M_{300}^{\text{th}} = \frac{4\pi \Sigma_0^{\text{th}} r_u^2}{\xi_h \sqrt{2}} = 1.82 \times 10^7 M_\odot, \quad (28)$$

which is in very good agreement with the observational value.

IV. A CURIOUS CONNECTION BETWEEN ATOMIC AND COSMOLOGICAL SCALES

A. The surface density of the electron

The classical radius of the electron r_e can be obtained qualitatively by writing that the electrostatic energy of the electron, e^2/r_e , is equal to its rest-mass energy $m_e c^2$. Recalling the value of the charge of the electron $e = 4.80 \times 10^{-13} \text{ g}^{1/2} \text{ m}^{3/2} \text{ s}^{-1}$ and its mass $m_e = 9.11 \times 10^{-28} \text{ g}$, we obtain $r_e = e^2/m_e c^2 = 2.82 \times 10^{-15} \text{ m}$. As a result, the surface density of the electron is⁴

$$\Sigma_e = \frac{m_e}{r_e^2} = \frac{m_e^3 c^4}{e^4} = 115 \text{ g/m}^2 = 54.9 M_\odot/\text{pc}^2, \quad (29)$$

which is of the same order of magnitude as the surface density of dark matter halos from Eq. (22). This coincidence is amazing in view of the different scales (atomic versus cosmological) involved. More precisely, we find $\Sigma_e = \sigma \Sigma_0^{\text{th}}$ with $\sigma \simeq 0.413$. Of course, the value of σ depends on the precise manner used to define the surface density of the electron, or its radius, but the important point is that this number is of order unity.

B. Relation between α and B

By matching the two formulae (23) and (29), writing $\Sigma_e = \sigma \Sigma_0^{\text{th}}$, we get

$$\Lambda = \frac{32\pi^2}{B\xi_h^2 \sigma^2} \frac{m_e^6 c^6 G^2}{e^8} = \frac{32\pi^2}{B\xi_h^2 \sigma^2 \alpha^4} \frac{m_e^6 c^2 G^2}{\hbar^4}, \quad (30)$$

⁴ We note that the Thomson cross-section $\sigma = (8\pi/3)(e^2/m_e c^2)^2$ can be written as $\sigma = (8\pi/3)r_e^2$ giving a physical meaning to the classical electron radius r_e . We also note that r_e can be written as $r_e = \alpha \hbar/m_e c = \alpha \lambda_C$ where $\lambda_C = \hbar/m_e c$ is the Compton wavelength of the electron and α is the fine-structure constant α [see Eq. (A1)]. Similarly, we can write $\Sigma_e = (1/\alpha^2)m_e^3 c^2/\hbar^2$.

where we have introduced the fine-structure constant α in the second equality (see Appendix A). This expression provides a curious relation between the cosmological constant, the mass of the electron, and its charge. This relation is similar to Weinberg's empirical relation (see Appendix B) which can be written as [combining Eqs. (15) and (B3)]

$$\Lambda = 192\pi^2 \mu^2 \Omega_{\text{de},0} \frac{m_e^6 c^6 G^2}{e^8}, \quad (31)$$

where $\mu \simeq 3.42$. Note that in our formula (30), Λ appears two times: on the left hand side and in B (which depends logarithmically on Λ). This will have important consequences in the following.

Böhmer and Harko [27], by a completely different approach, found a similar relation⁵

$$\Lambda = \nu \frac{\hbar^2 G^2 m_e^6 c^8}{e^{12}} = \frac{\nu}{\alpha^6} \frac{G^2 m_e^6 c^2}{\hbar^4}, \quad (32)$$

where $\nu \simeq 0.816$ is of order unity. Their result can be obtained as follows. They first introduced a minimum mass $m_\Lambda \sim \hbar\sqrt{\Lambda}/c^2 = 2.08 \times 10^{-33} \text{ eV}/c^2$ interpreted as being the mass of the elementary particle of dark energy, called the cosmon. Then, they defined a radius R by the relation $m_\Lambda \sim \rho_\Lambda R^3$ where $\rho_\Lambda = \Lambda/8\pi G$ is the cosmological density considered as being the lowest density in the Universe. Finally, they remarked that $R \sim 8.54 \times 10^{-15} \text{ m}$ has typically the same value as the classical radius of the electron $r_e = e^2/m_e c^2 = 2.82 \times 10^{-15} \text{ m}$. Matching R and r_e leads to the scaling of Eq. (32). We have then added a prefactor ν and adjusted its value in order to exactly obtain the measured value of the cosmological constant [5]. Since the approach of Böhmer and Harko [27] is essentially qualitative, and depends on the precise manner used to define the radius of the electron, their result can be at best valid up to a constant of order unity.

We would like now to compare the estimates from Eqs. (30) and (32). At that stage, we can have two points of view. If we consider that comparing the prefactors is meaningless because our approach can only provide "rough" orders of magnitude, we conclude that Eqs. (30) and (32) are equivalent, and that they are also equivalent to Weinberg's empirical relation (B3). Alternatively, if we take the prefactors seriously into account (in particular the presence of B which depends on Λ) and match the formulae (30) and (32), we find an interesting relation between the fine-structure constant α and the logotropic constant B :

$$\alpha = \left(\frac{\nu}{32}\right)^{1/2} \frac{\xi_h \sigma}{\pi} \sqrt{B} \simeq 0.123 \sqrt{B}. \quad (33)$$

Therefore, the fine-structure constant (electron charge normalized by the Planck charge) is determined by the

logotropic constant B (logarithm of the cosmological density normalized by the Planck density) by a relation of the form $\alpha \propto B^{1/2}$. This makes a connection between atomic scales and cosmological scales. This also suggests that the famous numbers 137 and 123 (see Appendix A) are related to each other, or may even represent the same thing. From Eq. (33), we have⁶

$$137 \simeq 12.3 \sqrt{123}. \quad (34)$$

Remark: the logotropic constant B is related to the effective cosmological constant Λ by [see Eq. (18)]

$$B = \frac{1}{\ln\left(\frac{8\pi c^5}{G\hbar\Lambda}\right)}. \quad (35)$$

Using Eqs. (33) and (35), we can express the fine-structure constant α as a function of the effective cosmological constant Λ or, using Eq. (21), as a function of the age of the Universe $t_\Lambda = 1/H_0$. We get

$$\alpha = \frac{0.123}{\ln\left(\frac{8\pi c^5}{G\hbar\Lambda}\right)^{1/2}} = \frac{0.123}{\sqrt{2} \ln\left[\left(\frac{8\pi}{3\Omega_{\text{de},0}}\right)^{1/2} \frac{t_\Lambda}{t_P}\right]^{1/2}}. \quad (36)$$

We emphasize the scaling $1/\alpha \propto (\ln t_\Lambda)^{1/2}$. It is interesting to note that similar relations have been introduced in the past from pure numerology (see [29], P. 428). These relations suggest that the fundamental constants may change with time as argued by Dirac [30, 31].

C. The mass and the charge of the electron in terms of B

Using Eqs. (32), (33) and (35), we find that the mass and the charge of the electron (or the fine-structure constant) are determined by the cosmological constant Λ according to

$$m_e = \frac{1}{\nu^{1/6}} \alpha \left(\frac{\Lambda \hbar^4}{G^2 c^2}\right)^{1/6} = 1.03 \alpha \left(\frac{\Lambda \hbar^4}{G^2 c^2}\right)^{1/6} = 9.11 \times 10^{-28} \text{ g}, \quad (37)$$

$$\alpha = \frac{e^2}{\hbar c} = \left(\frac{\nu}{32}\right)^{1/2} \frac{\xi_h \sigma}{\pi} \frac{1}{\ln\left(\frac{8\pi c^5}{G\hbar\Lambda}\right)^{1/2}} = \frac{0.123}{\ln\left(\frac{8\pi c^5}{G\hbar\Lambda}\right)^{1/2}} = \frac{1}{137}. \quad (38)$$

Alternatively, using Eqs. (9), (18), (30) and (33), the normalized mass and the normalized charge of the electron

⁵ A closely related formula, involving the Hubble constant instead of the cosmological constant, was first found by Stewart [28] in 1931 by trial and error.

⁶ We note that the prefactors in Eqs. (33) and (34) appear to be close to 123/1000 and 123/10, where the number 123 appears again (!). We do not know whether this coincidence is fortuitous or if it bears a deeper significance than is apparent at first sight.

are determined by the logotropic constant B according to

$$\begin{aligned} \frac{m_e}{M_P} &= \left(\frac{8\pi}{\nu}\right)^{1/6} \left(\frac{\nu}{32}\right)^{1/2} \frac{\xi_h \sigma}{\pi} \sqrt{B} e^{-1/(6B)} \\ &= 0.217 \sqrt{B} e^{-1/(6B)} = 4.18 \times 10^{-23}, \end{aligned} \quad (39)$$

$$\frac{e^2}{q_P^2} = \left(\frac{\nu}{32}\right)^{1/2} \frac{\xi_h \sigma}{\pi} \sqrt{B} = 0.123 \sqrt{B} = 7.29 \times 10^{-3}, \quad (40)$$

where $M_P = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5} \text{ g}$ is the Planck mass and $q_P = (\hbar c)^{1/2} = 5.62 \times 10^{-12} \text{ g}^{1/2} \text{ m}^{3/2} \text{ s}^{-1}$ is the Planck charge. These relations suggest that the mass and the charge of the electron (atomic scales) are determined by the effective cosmological constant Λ or B (cosmological scales). We emphasize the presence of the exponential factor $e^{-1/(6B)}$ in Eq. (39) explaining why the electron mass is much smaller than the Planck mass while the electron charge is comparable to the Planck charge [see Eq. (40)].

D. A prediction of B

If we match Eqs. (23) and (B2), or equivalently Eqs. (30) and (31), we obtain

$$B^{\text{app}} = \frac{1}{6\lambda^2 \xi_h^2 \Omega_{\text{de},0}}. \quad (41)$$

Taking $\lambda^{\text{app}} = 1$ (since we cannot predict its value) and $\Omega_{\text{de},0}^{\text{th}} = e/(1+e)$ [see Eq. (11)], we get $B^{\text{app}} = 6.67 \times 10^{-3}$ instead of $B = 3.53 \times 10^{-3}$. We recall that the value of B was obtained in Sec. II E from the observations. On the other hand, Eq. (41) gives the correct order of magnitude of B without any reference to observations, up to a dimensionless constant $\lambda \simeq 1.41$ of order unity. Considering that B is predicted by Eq. (41) implies that we can predict the values of Λ , H_0 , α , m_e and e without reference to observations, up to dimensionless constants $\lambda \simeq 1.41$, $\nu \simeq 0.816$ and $\sigma \simeq 0.413$ of order unity. We note, however, that even if these dimensionless constants (λ , ν , σ) are of order unity, their precise values are of importance since B usually appears in exponentials like in Eqs. (18), (21) and (39).

V. CONCLUSION

In this paper, we have developed the logotropic model introduced in [1, 2]. In this model, dark matter corresponds to the rest mass energy of a dark fluid and dark energy corresponds to its internal energy. The Λ CDM model may be interpreted as the semiclassical limit $\hbar \rightarrow 0$ of the logotropic model. We have first recalled that the logotropic model is able to predict (without free parameter) the universal value of the surface density of dark

matter halos Σ_0 , their mass-radius relation $M_h - r_h$, the Tully-Fisher relation $M_b \sim v_h^4$ and the value of the mass M_{300} of dSphs. Then, we have argued that it also predicts the value of the present fraction of dark energy $\Omega_{\text{de},0}$. This arises from a sort of ‘‘strong cosmic coincidence’’ but this could also correspond to a fixed point of the model. Finally, we have observed that the surface density of the dark matter halos Σ_0 is of the same order as the surface density of the Universe Σ_Λ and the surface density of the electron Σ_e . This makes an empirical connection between atomic physics and cosmology. From this connection, we have obtained a relation between the fine-structure constant $\alpha \sim 1/137$ and the logotropic constant $B \sim 1/123$. We have also expressed the mass m_e and the charge $-e$ of the electron as a function of B (or as a function of the effective cosmological constant Λ). Finally, we have obtained a prediction of the order of magnitude of B independent from the observations. In a sense, our approach which expresses the mass and the charge of the electron in terms of the cosmological constant is a continuation of the program initiated by Eddington [32] in his quest for a ‘*Fundamental Theory*’ of the physical world in which the basic interaction strengths and elementary particle masses would be predicted entirely combinatorially by simple counting processes [6]. In the Appendices, we try to relate these interconnections to a form of holographic principle [8] (of course not known at the time of Eddington) stating that the entropy of the electron, the dark matter halos, and the Universe scales like their area as in the case of black holes [9, 10].

This paper has demonstrated that physics is full of ‘‘magic’’ and mysterious relations that are still not fully understood (one of them being the empirical Weinberg relation). Hopefully, a contribution of this paper is to reveal these ‘‘mysteries’’ and propose some tracks so as to induce further research towards their elucidation.

Appendix A: The constants α and B

There are two famous numbers in physics, 137 and 123, which respectively apply to atomic and cosmological scales.

At the atomic level, the fine-structure constant α , also known as Sommerfeld’s constant [29], is a dimensionless physical constant characterizing the strength of the electromagnetic interaction between elementary charged particles. Its value is

$$\alpha = \frac{e^2}{\hbar c} = \frac{e^2}{q_P^2} \simeq \frac{1}{137} \simeq 7.30 \times 10^{-3}. \quad (A1)$$

It can be seen as the square of the charge $e = 4.80 \times 10^{-13} \text{ g}^{1/2} \text{ m}^{3/2} \text{ s}^{-1}$ of the electron normalized by the Planck charge $q_P = (\hbar c)^{1/2} = 5.62 \times 10^{-12} \text{ g}^{1/2} \text{ m}^{3/2} \text{ s}^{-1}$. The quantum theory does not predict its value. The number $1/\alpha \simeq 137$ intrigued a lot of famous researchers including Born, Dirac, Eddington, Feynman, Hawking,

Heisenberg, Landé, and Pauli among others [29]. Feynman writes [33]: *It's one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding by man. You might say the "hand of God" wrote that number, and "we don't know how He pushed his pencil."*

At the cosmological level, there is another famous number

$$B = \frac{1}{\ln(\rho_P/\rho_\Lambda)} \simeq \frac{1}{123 \ln(10)} \simeq 3.53 \times 10^{-3}. \quad (\text{A2})$$

It can be seen as the inverse of the logarithm of the cosmological - or dark energy - density $\rho_\Lambda = \Lambda/8\pi G = 5.96 \times 10^{-24} \text{ g m}^{-3}$ (where $\Lambda = 1.00 \times 10^{-35} \text{ s}^{-2}$ is the cosmological constant), normalized by the Planck density $\rho_P = c^5/\hbar G^2 = 5.16 \times 10^{99} \text{ g m}^{-3}$. This number appeared in connection to the so-called cosmological constant problem [34, 35], i.e., the fact that there is a difference of 123 orders of magnitude ($\rho_P/\rho_\Lambda \sim 10^{123}$) between the Planck density ρ_P and the cosmological density ρ_Λ interpreted as the vacuum energy.

We have suggested in this paper that the two dimensionless constants α and B , or the two numbers 137 and 123, are related to each other [see Eqs. (33) and (34)] and that, in some sense, they correspond to the same thing. If this idea is correct, it would yield a fascinating connection between atomic and cosmic physics.

Appendix B: Surface density of the Universe, surface density of the electron and Weinberg's empirical relation

Using qualitative arguments, let us determine the surface density of the Universe. The Hubble time (\sim age of the Universe) is $t_\Lambda = 1/H_0 = 14.4$ billion years. The Hubble radius (\sim radius of the visible Universe) is $R_\Lambda = ct_\Lambda = c/H_0 = 1.37 \times 10^{26} \text{ m}$. The present density of the Universe is $\epsilon_0/c^2 = 3H_0^2/8\pi G = 8.62 \times 10^{-24} \text{ g m}^{-3}$. The Hubble mass (\sim mass of the Universe) is $M_\Lambda = (4/3)\pi(\epsilon_0/c^2)R_\Lambda^3 = c^3/2GH_0 = 9.20 \times 10^{55} \text{ g}$. Combining these relations, we find that the surface density of the Universe is

$$\Sigma_\Lambda = \frac{M_\Lambda}{4\pi R_\Lambda^2} = \frac{cH_0}{8\pi G} = 392 \text{ g m}^{-2} = 188 M_\odot/\text{pc}^2. \quad (\text{B1})$$

It can be written as $\Sigma_\Lambda = cH_0/\kappa c^4$ where $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant (which includes the 8π factor). Using Eq. (15), we obtain

$$\Sigma_\Lambda = \frac{1}{8\pi\sqrt{3\Omega_{\text{de},0}}} \frac{c\sqrt{\Lambda}}{G}. \quad (\text{B2})$$

This relation shows that the surface density of the Universe provides the correct scale for the surface density of dark matter halos [see Eq. (23)]. We have $\Sigma_\Lambda = \lambda\Sigma_0^{\text{th}}$ with $\lambda \simeq 1.41$.

Therefore, the surface density of the Universe is of the same order as the surface density of the dark matter halos which is also of the same order as the surface density of the electron (as we have previously observed). We have $\Sigma_\Lambda = \mu\Sigma_e$ with $\mu = \lambda/\sigma \simeq 3.42$. Matching Eqs. (29) and (B1), we get

$$m_e = \left(\frac{e^4 H_0}{8\pi\mu G c^3} \right)^{1/3}. \quad (\text{B3})$$

This relation expresses the mass of the electron as a function of its charge and the Hubble constant. This mysterious relation is mentioned in the book of Weinberg [11] where it is obtained from purely dimensional arguments.⁷ He observed that the term in the right hand side of Eq. (B3) has the dimension of a mass and that this mass, $1.37 \times 10^{-27} \text{ g}$ (with $\mu^{\text{app}} = 1$), is of the order of the mass of the electron. The fact that relation (B3) expresses the commensurability of the surface density of the Universe and the surface density of the electron, as we observe here, may help elucidating its physical meaning (see Appendix C 4).

Remark: If the dark matter halos resulted from the balance between the gravitational attraction and the repulsion due to the dark energy, they would have a typical density $M_h/r_h^3 \sim \rho_\Lambda$. Actually, such an equilibrium is unstable as is well-known in the case of the Einstein static Universe. Therefore, the radius of dark matter halos must satisfy the constraint $r_h < (M_h/\rho_\Lambda)^{1/3}$. Now, we have seen that their mass-radius relation scales as $M_h \sim (c\sqrt{\Lambda}/G)r_h^2$. We then find that the constraint $r_h < (M_h/\rho_\Lambda)^{1/3}$ is satisfied provided that $M_h < c^3/G\sqrt{\Lambda}$. Since the upper bound is of the order of the mass of the Universe, $M_\Lambda \sim c^3/G\sqrt{\Lambda}$, we conclude that the size of the dark matter halos is always much smaller than the critical size $(r_h)_{\text{crit}} = (M_h/\rho_\Lambda)^{1/3}$ as required for stability reasons.

Appendix C: Analogy with black hole thermodynamics

1. Black hole entropy

The Bekenstein-Hawking [9, 10] entropy of a Schwarzschild black hole is given by

$$S_{\text{BH}} = \frac{1}{4}k_B \frac{A}{l_P^2} = \frac{k_B \pi c^3 R^2}{G\hbar}, \quad (\text{C1})$$

⁷ Weinberg considers this relation as "so far unexplained" and having "a real though mysterious significance". Similar relations have been obtained in the past by Stewart [28], Eddington [32] and others from purely heuristic arguments or from dimensional analysis [6, 29, 36]. Their goal was to express the mass of the elementary particles in terms of the fundamental constants of Nature.

where $A = 4\pi R^2$ is the area of the event horizon of the black hole and $l_P = (G\hbar/c^3)^{1/2} = 1.62 \times 10^{35}$ m is the Planck length. The radius of a Schwarzschild black hole is connected to its mass by

$$R = \frac{2GM}{c^2}. \quad (\text{C2})$$

The Hawking temperature [10] of a Schwarzschild black hole is

$$k_B T = \frac{\hbar c^3}{8\pi GM} = \frac{\hbar c}{4\pi R}. \quad (\text{C3})$$

The black hole entropy (C1) can be obtained from the Hawking temperature (C3) by using the thermodynamic relation $T^{-1} = dS_{\text{BH}}/d(Mc^2)$. If we consider a Planck black hole of radius l_P and mass M_P , we find that its temperature is of the order of the Planck temperature $T_P = M_P c^2/k_B = 1.42 \times 10^{32}$ K and its entropy $S_{\text{BH}}/k_B \sim 1$.

2. Analogy between the Universe and a black hole

Using the results of Appendix B, we note that the radius of the Universe is related to its mass by

$$R_\Lambda = \frac{2GM_\Lambda}{c^2}. \quad (\text{C4})$$

This expression coincides with the mass-radius relation (C2) of a Schwarzschild black hole. This coincidence has sometimes led people to say that the Universe is a black hole, or that we live in a black hole, although this analogy is probably too naive. Nevertheless, at least on a purely dimensional basis, we can use the analogy with black holes to define the entropy and the temperature of the Universe. In this manner, we get a temperature scale (temperature on the horizon)

$$k_B T_\Lambda = \frac{\hbar c}{4\pi R_\Lambda} = \frac{\hbar H_0}{4\pi} \sim \hbar \sqrt{\Lambda}. \quad (\text{C5})$$

Its value is $T_\Lambda \sim 2.41 \times 10^{-29}$ K. The temperature can be written as

$$k_B T_\Lambda = \frac{2\hbar a_\Lambda}{c}, \quad (\text{C6})$$

where

$$a_\Lambda = G\Sigma_\Lambda = \frac{GM_\Lambda}{4\pi R_\Lambda^2} = \frac{c^2}{8\pi R_\Lambda} = \frac{cH_0}{8\pi} \sim c\sqrt{\Lambda} \quad (\text{C7})$$

is the surface gravity of the Universe (similar relations apply to black holes). We note that $\Sigma_\Lambda = c^2/8\pi GR_\Lambda$. We can also write

$$k_B T_\Lambda = m_\Lambda c^2, \quad (\text{C8})$$

with

$$m_\Lambda \sim \frac{\hbar\sqrt{\Lambda}}{c^2} = 2.08 \times 10^{-33} \text{ eV}/c^2. \quad (\text{C9})$$

This mass scale is often interpreted as the smallest mass of the bosons predicted by string theory [37] or as the upper bound on the mass of the graviton [38].⁸ It can be contrasted from the mass scale

$$M_\Lambda \sim \frac{c^3}{G\sqrt{\Lambda}} = 7.16 \times 10^{88} \text{ eV}/c^2, \quad (\text{C10})$$

which is usually interpreted as the mass of the Universe. Thus m_Λ and M_Λ represent fundamental lower and upper mass scales. Their ratio is

$$\frac{M_\Lambda}{m_\Lambda} \sim \frac{c^5}{G\hbar\Lambda} \sim \frac{\rho_P}{\rho_\Lambda} \sim e^{1/B} \sim 10^{123}, \quad (\text{C11})$$

which exhibits the famous number 123 (see Appendix A). On the other hand, our analogy between the Universe and a black hole leads to an entropy scale (entropy on the Hubble horizon):

$$S_\Lambda = \frac{k_B \pi c^3 R_\Lambda^2}{G\hbar} = \frac{k_B \pi c^5}{G\hbar H_0^2} \sim \frac{k_B c^5}{G\hbar\Lambda}. \quad (\text{C12})$$

We note that the entropy of the Universe can be written as

$$S_\Lambda/k_B \sim \frac{M_\Lambda}{m_\Lambda} \sim \frac{\Sigma_\Lambda R_\Lambda^2}{m_\Lambda} \sim e^{1/B} \sim 10^{123}. \quad (\text{C13})$$

This entropy may be identified with the total entropy of the logotropic dark fluid (see the Appendix of [3] and Appendix E). It can be compared to the entropy of radiation [39]:

$$\begin{aligned} S_{\text{rad}}/k_B &= \frac{4}{3} \left(\frac{3\Omega_{\text{rad},0}}{8\pi} \right)^{3/4} \left(\frac{\pi^2}{15} \right)^{1/4} \frac{1}{(H_0 t_P)^{3/2}} \\ &= 5.64 \times 10^{87}, \end{aligned} \quad (\text{C14})$$

obtained by using Eq. (E2) with $P_{\text{rad}} = \epsilon_{\text{rad}}/3$, $\epsilon_{\text{rad}} = \sigma T^4$ with $\sigma = \pi^2 k_B^4/15c^3\hbar^3$ (Stefan-Boltzmann constant), $\epsilon_{\text{rad}} = \Omega_{\text{rad},0}\epsilon_0/a^4$ and $\Omega_{\text{rad},0} = 9.24 \times 10^{-5}$. They differ by about 36 orders of magnitude.

Remarks: We note that $T_\Lambda S_\Lambda = (1/2)M_\Lambda c^2$ so the free energy of the Universe is $F_\Lambda = M_\Lambda c^2 - T_\Lambda S_\Lambda = (1/2)M_\Lambda c^2$. On the other hand, using Eqs. (18) and (39), we obtain the relations

$$\frac{m_\Lambda}{M_P} \sim e^{-1/(2B)} = 3.40 \times 10^{-62}, \quad (\text{C15})$$

$$\frac{m_e}{m_\Lambda} \sim \sqrt{B} e^{1/(3B)} = 5.66 \times 10^{39}. \quad (\text{C16})$$

Since $m_\Lambda \sim \rho_\Lambda r_e^3$ (see Sec. IV B) we have $m_e/m_\Lambda \sim \rho_e/\rho_\Lambda$. The gravitational radius of the cosmon is $r_\Lambda = 2Gm_\Lambda/c^2 \sim G\hbar\sqrt{\Lambda}/c^4 = 2.75 \times 10^{-96}$ m.

⁸ It is simply obtained by equating the Compton wavelength of the particle $\lambda_c = \hbar/mc$ with the Hubble radius $R_\Lambda = c/H_0$ (the typical size of the visible Universe) giving $m_\Lambda = \hbar H_0/c^2$. Using Eq. (15), we obtain Eq. (C9). By comparison, if we identify the Compton wavelength $\lambda_c = \hbar/mc$ with the Schwarzschild radius $r_S \sim Gm/c^2$ we get the Planck mass $M_P = (\hbar c/G)^{1/2}$.

3. Entropy of logotropic dark matter halos

Let us define the entropy of a logotropic dark matter halo by

$$S \sim k_B N \sim k_B \frac{M_h}{m_\Lambda}, \quad (\text{C17})$$

where M_h is the halo mass and m_Λ is the mass of the hypothetical particle composing the logotropic dark fluid. Using the mass-radius relation $M_h \sim \Sigma_0 r_h^2$ of a logotropic dark matter halo, where $\Sigma_0 \sim c\sqrt{\Lambda}/G$ is the universal surface density given by Eq. (23), we get

$$S \sim \frac{k_B c \sqrt{\Lambda} r_h^2}{G m_\Lambda}. \quad (\text{C18})$$

Interestingly, the entropy given by Eq. (C18) scales like the surface r_h^2 of the object, similarly to the black hole entropy (C1).⁹ This may be connected to a form of holographic principle [8]. Matching the formulae (C1) and (C18), we find that m_Λ corresponds to the mass given by Eq. (C9). Inversely, if we assume from the start that the logotropic dark fluid is composed of particles of that mass (cosmons), we find that the entropy of dark matter halos coincides with the entropy of black holes.¹⁰ On the other hand, since the surface density of the Universe is of the same order as the surface density of dark matter halos, the previous formulae also apply to the Universe as a whole and return the results of Appendix C 2. This may be a form of justification, for reasons of self-consistency, of Eq. (C17).

Remark: If we alternatively define the entropy of dark matter halos by $S \sim k_B M_h / m_e$ where m_e is the electron mass and use $M_h \sim \Sigma_0 r_h^2$ with $\Sigma_0 \sim \Sigma_e$, where Σ_e is the surface density of the electron given by Eq. (29), we obtain

$$S \sim k_B \frac{r_h^2}{r_e^2}, \quad (\text{C19})$$

which is similar to the black hole entropy formula (C1) where the Planck length l_P is replaced by the classical radius of the electron r_e . It is not clear, however, if this formula is physically relevant.

4. Postulates: entropic principles

We can find a form of explanation of the different relations found in this paper by making the following two

⁹ Inversely, a manner to understand why the surface density of the dark matter halos has a universal value is to argue that their entropy given by Eq. (C17) should scale like r_h^2 (see Appendix C 4).

¹⁰ Of course, we are not claiming that dark matter halos are black holes since they obviously do not fulfill the Schwarzschild relation (C2). However, they may have the same entropy as black holes expressed in terms of r_h [see Eq. (C1)].

postulates.

Postulate 1: We postulate that the entropy of the electron, the entropy of dark matter halos and the entropy of the Universe (and possibly other objects) is given by

$$S \sim \frac{k_B c^3 R^2}{G \hbar}, \quad (\text{C20})$$

like the Bekenstein-Hawking [9, 10] entropy of black holes (C1), where R is the radius of the corresponding object. This may be connected to a form of holographic principle [8] stating that the entropy is proportional to the area (instead of the volume). Therefore,

$$S_e/k_B \sim \frac{c^3 r_e^2}{G \hbar} \quad (\text{electron}) \quad (\text{C21})$$

$$S/k_B \sim \frac{c^3 r_h^2}{G \hbar} \quad (\text{dark matter}) \quad (\text{C22})$$

$$S_\Lambda/k_B \sim \frac{c^3 R_\Lambda^2}{G \hbar} \quad (\text{Universe}) \quad (\text{C23})$$

Postulate 2: We postulate that the entropy of the electron, the entropy of dark matter halos and the entropy of the Universe (and possibly other objects) is also given by¹¹

$$S \sim k_B \frac{M}{m_\Lambda}, \quad (\text{C24})$$

where M is the mass of the corresponding object and m_Λ is the mass defined by Eq. (C9). Therefore,

$$S_e/k_B \sim \frac{m_e}{m_\Lambda} \sim \frac{\Sigma_e r_e^2}{m_\Lambda} \sim 10^{39} \quad (\text{electron}) \quad (\text{C25})$$

$$S/k_B \sim \frac{M_h}{m_\Lambda} \sim \frac{\Sigma_0 r_h^2}{m_\Lambda} \quad (\text{dark matter}) \quad (\text{C26})$$

$$S_\Lambda/k_B \sim \frac{M_\Lambda}{m_\Lambda} \sim \frac{\Sigma_\Lambda R_\Lambda^2}{m_\Lambda} \sim 10^{123} \quad (\text{Universe}) \quad (\text{C27})$$

The comparison of Eqs. (C20) and (C24) directly implies that the surface density of the electron, the surface density of *all* the dark matter halos, and the surface density of the Universe is (approximately) the same and has the typical value

$$\Sigma \sim \frac{M}{R^2} \sim \frac{m_\Lambda c^3}{G \hbar} \sim \frac{m_\Lambda}{M_P} \Sigma_P \sim \frac{c \sqrt{\Lambda}}{G}, \quad (\text{C28})$$

¹¹ Note that this relation is *not* satisfied by black holes since $M_{\text{BH}} \propto R$ while $S_{\text{BH}} \propto R^2$.

where $\Sigma_P = (c^7/\hbar G^3)^{1/2} = 8.33 \times 10^{64} \text{ g m}^{-2}$ is the Planck surface density. Then, comparing this universal value with the surface density of the electron [see Eq. (29)], we obtain the Weinberg relation

$$\Lambda \sim \frac{m_e^6 G^2 c^6}{e^8}. \quad (\text{C29})$$

Remark: we have introduced the entropy of an electron [see Eqs. (C21) and (C25)] by analogy with the black hole entropy. If these ideas are physically relevant, a notion of thermodynamics for the electron (assuming that it is made of 10^{39} subparticles of mass m_Λ) should be developed. Again, the analogy with black holes (although, of course, an electron is not a black hole) might be useful. In particular, from the entropy $S_e \sim k_B m_e/m_\Lambda$, we may define the temperature of the electron by $1/T_e = dS_e/d(m_e c^2)$ and get $k_B T_e \sim m_\Lambda c^2 \sim \hbar \sqrt{\Lambda} \sim k_B T_\Lambda$. In this sense, the temperature of the electron turns out to coincide with the temperature of the Universe. The same argument applies to dark matter halos (with $S \sim k_B M_h/m_\Lambda$ and $1/T = dS/d(M_h c^2)$) leading again to $T \sim T_\Lambda$. Interestingly, similar relations are obtained in Appendix E from different arguments.

Appendix D: Large numbers and coincidences

The ratio between the electric radius of the electron $r_e = e^2/m_e c^2$ and its gravitational radius $r_g = 2Gm_e/c^2$ is of the order of $e^2/Gm_e^2 = 4.17 \times 10^{42}$. This dimensionless number was computed by Weyl in 1919 [6, 40]. He was the first to notice the presence of large dimensionless numbers in Nature. This led Eddington [32] and others to try to relate such large numbers to cosmological quantities. In particular, Eddington evaluated the total number of particles in the Universe and found $N_p \sim 10^{79}$. He then tried to relate the basic interaction strengths and elementary particle masses to this number. For example, it was observed by different authors that the following quantities are of the same order of magnitude (see Ref. [6], P. 224-231):

$$\frac{m_e c^2}{\hbar H_0} \sim \frac{e^2}{Gm_e^2} \sim \sqrt{N_p} \sim \left(\frac{M_P}{m_e}\right)^2 \sim 10^{40}. \quad (\text{D1})$$

These coincidences can be easily understood from our results (39) and (40) which express the mass and the charge of the electron in terms of the cosmological constant. For convenience, we shall replace the Eddington

number by¹²

$$N_e = \frac{M_\Lambda}{m_e} \sim e^{2/(3B)} \sim 10^{80}. \quad (\text{D2})$$

Combining our results, we find

$$\frac{m_e c^2}{\hbar H_0} \sim \frac{m_e}{m_\Lambda} \sim e^{1/(3B)} \sim 10^{40}, \quad (\text{D3})$$

$$\frac{e^2}{Gm_e^2} \sim e^{1/(3B)} \sim 10^{40}, \quad (\text{D4})$$

$$\frac{M_P}{m_e} \sim e^{1/(6B)} \sim 10^{20}, \quad (\text{D5})$$

leading to the relations from Eq. (D1). We also note that

$$\frac{1}{m_e^4} \left(\frac{\hbar c}{G}\right)^2 \sim \left(\frac{M_P}{m_e}\right)^4 \sim 10^{80} \sim N_e, \quad (\text{D6})$$

which is one of the ‘‘coincidences’’ pointed out by Chandrasekhar [41].

In a sense, these results arise from the Weinberg relation (B3) that has been found by different authors (see footnote 7). Nevertheless we believe that our approach is original and may bring new light on the subject. In particular, we have proposed a form of common explanation of these different ‘‘coincidences’’ in terms of entropic principles (see Appendix C 4).

Appendix E: Thermodynamics of the logotropic dark fluid

Let us try to relate the results of the previous Appendices to the thermodynamics of the logotropic dark fluid.

We assume that the Universe is filled with a dark fluid at temperature T . From the first principle of thermodynamics, one can derive the thermodynamic equation [11]:

$$\frac{dP}{dT} = \frac{1}{T}(\epsilon + P). \quad (\text{E1})$$

If the dark fluid is described by a barotropic equation of state of the form $P = P(\epsilon)$, Eq. (E1) can be integrated

¹² The Eddington number corresponds typically to the number of protons in the Universe, $N \sim M_\Lambda/m_p$, where m_p is the proton mass (in Eq. (D2) we have replaced m_p by m_e since we have not introduced the proton mass previously). Note that the Eddington number was introduced before dark matter and dark energy were discovered. If the dark fluid is made of cosmons of mass m_Λ , the number of particles in the Universe is $N_\Lambda = M_\Lambda/m_\Lambda \sim 10^{123}$ giving another interpretation to the famous number 123 (see Appendix A). This number N_Λ should supersede the Eddington number N_p .

to obtain the relation $T = T(\epsilon)$ between the temperature and the energy density. On the other hand, the entropy of the dark fluid in a volume a^3 is given by [11]

$$S = \frac{a^3}{T}(P + \epsilon). \quad (\text{E2})$$

From the Friedmann equations, one can show that the entropy of the Universe is conserved: $dS/dt = 0$ [11].

The previous results are general. Let us now apply them to the logotropic dark fluid. According to Eqs. (5) and (6), the equation of state $P = P(\epsilon)$ of the logotropic dark fluid is given by the inverse of [2, 3]

$$\epsilon = \rho_P e^{P/A} c^2 - P - A. \quad (\text{E3})$$

Eq. (E1) with Eq. (E3) can be easily integrated to give

$$T = \frac{\rho_P c^2}{K} \left(1 - \frac{A}{\rho c^2} \right), \quad (\text{E4})$$

where K is a constant of integration and we have used Eq. (5). Substituting Eqs. (E3) and (E4) into Eq. (E2), and using Eqs. (3) and (5), we find that

$$S = K \frac{\rho_0}{\rho_P}. \quad (\text{E5})$$

We explicitly check on this expression that the entropy of the Universe is conserved. Furthermore, since the entropy is positive, we must have $K > 0$. Considering Eq. (E4), we note that the temperature is positive when $\rho > \rho_M = A/c^2$ and negative when $\rho < \rho_M = A/c^2$, that is to say when the Universe becomes phantom [1, 2].¹³

We can determine the constant K by assuming that the entropy of the logotropic dark fluid is given by

$$S \sim k_B \frac{M_\Lambda}{m_\Lambda} \sim 10^{123} k_B \quad (\text{E6})$$

as in Appendix C. Noting that the “true” entropy is obtained by multiplying Eq. (E2) by R_Λ^3 (since we have taken $a_0 = 1$ at the present time), and comparing Eqs. (E5) and (E6), we obtain

$$K \sim k_B \frac{\rho_P}{m_\Lambda}. \quad (\text{E7})$$

As a result, the temperature of the logotropic dark fluid is given by

$$k_B T \sim m_\Lambda c^2 \left(1 - \frac{B \rho_\Lambda}{\rho} \right), \quad (\text{E8})$$

¹³ This is a general result [42] which can be obtained from Eq. (E2) using the fact that the entropy is constant and positive. We see on Eq. (E2) that the sign of the temperature coincides with the sign of $P + \epsilon$. As a result, the temperature is positive in a normal Universe ($P > -\epsilon$) and negative in a phantom Universe ($P < -\epsilon$).

where we have used Eq. (12). In the “early” Universe $\rho \gg \rho_\Lambda$ we find that¹⁴

$$T \simeq m_\Lambda c^2 / k_B = 2.41 \times 10^{-29} \text{ K}. \quad (\text{E9})$$

In the late Universe $\rho \ll \rho_\Lambda$ we find that

$$k_B T \sim -m_\Lambda c^2 B \rho_\Lambda / \rho \propto -a^3. \quad (\text{E10})$$

Remark: In Ref. [3] we have shown that the logotropic constant B could be interpreted as a dimensionless logotropic temperature

$$B = \frac{k_B T_L}{m_\Lambda c^2} \quad (\text{E11})$$

in a generalized thermodynamical framework [1, 2]. This shows that at least two temperatures exist for the logotropic dark fluid, a time-varying temperature T and a constant temperature T_L . They become equal when

$$\frac{\rho_*}{\rho_\Lambda} \sim \frac{B}{1-B} \sim 3.54 \times 10^{-3}, \quad (\text{E12})$$

corresponding to

$$a_* \sim \left(\frac{\Omega_{m,0}}{\Omega_{de,0}} \frac{1-B}{B} \right)^{1/3} \sim 5.01. \quad (\text{E13})$$

Appendix F: The mass of the bosonic dark matter particle

It has been suggested that dark matter may be made of bosons (like ultralight axions) in the form of Bose-Einstein condensates (BECs).¹⁵ We can use the results of the present paper to predict the mass m of the bosonic dark matter particle in terms of the cosmological constant Λ . We assume that the smallest and most compact dark matter halo that is observed corresponds to the ground state of a self-gravitating BEC (to fix the ideas we assume that this halo is the dSphs Fornax with a mass $M \sim 10^8 M_\odot$ and a radius $R \sim 1$ kpc). For non-interacting bosons, it can be shown by solving the Gross-Pitaevskii-Poisson equations (see, e.g., Sec. III.B.1 of

¹⁴ We recall that the logotropic model, which is a unification of dark matter and dark energy, is not valid in the very early Universe corresponding to the big bang, the inflation era, and the radiation era. Therefore, the temperature $m_\Lambda c^2$ corresponds to the temperature of the dark fluid in the matter era, i.e., when the rest-mass energy of the dark fluid overcomes its internal energy (see Sec. II). We emphasize that the temperature T of the logotropic dark fluid is different from the temperature of radiation and of any other standard temperature. We also note that the corresponding temperature in the Λ CDM model is not defined since Eq. (E1) breaks down when $P = 0$.

¹⁵ See, e.g., the bibliography of Refs. [43, 44] for an exhaustive list of references. The possible connections between the BECDM model and the logotropic model will be investigated in a future paper [45].

[46]) that the mass $(M_h)_{\min}$, the radius $(r_h)_{\min}$ and the central density $(\rho_0)_{\max}$ of this ultracompact halo (ground state) are related to each other by the relations

$$M_h = 1.91 \rho_0 r_h^3 \quad \text{and} \quad M_h r_h = 1.85 \frac{\hbar^2}{G m^2}. \quad (\text{F1})$$

As a result, its surface density is given by

$$\Sigma_0 = 0.153 \frac{G^2 m^4 M_h^3}{\hbar^4}. \quad (\text{F2})$$

On the other hand, the minimum mass of dark matter halos may be obtained from a quantum Jeans instability theory (see, e.g., Ref. [44]) giving the result

$$M_J = \frac{1}{6} \pi \left(\frac{\pi^3 \hbar^2 \rho_{\text{dm},0}^{1/3}}{G m^2} \right)^{3/4}. \quad (\text{F3})$$

For usually considered values of the boson mass, of the order of $m \sim 10^{-22}$ eV/c², the Jeans mass $M_J \sim 10^7 M_\odot$ from Eq. (F3) is indeed of the order of magnitude of the minimum mass $(M_h)_{\min} \sim 10^8 M_\odot$ of observed dark matter halos. There may be, however, a numerical factor of order 10 between M_J and $(M_h)_{\min}$. For that reason, we write $(M_h)_{\min} = \chi M_J$ with $\chi \sim 10$. Using Eq. (F3) with $\rho_{\text{dm},0} = (\Omega_{\text{dm},0}/\Omega_{\text{de},0})\rho_\Lambda = (\Omega_{\text{dm},0}/\Omega_{\text{de},0})(\Lambda/8\pi G)$, we get

$$(M_h)_{\min} = \chi \frac{\pi^3}{6} \left(\frac{\Omega_{\text{dm},0}}{8\Omega_{\text{de},0}} \right)^{1/4} \frac{\hbar^{3/2} \Lambda^{1/4}}{G m^{3/2}}. \quad (\text{F4})$$

Substituting this expression into Eq. (F2), we obtain

$$\Sigma_0 = 0.153 \chi^3 \frac{\pi^9}{216} \left(\frac{\Omega_{\text{dm},0}}{8\Omega_{\text{de},0}} \right)^{3/4} \frac{\hbar^{1/2} \Lambda^{3/4}}{G m^{1/2}}. \quad (\text{F5})$$

Comparing this expression with Eq. (23), we predict that the mass of the bosonic particle is given by

$$m = \chi^6 \frac{0.0234 \pi^{20}}{1458 B \xi_h^2} \left(\frac{\Omega_{\text{dm},0}}{8\Omega_{\text{de},0}} \right)^{3/2} \frac{\hbar \sqrt{\Lambda}}{c^2} = 15397 \chi^6 \frac{\hbar \sqrt{\Lambda}}{c^2}. \quad (\text{F6})$$

We see that the mass of the bosonic dark matter particle is equal to the mass scale $m_\Lambda \sim 10^{-33}$ eV/c² given by Eq. (C9) multiplied by a huge numerical factor of order 10^{10} (for $\chi \sim 10$). This gives $m \sim 10^{-23}$ eV/c² which is the correct order of magnitude of the mass of ultralight axions usually advocated [47]. We note that this result has been obtained independently from the observations, except for the value of Λ and the other fundamental constants (Planck scales).

Appendix G: The proton mass

The maximum mass of neutron stars due to general relativity has been obtained by Oppenheimer and Volkoff

[48] using the equation of state of an ideal Fermi gas at $T = 0$. It is given by

$$M_{\text{max}}^{\text{OV}} = 0.384 \left(\frac{\hbar c}{G} \right)^{3/2} \frac{1}{m_n^2} = 0.710 M_\odot, \quad (\text{G1})$$

where $m_n = 1.675 \times 10^{-24}$ g is the neutron mass. It is not well-known¹⁶ that, at the same period, Zwicky [50, 51] also attempted to determine the maximum mass of neutron stars interpreted as Schwarzschild's mass. He used heuristic arguments and obtained an expression of the form

$$M_{\text{max}}^{\text{Zwicky}} = k m_n \left(\frac{e^2}{G m_p m_e} \right)^{3/2} = 91 k M_\odot, \quad (\text{G2})$$

where $m_p = 1.673 \times 10^{-24}$ g is the proton mass and k is a dimensionless number assumed to be of order unity. Matching these two expressions (G1) and (G2), we get $k = 7.80 \times 10^{-3}$ and

$$\alpha = \frac{e^2}{\hbar c} = 13.4 \frac{m_p m_e}{m_n^2} \simeq 13.4 \frac{m_e}{m_p}. \quad (\text{G3})$$

Interestingly, this equation provides a relation between the fine-structure constant α and the ratio m_e/m_p between the electron mass and the proton mass (we have used $m_n \simeq m_p$). This type of relationships has been proposed in the past by several authors [29] using heuristic arguments or pure numerology. Since we have related the electron mass and the electron charge (hence the fine-structure constant) to the cosmological constant (see Sec. IV C), using Eq. (G3) we can also relate the proton mass to the cosmological constant. Using Eqs. (37), (38) and (G3), we find that the mass of the proton is determined by the cosmological constant Λ according to

$$m_p = \frac{13.4}{\nu^{1/6}} \left(\frac{\Lambda \hbar^4}{G^2 c^2} \right)^{1/6} = 13.9 \left(\frac{\Lambda \hbar^4}{G^2 c^2} \right)^{1/6} = 1.67 \times 10^{-24} \text{ g}. \quad (\text{G4})$$

Alternatively, using Eqs. (39), (40) and (G3), the normalized mass of the proton is determined by the logarithmic constant B according to

$$\begin{aligned} \frac{m_p}{M_P} &= 13.4 \left(\frac{8\pi}{\nu} \right)^{1/6} e^{-1/(6B)} \\ &= 23.7 e^{-1/(6B)} = 7.68 \times 10^{-20}. \end{aligned} \quad (\text{G5})$$

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